Research Article

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Multiplicity of solutions for a class of critical Schrödinger-Poisson systems on the Heisenberg group

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Abstract: We deal with multiplicity of solutions to the following Schrödinger-Poisson-type system in this article:

$$\begin{cases}\Delta_H u - \mu_1\phi_1 u = |u|^2 u + F_u(\xi, u, v), & \text{in } \Omega,\\ -\Delta_H v + \mu_2\phi_2 v = |v|^2 v + F_v(\xi, u, v), & \text{in } \Omega,\\ -\Delta_H\phi_1 = u^2, \quad -\Delta_H\phi_2 = v^2, & \text{in } \Omega,\\ \phi_1 = \phi_2 = u = v = 0, & \text{on } \partial\Omega,\end{cases}$$

where Δ_H is the Kohn-Laplacian and Ω is a smooth bounded region on the first Heisenberg group $\mathbb{H}^1$, μ_1, and μ_2 are some real parameters, and $F = F(x, u, v)$, $F_u = \frac{\partial F}{\partial u}$, $F_v = \frac{\partial F}{\partial u}$ satisfying natural growth conditions. By the limit index theory and the concentration compactness principles, we prove that the aforementioned system has multiplicity of solutions for $\mu_1, \mu_2 < |\Omega|^{-\frac{1}{2}}S$, where S is the best Sobolev constant. The novelties of this article are the presence of critical nonlinear term, and the system is set on the Heisenberg group.

Keywords: Schrödinger-Poisson-type system, Heisenberg group, limit index, variational methods, concentration compactness principles

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1 Introduction and main results

In this article, we are interested in the existence of multiplicity of solutions for the following Schrödinger-Poisson-type system on the Heisenberg group of the form:

$$\begin{cases}\Delta_H u - \mu_1\phi_1 u = |u|^2 u + F_u(\xi, u, v), & \text{in } \Omega,\\ -\Delta_H v + \mu_2\phi_2 v = |v|^2 v + F_v(\xi, u, v), & \text{in } \Omega,\\ -\Delta_H\phi_1 = u^2, \quad -\Delta_H\phi_2 = v^2, & \text{in } \Omega,\\ \phi_1 = \phi_2 = u = v = 0, & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{H}^1$ is a smooth bounded domain, $\mu_1, \mu_2 \in \mathbb{R}$ is a real parameter, $Q = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, and in the case under consideration $n = 1$ and $Q = 4$, $Q^* = 2Q/(Q - 2) = 4$ is the Sobolev critical exponent. Problem (1.1) can be thought of a very degenerate elliptic system in $\mathbb{R}^3$ or a Schrödinger-

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Poisson system with the Kohn-Laplacian Δ_H. As for the function F, we assume $F \in C^1(\Omega \times \mathbb{H}^1)$ and satisfies the following assumptions:

(F_1) $F \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R}^+)$ and $F(\xi, s, t) = F(\xi, -s, -t)$ for all $(\xi, s, t) \in \Omega \times \in\mathbb{R}^2$;

(F_2) $\lim_{|t|\to\infty} \frac{F_t(\xi, s, t)}{|t|} = 0$ uniformly for $\xi \in \Omega$ and $s \in \mathbb{R}$;

(F_3) $sF_s(\xi, s, t) \geq 0$ for all $(\xi, s, t) \in \overline{\Omega} \times \mathbb{R}^2$.

The condition (F_2) implies that

$$F_v(\xi, u, v)v = o(|v|^2).$$

Thus, for any $\varepsilon > 0$ and fixed u, there exist $a(\varepsilon), b(\varepsilon) > 0$ such that

$$|F(\xi, 0, v)| \leq a(\varepsilon) + \varepsilon\, |v|^2 \tag{1.2}$$

and

$$|F_v(\xi, u, v)v| \leq b(\varepsilon) + \varepsilon\, |v|^2. \tag{1.3}$$

Hence, by (1.2) and (1.3), we have

$$|F(\xi, 0, v) - \beta F_v(\xi, u, v)v| \leq c(\varepsilon) + \varepsilon\, |v|^2 \tag{1.4}$$

for any constants β, fixed u, and $c(\varepsilon) > 0$.

Furthermore, we assume that $F(\xi, u, v)$ also fulfills the following assumption:

(F_4) There exist $2 < p < 4$, $\sigma > 0$, and $\tau \geq 0$ such that $F(\xi, s, t) \geq \sigma\, |t|^p - \tau$ for all $(\xi, s, t) \in \overline{\Omega} \times \mathbb{R}^2$.

The novelty of our work is that we link several different phenomena to one problem. The characteristics of this article are as follows:

(1) We emphasize that no results are available in the current literature for the critical Schrödinger-Poisson system on the Heisenberg group. In this regard, the results demonstrated in this article are completely new.
(2) The difficulty in solving Problem (1.1) is the lack of compactness, which can be illustrated by the fact that the embedding of $S_0^1(\Omega)$ into $L^{Q^*}(\Omega)$ is no longer compact.

Before describing the main results of this article, we give some concepts about the Heisenberg group. Let $\mathbb{H}^1 = (\mathbb{R}^3, \circ)$ be the first Heisenberg group. If $\xi = (x, y, t) \in \mathbb{H}^1$, then the group law is defined by:

$$\xi \circ \xi' = \tau_\xi(\xi') = (x + x', y + y', t + t' + 2(x'y - y'x)), \quad \forall \xi, \xi' \in \mathbb{H}^1.$$

A natural group of dilations on $\mathbb{H}^1$ is given by $\delta_s(\xi) = (sx, sy, s^2 t)$ for any positive number s. Hence, $\delta_s(\xi_0 \circ \xi) = \delta_s(\xi_0) \circ \delta_s(\xi)$ and the number $Q = 4$ is the homogeneous dimension of $\mathbb{H}^1$. The gauge norm $|\cdot|_H$ in $\mathbb{H}^1$ is defined as:

$$|\xi|_H = [(x^2 + y^2)^2 + t^2]^{\frac{1}{4}}$$

for any $\xi \in \mathbb{H}^1$. The norm $|\cdot|_H$ is homogeneous of degree one with respect to the dilation δ_λ. The left-invariant distance d_H on $\mathbb{H}^1$ is accordingly defined by:

$$d_H(\xi_0, \xi) = |\xi^{-1} \circ \xi_0|_H,$$

where $\xi^{-1} = -\xi$. Also, the Heisenberg ball of radius r centered at ξ_0 is the set

$$B_H(\xi_0, r) = \{\xi \in \mathbb{H}^1 : d_H(\xi_0, \xi) < r\}.$$

The natural volume in $\mathbb{H}^1$ is the Haar measure, which coincides with the Lebesgue measure L^3 in $\mathbb{R}^3$; then, $|B_H(\xi_0, r)| = \alpha_Q r^Q$, where $\alpha_Q = |B_H(0, 1)|$. A basis for the Lie algebra of left-invariant vector fields on $\mathbb{H}^1$ is given by:

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t} \quad \text{and} \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t},$$

and the horizontal gradient $\nabla_H = (X, Y)$. Let $\mathrm{div}_H(v_1, v_2) = Xv_1 + Yv_2$ for any vector-valued function (v_1, v_2). Then, the Kohn-Laplacian Δ_H is denoted by $\Delta_H u = \mathrm{div}_H(\nabla_H u)$. It is well known that Δ_H is a very degenerate elliptic operator and Bony's maximum principle is satisfied (see [1]). For more detailed settings about the Heisenberg group, we can refer to [2].

In recent years, the geometric analysis of the Heisenberg group has attracted much attention of many scholars due to its important applications in quantum mechanics, partial differential equations, and other fields. The analysis of the Heisenberg group is very interesting because this space is topologically Euclidean, but analytically non-Euclidean, so there are some basic analytical ideas. The Schrödinger-Poisson system is a standard model in quantum mechanics to describe electron motion on a positive charge background (see [3,4]). The investigation of Problem (1.1) is motivated by the existence of several recent mathematical research. To be more precise, in the Euclidean case, many scholars have studied the following Schrödinger-Poisson system:

$$\begin{cases} -\Delta u + \varepsilon q\phi f(u) = \eta\ |u|^{p-2}u, & \text{in } D, \\ -\Delta\phi = 2qF(u), & \text{in } D, \\ \phi = u = 0, & \text{on } \partial D, \end{cases} \tag{1.5}$$

where $D \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial D, 1 < p < 5$, $q > 0, \varepsilon, \eta = \pm 1, f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and $F(t) = \int_0^t f(s)\mathrm{d}s$. If $f(s) = s$, many researchers have come up with interesting results (see [5–8]). In particular, Azzollini et al. in [9] proved the existence and nonexistence results of Problem (1.5) when f is subcritical and critical. And just recently, Lei and Suo [10] obtained two positive solutions for the following system:

$$\begin{cases} -\Delta u + \lambda\phi u = \lambda|u|^{q-2}u + |u|^4 u, & \text{in } D, \\ -\Delta\phi = u^2, & \text{in } D, \\ \phi = u = 0, & \text{on } \partial D, \end{cases}$$

where $q \in (1, 2)$ and $\lambda \in \mathbb{R}^+$ is small enough. In addition, the Schrödinger-Poisson systems with critical growth on $\mathbb{R}^3$ were also investigated extensively and we refer the readers to [11–13]. In [14], the authors studied the following Schrödinger-Poisson-type system

$$\begin{cases} -\Delta_H u + \mu\phi u = \lambda|u|^{q-2}u + |u|^2 u, & \text{in } \Omega, \\ -\Delta_H\phi = u^2, & \text{in } \Omega, \\ u = \phi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where $1 < q < 2$, by the Green's representation formula and the critical point theory, they obtained at least two positive solutions and a positive ground-state solution.

As we know, the limit index theory proposed by Li [15] is one of the most effective ways to study the existence of infinite solutions of equations in a Euclidean setting. For example, Song and Shi [16] considered the noncooperative critical nonlocal system with the limit index theory. Baldelli et al. [17] proved existence results in $\mathbb{R}^N$ for an elliptic system of (p, q)-Laplacian type involving a critical term, nonnegative weights, and a positive parameter λ. Not long after, they studied elliptic systems of (p, q)-Laplacian involving a critical term and a subcritical term in [18]. In particular, nonnegative nontrivial weights satisfying some symmetry conditions with respect to a certain group are included in the nonlinearity.

Inspired by the aforementioned literatures, this article mainly studies the existence of multiplicity solutions for Problem (1.1). To the best of our knowledge, this article first deals with this kind of Schrödinger-Poisson system with the Kohn-Laplacian. Furthermore, although some properties are similar between Kohn-Laplacian Δ_H and the classical Laplacian Δ, the similarities may be deceitful (see, e.g., [19]). In addition, the critical exponent Q^* is equal to 4 on $\mathbb{H}^1$, while 2^* is equal to 6 on $\mathbb{R}^3$, which has created us some obstacles in proving the existence of solutions to Problem (1.1). In order to overcome these difficulties, we will use concentration compactness principles on the Heisenberg group.

Definition 1.1. We say that $(u, v) \in S_0^1(\Omega) \times S_0^1(\Omega)$ is a (weak) solution of Problem (1.1), if

$$
\begin{aligned}
&-\int_\Omega \nabla_H u \nabla_H \tilde{u} d\xi - \mu_1 \int_\Omega \phi_{1,u} u\tilde{u} d\xi + \int_\Omega \nabla_H v \nabla_H \tilde{v} d\xi + \mu_2 \int_\Omega \phi_{2,v} v\tilde{v} d\xi \\
&= \int_\Omega |u|^2 u\tilde{u} d\xi + \int_\Omega |v^2| v\tilde{v} d\xi + \int_\Omega F_u(\xi, u, v)\tilde{u} d\xi + \int_\Omega F_v(\xi, u, v)\tilde{v} d\xi
\end{aligned}
$$

for any $(\tilde{u}, \tilde{v}) \in S_0^1(\Omega) \times S_0^1(\Omega)$.

The main result of Problem (1.1) is as follows.

Theorem 1.1. Let $(F_1) - (F_4)$ *hold. Then, there exists* $k_0 > 1$ *such that Problem* (1.1) *possesses at least* $k_0 - 1$ *pair nontrivial solutions in* $S_0^1(\Omega) \times S_0^1(\Omega)$.

Remark 1.1. Let $E = S_0^1(\Omega)$, $X = E \times E$, $\|u\|_{S_0^1(\Omega)} = (\int_\Omega |\nabla_n u|^2 d\xi)^{\frac{1}{2}}$ with the norm $\|(u, v)\| := (\|u\|^2 + \|v\|^2)^{\frac{1}{2}}$, $|\cdot|_p$ be the usual norm in $L^p(\Omega)$, $L_2^p(\Omega) = L^p(\Omega) \times L^p(\Omega)$ with the norm $|(u, v)|_p := (|u|_p^p + |v|_p^p)^{\frac{1}{p}}$. Weak (resp. strong) convergence is denoted by $\rightharpoonup$ (resp., $\rightarrow$), and c_i is a positive constant and can be determined in concrete conditions.

This article is organized as follows. In Section 2, we present some necessary preliminary knowledge on the Heisenberg group functional setting and collect some properties about the Folland-Stein space $S_0^1(\Omega)$. In Section 3, we prove some preliminary lemmas and Theorem 1.1. Section 4 is devoted to the proofs of Theorem 1.1.

2 Sobolev spaces and limit index theory

This section will be divided into two parts. First, we briefly review the definitions and list some basic properties of Sobolev spaces. Second, we recall the limit index theory of Li [15].

2.1 Sobolev spaces

In this section, we first give some basic results for our space $S_0^1(\Omega)$ which will be used later. The Folland-Stein space $S_0^1(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm:

$$\|u\|^2_{S_0^1(\Omega)} = \int_\Omega |\nabla_H u|^2 d\xi.$$

For simplicity's sake, we use the notation $\|u\| = \|u\|_{S_0^1(\Omega)}$ and $\|\cdot\|_p$ denotes the usual L^p-norm, that is,

$$\|u\|_p^p = \int_\Omega |u|^p d\xi \quad \text{for all } u \in L^p(\Omega).$$

By [20], the Folland-Stein space $(S_0^1(\Omega), \|\cdot\|)$ is a Hilbert space and the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact when $1 \le p < Q^*$, while it is only continuous if $p = Q^*$. In particular, Jerison and Lee [21] proved that the best Sobolev constant

$$S = \inf_{u \in S_0^1(\mathbb{H}^1)} \frac{\int_{\mathbb{H}^1} |\nabla_H u|^2 d\xi}{\left(\int_{\mathbb{H}^1} |u|^{Q^*} d\xi\right)^{\frac{2}{Q^*}}} \tag{2.1}$$

is achieved by the C^∞ function:

$$U(x,y,t) = \frac{c_0}{\sqrt{(1+x^2+y^2)^2+t^2}},$$

where c_0 is a suitable positive constant. In other words, the function U is a positive solution to the following equation:

$$-\Delta_H u = u^3, \quad u \in S_0^1(\mathbb{H}^1) \tag{2.2}$$

and

$$\int_{\mathbb{H}^1} |\nabla_H U|^2 d\xi = \int_{\mathbb{H}^1} |U|^4 d\xi = S^2.$$

Furthermore, for any $\varepsilon > 0$, the scaling function

$$U_\varepsilon(\xi) = \varepsilon^{-1} U(\delta_{\varepsilon^{-1}}(\xi))$$

is alone a solution of (2.2). This deep result of Jerison and Lee [21] is the Kohn-Laplacian counterpart of a celebrated theorem of Talenti [22] for the classical Laplace operator. Since $S_0^1(\Omega)$ is separable and reflexive space, there exists the basis $\{e_n\}_{n=1}^{\infty} \subset S_0^1(\Omega)$, which is characterized by the relationship:

$$e_n(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m, \end{cases}$$

(see also [23] for more details). Furthermore, it is easy to see that the following attributes held:

$$E_n = \operatorname{span}\{e_1, \ldots, e_n\}, \quad E_n^{\perp} = \overline{\operatorname{span}\{e_{n+1}, \ldots\}}.$$

Let $P_n : S_0^1 \to E_n$ be the projector corresponding to decomposition $E = E_n \oplus E_n^{\perp}$.

2.2 Limit index theory

Next, we review the limit index theory of Li [15]. To this end, we introduce the following definitions.

Definition 2.1. [15] The action of a topological group G on a normed space Z is a continuous map

$$G \times Z \to Z : [g, z] \mapsto gz$$

such that

$$1 \cdot z = z, \quad (gh)z = g(hz) \quad z \mapsto gz \quad \text{is linear}, \quad \forall g, h \in G.$$

The action is isometric if

$$||gz|| = ||z||, \quad \forall g \in G, \quad z \in Z.$$

And in this case, Z is called the G-space.

The set of invariant points is defined by:

$$\operatorname{Fix}(G) := \{z \in Z : gz = z, \quad \forall g \in G\}.$$

A set $A \subset Z$ is invariant if $gA = A$ for every $g \in G$. A function $\varphi : Z \to R$ is invariant $\varphi \circ g = \varphi$ for every $g \in G$, $z \in Z$. A map $f : Z \to Z$ is equivariant if $g \circ f = f \circ g$ for every $g \in G$.

Suppose that Z is a G-Banach space, that is, there is a G isometric action on Z. Let

$$\Sigma := \{A \subset Z : A \quad \text{is closed and } gA = A, \forall g \in G\}$$

be a family of all G-invariant closed subsets of Z, and let

$$\Gamma := \{h \in C^0(Z, Z) : h(gu) = g(hu), \quad g \in G\}$$

be the class of all G-equivariant mappings of Z. Finally, we call the set

$$O(u) \coloneqq \{gu : g \in G\}$$

the G-orbit of u.

Definition 2.2. [15] An index for (G, Σ, Γ) is a mapping $i : \Sigma \to \mathcal{Z}_+ \cup \{+\infty\}$ (where $\mathcal{Z}_+$ is the set of all nonnegative integers) such that for all $A, B \in \Sigma$, and $h \in \Gamma$, the following conditions are satisfied:
(1) $i(A) = 0 \Leftrightarrow A = \emptyset$;
(2) (Monotonicity) $A \subset B \Rightarrow i(A) \le i(B)$;
(3) (Subadditivity) $i(A \cup B) \le i(A) + i(B)$;
(4) (Supervariance) $i(A) \le i(\overline{h(A)}), \forall h \in \Gamma$;
(5) (Continuity) If A is compact and $A \cap \mathrm{Fix}(G) = \emptyset$, then $i(A) < +\infty$ and there is a G-invariant neighborhood N of A such that $i(\overline{N}) = i(A)$;
(6) (Normalization) If $x \notin \mathrm{Fix}(G)$, then $i(O(x)) = 1$.

Definition 2.3. [24] An index theory is said to satisfy the d-dimensional property if there is a positive integer d such that

$$i(V^{dk} \cap S_1) = k$$

for all dk-dimensional subspaces $V^{dk} \in \Sigma$ such that $V^{dk} \cap \mathrm{Fix}(G) = \{0\}$, where S_1 is the unit sphere in Z.

Suppose that U and V are G-invariant closed subspaces of Z such that

$$Z = U \oplus V,$$

where V is infinite dimensional and

$$V = \overline{\bigcup_{j=1}^{\infty} V_j},$$

where V_j is a dn_j-dimensional G-invariant subspace of V, $j = 1, 2, \cdots$, and $V_1 \subset V_2 \subset \cdots \subset V_n \subset \cdots$. Let

$$Z_j = U \oplus V_j$$

and $\forall A \in \Sigma$, we and let

$$A_j = A \oplus Z_j.$$

Definition 2.4. [15] Let i be an index theory satisfying the d-dimensional property. A limit index with respect to (Z_j) induced by i is a mapping

$$i^\infty : \Sigma \to \mathcal{Z} \cup \{-\infty, +\infty\}$$

given by

$$i^\infty(A) = \limsup_{j\to\infty}(i(A_j) - n_j).$$

Proposition 2.1. [15] *Let $A, B \in \Sigma$. Then i^∞ satisfies*:
(1) $A = \emptyset \Rightarrow i^\infty = -\infty$;
(2) (*Monotonicity*) $A \subset B \Rightarrow i^\infty(A) \le i^\infty(B)$;
(3) (*Subadditivity*) $i^\infty(A \cup B) \le i^\infty(A) + i^\infty(B)$;
(4) *If* $V \cap \mathrm{Fix}(G) = \{0\}$, *then* $i^\infty(S_\rho \cap V) = 0$, *where* $S_\rho = \{z \in Z : \|z\| = \rho\}$;
(5) *If* Y_0 *and* $\widetilde{Y_0}$ *are* G-*invariant closed subspaces of* V *such that* $V = Y_0 \oplus \widetilde{Y_0}$, $\widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim(Y_0) = dm$, *then* $i^\infty(S_\rho \cap Y_0) \ge -m$.

Definition 2.5. [25] A functional $I \in C^1(Z, R)$ is said to satisfy the condition $(\mathrm{PS})_c^*$ if any sequence $\{u_{n_k}\} \subseteq Z_{n_k}$ such that

$$I(u_{n_k}) \to c, \quad dI_{n_k}(u_{nk}) \to 0, \quad \text{as } k \to \infty$$

possesses a convergent subsequence, where Z_{n_k} is the n_k-dimensional subspace of Z, $I_{n_k} = I\,|_{Z_{n_k}}$.

Theorem 2.1. [15] *Assume that*

(B_1) $I \in C^1(Z, R)$ *is G-invariant;*
(B_2) *There are G-invariant closed subspaces U and V such that V is infinite dimensional and* $Z = U \oplus V$;
(B_3) *There is a sequence of G-invariant finite dimensional subspaces*

$$V_1 \subset V_2 \subset \cdots \subset V_j \subset \cdots, \quad \dim(V_j) = dn_j,$$

such that $V = \overline{\cup_{j=1}^{\infty} V_j}$;
(B_4) *There is an index theory i on Z satisfying the d-dimensional property;*
(B_5) *There are G-invariant subspaces* Y_0, $\widetilde{Y_0}$, *and* Y_1 *of V such that* $V = Y_0 \oplus \widetilde{Y_0}$, $Y_1, \widetilde{Y_0} \subset V_{j_0}$ *for some* j_0 *and* $\dim(\widetilde{Y_0}) = dm < dk = \dim(Y_1)$;
(B_6) *There are* α *and* β, $\alpha < \beta$ *such that* f *satisfies* $(\mathrm{PS})_c^*$, $\forall c \in [\alpha, \beta]$;
(B_7)

$$\begin{cases} (a) & \textit{either } \mathrm{Fix}(G) \subset U \oplus Y_1, \quad \textit{or} \quad \mathrm{Fix}(G) \cap V = \{0\}, \\ (b) & \textit{there is } \rho > 0 \quad \textit{such that} \quad \forall u \in Y_0 \cap S_\rho, f(z) \geq \alpha, \\ (c) & \forall z \in U \oplus Y_1, \quad f(z) \leq \beta, \end{cases}$$

if i^∞ *is the limit index corresponding to i, then the numbers*

$$c_j = \inf_{i^\infty(A) \geq j} \sup_{z \in A} f(u), \quad -k+1 \leq j \leq -m,$$

are the critical values of f, and $\alpha \leq c_{-k+1} \leq \cdots \leq c_{-m} \leq \beta$. *Moreover, if* $c = c_l = \cdots = c_{l+r}$, $r \geq 0$, *then* $i(\mathbb{K}_c) \geq r + 1$, *where* $\mathbb{K}_c = \{z \in Z : df(z) = 0, f(z) = c\}$.

3 Verification of $(\mathrm{PS})_c$ condition

Before we begin, let us set a few facts straight. We denote by B_ρ the closed ball of radius ρ centered at zero in the Folland-Stein space $S_0^1(\Omega)$, and by S_ρ its relative boundary, that is,

$$B_\rho = \{u \in S_0^1(\Omega) : \|u\| \leq \rho\}, \quad S_\rho = \{u \in S_0^1(\Omega) : \|u\| = \rho\}.$$

Similar to the proof of Lemma 3.1 in [14], we have the following lemma.

Lemma 3.1. *Let* $u, v \in S_0^1(\Omega)$. *Then, there exists a unique nonnegative function* $\phi_{1,u}, \phi_{2,v} \in S_0^1(\Omega)$ *satisfying*

$$\begin{cases} -\Delta_H \phi_i = u^2, & \textit{in } \Omega, \\ \phi_i = 0, & \textit{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $i = 1, 2$. *Moreover,* $\phi_{1,u}, \phi_{2,v} > 0$ *if* $u, v \neq 0$. *And the following properties hold:*
(a) *For any nonzero constant s, then* $\phi_{1,su} = s^2\phi_{1,u}, \phi_{1,sv} = s^2\phi_{2,v}$. *Moreover,*

$$\int_\Omega \phi_{1,u} u^2 \mathrm{d}\xi = \int_\Omega |\nabla_H \phi_{1,u}|^2 \mathrm{d}\xi \leq S^{-1}\|u\|^4_{\frac{8}{3}} \leq S^{-3}\,|\Omega|^{\frac{1}{2}}\|u\|^4 \tag{3.2}$$

and

$$\int_\Omega \phi_{2,v} v^2 \mathrm{d}\xi = \int_\Omega |\nabla_H \phi_{2,v}|^2 \mathrm{d}\xi \leq S^{-1}\|v\|^4_{\frac{8}{3}} \leq S^{-3}\,|\Omega|^{\frac{1}{2}}\|v\|^4. \tag{3.3}$$

(*b*) *Let* $u_n \rightharpoonup u$, $v_n \rightharpoonup v$ *in* $S_0^1(\Omega)$. *Then,* $\phi_{1,u_n} \to \phi_{1,u}$ *and* $\phi_{2,v_n} \to \phi_{2,v}$ *in* $S_0^1(\Omega)$. *Moreover,*

$$\int_\Omega \phi_{1,u_n} u_n \tilde{u} \mathrm{d}\xi \to \int_\Omega \phi_{1,u} u \tilde{u} \mathrm{d}\xi \quad \text{for every } \tilde{u} \in S_0^1(\Omega) \tag{3.4}$$

and

$$\int_\Omega \phi_{2,v_n} v_n \tilde{v} \mathrm{d}\xi \to \int_\Omega \phi_{2,v} v \tilde{v} \mathrm{d}\xi \quad \text{for every } \tilde{v} \in S_0^1(\Omega). \tag{3.5}$$

Using Lemma 3.1, we have that $(u, \phi_1), (v, \phi_2) \in S_0^1(\Omega) \times S_0^1(\Omega)$ is a solution of Problem (1.1) if and only if $\phi_1 = \phi_{1,u}$, $\phi_2 = \phi_{2,v}$ and $(u, v) \in S_0^1(\Omega) \times S_0^1(\Omega)$ is a solution of the following nonlocal problem:

$$\begin{cases} \Delta_H u - \mu_1 \phi_{1,u} u = |u|^2 u + F_u(\xi, u, v), & \text{in } \Omega, \\ -\Delta_H v + \mu_2 \phi_{2,v} v = |v|^2 u + F_v(\xi, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases}$$

Now, we define the functional I:

$$\begin{aligned} I(u, v) = &-\frac{1}{2}\int_\Omega |\nabla_H u|^2 \mathrm{d}\xi - \frac{\mu_1}{4}\int_\Omega \phi_{1,u} u^2 \mathrm{d}\xi + \frac{1}{2}\int_\Omega |\nabla_H v|^2 \mathrm{d}\xi \\ &+ \frac{\mu_2}{4}\int_\Omega \phi_{2,v} v^2 \mathrm{d}\xi - \frac{1}{4}\int_\Omega |u|^4 \mathrm{d}\xi - \frac{1}{4}\int_\Omega |v|^4 \mathrm{d}\xi - \int_\Omega F(\xi, u, v) \mathrm{d}\xi. \end{aligned} \tag{3.6}$$

Under the assumptions (F_1)–(F_3), it is easy to prove $I \in C^1(S_0^1(\Omega) \times S_0^1(\Omega), \mathbb{R})$ (see [25]). Therefore, in accordance with the aforementioned argument, we will devote ourselves to the existence critical point of the functional I by using the critical point theory.

Similar to the proof in [25,26], it is easy to obtain the following results.

Lemma 3.2. *Assume* $1 \le \theta_1, \theta_2, \theta < \infty$, $I \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$, *and*

$$I(\xi, u, v) \le C\left(|u|^{\frac{\theta_1}{\theta}} + |v|^{\frac{\theta_2}{\theta}}\right).$$

Then, for every $(u, v) \in L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$, $I(\cdot, u, v) \in L^\theta(\Omega)$, *the operator*

$$T : (u, v) \mapsto I(\xi, u, v)$$

is a continuous map from $L^{\theta_1}(\Omega) \times L^{\theta_2}(\Omega)$ *to* $L^\theta(\Omega)$.

Lemma 3.3. *Suppose that* $F(\xi, u, v)$ *satisfies conditions* (F_1)–(F_3). *Then,*

(*i*) $I \in C^1(S_0^1(\Omega), \mathbb{R})$, *with*

$$\begin{aligned} \langle \mathrm{d}I(u, v), (\tilde{u}, \tilde{v})\rangle = &-\int_\Omega \nabla_H u \nabla_H \tilde{u} \mathrm{d}\xi - \mu_1 \int_\Omega \phi_{1,u} u \tilde{u} \mathrm{d}\xi + \int_\Omega \nabla_H v \nabla_H \tilde{v} \mathrm{d}\xi + \mu_2 \int_\Omega \phi_{2,v} v \tilde{v} \mathrm{d}\xi \\ &- \int_\Omega |u|^2 u \tilde{u} \mathrm{d}\xi - \int_\Omega |v|^2 v \tilde{v} \mathrm{d}\xi - \int_\Omega F_u(\xi, u, v) \tilde{u} \mathrm{d}\xi - \int_\Omega F_v(\xi, u, v) \tilde{v} \mathrm{d}\xi. \end{aligned}$$

(*ii*) *A critical point of I is a weak solution of System* (1.1).

Now, set

$$\begin{gathered} X = U \oplus V, \quad U = S_0^1 \times \{0\}, \quad V = \{0\} \times S_0^1, \\ Y_0 = \{0\} \times E_1^\perp, \quad V = Y_0 \oplus \tilde{Y}_0, \\ Y_1 = \{0\} \times E_{k_0}, \quad E_{k_0} = \operatorname{span}\{e_1, \ldots, e_{k_0}\}, \end{gathered}$$

then $\dim \widetilde{Y}_0 = 1, \dim Y_1 = k_0$. Define a group action $G_2 = \{1, \tau\} \cong \mathcal{Z}_2$ by setting $\tau(u, v) = (-u, -v)$, then Fix $G = \{0\} \times \{0\}$ (also denote by $\{0\}$). It is clear that U and V are G-invariant closed subspaces of X, and Y_0, $\widetilde{Y}_0$, and Y_1 are the G-invariant subspace of V. Set

$$\Sigma = \{A \subset X \backslash \{0\} : A \quad \text{is closed in } X \quad \text{and} \quad (u, v) \in A \Rightarrow (-u, -v) \in A\}.$$

Define an index γ on Σ by:

$$\gamma(A) = \begin{cases} \min\{N \in \mathcal{Z} : \exists h \in C(A, \mathbb{R}^N \backslash \{0\}) \quad \text{such that } h(-u, -v) = h(u, v)\}, \\ 0, \quad \text{if } A = \emptyset, \\ +\infty, \quad \text{if such } h \text{ does not exist.} \end{cases}$$

Then, we have the following proposition from [26]: γ is an index satisfying the properties given in Definition 2.2. Moreover, γ satisfies the one-dimensional property. According to Definition 2.4, we can obtain a limit index γ^∞ with respect to (X_n) from γ. Now, we turn to prove the Palais-Smale condition using the following concentration-compactness principle on the Heisenberg group.

Lemma 3.4. [27, Lemma 3.5] *Suppose that $u_n \subset E$ verifies*

$$\begin{cases} u_n(\xi) \to u(\xi) \quad a.e. \text{ in } \Omega, \\ |\nabla_H u_n|^2 \rightharpoonup \mathrm{d}\mu \quad in \quad \mathcal{N}(\Omega), \\ |u_n|^4 \rightharpoonup \mathrm{d}\nu \quad in \ \mathcal{N}(\Omega), \end{cases}$$

where $\mathcal{N}(\Omega)$ is the space of all finite Radon measures on $\Omega \subset \mathbb{H}^1$, J is an at most countable index set, which can be empty, and family $\{\xi_j, j \in J\}$ of points in $\bar{\Omega}$ such that
(a) $\mathrm{d}\mu \geq |\nabla_H u|^2 + \sum_{j \in J} \mu_j \delta_{\xi_j}$,
(b) $\mathrm{d}\nu = |u|^4 + \sum_{j \in J} \nu_j \delta_{\xi_j}$,

where $\xi_j \in \Omega$ and δ_{ξ_j} is the Dirac mass at ξ_j. Moreover, we have

$$\mu_j, \nu_j, \geq 0, \quad \mu_j \geq S \nu_j^{\frac{1}{2}} \quad for \ any \ j \in J.$$

Lemma 3.5. *Let (F_1)–(F_3) hold. Then, the functional J satisfies the local* $(\mathrm{PS})_c$ *condition in*

$$c \in \left(-\infty, \frac{1}{8} S^2 - c\left(\frac{1}{8} |\Omega|^{-\frac{1}{2}} S\right) |\Omega|\right),$$

in the following sense: if

$$I(u_{n_k}, v_{n_k}) \to c \in \left(-\infty, \frac{1}{8} S^2 - c\left(\frac{1}{8} |\Omega|^{-\frac{1}{2}} S\right) |\Omega|\right), \quad \mathrm{d}I_{n_k}(u_{n_k}, v_{n_k}) \to 0 \quad as \ k \to \infty,$$

where $I_{n_k} = I|_{X_{n_k}}$ with $X_{n_k} = E_{n_k} \times E_{n_k}$. Then, $\{(u_{n_k}, v_{n_k})\}_k$ contains a subsequence converging strongly in X.

Proof. We first show that $\{(u_{n_k}, v_{n_k})\}_k$ is bounded in X. Note that by (F_1) and (F_3), we have

$$\begin{aligned} o(1)\|u_{n_k}\| &= \langle -\mathrm{d}I_{n_k}(u_{n_k}, v_{n_k}), (u_{n_k}, 0)\rangle \\ &= \int_\Omega |\nabla_H u_{nk}|^2 \mathrm{d}\xi + \mu_1 \int_\Omega \phi_{1,u_{nk}} |u_{nk}|^2 \mathrm{d}\xi + \int_\Omega |u_{nk}|^4 \mathrm{d}\xi + \int_\Omega F_u(\xi, u_{nk}, v_{nk}) u_{nk} \mathrm{d}\xi \\ &\geq \|u_{n_k}\|^2 + \mu_1 \int_\Omega \phi_{1,u_{nk}} |u_{nk}|^2 \mathrm{d}\xi + \int_\Omega |u_{nk}|^4 \mathrm{d}\xi \\ &\geq \|u_{n_k}\|^2. \end{aligned} \tag{3.7}$$

This fact implies that $\|u_{n_k}\|$ is bounded in $S_0^1(\Omega)$. On the one hand, we have

$$
\begin{aligned}
c + o(1)\|v_{n_k}\| &= I_{n_k}(0, v_{n_k}) - \frac{1}{4}\langle dI_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle \\
&= \frac{1}{2}\int_\Omega |\nabla_H v_{nk}|^2 d\xi + \frac{\mu_2}{4}\int_\Omega \phi_{2,v_{nk}} v_{nk}^2 d\xi - \frac{1}{4}\int_\Omega |v_{nk}|^4 d\xi - \int_\Omega F(\xi, 0, v_{nk}) d\xi \\
&\quad - \frac{1}{4}\int_\Omega |\nabla_H v_{nk}|^2 d\xi - \frac{\mu_2}{4}\int_\Omega \phi_{2,v_{nk}} v_{nk}^2 d\xi + \frac{1}{4}\int_\Omega |v_{nk}|^4 d\xi + \frac{1}{4}\int_\Omega F_v(\xi, u_{nk}, v_{nk}) v_{nk} d\xi \\
&= \frac{1}{4}\int_\Omega |\nabla_H v_{nk}|^2 d\xi - \left(\int_\Omega F(\xi, 0, v_{nk}) d\xi - \frac{1}{4}\int_\Omega F_v(\xi, u_{nk}, v_{nk}) v_{nk} d\xi\right) \\
&\geq \frac{1}{4}\ \|v_{n_k}\|^2 - \int_\Omega \left[F(\xi, 0, v_{n_k}) - \frac{1}{4} F_v(\xi, u_{n_k}, v_{n_k}) v_{n_k}\right] d\xi.
\end{aligned}
$$

Note that the embedding $S_0^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in [1, 4)$ is compact. Thus, by (1.4) and (2.1), we have

$$
\int_\Omega \left[F(\xi, 0, v_{n_k}) - \frac{1}{4} F_v(\xi, u_{n_k}, v_{n_k}) v_{n_k}\right] d\xi \leq c(\varepsilon)|\Omega| + \varepsilon|\Omega|^{\frac{1}{2}} S^{-1}\|v_{n_k}\|^2. \tag{3.8}
$$

Setting $\varepsilon = \frac{1}{8}\ |\Omega|^{-\frac{1}{2}} S$ in (3.8) and together with (3.8), we obtain

$$
\|v_{n_k}\|^2 \leq C_1 + o(1)\|v_{n_k}\|, \tag{3.9}
$$

and C_1 is a some positive number. Thus, (3.9) implies that $\{v_{n_k}\}_k$ is bounded in $S_0^1(\Omega)$. Hence, $\|u_{n_k}\| + \|v_{n_k}\|$ is bounded.

Next, we prove that $\{(u_{n_k}, v_{n_k})\}_k$ contains a subsequence converging strongly in X. We note that $\{u_{n_k}\}_k$ is bounded in $S_0^1(\Omega)$. Hence, up to a subsequence, $u_{n_k} \rightharpoonup u$ weakly in $S_0^1(\Omega)$ and $u_{n_k}(\xi) \to u(\xi)$, a.e. in Ω. We claim that $u_{n_k} \to u$ strongly in $S_0^1(\Omega)$. In fact, note that

$$
\begin{aligned}
&\|u_{n_k} - u\|^2 + \mu_1 \int_\Omega \phi_{1,u_{nk}}\ |u_{n_k} - u|^2 d\xi + \int_\Omega F_u(\xi, u_{n_k} - u, v_{n_k})(u_{n_k} - u) d\xi \\
&\leq \langle -dI_{n_k}(u_{n_k} - u, v_{n_k}), (u_{n_k} - u, 0)\rangle \to 0 \quad \text{as } k \to \infty
\end{aligned}
$$

and condition (F_3) implies that

$$
u_{n_k} \to u \quad \text{strongly in } S_0^1(\Omega). \tag{3.10}
$$

In the following, we will prove that there exists $v \in S_0^1(\Omega)$ such that

$$
v_{n_k} \to v \quad \text{strongly in } S_0^1(\Omega). \tag{3.11}
$$

We know that $\{v_{n_k}\}_k$ is also bounded in $S_0^1(\Omega)$. Therefore, by Lemma 3.4, we assume that there exist two positive measures μ and ν on Ω such that

$$
\begin{aligned}
&v_{nk}(\xi) \to v(\xi), \quad \text{a.e. in } \Omega, \\
&|\nabla_H v_{nk}|^2 \rightharpoonup d\mu \geq |\nabla_H v|^2 + \sum_{j\in J} \mu_j \delta_{\xi_j}, \\
&|v_{nk}|^4 \rightharpoonup d\nu = |v|^4 + \sum_{j\in J} \nu_j \delta_{\xi_j}, \\
&\mu_j, \nu_j \geq 0, \quad \mu_j \geq S\nu_j^{\frac{1}{2}},
\end{aligned}
$$

where j is an at most countable index set, $\xi_j \in \Omega$, and δ_{ξ_j} is the Dirac mass at ξ_j.

Now, we claim that $J = \varnothing$. In fact, we assume that there exists $j \in J$ such that $\mu_j \neq 0$. Then, for $\varepsilon > 0$ small enough, it follows from Lemma 3.2 of [28] that there exists a cut-off function $\psi_{\varepsilon,j} \in C_0^\infty(B_H(\xi_j, \varepsilon))$ such that $0 \leq \psi_{\varepsilon,j}(\xi) \leq 1$ and

$$\begin{cases}\psi_{\varepsilon,j}(\xi)=1, & \text{in } B_H\left(\xi_j,\dfrac{\varepsilon}{2}\right),\\ \psi_{\varepsilon,j}(\xi)=0, & \text{in } \Omega\backslash B_H(\xi_j,\varepsilon),\\ |\nabla_H\psi_{\varepsilon,j}(\xi)|\le \dfrac{4}{\varepsilon}.\end{cases}$$

Then, by the boundedness of $\{\psi_{\varepsilon,j}(\xi)u_n\}$, we have $\langle dI_{n_k}(u_{n_k},v_{n_k}),(0,v_{n_k}\psi_{\varepsilon,j}(\xi))\rangle\to 0$, i.e.,

$$\begin{aligned}&\int_\Omega|\nabla_H v_{nk}|^2\psi_{\varepsilon,j}d\xi+\int_\Omega\nabla_H v_{nk}\nabla_H\psi_{\varepsilon,j}v_{nk}d\xi+\mu_2\int_\Omega\phi_{2,v_{nk}}v_{nk}\psi_{\varepsilon,j}(\xi)v_{nk}d\xi\\ &=\int_\Omega v_{nk}^3\psi_{\varepsilon,j}(\xi)v_{nk}d\xi+\int_\Omega F_{v_{nk}}(\xi,u_{nk},v_{nk})\psi_{\varepsilon,j}(\xi)v_{nk}d\xi+o(1),\end{aligned}\tag{3.12}$$

where $o(1)=0$ as $n\to\infty$. It follows from (a) of Lemma 3.1 and (3.10) that

$$\begin{aligned}\left|\int_\Omega\phi_{2,v_{nk}}v_{nk}\psi_{\varepsilon,j}(\xi)v_{nk}d\xi\right|&\le\int_{B_H(\xi_j,\varepsilon)}\phi_{2,v_{nk}}v_{nk}^2d\xi\le S^{-1}\left(\int_{B_H(\xi_j,\varepsilon)}|v_{nk}|^{\frac{8}{3}}d\xi\right)^{3/2}\\ &\le S^{-1}|B_H(\xi_j,\varepsilon)|^{1/2}\int_{B_H(\xi_j,\varepsilon)}|v_{nk}|^4d\xi\\ &\le\alpha_Q\varepsilon^2S^{-1}\int_\Omega|v_{nk}|^4d\xi\\ &\le\alpha_Q\varepsilon^2S^{-3}||v_{nk}||^4.\end{aligned}$$

Hence,

$$\lim_{\varepsilon\to0}\lim_{k\to\infty}\int_\Omega\phi_{2,v_{nk}}v_{nk}\psi_{\varepsilon,j}(\xi)v_{nk}d\xi=0.\tag{3.13}$$

In addition, it follows from (3.10) and the Hölder inequality that:

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{k\to\infty}\left|\int_\Omega v_{nk}\nabla_H v_{nk}\nabla_H\psi_{\varepsilon,j}d\xi\right|\\ &\le\lim_{\varepsilon\to0}\lim_{k\to\infty}\left(\int_{B_H(\xi_j,\varepsilon)}|\nabla_H v_{nk}|^2d\xi\right)^{\frac12}\left(\int_{B_H(\xi_j,\varepsilon)}|v_{nk}|^2|\nabla_H\psi_{\varepsilon,j}|^2d\xi\right)^{\frac12}\\ &\le C\lim_{\varepsilon\to0}\left(\int_{B_H(\xi_j,\varepsilon)}|v|^4d\xi\right)^{\frac14}\left(\int_{B_H(\xi_j,\varepsilon)}|\nabla_H\psi_{\varepsilon,j}|^4d\xi\right)^{\frac14}\\ &=0.\end{aligned}\tag{3.14}$$

It follows from (3.13) that

$$\lim_{\varepsilon\to0}\lim_{k\to\infty}\int_\Omega|\nabla_H v_{nk}|^2\psi_{\varepsilon,j}d\xi\ge\lim_{\varepsilon\to0}\left(\mu_j+\int_{B_H(\xi_j,\varepsilon)}|\nabla_H v|^2\psi_{\varepsilon,j}d\xi\right)=\mu_j\tag{3.15}$$

and

$$\lim_{\varepsilon\to0}\lim_{k\to\infty}\int_\Omega\psi_{\varepsilon,j}(\xi)|v_{nk}|^4d\xi=\lim_{\varepsilon\to0}\left(\nu_j+\int_{B_H(\xi_j,\varepsilon)}|v|^4d\xi\right)=\nu_j.\tag{3.16}$$

On the other hand, by using the definition of $\psi_{\varepsilon,j}$ and Vitali's convergence theorem, we obtain

$$\int_{B(\xi_j,\varepsilon)}F_v(\xi,u_{n_k},v_{n_k})v_{n_k}\psi_{\varepsilon,j}d\xi\to\int_{B(\xi_j,\varepsilon)}F_v(\xi,u,v)v\psi_{\varepsilon,j}d\xi\tag{3.17}$$

as $k \to \infty$. We also observe that the integral goes to 0 as $\varepsilon \to 0$. So, by (3.16), we conclude that $\nu_j \geq \mu_j$. Then, by (3.9), we obtain $\mu_j \geq S\mu_j^{\frac{1}{2}}$, which implies that

$$\mu_j = 0 \quad \text{or} \quad \mu_j \geq S^2.$$

If the second case $\nu_j \geq S^2$ holds, then by (3.8) and (3.13), we have

$$\begin{aligned} c &= \lim_{k\to\infty}\left(I_{n_k}(0, v_{n_k}) - \frac{1}{4}\langle \mathrm{d}I_{n_k}(u_{n_k}, v_{n_k}), (0, v_{n_k})\rangle\right) \\ &= \lim_{k\to\infty}\left\{\frac{1}{4}\ \|v_{n_k}\|^2 - \lim_{k\to\infty}\int_\Omega\left[F(\xi, 0, v_{n_k}) - \frac{1}{4}F_v(\xi, u_{n_k}, v_{n_k})v_{n_k}\right]\mathrm{d}\xi\right\} \\ &\geq \frac{1}{8}\int_\Omega \mathrm{d}\mu - c\left(\frac{1}{8}\ |\Omega|^{-\frac{1}{2}}S\right)|\Omega| \\ &\geq \frac{1}{8}S^2 - c\left(\frac{1}{8}\ |\Omega|^{-\frac{1}{2}}S\right)|\Omega|, \end{aligned} \tag{3.18}$$

which contradicts (3.5). Consequently, $J = \varnothing$. Furthermore, we have

$$\int_\Omega |v_{n_k}|^4 \mathrm{d}\xi \to \int_\Omega |v|^4 \mathrm{d}\xi, \tag{3.19}$$

as $k \to \infty$.

From Lemma 3.6 and (3.18), we obtain

$$\int_\Omega \nabla_H v_{nk} \nabla_H \varphi \mathrm{d}\xi + \mu_2 \int_\Omega \phi_{2,v_{nk}} v_{nk} \varphi \mathrm{d}\xi - \int_\Omega v_{nk}^3 \varphi \mathrm{d}\xi - \int_\Omega F_v(\xi, u_{nk}, v_{nk})\varphi \mathrm{d}\xi = o(1). \tag{3.20}$$

Let $\varphi = v$ in (3.19); then,

$$\|v\|^2 + \mu_2 \int_\Omega \phi_{2,v} v^2 \mathrm{d}\xi - \int_\Omega v^4 \mathrm{d}\xi - \int_\Omega F_v(\xi, u, v)v \mathrm{d}\xi = 0. \tag{3.21}$$

By (3.9), (3.18), and Lemmas 3.1 and 3.6, we also have

$$\lim_{k\to\infty} \|v_{nk}\|^2 + \mu_2 \int_\Omega \phi_{2,v} v^2 \mathrm{d}\xi - \int_\Omega v^4 d\xi - \int_\Omega F_v(\xi, u, v)v \mathrm{d}\xi = 0. \tag{3.22}$$

From (3.20) and (3.21), one has

$$\lim_{k\to\infty} \|v_{nk}\| = \|v\|.$$

So, from the uniform convexity of $S_0^1(\Omega)$, we obtain $v_{nk} \to v$ in $S_0^1(\Omega)$. Thus, we have proved that $\{v_{n_k}\}_k$ strongly converges to v in $S_0^1(\Omega)$. □

4 Proof of Theorem 1.1

In this section, we give the proof of Theorem 1.1. For this, we shall verify the conditions of Theorem 2.1. Obviously, (B_1), (B_2), and (B_4) in Theorem 2.1 are satisfied. Set $V_j = E_j = \mathrm{span}\{e_1, e_2, \ldots, e_j\}$, then (B_3) is also satisfied. Since $1 = \dim \widetilde{Y}_0 < k_0 = \dim Y_1$, (B_5) is satisfied. In the following, we verify the conditions in (B_7). Since $\mathrm{fix}\, G \cap V = 0$, that is, (a) of (B_7) holds. It remains to verify (b) and (c) of (B_7). Choose a real number α such that

$$\alpha < \min\left\{0, \frac{1}{8}S^2 - c\left(\frac{1}{8}\ |\Omega|^{-\frac{1}{2}}S\right)|\Omega|, \frac{S^2}{16} - b(\varepsilon)|\Omega|\right\}. \tag{4.1}$$

(i) If $(0, v) \in Y_0 \cap S_\rho$ (where ρ is to be determined), then by (1.2), we obtain

$$\begin{aligned} I(0, v) &= \frac{1}{2}\int_\Omega |\nabla_H v|^2 d\xi + \frac{\mu_2}{4}\int_\Omega \phi_{2,v} v^2 d\xi - \frac{1}{4}\int_\Omega |v|^4 d\xi - \int_\Omega F(\xi, 0, v) d\xi \\ &\geq \left(\frac{1}{2} - \varepsilon\, |\Omega|^{\frac{1}{2}} S^{-1}\right)\|v\|^2 - \frac{1}{4}S^{-2}\|v\|^4 - b(\varepsilon)|\Omega| \\ &= \frac{1}{4}\|v\|^2 - \frac{1}{4}S^{-2}\|v\|^4 - b\left(\frac{1}{4}\, |\Omega|^{-\frac{1}{2}} S\right)|\Omega|, \end{aligned} \tag{4.2}$$

where $\varepsilon = \frac{1}{4}\, |\Omega|^{-\frac{1}{2}} S$. Since

$$\max_{t \in \mathbb{R}}\left(\frac{1}{4}t^2 - \frac{1}{4}S^{-2}t^4 - b\left(\frac{1}{4}\, |\Omega|^{-\frac{1}{2}} S\right)|\Omega|\right) = \frac{S^2}{16} - b(\varepsilon)|\Omega|.$$

Therefore, there exists $\rho > 0$ such that $I(0, v) \geq \alpha$ for every $\|v\| = \rho$, that is, (b) of (B_7) holds.

(ii) For each $(u, v) \in U \oplus Y_1$.

Note that

$$\int_\Omega \phi_{2,v} v^2 d\xi \leq \left(\int_\Omega |\phi_{2,v}|^4 d\xi\right)^{\frac{1}{4}}\left(\int_\Omega |v|^{\frac{8}{3}} d\xi\right)^{\frac{3}{4}} \leq S^{-1}\, |\Omega|^{\frac{1}{2}}\int_\Omega |v|^4 d\xi.$$

Thus, by (F_4) and $\mu_2 < |\Omega|^{-\frac{1}{2}} S$, we have

$$\begin{aligned} I(u, v) &= -\frac{1}{2}\int_\Omega |\nabla_H u|^2 d\xi - \frac{\mu_1}{4}\int_\Omega \phi_1 u^2 d\xi + \frac{1}{2}\int_\Omega |\nabla_H v|^2 d\xi + \frac{\mu_2}{4}\int_\Omega \phi_{2,v} v^2 d\xi \\ &\quad - \frac{1}{4}\int_\Omega |u|^4 d\xi - \frac{1}{4}\int_\Omega |v|^4 d\xi - \int_\Omega F(\xi, u, v) d\xi \\ &\leq \frac{1}{2}\int_\Omega |\nabla_H v|^2 d\xi - \left(\frac{1}{4} - \frac{\mu_2}{4}S^{-1}\, |\Omega|^{\frac{1}{2}}\right)\int_\Omega |v|^4 d\xi - \sigma\, |v|_2^2 + \tau|\Omega| \\ &\leq \frac{1}{2}\int_\Omega |\nabla_H v|^2 d\xi - \sigma\, |v|_p^p + \tau|\Omega| \\ &\leq \frac{C_0}{2}\|v\|^2 - \sigma\, |v|_p^p + \tau|\Omega|. \end{aligned} \tag{4.3}$$

Since all norms are equivalent on the finite-dimensional space Y_1, we obtain

$$I(u, v) \leq \frac{C_0 C_*}{2}\|v\|_p^2 - \sigma\, |v|_p^p + \tau|\Omega|. \tag{4.4}$$

Put

$$r = \min\left\{\int_\Omega |v|^p d\xi : v \in E_{k_0}\right\}.$$

Taking

$$\sigma \geq \frac{C_0 C_*}{2r^{\frac{p}{p-2}}},$$

we have

$$\frac{C_0 C_*}{2} - \sigma\, |v|_p^2 \leq \frac{C_0 C_*}{2} - \sigma r^{\frac{p}{p-2}} \leq 0. \tag{4.5}$$

It follows from (4.3)–(4.5) and (F_4) that

$$I(u, v) \leq \tau|\Omega| \leq \min\left\{0, \frac{1}{8}S^2 - c\left(\frac{1}{8}\, |\Omega|^{-\frac{1}{2}} S\right)|\Omega|\right\}.$$

Let $\beta = \tau|\Omega|$, so we obtain (c) in (B_7). By Lemma 3.7, for any $c \in [\alpha, \beta]$, $I(u, v)$ satisfies the condition of $(PS)_c^*$, then (B_6) in Theorem 2.1 holds. Finally, according to Theorem 2.1, we have that:

$$c_j = \inf_{i^\infty(A) \geq j_{z\in A}} \sup_{z\in A} I(u, v), \quad -k_0 + 1 \leq j \leq -1,$$

are the critical values of I, $\alpha \leq c_{-k_0+1} \leq \cdots \leq c_{-1} \leq \beta < 0$ and I has at least $k_0 - 1$ pair critical points. The proof is thus complete.

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