DE GRUYTER Open Mathematics 2023; 21: 20230105

Review Article

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A comprehensive review on fractional-order optimal control problem and its solution

https://doi.org/10.1515/math-2023-0105
received September 16, 2022; accepted July 27, 2023

Abstract: This article presents a comprehensive literature survey on fractional-order optimal control problems. Fractional-order differential equation is extensively used nowadays to model real-world systems accurately, which exhibit fractal dimensions, memory effects, as well as chaotic behaviour. These versatile features attract engineers to concentrate more on this, and it is widely used in the broad domain of science and technology. The mentioned numerical tools take the necessary optimal conditions into account, which makes it a two-point boundary value problem of non-integer order. In this review article, some numerical approaches for the approximation have been stated for obtaining the solution to fractional optimal control problems (FOCPs). Here, few numerical approaches including Grunwald-Letnikov approximation, Adams type predictor-corrector method, generalized Euler's method, Caputo-Fabrizio method Bernoulli and Legendre polynomials method, Legendre operational method, and Ritz's and Jacobi's method are treated as an advanced method to obtain the solution of FOCP. Fractional delayed optimal control is selected for our investigation. It refers to a type of control problem where the control action is delayed by a fractional amount of time. In other words, the control input at a given time depends not only on the current state of the system but also on its past state at fractional times. The fractional delayed optimal control problem is formulated as an optimization problem that seeks to minimize a cost function subject to a set of constraints that represent the dynamics of the system and the fractional delay in the control input. The solution to this problem typically involves the use of fractional polynomials types, i.e. Chebyshev and Bassel polynomials.

Keywords: fractional order optimal control problem, two-point boundary value problem, fractional order differential equation, fractional delay differential equation, numerical methods

MSC 2020: 26A33, 35G15

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1 Introduction

Fractional calculus (FC) was born almost 300 years ago, which came into the picture when L. Hospital raised the question in his letter to Leibnitz in 1695. FC is nothing but a generalization of integer-order calculus to the calculus of non-integer as well as arbitrary order. It is seen that this non-integer order calculus is contemporary to classical calculus. Differential equations of non-integer order are used as an alternative to the nonlinear integer-order differential equation. Due to the difficulty and lack of research, the application of fractional-order calculus was unexplored for a long period. But in recent years, FC has proved its versatility as it finds its application to real-life problems, which are obtained from the literature [1–7]. It is founded that the fractional differential equation finds its application in many fields as it is proven for being a strong tool to describe a nonlinear phenomenon which makes it useful in the areas of application in physics, irreversibility, chemistry materials, biology, bioengineering, control theory, ecology, sociology, signal processing, robotics, irreversibility, GA, percolation, modelling as well as identification of systems, telecommunication, electronics, finance, engineering, as well as various fields of applied sciences [8–47]. Fractional order differential equations (FDEs) are used to model large number of systems including the physical phenomenon-like mechanics of viscoelastic material; phenomenon of acoustic dissipation; effects of boundary layer, which are found in duct; field of biomedical engineering; phenomena of power law, which is observed in fluid as well as in network which is not simple; algometric laws of scaling, which are found in applied mathematics when it deals with ecology as well as epidemiology, epidemics, random walk in continuous time, polarization in dielectrics, pervious media, field of quantitative finance, field of quantum and complex systems and their evolution, anomalous slow diffusion, Levy statistics, Brownian motion, non-integer order image as well as signal processing, polarization in electrode-electrolyte, waves which are electromagnetic in nature, motion of filters, fractional-order phase-locked loops and phenomenon, which is non-local in nature and have explained to provide a clear elucidation of the events which are under research than the models with the derivative of integer order [48–55]. It has been also found that theoretical study, practical implementation as well as numerical methods have been developed at a rapid pace in this field [33,56,57]. A detailed survey along with the formulas of FC has been given in the study by Valério et al. [58].

In recent years, a lot of studies are carried out on the approximation of FDE. Many researchers like Diethelm et al. [59] and Li et al. [60] have already shown some results of numerical approximation of FDE. When looking for the numerical approximation for the solution of FDE and fractional partial differential equation (FPDE), there are the Adomian decomposition method (ADM), homotopy perturbation method (HPM), and differential transformation method [61,62]. Practical FC problems have great importance, and their numerical solution is very important, which enthusiastic researchers may check [5,63–65]. There are other numerical methods like the finite element method, finite difference method, as well as the spectral method which were used to solve FDE numerically [66–78]. The Adams-Bash for the method is proved to be an accurate and efficient method that gives the solution numerically and that is closer to the accurate solution [79]. First, the method was developed for integer-order differentiation using the first principle of calculus, and later it was extended for Caputo derivative of non-integer order. Many studies are carried out on various definitions of non-integer derivatives, starting from Riemann-Liouville (RL) to Atangana-Baleanu (AB) [80–82]. Various methods have been shown which have been taken in space [83–88] as well as time to solve the problem numerically [81,89,90].

The optimal control problem (OCP) deals with the maximization or minimization of a cost function that is subjected to constraints that are dynamic on state variables as well as control variables. When FDEs are used as a constraint of dynamic nature, then fractional optimal control problems (FOCPs) are made. Fractional-order optimal control theory is a very younger topic in the field of applied mathematics. A FOCP can be demonstrated easily with old as well as new definitions of fractional derivatives, but the mostly used derivatives are given by RL and Caputo. In this article, all the definitions of fractional derivatives have been shown as well as all the approximation methods have been shown for FOCP. It is seen that a few years ago, many analytical methods were used to solve nonlinear phenomena which are described by using nonlinear FDE. Liao showed the homotopy analysis method [91], and He showed HPM [92,93] and the variational iteration method (VIM) [94,95]. Adomian [96], Siddiqui et al. [97], as well as Wazwaz and El-Sayed showed ADM [96–98].

The modified ADM was shown by Jin and Liu [99], Hosseini and Nasabzadeh [100], and Hasan and Zhu [101]. But there are fewer research papers on the analytical solution of FOCP, which makes the topic very important to develop an analytical method for solving FOCP, and this was shown by Özdemir et al. who were able to solve two-dimensional FOCP by using the variable separation method [102]. Another three-dimensional fractional diffusion problem in spherical coordinates was solved by Povstenko [103] and Qi and Liu [104]. Some other numerical methods, which were used to solve FOCP, were presented in previous studies [105–121]. Recently, a detailed survey has been done on the fuzzy fractional differential equation and its optimal control, which was presented by Agarwal et al. [122].

It is seen that the numerical approximations for the solution of FOCP are classified as indirect and direct methods. First, when we come to the indirect methods, it can be said that these methods are constructed from Pontryagin's maximum principle (PMP) which needs the solution of terminal value problems numerically and which comes from the necessary conditions of constrained OCP [123]. When it comes to the direct optimization methods, the continuous problem of infinite dimension is converted into a problem of nonlinear programming of finite dimension based on a few parameterizations of the state as well as control variables. In the case of the direct method of optimization where the initial guess is required for getting the quantities of the state as well as control from intuition. In the indirect methods, first we optimize and then discretize the OCP; however, in the case of direct methods of optimization, first we discretize and then optimize the OCP. Indirect methods are advantageous over direct methods as it gives a better and more accurate numerical solution of OCP. The first thing is satisfying the necessary condition which comes from the calculus of variation (COV) and PMP. Here, it is seen that first, we impose PMP on our FOCP which makes the problem a two-point boundary value problem (TPBVP), and in this way, if one can find the solution of FDE, then the approximation of the solution of the main problem can be obtained. The necessary and sufficient conditions for solving the FOCP, PMP, as well as COV have a very crucial role. There are many ways to solve TPBVP such as integrating the given problem in a specified interval in such a way that the terminal conditions are included in the problem. Then, Gauss quadrature rules can be used, which is a highly accurate method for smooth data. Another method operational matrices of integration (OMI) can be used for polynomials and orthogonal functions, but the accuracy decreases. In general, the Gauss quadrature rule is used to deal with complicated nonlinear problems on the other hand OMI is used to solve simple nonlinear problems [124]. Examples of these matrices are Laguerre polynomials [125], Hermite polynomials [126], Bessel functions [127], Muntz-Legendre polynomials [128], Walsh functions [129], Bernstein polynomials [130], Block-pulse functions [131], Fourier Series [132], and Chebyshev polynomials [133]. A numerical scheme where hybrid Chelyshkov functions are used to solve FOCP is shown in the study by Mohammadi et al. [134]. A new FOCP in the presence of a non-singular kernel has been discussed in the study by Yıldız et al. [135]. In the study by Ezz-Eldien et al. [136], a numerical scheme has used Legendre orthonormal polynomials to solve FOCP. The inclusion of Mittag-Leffler's (ML) non-singular kernel to formulate FOCP has been shown in the studies by Baleanu et al. and Jafari et al. [137,138]. In the presence of external persistent disturbances, a nonlinear FOCP has been formulated [139]. Moradi et al. [140] used Chelyskov wavelets to solve FOCP with delay. Bhrawy et al. [141] used Chebyshev-Legendre operational technique to solve FOCP. Abdelhakem et al. [142] used shifted Chebyshev scheme to solve FOCP.

Another popular method for obtaining the solution of a TPBVP is the direct method. Here, the TPBVP cannot be integrated directly, but the problem can be directly solved by the solver by using approximation theory like collocation theory but collocation or Galerkin schemes are not being used by some new solvers because of their bases and the completeness of it [143]. These new methods are more accurate and faster for solving linear delay of higher order and Fredholm integrodifferential equations. Galerkin and collocation schemes are used to solve linear, nonlinear differential equations as well as a hyperbolic partial differential equation, Fredholm and Volterra integrodifferential delay equations [144–147]. It is seen that the direct solvers have more accuracy and efficiency than the schemes which deal with the forms of an integral [148].

The fractional operators are subjected to non-locality and weak singularity which results in expensive computational costs as well as storage of numerical schemes to solve FDEs which do not occur in the case of solving the ordinary differential equation. To overcome these problems, Richard's extrapolation [72,149,150] and short memory principles [151,152] can be used. Obtaining ADI and higher order methods for the problems of two or more dimensional can convert a higher dimensional problem to a problem of one dimension

[73,74,153]. The general formulation of FOCP was shown by Agrawal [154,155] and Almeida and Torres [156], and the necessary conditions are obtained for FOCP with RL and Caputo derivative of fractional order. Some other numerical approximation methods for FOCP were also presented in previous studies [88,99,134,157–162].

Fractional-order optimal control problems have several advantages over their classical integer-order counterparts. Some of these advantages are as follows:

1. Greater flexibility: Fractional-order optimal control problems allow for more flexibility in the design of control systems. This is because the order of the fractional derivative can be tuned to achieve specific performance objectives, such as faster response times, improved stability, and better disturbance rejection.
2. Better accuracy: Fractional-order models can provide better accuracy than integer order models when describing systems with non-integer behaviour. This is because many real-world systems exhibit non-integer dynamics, and modelling them using fractional-order methods can lead to more accurate results.
3. Improved robustness: Fractional-order control systems have been shown to be more robust to parameter uncertainties and disturbances than integer-order control systems. This is because the fractional-order models can capture the system's complex behaviour and provide better control over it.
4. Reduced energy consumption: Fractional-order optimal control problems can help to reduce energy consumption in control systems. This is because they allow for the optimization of the control inputs more efficiently, leading to reduced energy usage and improved sustainability.
5. Increased system lifespan: Fractional-order optimal control problems can help to increase the lifespan of control systems. This is because they provide better control over the system dynamics, leading to reduced wear and tear on the system components and improved reliability.

Generally, fractional-order optimal control problems offer several advantages over classical integer-order control problems, making them a valuable tool for the control system design in a wide range of applications.

This review article is organized as follows: In Section 2, different definitions of non-integer-order derivatives and the numerical schemes for solving FOCP have been shown. In Section 3, some methods have been shown to approximate RL, Caputo, Caputo-Fabrizio (CF), and AB fractional derivatives. In Section 4, different methods to solve FOCP have been shown. In Section 5, fractional delay differential equation (FDDE) is discussed, and Section 6 ends with a conclusion and discussions.

2 Mathematical preliminaries

In this section, the different definitions of FC and the symbols described are extensively used for numerical approximation. The definitions of fractional derivatives are not unique. Generally, the Grunwald-Letnikov (GL) derivative, the RL derivative, and the Caputo derivative are used [45,163–166]. But as the research progresses, other definitions of fractional derivatives like the CF fractional derivative and the AB fractional derivative are used [80].

2.1 Cauchy's fractional integral

This is an iterated integral, which can be written as a weighted integral with a weighting function as follows:

$$D^{-n}f(t) = \int \dots \int_0^t f(y)\mathrm{d}y \dots \mathrm{d}y = \int_0^t \frac{f(y)(t-y)^{n-1}}{(n-1)!}\mathrm{d}y. \tag{1}$$

2.2 Gamma function

The gamma function is utilized for generalizing the factorial notation as $z!$, for all real numbers such that $\Gamma(z) = (z-1)!$ for $z \in N$. If z is a complex number with a positive real part (i.e. $\mathrm{Re}(z) > 0$), then

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \mathrm{d}t. \tag{2}$$

This is called the gamma function, and its property is as follows:

$$\Gamma(z+1) = z\Gamma(z), \tag{3}$$

it is seen that when $z \in N^+$, then $\Gamma(z) = (z-1)!$.

2.3 RL fractional derivative

The varied form of RL fractional integral operator is nothing but the generality of the equation of n-fold integral given by Monje et al. [43]. The fractional derivative according to RL definition for the order of α, where $\alpha > 0$ of the mentioned function $f(t)$, is written as follows:

$$D^{\alpha}_{t_0,t} = I^{-\alpha}_{t_0,t} = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f(s) \mathrm{d}s. \tag{4}$$

After writing this, we are going to write left- and right-hand RL derivatives of the fractional order α of $f(t)$with as $\alpha > 0$:

$${}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} f(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n}[D^{-(n-\alpha)}_{t_0,t} f(t)] = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f(s) \mathrm{d}s \tag{5}$$

and

$${}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f} f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_{t}^{t_f} (s-t)^{n-\alpha-1} f(s) \mathrm{d}s. \tag{6}$$

In these two expressions, n denotes a non-negative integer, and $n-1 \le \alpha < n$.

2.4 Caputo fractional derivative

Left- and right-hand Caputo derivative of non-integer order α of $f(t)$ with $\alpha > 0$ is written as follows:

$${}^{C}_{t_0}D^{\alpha}_{t} f(t) = D^{-(n-\alpha)}_{t_0,t}[f^{(n)}(t)] = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \tag{7}$$

$${}^{C}_{t}D^{\alpha}_{t_f} f(t) = D^{-(n-\alpha)}_{t,t_f}[f^{(n)}(t)] = \frac{1}{\Gamma(n-\alpha)} \int_{t}^{t_f} (s-t)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s. \tag{8}$$

In these two expressions, n is a positive integer, and $n-1 \le \alpha < n$.

The description of fractional-order derivative (FD) is not single, and the different descriptions are not equivalent, and the difference and their relations were presented in previous studies [45,163–165,167]. The relationship between left- and right-hand RL derivative and Caputo derivative is created as follows:

$$ {}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t}f(t) = {}^{C}_{t_0}D^{\alpha}_{t}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(t_0)(t-t_0)^{k-\alpha}}{\Gamma(k+1-\alpha)}, \tag{9} $$

$$ {}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f}f(t) = {}^{C}_{t_0}D^{\alpha}_{t}f(t) + \sum_{k=0}^{n-1}\frac{f^{(k)}(t_f)(t_f-t)^{k-\alpha}}{\Gamma(k+1-\alpha)}. \tag{10} $$

The function $f\epsilon C^{n-1}[t_0, t]$ and $f^{(n)}$ is integrable in the interval $[t_0, t]$ and when the order α lies between 0 and 1, and then we can write as follows:

$$ {}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} = {}^{C}_{t_0}D^{\alpha}_{t} + \frac{f(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}, \tag{11} $$

$$ {}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f}f(t) = {}^{C}_{t_0}D^{\alpha}_{t}f(t) + \frac{f(t_f)(t_f-t)^{-\alpha}}{\Gamma(1-\alpha)}. \tag{12} $$

The RL derivative and GL derivative are related as shown below.

If the function $f\in C^n[t_0, t]$, then the RL derivative and GL derivative are equal to each other, i.e.

$$ {}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t}f(t) = {}^{\mathrm{GL}}_{t_0}D^{\alpha}_{t}f(t). \tag{13} $$

In De Oliveira and Tenreiro Machado [168], fractional integrability and differentiability of a given function have been discussed.

Lemma 1. *Consider a continuous function $f\in AC^n$ and the order of the derivative α, where $\alpha > 0$ and $n = [\alpha] + 1$, then*

$$ I^{\alpha}_{t_0,t}{}^{C}_{t_0}D^{\alpha}_{t}f(t) = f(t) - \sum_{j=0}^{n-1}\frac{f^{(j)}(t_0)}{j}(t-t_0)^{j}, \tag{14} $$

$$ I^{\alpha}_{t,t_f}{}^{C}_{t}D^{\alpha}_{t_f} = f(t) - \sum_{j=0}^{n-1}\frac{(-1)^{j}f^{(j)}(t_f)}{j}(t_f-t)^{j}. \tag{15} $$

In the case when the order$\alpha$is in between 0 to 1, then it can be written as follows:

$$ I^{\alpha}_{t_0,t}{}^{C}_{t_0}D^{\alpha}_{t}f(t) = f(t) - f(t_0), \tag{16} $$

$$ I^{\alpha}_{t,t_f}{}^{C}_{t}D^{\alpha}_{t_f}f(t) = f(t) - f(t_f). \tag{17} $$

Lemma 2. *If a continuous function f lies in $[t_0, t_f]$ and if the order is α, then it can be said that*

$$ {}^{C}_{t_0}D^{\alpha}_{t}I^{\alpha}_{t_0,t}f(t) = f(t), \tag{18} $$

$$ {}^{C}_{t}D^{\alpha}_{t_f}I^{\alpha}_{t,t_f}f(t) = f(t). \tag{19} $$

2.5 Generalized Taylor's formula

Let ${}^{C}_{t_0}D^{i\alpha}_{t}f(t)\in C([t_0, t_f])$ for $i = 0, 1, 2 \ldots (N+1)$ and α lies between 0 and 1. Then it can be said that

$$ f(t) = \sum_{k=0}^{N}\frac{({}^{C}_{t_0}D^{k\alpha}_{t}f)(t_0)}{\Gamma(k\alpha+1)}(t-t_0)^{k\alpha} + R_{N,t_0}(t), \tag{20} $$

where

$$R_{N,t_0}(t) = \frac{({}^{C}_{t_0}D^{(N+1)\alpha}_{t} f)(\xi)(t - t_0)^{(N+1)\alpha}}{\Gamma((N+1)\alpha + 1)}. \tag{21}$$

With the constraints, $t_0 \le \xi \le t$ for all $t \in [t_0, t_f]$, and another condition is ${}^{C}_{t_0}D^{N\alpha}_{t} \coloneqq {}^{C}_{t_0}D^{\alpha}_{t}\,{}^{C}_{t_0}D^{\alpha}_{t} \cdots {}^{C}_{t_0}D^{\alpha}_{t}$ (N number of times).

2.6 GL fractional derivative

Left- and right-hand GL derivative of order α of $f(t)$ with $\alpha > 0$ is written as follows:

$$ {}^{\mathrm{GL}}_{t_0}D^{\alpha}_{t} f(t) = \lim_{\Delta h \to 0} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t - jh), \tag{22}$$

$$h = \frac{t - t_0}{N}$$

$$ {}^{\mathrm{GL}}_{t}D^{\alpha}_{t_f} f(t) = \lim_{\Delta h \to 0} h^{-\alpha} \sum_{j=0}^{N} (-1)^j \binom{\alpha}{j} f(t + jh), \tag{23}$$

$$h = \frac{t_f - t}{N}$$

where h is the step size.

2.7 CF fractional derivative

CF derivative of $\alpha \in [0,1]$ and $a \in -\infty, t), f \in H^1(a, b), b > a$ [122]

$$ {}^{\mathrm{CF}}_{a}D^{\alpha}_{t} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f(s) \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right] \mathrm{d}s. \tag{24}$$

Here, $M(\alpha)$ is a normalized function such that it is written as $M(0) = M(1) = 1$.

As per the definition of Caputo derivative and CF derivative, the derivative of a fixed term is zero in both cases, but the only difference between the two definitions is that there is no singularity of the new kernel for $t = s$.

2.8 AB fractional derivative in the meaning of Caputo

AB derivative in the sense of Caputo (ABC) [122] of $\alpha \in [0,1]$ and $a \in [-\infty, t); f \in H^1(a, b), b > a$.

$$ {}^{\mathrm{ABC}}_{a}D^{\alpha}_{t}[f(t)] = \frac{M(\alpha)}{(1-\alpha)} \int_a^t f'(s) E_\alpha\left[-\alpha \frac{(t-s)^\alpha}{1-\alpha}\right] \mathrm{d}s, \tag{25}$$

where $M(\alpha)$ is the same as it is used in the expression of CF derivative, and E_α is the generalized ML function, which is used to solve equation (26) [122]:

$$\frac{\mathrm{d}^\alpha y}{\mathrm{d}t^\alpha} = ay, \tag{26}$$

and the generalized ML function is written as follows [122]:

$$E_\alpha(-t^\alpha) = \sum_{s=0}^{\infty} \frac{(-t)^{\alpha s}}{\Gamma(\alpha s + 1)}. \tag{27}$$

This new definition of FD describes the natural phenomenon as it provides accurate results. It is very easy to take the Laplace transform of this derivative which helps to deal with some real-world problems with the initial conditions accurately. It must be kept in mind that the original function cannot be recovered when $\alpha = 0$ excluding the case when the function disappears at the origin. To prevent this problem, this new definition of fractional derivative was proposed. Other types of fractional derivatives and integrals are presented in previous studies [163,169–176].

2.9 Advantages and disadvantages of different fractional derivatives

After discussing different FDs, the advantages and disadvantages of different FDs are presented in this section.

The advantages of different fractional derivatives are as follows:

1. While considering RL fractional derivative, it is not essential that an optional function be continuous at origin and differentiable.
2. Caputo's fractional derivative is advantageous as it includes initial as well as boundary conditions in the problem formulation [177]. In addition, Caputo's derivative of fixed is zero. It is considered to be the most appropriate FD to describe real-world problems.
3. AB FD has gained a lot of attention as being a new definition of FD it can depict the features of viscoelastic materials, which are orthodox in nature and thermal medium. It also depicts heterogeneities of material and several structures as well as media with non-identical scales. The non-locality, which is found in the new kernel, helps to describe the full memory within the structure as well as media with the non-identical scales as these cannot be expressed by using the classical FD as well as CF FD. CF FD helps to study the macroscopic behaviour of several materials which involves nonlocal exchanges, which are very crucial to defining the features and characteristics of the material [178]. AB FD is very handy to illustrate many real-world challenges of sciences, manufacturing, as well as technology.

Although these FDs have several advantages, they have a few disadvantages which are as follows:

1. When someone tries to model real-world problems with RL FDEs, then it was found that RL FD of a fixed value is not zero. Furthermore, if a function, which happens to be arbitrary, is a fixed value located at the origin, its FD has a singularity, i.e. it is exponential as well as has ML functions. So, RL FD lacks its application due to these disadvantages.
2. Caputo's FD always demands higher constraints of consistency for its differentiability for computing the FD of a function in the idea of Caputo, and hence, first, we need to determine its derivative. FD in the idea of Caputo is specified only for the differentiable functions, while the functions which do not have first-order derivatives could have FDs of all the orders less than one in the idea of RL.
3. For CF FD, the local kernel is used, and its derivative for $\alpha = 0$ does not provide the initial function. Besides this, the anti-derivative which is associated is not fractional [179].

3 Approximation of different fractional derivatives

For presenting a detailed demonstration of numerical approximations of different fractional derivatives, some symbols are used, and the symbol carries their meaning as a single-valued function $f(t)$ is defined in $[t_0, t_f]$, and the uniform time step is given by

$$\Delta t = \frac{t_f - t_0}{N},$$

where N is taken as a positive integer, and the grid point is considered as follows:

$$t_k = t_0 + k\Delta t, \tag{28}$$

where $k = 0,1,2, \ldots, N$, and approximation of $f(t_k)$ is considered as f_k.

3.1 Approximation of RL derivative

For a large number of functions, GL as well as RL derivatives are equivalent, and it is also especially observed when they are used in the applications. So, it can be said that RL derivative can be used to formulate a specific problem, and on the other hand, GL derivative is used to find the solution numerically [51]. Here, some methods are shown to approximate RL derivative, and the analysis is also shown.

The FD of order α is approximated in many ways. Here, $0 < \alpha \le 2$, and left- and right-hand RL derivatives can be easily approximated by equations (22) and (23), i.e. GL approximation. RL left- and right-hand derivative is approximated as follows:

$${}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} f(t_k) \approx {}^{\mathrm{RL}}_{t_0}\tilde{D}^{\alpha}_{t} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{k} \omega_j^{\alpha} f(t_{k-j}), \tag{29}$$

$${}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f} f(t_k) \approx {}^{\mathrm{RL}}_{t}\tilde{D}^{\alpha}_{t_f} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{nT-k} \omega_j^{\alpha} f(t_{k+j}), \tag{30}$$

where

$$\omega_j^{\alpha} = (-1)^j \binom{\alpha}{j}.$$

This approximation of RL derivative through standard GL approximation as (29) and (30) leads to an unstable numerical method to solve FDE [180]. When the order of the derivative in equations (29)–(30), α, lies between 1 and 2, the RL derivative is approximated by shifted GL approximation and the approximation leads to a stable numerical scheme for solving FDE. So, left- and right-hand RL derivatives can be approximated as follows:

$${}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} f(t_k) \approx {}^{\mathrm{RL}}_{t_0}\tilde{D}^{\alpha}_{t} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{k+1} \omega_j^{\alpha} f(t_{k-j+1}), \tag{31}$$

$${}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f} f(t_k) \approx {}^{\mathrm{RL}}_{t}\tilde{D}^{\alpha}_{t_f} f(t_k) = \Delta t^{-\alpha} \sum_{j=0}^{nT-k+1} \omega_j^{\alpha} f(t_{k+j-1}). \tag{32}$$

$L1$ and $L2$ scheme is another alternative way to approximate RL derivative of order α when α lies between 0 and 1 [181,182]. When α lies between 1 and 2, then $L2C$ scheme can be used [181,183]. It is the fact that RL derivative can be expanded in a power series, and it includes the derivate of integer order only. If an analytic function $f(.)$ is considered [184], then it can be written as follows:

$${}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} f(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(t - t_0)^{k-\alpha}}{\Gamma(k + 1 - \alpha)} x^k(t), \tag{33}$$

where

$$\binom{\alpha}{k} = \frac{(-1)^{k-1} \alpha \Gamma(k - \alpha)}{\Gamma(1 - \alpha)\Gamma(k + 1)}. \tag{34}$$

The drawback of using equation (33) in numerical approximation is when the error is small, then the huge number of terns needs to be summed up and the given function has a derivative of higher order, and this is not appropriate in the case of OCP. To overcome this drawback, another efficient numerical approximation has been used [185,186]. The numerical scheme is stated for RL derivative as follows:

It is taken that $\alpha \in (0,1)$ and $f(.) \in C^2[t_0, t_f]$, then it can be written as follows:

$$ {}^{\mathrm{RL}}_{t_0}D^{\alpha}_{t} f(t) \simeq A(\alpha, N)(t - t_0)^{-\alpha} f(t) + B(\alpha, N)(t - t_0)^{1-\alpha}\dot{f}(t) - \sum_{p=2}^{N} (\alpha, N)(t - t_0)^{1-p-\alpha} CV_p(t) - \frac{f(t_0)(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)}. \tag{35} $$

Here, N is taken as $N \geq 2$ and $V_p(t)$ is known as the solution of the given scheme, and it is written as follows:

$$ \left.\begin{aligned} \dot{V}_p(t) &= (1 - p)(t - t_0)^{p-2} f(t) \\ V_p(t_0) &= 0, p = 2, 3, 4, \ldots, N \end{aligned}\right\}. \tag{36} $$

There is another equation for the right-hand RL derivative:

$$ {}^{\mathrm{RL}}_{t}D^{\alpha}_{t_f} \simeq A(\alpha, N)(t_f - t)^{-\alpha} f(t) - B(\alpha, N)(t_f - t)^{1-\alpha}\dot{f}(t) + \sum_{p=2}^{N} C(\alpha, N)(t_f - t)^{1-p-\alpha} W_p(t) - \frac{f(t_f)(t_f - t)^{-\alpha}}{\Gamma(1 - \alpha)}. \tag{37} $$

Here, $W_p(t)$ is considered the solution of the given differential equation, and it is written as follows:

$$ \begin{cases} \dot{W}_p(t) = -(1 - p)(t_f - t)^{p-2} f(t), \\ W_p(t_f) = 0, p = 2, 3, 4, \ldots, N. \end{cases} \tag{38} $$

Here, $A(\alpha, N)$, $B(\alpha, N)$, and $C(\alpha, N)$ can be written as follows:

$$ A(\alpha, N) = \frac{1}{\Gamma(1 - \alpha)}\left[1 + \sum_{p=2}^{N} \frac{\Gamma(p - 1 + \alpha)}{\Gamma(\alpha)(p - 1)!}\right], \tag{39} $$

$$ B(\alpha, N) = \frac{1}{\Gamma(2 - \alpha)}\left[1 + \sum_{p=2}^{N} \frac{\Gamma(p - 1 + \alpha)}{\Gamma(\alpha - 1)p!}\right], \tag{40} $$

$$ C(\alpha, N) = \sum_{p=2}^{N} \frac{1}{\Gamma(2 - \alpha)\Gamma(\alpha - 1)} \frac{\Gamma(p - 1 + \alpha)}{(p - 1)!}, \tag{41} $$

The boundedness of the error is expressed as follows:

$$ |E_{tr}(t)| \leq \max_{\tau \in [t_0, t]} \lceil \ddot{f}(\tau) \rceil \frac{\exp((1 - \alpha)^2 + 1 - \alpha)}{\Gamma(2 - \alpha)(1 - \alpha)N^{(1-\alpha)}} (t - t_0)^{(2-\alpha)}. \tag{42} $$

More details have been given in [186]. By using equation (9), same deduction equation was obtained for Caputo's derivative.

3.2 Approximation of Caputo derivative

Here, the considered FDE is

$$ {}^{C}_{0}D^{\alpha}_{t} y(t) = f(t, y(t)). \tag{43} $$

After applying the first principle of calculus to this fractional-order differential equation, we can write equation (43) as an integral equation as follows:

When $t = t_{n+1}$ and $n = 0,1,2, \ldots$, we can write

$$ y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t f(\lambda, y(\lambda))(t - \lambda)^{(\alpha-1)} \mathrm{d}\lambda, \tag{44} $$

$$y(t_{n+1}) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t, \tag{45}$$

$$y(t_n) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t. \tag{46}$$

After subtracting (46) from (45), we obtain the following equation:

$$y(t_{n+1}) = y(t_n) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t + \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t. \tag{47}$$

This equation can be created as follows:

$$y(t_{n+1}) = y(t_n) + A_{\alpha,1} + A_{\alpha,2}, \tag{48}$$

$$A_{\alpha,1} = \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t,$$

and

$$A_{\alpha,2} = \frac{1}{\Gamma(\alpha)} \int_0^{t_n} (t_n - t)^{(\alpha-1)} f(t, y(t))\mathrm{d}t. \tag{49}$$

Utilizing the Lagrange interpolating $f(t, y(t))$ is approximated as follows:

$$f(t, y) = \frac{f(t_n, y_n)}{h}(t - t_{n-1}) - \frac{f(t_{n-1}, y_{n-1})}{h}(t - t_n), \tag{50}$$

So,

$$\begin{aligned} A_{\alpha,1} = \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)(t - t_n)\mathrm{d}t = \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \int_0^{t_{n+1}} y^{(\alpha-1)}(t_{n+1} - y - t_{n-1})\mathrm{d}y \\ - \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha)} \int_0^{t_{n+1}} y^{(\alpha-1)}(t_{n+1} - y - t_n)dy = \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \left\{ \frac{2h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\} \\ - \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha)} \left\{ \frac{h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\}. \end{aligned} \tag{51}$$

So, $A_{\alpha,1}$ can be written as follows:

$$A_{\alpha,1} = \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \left\{ \frac{2h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\} - \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha)} \left\{ \frac{h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\}. \tag{52}$$

In the same way, $A_{\alpha,2}$ can be deduced, and it can be written as follows:

$$A_{\alpha,2} = \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \left\{ \frac{ht_n^{\alpha}}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right\} + \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha+1)} t_n^{\alpha+1}. \tag{53}$$

So, the numerical approximation of FDE is written as follows:

$$\begin{aligned} y(t_{n+1}) = y(t_n) + \frac{f(t_n, y_n)}{h\Gamma(\alpha)} \left\{ \frac{2h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{ht_n^{\alpha}}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right\} \\ + \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha)} \left\{ \frac{h}{\alpha} t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1} \right\}. \end{aligned} \tag{54}$$

When the function is bounded, then the solution of the FDE is obtained from the following theorem.

Theorem 1. *Let*

$$ {}_0^C D_t^{\alpha} y(t) = f(t, y(t)). \tag{55}$$

This is a FDE, and for the bounded function $f(t, y(t))$, the computational approximation of $y(t)$ is given as follows:

$$\begin{aligned} y(t_{n+1}) = y(t_n) &+ \frac{f(t_n, y_n)}{h\Gamma(\alpha)}\left\{\frac{2h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{ht_n^{\alpha}}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1}\right\} \\ &+ \frac{f(t_{n-1}, y_{n-1})}{h\Gamma(\alpha)}\left\{\frac{h}{\alpha}t_{n+1}^{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t_n^{\alpha}}{\alpha+1}\right\} + R_n^{\alpha}(t), \end{aligned} \tag{56}$$

where

$$R_n^{\alpha}(t) < \frac{h^{3+\alpha}M}{12\Gamma(\alpha+1)}\{(n+1)^{\alpha} + n^2\}. \tag{57}$$

More details are found by Atangana and Owolabi [80].

It is observed that RL and Caputo's derivatives are correlated to equation (9). So, the Caputo derivative can be approximated by $L1$, $L2$, and $L2C$. In the study by Odibat [187], FDE where Caputo's fractional derivative was taken has been solved and discussed. There are other methods which were used to obtain the approximation of Caputo's derivative [188–190]. An algorithm was proposed for the approximation of Caputo's derivative [191]. For the second order, another difference scheme was deduced [192]. In the studies by Podlubny et al. and Pooseh et al. [169,193], the matrix approach was shown and that was used to find the solution to FDE. In this method, a matrix named triangular strip was taken to estimate the derivative and integral of arbitrary order. The conventional step-by-step technique was replaced by this method, and this matrix approach is very simple to implement. The extended version of this method can be used for dealing with nonlinear differential equations, and it was discussed in the studies by Podlubny et al. and Pooseh et al. [169,193].

3.3 Approximation of CF derivative

Here, the considered FDE is

$$ {}_0^{\mathrm{CF}} D_t^{\alpha} = f(t, y(t)). \tag{58}$$

It is written as follows:

$$\frac{M(\alpha)}{1-\alpha}\int_0^t y'(\tau)\exp\left[-\frac{\alpha}{1-\alpha}(t-\tau)\right]\mathrm{d}\tau = f(t, y(t)). \tag{59}$$

After applying the first principle of calculus to this FDE, equation (58) is redrafted as follows:

$$y(t) - y(0) = \frac{1-\alpha}{M(\alpha)}f(t, y(t)) + \frac{\alpha}{M(\alpha)}\int_0^t f(\tau, y(\tau))\mathrm{d}\tau, \tag{60}$$

$$y(t_{n+1}) - y(0) = \frac{1-\alpha}{M(\alpha)}f(t_n, y(t_n)) + \frac{\alpha}{M(\alpha)}\int_0^{t_{n+1}} f(t, y(t))\mathrm{d}t, \tag{61}$$

$$y(t_n) - y(0) = \frac{1-\alpha}{M(\alpha)}f(t_{n-1}, y(t_{n-1})) + \frac{\alpha}{M(\alpha)}\int_0^{t_n} f(t, y(t))\mathrm{d}t. \tag{62}$$

Finally, we obtain the following equation:

$$y(t_{n+1}) - y(t_n) = \frac{1-\alpha}{M(\alpha)}\{f(t_n, y_n) - f(t_{n-1}, y_{n-1})\} + \frac{\alpha}{M(\alpha)}\int_{t_n}^{t_{n+1}} f(t, y(t))\mathrm{d}t. \tag{63}$$

The given integral

$$\int_{t_n}^{t_{n+1}} f(t, y(t))\mathrm{d}t = \int_{t_n}^{t_{n+1}} \left\{\frac{f(t_n, y_n)}{h}(t - t_{n-1}) - \frac{f(t_{n-1}, y_{n-1})}{h}(t - t_n)\right\}\mathrm{d}t = \frac{3h}{2}f(t_n, y_n) - \frac{h}{2}f(t_{n-1}, y_{n-1}). \tag{64}$$

So, the numerical approximation can be written as follows:

$$y_{n+1} = y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right)f(t_n, y_n) + \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right)f(t_{n-1}, y_{n-1}). \tag{65}$$

This numerical approximation of the CF derivative has been done using the Adams-Bashforth method (ABM). For analyzing convergence as well as stability, the following theorems have been stated [80].

Theorem 2. *Let $y(t)$ be the solution to the FDE*

$${}_{0}^{\mathrm{CF}}D_t^{\alpha} = f(t, y(t)). \tag{66}$$

A function f is a continuous function, and then the numerical approximation of the FDE is written as follows:

$$y_{n+1} = y_n + \left(\frac{1-\alpha}{M(\alpha)} + \frac{3\alpha h}{2M(\alpha)}\right)f(t_n, y_n) + \left(\frac{1-\alpha}{M(\alpha)} + \frac{\alpha h}{2M(\alpha)}\right)f(t_{n-1}, y_{n-1}) + R_\alpha^n. \tag{67}$$

Here, $\|R_\alpha^n\| \le M$

Theorem 3. *Let $y(t)$ be the solution to the FDE*

$${}_{0}^{\mathrm{CF}}D_t^{\alpha} = f(t, y(t)) \quad \text{for every} \quad n \in N, \tag{68}$$

$$\|y_{n+1} - y_n\|_\infty < \frac{1-\alpha}{M(\alpha)}f(t_n, y_n) - f(t_{n-1}, y_{n-1})_\infty + \frac{\alpha h^{n+1}(n+1)!}{4M(\alpha)}, \tag{69}$$

where $\|f(t_n, y_n) - f(t_{n-1}, y_{n-1})\|_\infty \to 0$ *if* $n \to \infty$ so $\|y_{n+1} - y_n\|_\infty \to 0$ *as* $n \to \infty$.

3.4 Approximation of AB derivative in Caputo meaning

Here, the contemplated FDE is expressed as follows:

$${}_{0}^{\mathrm{ABC}}D_t^{\alpha} y(t) = f(t, y(t)). \tag{70}$$

After applying the first principle of calculus to this FDE, equation (70) is rewritten as follows:

$$y(t) - y(0) = \frac{1-\alpha}{\mathrm{ABC}(\alpha)}f(t, y(t)) + \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau))\mathrm{d}\tau, \tag{71}$$

$$y(t_{n+1}) - y(0) = \frac{1-\alpha}{\mathrm{ABC}(\alpha)}f(t_n, y_n) + \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)}\int_0^{t_{n+1}} (t_{n+1}-\tau)^{\alpha-1} f(t, y(t))\mathrm{d}t, \tag{72}$$

$$y(t_n) - y(0) = \frac{1-\alpha}{\mathrm{ABC}(\alpha)}f(t_{n-1}, y_{n-1}) + \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)}\int_0^{t_n} (t_n-\tau)^{\alpha-1} f(t, y(t))\mathrm{d}t. \tag{73}$$

Finally, we obtain the following obtained equation:

$$y(t_{n+1}) - y(t_n) = \frac{1-\alpha}{\mathrm{ABC}(\alpha)}\{f(t_n, y_n) - f(t_{n-1}, y_{n-1})\} + A_{\alpha,1} - A_{\alpha,2}. \tag{74}$$

$A_{\alpha,1}$ has been considered as follows:

$$A_{\alpha,1} = \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} f(t, y(t))\mathrm{d}t. \tag{75}$$

Another approximation has been taken as follows:

$$p(t) = \frac{t - t_{n-1}}{t_n - t_{n-1}} f(t_n, y_n) + \frac{t - t_{n-1}}{t_{n-1} - t_n} f(t_{n-1}, y_{n-1}). \tag{76}$$

So,

$$A_{\alpha,1} = \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} \left\{ \frac{t - t_{n-1}}{h} f(t_n, y_n) - \frac{t - t_n}{h} f(t_n, y_n) \right\}, \tag{77}$$

$$A_{\alpha,1} = \frac{\alpha f(t_n, y_n)}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left\{ \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} f(t - t_{n-1}) \right\} \mathrm{d}t - \frac{\alpha f(t_{n-1}, y_{n-1})}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left\{ \int_0^{t_{n+1}} (t_{n+1} - t)^{\alpha-1} f(t - t_{n-1}) \right\} \mathrm{d}t, \tag{78}$$

$$A_{\alpha,1} = \frac{\alpha f(t_n, y_n)}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left\{ \frac{2ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\} - \frac{\alpha f(t_{n-1}, y_{n-1})}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left\{ \frac{ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\}. \tag{79}$$

Similarly, $A_{\alpha,2}$ can be obtained as follows:

$$A_{\alpha,2} = \frac{\alpha f(t_n, y_n)}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left\{ \frac{ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right\} - \frac{f(t_{n-1}, y_{n-1})}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h}. \tag{80}$$

So, the numerical approximation is expressed as follows:

$$\begin{aligned} y_{n+1} = y_n &+ f(t_n, y_n) \left\{ \frac{1-\alpha}{\mathrm{ABC}(\alpha)} + \frac{\alpha}{\mathrm{ABC}(\alpha)h} \left[\frac{2ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right] - \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left[\frac{ht_n^{\alpha}}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right] \right\} \\ &+ f(t_{n-1}, y_{n-1}) \left\{ \frac{\alpha-1}{\mathrm{ABC}(\alpha)} - \frac{\alpha}{h\Gamma(\alpha)\mathrm{ABC}(\alpha)} \left[\frac{ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t^{\alpha+1}}{h\Gamma(\alpha)\mathrm{ABC}(\alpha)} \right] \right\}. \end{aligned} \tag{81}$$

This numerical approximation of ABC has been done using the ABM. For analyzing convergence as well as stability, the following theorems have been stated.

Theorem 4. *Allow $y(t)$ to solve the FDE:*

$$ {}^{\mathrm{ABC}}_{0}D_t^{\alpha} y(t) = f(t, y(t)). \tag{82}$$

If f is continuous, then FDE is numerically approximated as follows:

$$\begin{aligned} y_{n+1} = y_n &+ f(t_n, y_n) \left\{ \frac{1-\alpha}{\mathrm{ABC}(\alpha)} + \frac{\alpha}{\mathrm{ABC}(\alpha)h} \left[\frac{2ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right] - \frac{\alpha}{\mathrm{ABC}(\alpha)\Gamma(\alpha)h} \left[\frac{ht_n^{\alpha}}{\alpha} - \frac{t_n^{\alpha+1}}{\alpha+1} \right] \right\} \\ &+ f(t_{n-1}, y_{n-1}) \left\{ \frac{\alpha-1}{\mathrm{ABC}(\alpha)} - \frac{\alpha}{h\Gamma(\alpha)\mathrm{ABC}(\alpha)} \left[\frac{ht_{n+1}^{\alpha}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} + \frac{t^{\alpha+1}}{h\Gamma(\alpha)\mathrm{ABC}(\alpha)} \right] \right\} + R_{\alpha}. \end{aligned} \tag{83}$$

Here, $\|R_{\alpha}^{n}\| \le M$.

Theorem 5. *If the function f satisfies a Lipchitz condition, then the condition of stability of the numerical method – the ABM, which is applied on ABC can be accomplished if*
$\|f(t_n, y_n) - f(t_{n-1}, y_{n-1})\|_{\infty} \to 0$ *as* $n \to \infty$.

For the verification of the theorem, interested readers can refer to the study by Atangana and Baleanu [80].

4 Computational techniques for solving FOCP

Here, in this portion, numerical methods for the estimation of non-integer order-constrained dynamic optimization problems with free final time have been addressed. When the order of the FD is α and two continuous as well as differentiable functions F and G such that $F, G : [t_0, t_f] \times R^2 \to R$ and FD of order $\alpha \in (0, 1)$. So, the generally used FOCP is written as follows:

$$\min J(u) = \int_{i_0}^{t_f} F(t, x(t), u(t)), \tag{84}$$

which is subjected to the control system:

$$A\dot{x}(t) + B{}_{t_0}^{C}D_t^{\alpha} x(t) = G(t, x(t), u(t)), \tag{85}$$

and the preliminary constraint is given as follows:

$$x(t_0) = x_0. \tag{86}$$

In the following equations, the variables $x(t)$ and $u(t)$ stand for state and control varying, and the variables A and B must not be zero. The initial condition $x(t_0)$ is considered as constant and the state with final time, i.e. $x(t_f)$ can be considered fixed or free. For solving the FOCP, sufficient and necessary conditions must be taken into account, and it was discussed by Pooseh et al. [194]. The objective of optimal control is to obtain the optimal control $u^*(t)$ in such a way that the cost function $J(u)$ is minimized and at the same time dynamic constraints must be satisfied. Here, some numerical methods are shown, which are extensively used for obtaining the solution of FOCP.

4.1 Indirect methods for solving FOCP

For obtaining the necessary conditions that are required to solve FOCP, the Hamiltonian function is written as follows:

$$H(x(t), u(t), \lambda(t), t) = F(t, x(t), u(t)) + \lambda^T(t)G(t, x(t), u(t)). \tag{87}$$

Here, in this equation, $\lambda \in R^n$ is a vector, and it is called Lagrange multiplier. It was found in the study by Yüzbaşı [195], and it was observed that if (x, u) is used for solving of FOCP, i.e. equations (84)–(86), then there exists $\lambda(t)$ for which the nonlinear TPBVP is satisfied by (x, u, λ), and it was shown by He [95] that

$$A\dot{\lambda}(t) - B{}_{t}^{RL}D_{t_f}^{\alpha}\lambda(t) = -\frac{\partial H}{\partial x}, \tag{88}$$

$$A\dot{x}(t) + B{}_{t_0}^{C}D_t^{\alpha} x(t) = \frac{\partial H}{\partial \lambda}, \tag{89}$$

and

$$\frac{\partial H}{\partial u} = 0, t \in [t_0, t_f] t \in [t_0, t_f], \tag{90}$$

$$x(t_0) = x_0, \lambda(t_f) = 0. \tag{91}$$

The Hamiltonian function is defined by equation (87). Also u can be found out by $\frac{\partial H}{\partial u} = 0$. So, again the system is written as follows:

$$A\dot{\lambda}(t) - B{}^{RL}_{t}D^{\alpha}_{t_f}\lambda(t) = M(t, x(t), \lambda(t)), \tag{92}$$

$$A\dot{x}(t) + B{}^{C}_{t_0}D^{\alpha}_{t}x(t) = N(t, x(t), \lambda(t)), \tag{93}$$

$$x(t_0) = x_0, \lambda(t_f) = 0. \tag{94}$$

Here, the functions $M(t, x(t), \lambda(t))$ and $N(t, x(t), \lambda(t))$ are established, and they are written in terms of x and λ, which was discussed in the study by Yüzbaşı [195]. It can be said that if $F(t, x(t), u(t))$ and $G(t, x(t), u(t))$ are considered as cambered functions, then the rewritten equations consist of the necessary as well as the sufficient condition for solving the FOCP. The estimate of $x(t)\lambda(t)$ must be done, which is involved in equations (92)–(94).

The computational technique used for resolving the TPBVP, which is described by equations (92)–(94), was the operational matrix approach of differentiation, which was used extensively for nonlinear problems [196–200]. In Clenshaw and Curtis [201], the indirect method was proposed to solve TPBVP (92)–(94) by using the truncated Bassel series as follows:

$$x(t) \approx x_N(t) = \sum_{n=0}^{N} X_n J_n(t), \tag{95}$$

$$\lambda(t) \approx \lambda_N(t) = \sum_{n=0}^{N} \Lambda_n J_n(t). \tag{96}$$

Here, X_n and Λ_n are Bassel coefficients, and $J_n(t)$ is Bassel polynomial [196], and it is written as follows:

$$J_n(t) = \sum_{k=0}^{\left[\frac{N-n}{2}\right]} \frac{(-1)^k}{k!(k+n)!}\left(\frac{t}{2}\right)^{2k+n}, 0 \le t \le \infty. \tag{97}$$

Here, N is considered the order of the approximation. This method is more advantageous compared to the matrix approach, and it was clearly described in previous studies [196–200]. The system is made in such a way that the time required for computation is less compared to the matrix approach, and finally, $2N$ algebraic equations can be easily solved as the whole system has been constructed as follows:

$$A\dot{x}_N(t_j) + B{}^{C}_{t_0}D^{\alpha}_{t_j}x_N(t_j) = N(t_j, x_N(j), \lambda_N(j)) \quad j = 1, 2, 3, \ldots, N, \tag{98}$$

$$A\dot{\lambda}_N(t_k) - B{}^{RL}_{t_k}D^{\alpha}_{t_f}\lambda_N(t_k) = M(t_k, x(t_k), \lambda(t_k)) \quad k = 0, 1, \ldots, N-1, \tag{99}$$

$$x_N(t_0) = x_0, \lambda_N(t_f) = 0. \tag{100}$$

To solve this system of equations, Newton's algorithm can be used, and it can be implemented through MATLAB and MAPLE using fsolve command after obtaining the solution of the nonlinear algebraic equations. The Bessel coefficients are substituted in equations (95) and (96), and after that, it shows the improved accuracy of the numerical scheme.

Another method used Chebyshev polynomials, and here, the fractional derivative was chosen as Caputo's derivative. Here, the first method was based on optimization and then discretization, and finally, conditions of optimality were approximated. But in the present method, discretization of the state equation was done based on the scheme developed by Sweilam et al. [202] for integrating the functions, which are non-singular, and it was based on the Rayleigh-Ritz method to evaluate state as well as control separately. The numerical approximation of the Caputo derivative, as well as RL Derivative, was given using Chebyshev polynomials [203]. For the numerical approximation, the grid points chosen were called Chebyshev–Gauss–Lobatto nods in the interval $[0, L]$ in such a way that

$$t_r = \frac{L}{2} - \frac{L}{2}\cos\left(\frac{\pi r}{N}\right) \qquad r = 0, 1, \ldots, N. \tag{101}$$

The numerical approximation x_N of x is expressed using shifted Chebyshev polynomials as follows:

$$x_N(t) = \sum_{n=0}^{N_1} a_n T_n^*(t) \tag{102}$$

and

$$a_n = \frac{2}{N}\sum_{r=0}^{N_2} x(t_r) T_n^*(t_r). \tag{103}$$

Here, $T^*(t)$ is called shifted Chebyshev polynomials, which were discussed in the study by Trefethen [203].

Now the approximation of Caputo's derivative of $x$at the time t_s is mentioned as follows:

$${}_{t_0}^{C}D_t^{\alpha} x_N(t_s) \cong \sum_{r=0}^{N} x(t_r) d_{s,r}^{\alpha} \quad \alpha > 0, \tag{104}$$

and $s, r = 0, 1, \ldots, N$.

$d_{s,r}^{\alpha}$ can be expressed as follows:

$$d_{s,r}^{\alpha} = \frac{4\theta_r}{N}\sum_{n=\alpha}^{N}\sum_{j=0}^{N}\sum_{k=\alpha}^{n} \frac{n\theta_n}{a_j}\frac{(-1)^{n-k}(n+k-1)!\Gamma(k-\alpha+1/2)T_n^*(t_r)T_j^*(t_s)}{L^{\alpha}\Gamma(k+1/2)(n-k)!\Gamma(k-\alpha-j+1)\Gamma(k-\alpha+j+1)}. \tag{105}$$

Here, $\theta_0 = \theta_N = \frac{1}{2}$ and $\theta_i = 1$ and $i = 1,2,\ldots, N-1$.

The approximation of the right-hand R-L derivative is given as follows:

$${}_{t}^{\mathrm{RL}}D_{t_f}^{\alpha} x(t) = \frac{x(t_f)}{\Gamma(1-\alpha)}(t_f - t)^{-\alpha} + \frac{J(t;x)}{\Gamma(1-\alpha)}. \tag{106}$$

Here, $J(t;x)$ is calculated as follows:

$$J(t;x) = \int_t^{t_f} (\tau - t)^{-\alpha} x(\tau)\mathrm{d}\tau, \; 0 < t < t_f. \tag{107}$$

The approximation of $x(t)$ after using shifted Chebyshev polynomials is expressed as follows:

$$x(t) \approx P_N(t) = \sum_{k=0}^{N_2} a_k T_k^*\left(\frac{2t}{t_f} - 1\right), \tag{108}$$

and a_k is obtained as follows:

$$a_k = \frac{2}{N}\sum_{j=0}^{N_2} x(t_j) T_k^*\left(\frac{2t_j}{t_f} - 1\right). \tag{109}$$

Here,

$$t_j = \frac{t_f}{2} - \left(\frac{t_f}{2}\right)\cos\left(\frac{\pi j}{N}\right) j = 0, 1, \ldots, N. \tag{110}$$

So, the right-hand RL derivative is approximated as follows:

$${}_{t}^{\mathrm{RL}}D_{t_f}^{\alpha} x(t) \approx \frac{x(t_f)}{\Gamma(1-\alpha)}(t_f - t)^{-\alpha} + \frac{J(t;P_N)}{\Gamma(1-\alpha)}. \tag{111}$$

Here, it was shown that the choice of Chebyshev base is advantageous as it does not deal with the Runge phenomenon for which a strong processor is needed. But it has been shown that by using Chebyshev polynomials, system (92)–(94) can be approximated. But it has been seen that Gauss-Legendre Quadrature nodes were not obtained. The distinction between Clenshaw-Curtis and Gauss-Legendre Quadrature is crucial when they deal with some numerical methods and their elements [204]. The extended numerical method was used as well, and a direct numerical method was used to solve FOCP by utilizing Jacobi polynomials for $(A, B) = (0,1)$ in (92)–(94), where the approximation of state and the co-state equation was done [120,155] as follows:

$$x(t) = \sum_{j=1}^{N} c_j P_j(t),\ \lambda(t) = \sum_{j=1}^{N} d_j P_j(t). \tag{112}$$

Here, $P_j(t)$ is considered as the shifted Legendre's polynomial, and it uses conditions of orthonormality and c_j and d_j are polynomial coefficients, $j = 1,2, ...,N$. Here, N is the no. of polynomials selected for solving the problem. This numerical scheme was developed on Legendre's polynomial based on an orthonormal basis. In previous studies [111,120,205,206], the operational matrix has been formed to represent Caputo's derivative, and it can be written as follows:

$$D^{\alpha} = \begin{bmatrix} D_{11} & D_{12} \ldots & D_{1(m+1)} \\ D_{21} & D_{22} \ldots & D_{2(m+1)} \\ D_{(m+1)1} & D_{(m+1)2} \ldots & D_{(m+1)(m+1)} \end{bmatrix}. \tag{113}$$

Here,

$$D_{ij} = \hat{B}_{i-1j-1,} 1 \le i, j \le m + 1, \tag{114}$$

$$\hat{B}_{ij} = \sqrt{(2i + 1)(2j + 1)} \sum_{\alpha}^{i} \sum_{l=0}^{j} \frac{(-1)^{i+k+j+l}(i + k)!(j + l)!}{(i - k)!k!\Gamma(k + 1 - \alpha)(j - l)!(l!)^2(l + k - \alpha + 1)}. \tag{115}$$

Here, the optimization problem is converted into algebraic equations to solve the given FOCP. For solving the problem at the very beginning, the given unknown function was expanded into modified Jacobi polynomials, and then the non-integer derivative of the given function, which is not known, was taken into the form of polynomials, which was named after Jacobi. The convergence of the scheme using Jacobi polynomials was discussed briefly in the study by Nemati et al. [207].

There is no obligation that the selection of the orthonormal polynomials is a basic function. Orthonormal polynomials were chosen because it was proven that they would reach a sparse matrix that is numerically stable also, and the characteristics of the polynomials help to obtain the expected matrices. Any orthonormal polynomials can be chosen to solve the problem, but if the properties of the orthonormal polynomials are modified, then it is advantageous while solving the problem numerically as it improves the efficiency of the numerical method.

Another numerical scheme was made which stood on another polynomial method for the numerical approximation, and it is known as Bernstein polynomials which were mentioned in the study by Yousefi et al. [208]. A novel operational matrix approach that used the Ritz procedure for approximating the fractional derivative of the given basis. The procedure, which is mentioned, is the same as it was described by Lotfi and Yousefi [111], where the Caputo derivative was taken, and it was approximated after using Bernstein polynomials and stated as follows:

$$D^{\alpha} = \begin{bmatrix} b_{00} & b_{01} \ldots & b_{0m} \\ b_{10} & b_{11} \ldots & b_{1m} \\ b_{m0} & b_{m1} & b_{mm} \end{bmatrix}. \tag{116}$$

Here,

$$b_{ij} = \sum_{k=0}^{\min(m-[\alpha]-1, m-i)} \sum_{q=0}^{m} \sum_{r=0}^{m-q} \sum_{l=0}^{\min(j,q)} (-1)^{2m-i-k-r+j} \times \frac{(2l + 1)}{\binom{m}{j}} \binom{m}{i} \binom{m - i}{k} \binom{m + l + 1}{m - q} \binom{m - l}{m - q} \binom{m - q}{r}$$
$$\times \frac{(m - k)!}{(2m - \alpha - r - k + 1)\Gamma(m - k - \alpha + 1)} 0 \le i, j \le m. \tag{117}$$

This new method of approximation is very efficient as well as flexible for solving FOCP, and after using this method, it was seen that the initial as well as boundary conditions are satisfied. This new method gives the higher term of the error of the operating matrix made for approximating Caputo's derivative and the matrix written for Bernstein polynomials, which was written in the form of Gram determinant, which is tending to zero.

Another method was proposed [209] after taking the Legendre multiwavelet basis and collocation method to numerically approximate the solution of the given FOCP. Here, the state $x(t)$ is written, which is defined in $[t_0, t_f]$ as follows:

$$x(t) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} c_{ij}\psi_{ij} = C^T\psi(t). \tag{118}$$

Here, k is denoted as a positive integer.

And

$$\psi_{nm} = \begin{cases} \sqrt{2m+1}\dfrac{2^{\frac{k}{2}}}{\sqrt{t_f - t_0}} P_m\left(\dfrac{2^k(t-t_0)}{t_f - t_0} - n\right), & \dfrac{n(t_f - t_0)}{2^k} + t_0 \le t < \dfrac{(n+1)(t_f - t_0)}{2^k} + t_0. \\ 0, \text{ otherwise} \end{cases} \tag{119}$$

Here, $m = 0, 1, 2, \ldots, M-1$, $n = 0, 1, 2, \ldots, 2^k - 1$.

$P_m(t)$ is defined as a shifted form of Legendre polynomials of the order m and c_{ij}, which is obtained as follows [210]:

$$c_{ij} = \int_{t_0}^{t_f} \psi_{ij}(t)x(t)\mathrm{d}t. \tag{120}$$

Herein, $m = 0, 1, \ldots, M-1$, $n = 0, 1, \ldots, 2^k - 1$.

This method is advantageous as it helps to enhance the accuracy of the numerical method. The approximation of state, co-state, and control using the Legendre multiwavelet basis is expressed as follows:

$$x(t) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} (t - t_0)cx_{ij}\psi_{ij}(t) + x_0, \tag{121}$$

$$\lambda(t) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} (t - t_f)c\lambda_{ij}\psi_{ij}(t), \tag{122}$$

$$u(t) = \sum_{i=0}^{2^k-1} \sum_{j=0}^{M} cu_{ij}\psi_{ij}(t). \tag{123}$$

Despite being an accurate and flexible method, it was seen that it provides few good and satisfactory results [209].

Another numerical method was presented, where $(A, B) = (0, 1)$ in equations (92)–(94) where fractional differential equation was rewritten as integral equations of Volterra as follows:

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t} (t-\tau)^{\alpha-1} N(\tau, x(\tau), \lambda(\tau))\mathrm{d}\tau, \tag{124}$$

$$\lambda(t) = -\frac{1}{\Gamma(\alpha)}\int_{t}^{t_f} (\tau - t)^{\alpha-1} M(\tau, x(\tau), \lambda(\tau))\mathrm{d}\tau. \tag{125}$$

These are known as forwarding and backward Volterra integral equations [210], and there are several methods or schemes mentioned in the literature for obtaining the solution of the initial value problem numerically as was shown in using the multistep method [211] and Adams method [212,213].

For obtaining the numerical approximation of the problem of equations (124) and (125), Agrawal [214] divided the interval of time $[t_0, t_f]$ in equal intervals of N in such a way that $Nh = 1$, and herein, h is the step size, and the state and co-state equations can be rewritten at the node t_i as follows:

$$x_i(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{ih}(ih - \tau)^{\alpha-1}N(\tau, x(\tau), \lambda(\tau))\mathrm{d}\tau, \tag{126}$$

$$\lambda_i(t) = -\frac{1}{\Gamma(\alpha)}\int_{ih}^{t_f}(\tau - ih)^{\alpha-1}M(\tau, x(\tau), \lambda(\tau))\mathrm{d}\tau. \tag{127}$$

Equations (126) and (127) can be reduced to a number of $2N$ linear equations consisting of unknowns $2N$ equations, that is easily solved by using the Gaussian elimination method and conjugate gradient method [120].

In the study by Özdemir [215], FOCP was solved for a system, which was distributed in the presence of a cylindrical coordinate system and where fractional derivative was taken as RL sense. The cost function is the function of the state as well as controlling factors and the given structure is defined in terms of FPDE. The numerical scheme involved in it is based on the separation of the variables to solve the problem. First, the FPDE was broken into two differential equations named FDE and the differential equation of Bessel. The space parameters are eliminated, and the FOCP is converted to a generalized form of state and control using the Eigen function for solving the problem numerically, and then the GL method was used for the numerical approximation of the given problem [102]. When the discretization of time was enhanced, then the solution was reaching convergence.

By using the same way, a new method was proposed by Sabouri et al. [216], where an artificial neural network with its weights, which were not known, was used to numerically approximate the FOCP. The extreme utilization of intelligent perceptron for approximating the functions, which were not linear, and it was able to approximate state, co-state, and control, which satisfies initial as well as boundary conditions. So, it is presented as follows:

$$\begin{cases} x_K(t, \psi_k) = A(t) + B(t)K(t, \psi_x), \\ \lambda_K(t, \psi_\lambda) = F(t) + G(t)K(t, \psi_\lambda), \\ u_K(t, \psi_u) = C(t) + D(t)K(t, \psi_u) \end{cases} \tag{128}$$

where $A(t), B(t), C(t), D(t), F(t),$ and $G(t)$ are the functions of a single and real variable, and it is also seen that the initial as well as boundary conditions are satisfied by the numerical approximation of x_K, λ_K, and u_K. If the preliminary constraint is carried as $x(0) = 0$, then it is required to select $A(t)$ and $B(t)$ in a way such that $x_K(0, \psi_x) = 0$ and $A(t)$ and $B(t)$ can be chosen as $A(t) = 0$ and $B(t) = t$. The weight vectors which are related to state, co-state, and control are considered as ψ_x, ψ_λ, and ψ_u, respectively. Then the approximated equation by the substituting equation (128) in equations (124) and (125) is as given as follows:

$$x_K(t, \psi_K) = x_0 + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - \tau)^{\alpha-1}N_K(\tau, x_K(t, \psi_K), \lambda_K(t, \psi_\lambda))\mathrm{d}\tau, \tag{129}$$

$$\lambda_K(t, \psi_\lambda) = -\frac{1}{\Gamma(\alpha)}\int_{t}^{t_f}(\tau - t)^{\alpha-1}M_K(\tau, x_K(t, \psi_K), \lambda_K(t, \psi_\lambda))\mathrm{d}\tau. \tag{130}$$

For solving these equations (129) and (130), squared residual error functions have been introduced, and it is presented as follows:

$$R_x(\Psi, t) = \left[x_K(t, \psi_x) - x_0 - \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t - \tau)^{\alpha-1}N_K(\tau, x_K(t, \psi_x), \lambda_K(t, \psi_\lambda))\mathrm{d}\tau\right]^2, \tag{131}$$

$$R_\lambda(\Psi,t)=\left[\lambda_K(t,\psi_\lambda)+\frac{1}{\Gamma(\alpha)}\int_t^{t_f}(\tau-t)^{\alpha-1}M_K(\tau,x_K(t,\psi_x),\lambda_K(t,\psi_\lambda))\mathrm{d}\tau\right]^2. \tag{132}$$

Here, $\Psi=(\psi_x,\psi_\lambda,\psi_u)$ is a vector consisting of the weights for approximating the functions given in equation (128). For solving this set of equations (131) and (132), the total interval $[t_0,t_f]$ into m number of subintervals by using computational integration procedures like the trapezoidal rule, and Simpson's rule the problem is transformed into an unrestrained optimization problem, which is written as follows:

$$\min_{\Psi} R(\Psi)=\sum_{i=1}^{m}(R_\lambda(\Psi,t_i)+R_x(\Psi,t_i)). \tag{133}$$

This problem can be easily solved using some optimization algorithms like the quasi-Newton method. Let $\Psi^*=(\psi_x^*,\psi_\lambda^*,\psi_u^*)$ be the optimized solution to the given problem of optimization mentioned in equation (133). The artificial neural networks are able to approximate globally, so the resulting weights are always in optimal values. This example was given to show the convergence of the resulting weights of the neural network. After putting the resulting optimal weights in the state, co-state, and control equations in equation (128), the final solution of the problem given in equations (84)–(86) is written as follows:

$$x_K(t,\psi_x^*)=A(t)+B(t)K(t,\psi_x^*), \tag{134}$$

$$\lambda_K(t,\psi_\lambda^*)=F(t)+G(t)K(t,\psi_\lambda^*), \tag{135}$$

$$u_K(t,\psi_u^*)=C(t)+D(t)K(t,\psi_u^*). \tag{136}$$

For getting more accuracy in the solution of the given problem, more neurons can be used easily to make a large neural network, and there are a lot of heuristic algorithms, which can be used. Intelligent neural networks are global approximations, so their value of the weights must be converged to some value, which is optimal.

Another iterative method has been proposed, known as VIM for solving equations (92)–(94) with $(A,B)=(0,1)$ for solving FDE using VIM, and the equation is taken as follows [217]:

$${}_{t_0}^{C}D_t^{\alpha}x(t)+L[x(t)]+[N(x,t)]=f(t). \tag{137}$$

Here, ${}_{t_0}^{C}D_t^{\alpha}$ is Caputo's derivative, L and N are linear and nonlinear operators, and a continuous function is denoted by $f(t)$, and equation (137) is rewritten like

$$x_{n+1}(t)=x_n(t)+\int_0^t\lambda(\tau)[{}_{t_0}^{C}D_t^{\alpha}x_n(\tau)-L[\tilde{x}_n(\tau)]-N[\tilde{x}_n]-f(\tau)]\mathrm{d}\tau,\ t>0,\ \alpha>0. \tag{138}$$

Here, λ is the Lagrange multiplier, and it is easily recognized by the variational theory of FC. $L[\tilde{x}_n(\tau)]$ and $N[\tilde{x}_n]$ are the variations, and there is some restriction on it such that both $\delta L[\tilde{x}_n(\tau)]$ and $\delta N[\tilde{x}_n]$ are equal, and their value is zero. The way for approximating the given problem is done by approximating the zeroth term included in the equation which is used for approximating the problem. For linear systems, the solving of the given problem is written like $x(t)=\lim_{n\to\infty}x_n(t)$, but it is observed that for the nonlinear system of the equation, this limit cannot be calculated directly, and it needs to be approximated. For approximating the solution of nonlinear problems, n must be taken as a large value, so that it will lead to $x(t)\approx x_n(t)$.

In the study by Tang et al. [218], state and co-state equations for the problem given in the equations (92)–(94) are approximated by using VIM, and it can be written as follows:

$$x_{n+1}(t)=x_n(t)-I_{t_0,t}^{\alpha}[{}_{t_0}^{C}D_t^{\alpha}x_n(t)-N(t,x_n(t),\lambda_n(t))], \tag{139}$$

$$\lambda_{n+1}(t)=\lambda_n(t)-I_{t,t_f}^{\alpha}[{}_{t}^{RL}D_{t_f}^{\alpha}\lambda_n(t)-M(t,x_n(t),\lambda_n(t))]. \tag{140}$$

At the very beginning of the iterative method of approximation, the initial approximation is taken first. The numerical approximation of the problem given in equations (139) and (140) must satisfy initial, final, as well as transversality conditions, so $x(t_0) = x_0, \lambda(t_f) = 0$. In everyone and all reiterations, approximations are used in the upcoming iteration, and the solution to the problem was approximated at the same time. The technique mentioned provides the convergence as well as the successive way to approximate the true solution of the given problem, and it was also shown that after obtaining the solution of the problem, it styles the classical solution of the problem in the order of the fractional derivative tends to one, i.e. it becomes an ordinary differential equation.

In the study by Tang et al. [218], unified framework, as well as an integral fractional pseudo-spectral method, was proposed for solving FOCP. Fractional pseudo spectral integration matrices are nothing but the generalization of classical pseudo spectral integration matrices, and after the generalization, fractional pseudo spectral integration matrices are accurate, efficient, and most importantly they are numerically stable, which makes it popular to approximate the OCP of fractional order numerically. First, the polynomial was taken which was known as Lagrange interpolating polynomials and then the fractional integral of the polynomial was transformed into the integral, which was known as the Jacobi-weight integral, and it was calculated easily by using the Jacobi-Gauss quadrature. These were taken with Lagrange interpolating polynomials, and their barycentric representation and the barycentric weights were taken for Radau, flipped Radau, as well as Gauss-type points for the polynomials known as Jacobi polynomials, which will result in accurate, stable, as well as a structured numerical method for solving the problem of fractional pseudo spectral integration matrices in the presence of the huge number of points which are of Jacobi type. In Biswas and Sen [219], fractional-order pseudo-state space approach was proposed, and the problem was solved by using G-L approximation.

In Biswas and Sen [220], a numerical method was proposed, and the problem was discretized using the GL approximation and then the problem of $2N$ equations with $2N$ unknowns was solved by using the Gaussian-elimination method or another linear solver [114]. For the numerical approximation, the entire time was divided into N subinterval. The time taken by node j is considered as $t_j = jh$, and the step size h was calculated as follows:

$$h = \frac{t_f - t_0}{N}. \tag{141}$$

After using this approximation, the approximated equation of the problem is given as follows:

$$h^\alpha \sum_{j=0}^{m} \omega_j^\alpha x_{m-j} = N(mh, x_m(t), \lambda_m(t)), m = 1, 2, \ldots, N, \tag{142}$$

$$-h^\alpha \sum_{j=0}^{N-m} \omega_j^\alpha \lambda_{m+j} = M(mh, x_m(t), \lambda_m(t)), m = N - 1, N - 2, \ldots, 0. \tag{143}$$

For the solution of Discrete-Time FOCP, readers can refer to previous studies [121,221–223].

4.2 Direct methods for solving FOCP

In the study by Almeida and Torres [224], Caputo derivative operator ${}_{t_0}^{C}D_t^\alpha$ can be approximated using equation (35), and with the relation mentioned in equations (9) and (10), it can be rewritten as follows:

$$\begin{aligned} A\dot{x}(t) + B\Bigg[&A(\alpha, N)(t - t_0)^{-\alpha}x(t) + B(\alpha, N)(t - t_0)^{(1-\alpha)}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{(1-p-\alpha)}V_p(t) \\ &- \frac{x(t_0)(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)}\Bigg] = G(t, x(t), u(t)). \end{aligned} \tag{144}$$

$$\dot{x}(t) = \frac{G(t, x(t), u(t)) - B\left[A(\alpha, N)(t - t_0)^{-\alpha}x(t) + B(\alpha, N)(t - t_0)^{(1-\alpha)}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{(1-p-\alpha)}V_p(t) - \frac{x(t_0)(t - t_0)^{-\alpha}}{\Gamma(1-\alpha)}\right]}{A + B \times B(\alpha, N)(t - t_0)^{(1-\alpha)}}. \tag{145}$$

After including the vector $\bar{V}(t) = (V_2(t), V_3(t), \ldots, V_N(t))$ in the equation, we obtain

$$\bar{G}(t, x, \bar{V}, u) = \frac{G(t, x(t), u(t)) - B\left[A(\alpha, N)(t - t_0)^{-\alpha}x(t) + B(\alpha, N)(t - t_0)^{(1-\alpha)}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha, N)(t - t_0)^{(1-p-\alpha)}V_p(t) - \frac{x(t_0)(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)}\right]}{A + B \times B(\alpha, N)(t - t_0)^{(1-\alpha)}}. \tag{146}$$

A newly formed OCP is written as follows:

$$\min \bar{J}(x, \bar{V}, u) = \int_{t_0}^{t_f} F(t, x(t), u(t)). \tag{147}$$

This is subjected to the dynamic constraints

$$\left.\begin{array}{c} \dot{x}(t) = \bar{G}(t, x, \bar{V}, u) \\ \dot{V}_p(t) = (1 - p)(t - t_0)^{(p-2)}x(t) p = 2, 3, \ldots, N, \end{array}\right\}. \tag{148}$$

Here, for this problem, the initial conditions are written as follows:

$$\left.\begin{array}{c} x(t_0) = x_0, \\ V_p(t_0) = 0, p = 2, 3, \ldots, N. \end{array}\right\}. \tag{149}$$

For solving this set of equations (147)–(149), the first Hamiltonian function needs to be formed and then PMP needs to be used to solve the OCP by solving an ordinary differential equation or one can use the discretization method of Euler for obtaining the approximation of the problem given in equations (147)–(149). It is discussed briefly in the study by Almeida and Torres [224].

$f(t) \in L^2[0,1]$ is expanded by using functions named rationalized Haar (RH) in such a way that it is written as follows:

$$f(t) = \sum_{r=0}^{\infty} a_r \mathrm{RH}(r, t) r = 1, 2, \ldots,. \tag{150}$$

The RH function was defined in the study by Marzban and Razzaghi [225] in [0,1), and a_r can be written as follows:

$$a_r = 2^i \int_0^1 f(t)\mathrm{RH}(r, t)\mathrm{d}t, \; r = 0, 1, \ldots, \tag{151}$$

When $r = 2^i + j - 1$, $i = 0, 1, 2, \ldots, j, j = 1, 2, \ldots, 2^i$, and when $i = j = 0$, then $r = 0$.

Let $i = 0,1,2, \ldots,\infty$, then RH functions up to its K terms of infinite series given in the equation (150) is expressed as follows:

$$f(t) \simeq \sum_{r=0}^{K-1} a_r \mathrm{RH}(r, t) = P^T \Phi_K(t).$$

Here, $K = 2^{(\alpha+1)}$, $\alpha = 0,1,2,\ldots,$ and RH functions vector Φ_K and its coefficient vector P can be expressed as follows:

$$P = [a_0, \; a_1, \ldots \; , a_{K-1}]^T; \Phi_K(t) = [\Phi_0(t), \; \Phi_1(t), \ldots, \; \Phi_{K-1}(t)]^T. \tag{152}$$

Here,

$$\Phi_r(t) = \mathrm{RH}(r, t), r = 0, 1, \ldots, K - 1. \tag{153}$$

Haar functions are very popular as it guarantees convergence when the functions are expanded, and for this positive ability of the functions, they are extensively used in Haar functions theory. In Hosseinpour and Nazemi [226], the direct method in terms of Haar wavelet was used to solve FOCP, where Caputo's derivative was taken in the problem. In this work, the Haar wavelet was used for transforming state as well as control variables in such a way that it would lead to parameters of nonlinear programming at the points of collocation. The fractional Caputo's derivative was taken, and the state and controlling factors were estimated by employing the Haar wavelet, and the number of collocation points was taken as K. The approximation using Haar wavelets is written as follows:

$$\left.\begin{array}{c} {}_{t_0}^{C}D_t^{\alpha}x(t) \approx C^T\Phi(t) \Rightarrow x(t) \approx C^T U_{K\times K}^{\alpha}\Phi(t) + x(t_0), \\ u(t) \approx D^T\Phi(t) \end{array}\right\}, \tag{154}$$

Here, $U_{K\times K}^{\alpha}$ is known as OMI of the fractional order of α.

So,

$$\left.\begin{array}{c} C^T = [c_1, \ c_2, \ \ldots, \ c_K]^T, \\ D^T = [d_1, \ d_2, \ \ldots, \ d_K]^T. \end{array}\right\} \tag{155}$$

It is observed that when the collocation method with Haar is used to solve FOCP directly, then the variables used in nonlinear programming must be taken as a vector consisting of an unknown coefficient for the FD to represent the state as well as control variables. After following this way, FOCP can be easily transformed into a problem of nonlinear programming, and it can be easily solved by using software named Lingo 11 [227]. It was also shown that the proposed method named orthogonal collocation helps to increase the convergence rate when the point of collocation enhances.

5 Numerical methods for solving fractional delayed optimal control problems (FDOCPs)

A FDOCP is considered a special problem. For example, FDDE is written as follows:

$$\left.\begin{array}{c} {}_{0}^{C}D_t^{\alpha}y(t) = f(t, y(t), y(t-\tau)), t \geq 0, m-1 < \alpha \leq m, \\ y(t) = \varphi(t), t \leq 0, \end{array}\right\}, \tag{156}$$

Here, $\varphi(t)$ is the initial value and m is an integer.

When the problem is having no delay and integer order, then it is written as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(t, y(t)), y(0) = y_0. \tag{157}$$

$t \in [0, T]$ and $t_j = jh, j = 0,1,2,\ldots, N$ and the step size $h = \frac{T}{N}$.

Take $y_j = y(t_j), j = 1, 2, \ldots, n$.

This ordinary differential equation can be approximated as follows:

$$y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\mathrm{d}s. \tag{158}$$

This integral is approximated numerically by using the trapezoidal quadrature formula and written as follows:

$$\int_{t_n}^{t_{n+1}} f(s, y(s))\mathrm{d}s \approx \frac{h}{2}(f(t_n), y(t_n)) + f(t_{n+1}, y(t_{n+1}))). \tag{159}$$

So, the solution of FDDE is approximated as follows:

$$y_{n+1} = y_n + [f(t_n, y(t_n) + f(t_{n+1}, y(t_{n+1})]. \tag{160}$$

This numerical scheme is named an implicit method, and here, the FDDE cannot be solved directly. Here is a predictor term y_{n+1}^{p}, which is obtained from an explicit method.

So, the predictor equation is presented as follows:

$$y_{n+1}^{p} = y_n + hf(t_n, y_n). \tag{161}$$

By using the one-step Adams-Bashforth-Moulton method, the problem given in equation (160) is approximated as follows:

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]. \tag{162}$$

For the fractional-order delay differential equation, the algorithm was used [228]. Hereafter, considering the delay, the equations can be written as follows.

5.1 When the delay is constant

If a constant delay(τ) is considered, then $(t_j - \tau)$ is not a grid point t_n for any n. If $(m - \delta)h = \tau$ and $0 \leq \delta < 1$. If δ is taken as zero, then $y(t_n - \tau)$ can be written as follows:

$$y(t_n - \tau) \approx \begin{cases} y_{n-m}, & \text{if} \quad n > m, \\ \varphi_n, & \text{if} \quad n \leq m. \end{cases} \tag{163}$$

It is seen that $0 < \delta < 1$, then we cannot calculate $y(t_n - \tau)$ directly. $(m - 1)h < \tau < mh$.

Here, $y(t_n - \tau)$ can be approximated by using v_{n+1} as follows:

$$v_{n+1} = \delta y_{n-m+2} + (1 - \delta)y_{n-m+1}. \tag{164}$$

From equation (164), it is observed that if m is greater than 1, then the equation can be solved directly. But when m is equal to 1 and δ is not equal to 0, then the predictor term must be included in the numerical approximation.

$$v_{n+1} = \delta y_{n+1}^{p} + (1 - \delta)y_n. \tag{165}$$

5.2 When the delay is time varying

When the delay τ is time varying $\tau = \tau(t)$, then the problem becomes more complicated. Here, $y(t_n - \tau)$ can be approximated by using v_{n+1}, and a linear interpolation term is taken to deal with the delay term. Let $\tau(t_{n+1}) = (m_{n+1} - \delta_{n+1})h$. Here, m_{n+1} is taken as a non-negative integer and $\delta_{n+1} \in [0,1)$, then

$$v_{n+1} = \delta_{n+1} y_{n-m_{n+1}+2} + (1 - \delta_{n+1})y_{n-m_{n+1}+1.} \tag{166}$$

It is seen that when the delay(τ) is constant for a specified value of step size and delay, the value of m needs to be checked whether it is greater than or equal to 1. But when the delay(τ) is time varying, then it is obvious that m must be time varying. Then in one-step value, m is equal to 1, and in another step value, m is greater than 1. When $m_{n+1} = 1$, then the right-hand side of the problem needs to be predicted, but when $m_{n+1} > 1$, then it is needed to go for some other steps. The problem can be rewritten using the Volterra integral equation based on the theory discussed in the previous studies [186,229–232], and the approximated equation is

$$y(t) = \sum_{k=0}^{m-1} \varphi(t)\frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s, y(s), y(s - \tau))}{(t - s)^{(1-\alpha)}} ds. \tag{167}$$

The method for solving the problem of FDDE was mentioned in the study by Wang [228]. For approximating the integral term on the right-hand side of equation (167), trapezoidal quadrature formula was used, and here, t_j nodes were taken, where $j = 0, 1, 2, \ldots, n + 1$ concerning the weight function taken as $(t_{n+1} - .)^{(\alpha-1)}$. So, the approximation can be written as follows:

$$\int_0^{t_{n+1}} (t_{k+1} - z)^{(\alpha-1)} g(z)\mathrm{d}z \approx \int_0^{t_{n+1}} (t_{n+1} - z)^{(\alpha-1)} \tilde{g}_{n+1}(z)\mathrm{d}z. \tag{168}$$

Here, $\tilde{g}_{n+1}(.)$ is the piecewise linear interpolation written for$g(.)$. Nodes were selected as t_j, where $j = 0,1,2,\ldots, n + 1$.

By using this, the integral can be approximated as follows:

$$\int_0^{t_{n+1}} (t_{n+1} - z)^{(\alpha-1)} \tilde{g}_{n+1}(z)dz = \sum_{j=0}^{n+1} a_{j,n+1} g(t_j), \tag{169}$$

where

$$a_{j,n+1} = \frac{h^\alpha}{\alpha(\alpha + 1)} \times \begin{cases} n^{(\alpha+1)} - (n - \alpha)(n + 1)^\alpha, j = 0, \\ (n - j + 2)^{(\alpha+1)} + (n - j)^{(\alpha+1)} - 2(n - j + 1)^{(\alpha+1)}, 1 \le j \le n. \\ 1, j = n + 1, \end{cases} \tag{170}$$

The approximated FDDE is

$$y_{n+1} = \sum_{k=0}^{m-1} \varphi_k \frac{t_{n+1}^k}{k!} + \frac{h^\alpha}{\Gamma(\alpha + 2)} f(t_{n+1}, y_{n+1}^p, v_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, y_j, v_j), \tag{171}$$

$$v_{n+1} = \begin{cases} \delta y_{n-m+2} + (1 - \delta) y_{n-m+1}, & \text{if} \quad m > 1, \\ \delta y_{n+1}^p + (1 - \delta) y_n, & \text{if} \quad m = 1, \end{cases} \tag{172}$$

The predictor equation is

$$y_{n+1}^p = \sum_{k=0}^{m-1} \varphi_k \frac{t_{n+1}^k}{k!} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, y_j, v_j), \tag{173}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n - j + 1)^\alpha - (n - j)^\alpha). \tag{174}$$

For different numerical schemes for solving FDOCP, readers can check [233,234], which is also discussed in short here.

It is seen that in a few papers, FDOCP was addressed where necessary condition and the condition of optimality was derived. The theory and application of delay differential equations were addressed [235], which can be any real-life applications in the field of power systems, biology, electronics, communication, manufacturing and chemical engineering, and life science [236–238]. In the study by Deng et al. [239], the fractional linear differential equation with multiple delays was mentioned, and the stability was discussed. In the study by Lima et al. [240], the authors relate fractional dynamics and signal delay. In Witayakiattilerd [241] and Wang et al. [242], FDOCP was addressed where the cost function was taken in the form of FDDE. In Rosenblueth [243], FDOCP with COVs was discussed. Fractional variational problem with delay was discussed in detail in the study by Baleanu et al. [244]. In the study by Mohammed and Wadi [233], generalized Hat functions were used to solve FDDE. In Kheyrinataj and Nazemi [234], FDDE was solved with equality and inequality constraints. In the study by Čermák et al. [245], stability and asymptotics of FDDE with constant delay were discussed. In the study by Vajrapatkul et al. [246], FDDE of CSOH was numerically solved using the predictor-corrector method.

It is seen that recently Bernstein polynomials were taken as a basis function for solving the problem of FDOCP where the delay was only present in the state, but in the study by Safaie et al. [247], the authors showed the numerical method to solve FDOCP where the delay was present in both state and control. Dehghan and Keyanpour [248] showed the way based on the problem of moments for obtaining the solution of FDOCP. In this problem, the cost function was taken as the function of state and control, and the dynamic of the scheme was illustrated by FDDE. Here, the R-L derivative was taken, and the order (α) of the derivative is between 0 and 1. When the nonlinear function is convexities, and the FDOCP is non-convex in nature then the absence of linearity in the control variable is easily described by the polynomials for solving the problem. Here, G-L approximation was used to solve FDDE. In Bhrawy and Ezz-Eldien [249], a computational method was presented depending on shifted Legendre orthonormal polynomials to solve FDDE where the delay is present in both states and in control. This proposed scheme is advantageous as few terms present in shifted Legendre orthonormal results in a good approximation of state as well as control variables.

Fractional integral inequalities refer to a class of mathematical inequalities that involve fractional integrals, which are generalizations of the usual integrals to non-integer orders. These inequalities are important in many areas of mathematics and its applications, such as analysis, geometry, probability theory, and physics. One of the most well-known fractional integral inequalities is the Hardy-Littlewood-Sobolev inequality. Another important example of a fractional integral inequality is the Poincaré inequality. Fractional integral inequalities have numerous applications in many areas of mathematics, including the theory of partial differential equations, harmonic analysis, and geometric measure theory [250–252].

6 Conclusion

In this work, a brief literature survey is presented to solve FOCP. Using the existing methods, the generalization of the numerical schemes that were developed for classical OCPs. Here, numerical methods like Adam's predictor-corrector method, Adam-Bashford-Moulton, as well as polynomial-based approximations like Jacobi, Legendre, Bernstein, and Bernoulli's polynomials were used. Finally, some numerical methods are discussed to solve FDOCPs as very limited work has been done on FDDE.

In general, the stability analysis of a fractional-order optimal control problem involves investigating the behaviour of the system in response to small perturbations in the initial conditions or the parameters of the problem. The stability of fractional-order optimal control problems can be analyzed using various methods, including Lyapunov stability theory, frequency domain analysis, and numerical simulation techniques. One approach to analyzing the stability of fractional-order optimal control problems is to use the concept of ML stability. A fractional-order system is considered ML stable if it exhibits exponential decay of the response with time. Another approach is to use the concept of fractional Lyapunov stability, which involves defining a fractional Lyapunov function that satisfies certain conditions for stability. Totally, the stability analysis of fractional-order optimal control problems is a complex and challenging problem, and more research is needed to fully understand the behaviour of these systems.

Based on the aforementioned discussion, we shall consider the stability of fractional-order optimal control problem as a future work.

Acknowledgment: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP2/251/44.

Funding information: The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research Project under grant number RGP2/251/44.

Author contributions: Conceptualization: Ramashis Banerjee and Shabir Ahmad. Methodology: Wasim Jamshed and Mohamed R. Eid. Formal analysis, investigation, and writing-original draft preparation: Kottakkaran Sooppy Nisar and Wasim Jamshed. Software and validation: Mohamed R. Eid and Assmaa Abd-Elmonem. Writing – review and editing: Rabha W. Ibrahim. Project administration: Sayed M El Din. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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