Research Article

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On the Waring-Goldbach problem for two squares and four cubes

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Abstract: Let N be a sufficiently large integer. In this article, it is proved that, with at most $O(N^{\frac{1}{12}+\varepsilon})$ exceptions, all even positive integers up to N can be represented in the form $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3$, where p_1, p_2, p_3, p_4, p_5, and p_6 are prime variables. This result constitutes a large improvement upon the previous result of Liu [*On a Waring-Goldbach problem involving squares and cubes*, Math. Slovaca. **69** (2019), no. 6, 1249–1262].

Keywords: Waring-Goldbach problem, Hardy-Littlewood method, exceptional set

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1 Introduction and main result

In 1938, Hua [1] proved that every sufficiently large integer N satisfying $N \equiv 5 \pmod{24}$ can be represented as the sum of five squares of primes, while every sufficiently large integer N satisfying $N \equiv 1 \pmod{2}$ can be represented as the sum of nine cubes of primes. Based on the significant results of Hua, it seems reasonable to conjecture that every sufficiently large integer satisfying some necessary congruence conditions can be written as the sum of four squares of primes or eight cubes of primes, i.e.

$$N = p_1^2 + p_2^2 + p_3^2 + p_4^2, \tag{1.1}$$

and

$$N = p_1^3 + p_2^3 + p_3^3 + p_4^3 + p_5^3 + p_6^3 + p_7^3 + p_8^3. \tag{1.2}$$

On the other hand, Hooley [2] introduced divisor sum techniques into the investigation of Waring's problem of mixed powers. In particular, Hooley's techniques provide an asymptotic formula for every sufficiently large integer N as the sum of two squares and four cubes of positive integers. Moreover, motivated by Hooley's result, it is reasonable to conjecture that every sufficiently large even integer N can be represented as follows:

$$N = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3, \tag{1.3}$$

where $p_1, \ldots, p_6$ are primes. Meanwhile, we can regard equation (1.3) as the hybrid conjecture of equations (1.1) and (1.2). But this expectation is probably far out of the reach of modern number theory techniques.

In 2016, Cai [3] gave an approximation to the conjecture (1.3) and proved that any sufficiently large even integer N can be written in the form $N = x^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3$, where $p_2, \ldots, p_6$ are primes and x is an

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almost-prime $\mathcal{P}_3$. Afterwards, Cai's studies [3] in this direction were subsequently generalized by Zhang and Li [4]. On the other hand, in 2019, Liu [5] considered the exceptional set of the conjecture (1.3) and showed that $E(N) \ll N^{\frac{1}{4}+\varepsilon}$ where $E(N)$ denotes the number of positive even integers n up to N, which cannot be represented as $p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3$.

In this article, we shall continue to consider the exceptional set of the problem (1.3) and improve the previous result.

Theorem 1.1. *Let $E(N)$ denote the number of positive even integers n up to N, which cannot be represented as*

$$n = p_1^2 + p_2^2 + p_3^3 + p_4^3 + p_5^3 + p_6^3. \tag{1.4}$$

Then, for any $\varepsilon > 0$, we have

$$E(N) \ll N^{\frac{1}{12}+\varepsilon}.$$

2 Preliminary and outline of the proof of Theorem 1.1

In order to better illustrate Lemmas 2.1 and 2.2, we first introduce some notations and definitions. When $C \subseteq \mathbb{N}$, we write $\overline{C}$ for the complement $\mathbb{N}\backslash C$ of C within $\mathbb{N}$. When a and b are non-negative integers, it is convenient to denote by $(C)_a^b$ the set $C \cap (a, b]$, and by $|C|_a^b$ the cardinality of $C \cap (a, b]$. Next, when $C, \mathcal{D} \subseteq \mathbb{N}$, we define

$$C + \mathcal{D} = \{c + d : c \in C \quad \text{and} \quad d \in \mathcal{D}\}.$$

It is convenient, when k is a natural number, to describe a subset Q of $\mathbb{N}$ as being a *high-density subset of the kth powers* when (i) one has $Q \subseteq \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε, whenever N is a natural number sufficiently large in terms of ε, then $|Q|_0^N > N^{1/k-\varepsilon}$. In addition, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having *complementary density growth exponent smaller than* θ when there exists a positive number δ with the property that, for all sufficiently large natural numbers N, one has $|\overline{\mathcal{R}}|_0^N < N^{\theta-\delta}$.

When q is a natural number and $\mathfrak{a} \in \{0, 1, \ldots, q-1\}$, we define $\mathcal{P}_{\mathfrak{a}} = \mathcal{P}_{\mathfrak{a},q}$ by

$$\mathcal{P}_{\mathfrak{a},q} = \{\mathfrak{a} + mq : m \in \mathbb{Z}\}.$$

In addition, we describe a set $\mathcal{L}$ as being a *union of arithmetic progressions modulo q* when, for some subset $\mathfrak{L}$ of $\{0, 1, \ldots, q-1\}$, one has

$$\mathcal{L} = \bigcup_{\mathfrak{l} \in \mathfrak{L}} \mathcal{P}_{\mathfrak{l},q}.$$

In such circumstances, given a subset C of $\mathbb{N}$ and integers a and b, it is convenient to write

$$\langle C \wedge \mathcal{L}\rangle_a^b = \min_{\mathfrak{l} \in \mathfrak{L}} |C \cap \mathcal{P}_{\mathfrak{l},q}|_a^b.$$

Let $\mathcal{L}$ be a union of arithmetic progressions modulo q, for some natural number q. When k is a natural number, we describe a subset Q of $\mathbb{N}$ as being a *high-density subset of the kth powers relative to* $\mathcal{L}$ when (i) one has $Q \subseteq \{n^k : n \in \mathbb{N}\}$, and (ii) for each positive number ε, whenever N is a natural number sufficiently large in terms of ε, then $\langle Q \wedge \mathcal{L}\rangle_0^N \gg_q N^{1/k-\varepsilon}$. In addition, when $\theta > 0$, we shall refer to a set $\mathcal{R} \subseteq \mathbb{N}$ as having *$\mathcal{L}$-complementary density growth exponent smaller than* θ when there exists a positive number δ with the property that, for all sufficiently large natural numbers N, one has $|\overline{\mathcal{R}} \cap \mathcal{L}|_0^N < N^{\theta-\delta}$.

Lemma 2.1. *Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be unions of arithmetic progressions modulo q, for some natural number q, and suppose that $\mathcal{N} \subseteq \mathcal{L} + \mathcal{M}$. Suppose also that $\mathcal{S}$ is a high-density subset of the squares relative to $\mathcal{L}$, and that $\mathcal{A} \subseteq \mathbb{N}$ has $\mathcal{M}$-complementary density growth exponent smaller than 1. Then, whenever $\varepsilon > 0$ and N is a natural number sufficiently large in terms of ε, one has*

$$|\overline{\mathcal{A}+\mathcal{S}}\cap\mathcal{N}|_{2N}^{3N}\ll_q N^{-\frac{1}{2}+\varepsilon}\,|\overline{\mathcal{A}}\cap\mathcal{M}|_N^{3N}.$$

Proof. See Theorem 2.2 of Kawada and Wooley [6]. □

Lemma 2.2. *Let $\mathcal{L}$, $\mathcal{M}$, and $\mathcal{N}$ be unions of arithmetic progressions modulo q, for some natural number q, and suppose that $\mathcal{N}\subseteq\mathcal{L}+\mathcal{M}$. Suppose also that C is a high-density subset of the cubes relative to $\mathcal{L}$, and that $\mathcal{A}\subseteq\mathbb{N}$ has $\mathcal{M}$-complementary density growth exponent smaller than θ, for some positive number θ. Then, whenever $\varepsilon>0$ and N is a natural number sufficiently large in terms of ε, without any condition on θ, one has*

$$|\overline{\mathcal{A}+C}\cap\mathcal{N}|_{2N}^{3N}\ll_q N^{-\frac{1}{3}+\varepsilon}\,|\overline{\mathcal{A}}\cap\mathcal{M}|_N^{3N}+N^{-1+\varepsilon}(|\overline{\mathcal{A}}\cap\mathcal{M}|_N^{3N})^2.$$

Proof. See Theorem 4.1 (a) of Kawada and Wooley [6]. □

In order to prove Theorem 1.1, we need the following proposition, whose proof will be given in Section 3.

Proposition 2.3. *Let $E_1(N)$ denote the number of positive integers n up to N, which satisfy $n\equiv 0 \pmod 2$ and cannot be represented as $p_1^2+p_2^3+p_3^3+p_4^3$. Then, for any $\varepsilon>0$, we have*

$$E_1(N)\ll N^{1-\frac{1}{12}+\varepsilon}.$$

The remaining part of this section is devoted to establishing Theorem 1.1 by using Lemmas 2.1 and 2.2 and Proposition 2.3.

Proof of Theorem 1.1. Let

$$\begin{aligned}
\mathcal{A}_1&=\{p_1^2+p_2^3+p_3^3+p_4^3:\ p_j\text{'s are primes},\ j=1,2,3,4\},\\
\mathcal{A}_2&=\{p_1^2+p_2^2+p_3^3+p_4^3+p_5^3:\ p_j\text{'s are primes},\ j=1,2,3,4,5\},\\
\mathcal{K}_j&=\{p^j: p\ \text{is a prime}\},\quad \mathcal{M}=\{n\in\mathbb{N}: n\equiv 0 \pmod 2\},\\
\mathcal{N}&=\mathcal{L}=\{n\in\mathbb{N}: n\equiv 1 \pmod 2\},\\
\mathscr{E}_1&=\{n\in\mathbb{N}: n\equiv 0 \pmod 2,\ n\neq p_1^2+p_2^3+p_3^3+p_4^3,\ p_j\text{'s are primes}\},\\
\mathscr{E}_2&=\{n\in\mathbb{N}: n\equiv 1 \pmod 2,\ n\neq p_1^2+p_2^2+p_3^3+p_4^3+p_5^3,\ p_j\text{'s are primes}\},\\
\mathscr{E}&=\{n\in\mathbb{N}: n\equiv 0 \pmod 2,\ n\neq p_1^2+p_2^2+p_3^3+p_4^3+p_5^3+p_6^3,\ p_j\text{'s are primes}\}.
\end{aligned}$$

Thus, we have $E_1(N)=|\mathscr{E}_1|_0^N$ and $E(N)=|\mathscr{E}|_0^N$. In addition, we write $E_2(N)=|\mathscr{E}_2|_0^N$. Then, $\mathcal{L}$ is an arithmetic progression modulo 2, and so are $\mathcal{M}$ and $\mathcal{N}$. In addition, there hold $\mathcal{N}\subseteq\mathcal{L}+\mathcal{M}$ and $\mathcal{M}\subseteq\mathcal{L}+\mathcal{N}$. Moreover, it follows from the prime number theorem in arithmetic progression that

$$\langle\mathcal{K}_2\wedge\mathcal{L}\rangle_0^N\gg N^{\frac{1}{2}}(\log N)^{-1}\quad\text{and}\quad\langle\mathcal{K}_3\wedge\mathcal{L}\rangle_0^N\gg N^{\frac{1}{3}}(\log N)^{-1}.$$

Therefore, $\mathcal{K}_2$ is a high-density subset of the squares relative to $\mathcal{L}$, while $\mathcal{K}_3$ is a high-density subset of the cubes relative to $\mathcal{L}$. By Proposition 2.3, it is easy to see that

$$|\overline{\mathcal{A}_1}\cap\mathcal{M}|_0^N=|\mathscr{E}_1|_0^N=E_1(N)\ll N^{1-\frac{1}{12}+\varepsilon},$$

and thus $\mathcal{A}_1$ has $\mathcal{M}$-complementary density growth exponent smaller than 1. From Lemma 2.1, we know that

$$|\mathscr{E}_2|_{2N}^{3N}=|\overline{\mathcal{A}_1+\mathcal{K}_2}\cap\mathcal{N}|_{2N}^{3N}\ll N^{-\frac{1}{2}+\varepsilon}\,|\overline{\mathcal{A}_1}\cap\mathcal{M}|_N^{3N}\ll N^{-\frac{1}{2}+\varepsilon}\cdot E_1(3N)\ll N^{\frac{1}{2}-\frac{1}{12}+\varepsilon}.$$

Let the integers N_j for $j\geqslant 0$ be determined by the iterative formula

$$N_0=\left\lceil\frac{1}{2}N\right\rceil,\quad N_{j+1}=\left\lceil\frac{2}{3}N_j\right\rceil\quad\text{for } j\geqslant 0, \tag{2.1}$$

where $\lceil N\rceil$ denotes the least integer not smaller than N. Moreover, we define J to be the least positive integer with the property that $N_J\leqslant 10$, then $J\ll\log N$. Therefore, there holds

$$E_2(N) \leqslant 10 + \sum_{j=1}^{J} |\mathscr{E}_2|_{2N_j}^{3N_j} \ll N^{\frac{1}{2}-\frac{1}{12}+\varepsilon}. \tag{2.2}$$

By equation (2.2), we know that

$$|\overline{\mathcal{A}_2} \cap \mathcal{N}|_0^N = |\mathscr{E}_2|_0^N = E_2(N) \ll N^{\frac{1}{2}-\frac{1}{12}+\varepsilon},$$

and thus $\mathcal{A}_2$ has $\mathcal{N}$-complementary density growth exponent smaller than $\frac{1}{2}$. From Lemma 2.2, we obtain

$$\begin{aligned}|\mathscr{E}|_{2N}^{3N} = |\overline{\mathcal{A}_2 + \mathcal{K}_3} \cap \mathcal{M}|_{2N}^{3N} &\ll N^{-\frac{1}{3}+\varepsilon}\,|\overline{\mathcal{A}_2} \cap \mathcal{N}|_N^{3N} + N^{-1+\varepsilon}(|\overline{\mathcal{A}_2} \cap \mathcal{N}|_N^{3N})^2\\ &\ll N^{-\frac{1}{3}+\varepsilon} \cdot E_2(3N) + N^{-1+\varepsilon}(E_2(3N))^2\\ &\ll N^{\frac{1}{6}-\frac{1}{12}+\varepsilon} \ll N^{\frac{1}{12}+\varepsilon}.\end{aligned}$$

Therefore, with the same notation of equation (2.1), we deduce that

$$E(N) \leqslant 10 + \sum_{j=1}^{J} |\mathscr{E}|_{2N_j}^{3N_j} \ll N^{\frac{1}{12}+\varepsilon},$$

which completes the proof of Theorem 1.1. □

3 Outline of the proof of Proposition 2.3

In this section, we shall give an outline of the proof of Proposition 2.3. Let N be a sufficiently large positive integer. For $k = 2, 3$, we define

$$f_k(\alpha) = \sum_{X_k < p \leqslant 2X_k} (\log p) e(p^k \alpha),$$

where $X_k = (N/16)^{\frac{1}{k}}$. Let

$$\mathscr{R}(n) = \sum_{\substack{n = p_1^2 + p_2^3 + p_3^3 + p_4^3 \\ X_2 < p_1 \leqslant 2X_2 \\ X_3 < p_i \leqslant 2X_3 \\ i = 2,3,4}} (\log p_1)(\log p_2)(\log p_3)(\log p_4).$$

Then, for any $Q > 0$, it follows from the orthogonality that

$$\mathscr{R}(n) = \int_0^1 f_2(\alpha) f_3^3(\alpha) e(-n\alpha) \mathrm{d}\alpha = \int_{\frac{1}{Q}}^{1+\frac{1}{Q}} f_2(\alpha) f_3^3(\alpha) e(-n\alpha) \mathrm{d}\alpha.$$

In order to apply the circle method, we set

$$P = N^{\frac{5}{36}-2\varepsilon}, \quad Q = N^{\frac{31}{36}+\varepsilon}. \tag{3.1}$$

By Dirichlet's lemma on rational approximation (e.g. see Lemma 2.1 of Vaughan [7]), each $\alpha \in [1/Q, 1 + 1/Q]$ can be written in the form

$$\alpha = \frac{a}{q} + \lambda, \quad |\lambda| \leqslant \frac{1}{qQ},$$

for some integers a, q with $1 \leqslant a \leqslant q \leqslant Q$, and $(a, q) = 1$. Then, we define the major arcs $\mathfrak{M}$ and minor arcs $\mathfrak{m}$ as follows:

$$\mathfrak{M} = \bigcup_{1 \leqslant q \leqslant P} \bigcup_{\substack{1 \leqslant a \leqslant q \\ (a,q)=1}} \mathfrak{M}(q, a), \quad \mathfrak{m} = \left[\frac{1}{Q}, 1 + \frac{1}{Q}\right] \Big\backslash \mathfrak{M}, \tag{3.2}$$

where

$$\mathfrak{M}(q,a)=\left[\frac{a}{q}-\frac{1}{qQ},\frac{a}{q}+\frac{1}{qQ}\right].$$

Then, one has

$$\mathscr{R}(n)=\left\{\int_{\mathfrak{M}}+\int_{\mathfrak{m}}\right\}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha.$$

In order to prove Proposition 2.3, we need the following two propositions, whose proofs will be given in Sections 4 and 6, respectively.

Proposition 3.1. *Let the major arcs* $\mathfrak{M}$ *be defined as in equation* (3.2) *with P and Q defined in equation* (3.1). *Then, for* $n\in(N/2,N]$ *and any* $A>0$, *there holds*

$$\int_{\mathfrak{M}}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha=\frac{1}{54}\mathfrak{S}(n)\mathfrak{J}(n)+O\left(N^{\frac{1}{2}}L^{-A}\right),$$

where $\mathfrak{S}(n)$ *is the singular series defined in equation* (4.1), *which is absolutely convergent and satisfies*

$$(\log\log n)^{-c^*}\ll\mathfrak{S}(n)\ll d(n)\tag{3.3}$$

for any integer n satisfying $n\equiv 0\pmod 2$ *and some fixed constant* $c^*>0$, *while* $\mathfrak{J}(n)$ *is defined by equation* (4.9) *and satisfies*

$$\mathfrak{J}(n)\asymp N^{\frac{1}{2}}.$$

For the properties (3.3) of singular series, we shall give the proof in Section 5.

Proposition 3.2. *Let the minor arcs* $\mathfrak{m}$ *be defined as in equation* (3.2) *with P and Q defined in equation* (3.1). *Then, we have*

$$\int_{\mathfrak{m}}|f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha\ll N^{2-\frac{1}{12}+\varepsilon}.$$

The remaining part of this section is devoted to establishing Proposition 2.3 by using Propositions 3.1 and 3.2.

Proof of Proposition 2.3. Let $\mathscr{U}(N)$ denote the set of integers $n\in(N/2,N]$ such that

$$\left|\int_{\mathfrak{m}}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|\gg N^{\frac{1}{2}}L^{-A}.$$

Then, we have

$$NL^{-2A}|\mathscr{U}(N)|\ll\sum_{n\in\mathscr{U}(N)}\left|\int_{\mathfrak{m}}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|^2\ll\sum_{\frac{N}{2}<n\leqslant N}\left|\int_{\mathfrak{m}}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|^2.\tag{3.4}$$

By Bessel's inequality, we have

$$\sum_{\frac{N}{2}<n\leqslant N}\left|\int_{\mathfrak{m}}f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha\right|^2\leqslant\int_{\mathfrak{m}}|f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha.\tag{3.5}$$

Combining equations (3.4) and (3.5) and Proposition 3.2, we have

$$|\mathscr{U}(N)| \ll N^{1-\frac{1}{12}+\varepsilon}.$$

Therefore, with at most $O(N^{1-1/12+\varepsilon})$ exceptions, all the integers $n \in (N/2, N]$ satisfy

$$\left|\int_{\mathfrak{m}} f_2(\alpha) f_3^3(\alpha) e(-n\alpha)\mathrm{d}\alpha\right| \ll N^{\frac{1}{2}} L^{-A},$$

from which, using Proposition 3.1, we deduce that, with at most $O(N^{1-1/12+\varepsilon})$ exceptions, all the positive integers $n \in (N/2, N]$ satisfying $n \equiv 0 \pmod 2$ can be represented in the form $p_1^2 + p_2^3 + p_3^3 + p_4^3$, where p_1, p_2, p_3, and p_4 are prime numbers. By a splitting argument, we obtain

$$E_1(N) \ll N^{1-\frac{1}{12}+\varepsilon}.$$

This completes the proof of Proposition 2.3. □

4 Proof of Proposition 3.1

In this section, we shall concentrate on proving Proposition 3.1. We first introduce some notations. For a Dirichlet character $\chi \bmod q$ and $k \in \{2, 3\}$, we define

$$C_k(\chi, a) = \sum_{h=1}^{q} \overline{\chi(h)} e\left(\frac{ah^k}{q}\right), \quad C_k(q, a) = C_k(\chi^0, a),$$

where χ^0 is the principal character modulo q. Let $\chi_2, \chi_3^{(1)}, \chi_3^{(2)}$, and $\chi_3^{(3)}$ be Dirichlet characters modulo q. Define

$$B(n, q, \chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C_2(\chi_2, a) C_3(\chi_3^{(1)}, a) C_3(\chi_3^{(2)}, a) C_3(\chi_3^{(3)}, a) e\left(-\frac{an}{q}\right),$$

$$B(n, q) = B(n, q, \chi^0, \chi^0, \chi^0, \chi^0),$$

and write

$$A(n, q) = \frac{B(n, q)}{\varphi^4(q)}, \quad \mathfrak{S}(n) = \sum_{q=1}^{\infty} A(n, q). \tag{4.1}$$

Lemma 4.1. *For $(a, q) = 1$ and any Dirichlet character $\chi \bmod q$, there holds*

$$|C_k(\chi, a)| \leqslant 2q^{1/2} d^{\beta_k}(q)$$

with $\beta_k = (\log k)/\log 2$.

Proof. See Problem 14 of Chapter VI of Vinogradov [8]. □

Lemma 4.2. *The singular series $\mathfrak{S}(n)$ satisfies equation* (3.3).

The proof of Lemma 4.2 is provided in Section 5.

Lemma 4.3. *Let $f(x)$ be a real differentiable function in the interval $[a, b]$. If $f'(x)$ is monotonic and satisfies $|f'(x)| \leqslant \theta < 1$. Then, we have*

$$\sum_{a<n\leqslant b} e^{2\pi i f(n)} = \int_a^b e^{2\pi i f(x)} \mathrm{d}x + O(1).$$

Proof. See Lemma 4.8 of Titchmarsh [9]. □

Lemma 4.4. *Let $\chi_2 \bmod r_2$ and $\chi_3^{(i)} \bmod r_3^{(i)}$ with $i = 1, 2, 3$ be primitive characters, $r_0 = [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}]$, and χ^0 be the principal character modulo q. Then, there holds*

$$\sum_{\substack{q\leqslant x\\ r_0|q}} \frac{1}{\varphi^4(q)} |B(n, q, \chi_2\chi^0, \chi_3^{(1)}\chi^0, \chi_3^{(2)}\chi^0, \chi_3^{(3)}\chi^0)| \ll r_0^{-1+\varepsilon} \log^{65} x. \tag{4.2}$$

Proof. By Lemma 4.1, we have

$$\begin{aligned}&|B(n, q, \chi_2\chi^0, \chi_3^{(1)}\chi^0, \chi_3^{(2)}\chi^0, \chi_3^{(3)}\chi^0)|\\ &\ll \sum_{\substack{a=1\\(a,q)=1}}^{q} |C_2(\chi_2\chi^0, a)C_3(\chi_3^{(1)}\chi^0, a)C_3(\chi_3^{(2)}\chi^0, a)C_3(\chi_3^{(3)}\chi^0, a)| \ll q^2\varphi(q)d^6(q).\end{aligned}$$

Therefore, the left-hand side of equation (4.2) is

$$\ll \sum_{\substack{q\leqslant x\\ r_0|q}} \frac{q^2\varphi(q)d^6(q)}{\varphi^4(q)} = \sum_{t\leqslant \frac{x}{r_0}} \frac{r_0^2 t^2 d^6(r_0 t)}{\varphi^3(r_0 t)} \ll r_0^{-1+\varepsilon}(\log x)\sum_{t\leqslant x}\frac{d^6(t)}{t} \ll r_0^{-1+\varepsilon}\log^{65} x.$$

This completes the proof of Lemma 4.4. □

Write

$$\begin{aligned} V_k(\lambda) &= \sum_{X_k<m\leqslant 2X_k} e(m^k\lambda),\\ W_k(\chi, \lambda) &= \sum_{X_k<p\leqslant 2X_k} (\log p)\chi(p)e(p^k\lambda) - \delta_\chi \sum_{X_k<m\leqslant 2X_k} e(m^k\lambda), \end{aligned} \tag{4.3}$$

where $\delta_\chi = 1$ or 0 according to whether χ is principal or not. Then, by the orthogonality of Dirichlet characters, for $(a, q) = 1$, we have

$$f_k\left(\frac{a}{q} + \lambda\right) = \frac{C_k(q, a)}{\varphi(q)} V_k(\lambda) + \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C_k(\chi, a)W_k(\chi, \lambda).$$

For $j = 1, 2, \dots, 8$, we define the sets $\mathscr{S}_j$ as follows:

$$\mathscr{S}_j = \begin{cases} \{2, 3, 3, 3\}, \text{ if } j = 1; & \{3, 3, 3\}, \text{ if } j = 3; \quad \{3, 3\}, \text{ if } j = 5; \quad \{3\}, \text{ if } j = 7;\\ \{2, 3, 3\}, \text{ if } j = 2; & \{2, 3\}, \text{ if } j = 4; \quad \{2\}, \text{ if } j = 6; \quad \emptyset, \text{ if } j = 8.\end{cases}$$

In addition, we write $\overline{\mathscr{S}_j} = \{2, 3, 3, 3\}\setminus\mathscr{S}_j$. Then, we have

$$\int_{\mathfrak{M}} f_2(\alpha)f_3^3(\alpha)e(-n\alpha)\mathrm{d}\alpha =: I_1 + 3I_2 + I_3 + 3I_4 + 3I_5 + I_6 + 3I_7 + I_8, \tag{4.4}$$

where

$$I_j = \sum_{q\leqslant P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} \left(\prod_{k\in\mathscr{S}_j} C_k(q, a)\right) e\left(-\frac{an}{q}\right) \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left(\prod_{k\in\mathscr{S}_j} V_k(\lambda)\right)\left(\prod_{k\in\overline{\mathscr{S}_j}} \sum_{\chi \bmod q} C_k(\chi, a)W_k(\chi, \lambda)\right) e(-n\lambda)\mathrm{d}\lambda.$$

In the following content of this section, we shall prove that I_1 produces the main term, while the others contribute to the error term.

For $k = 2, 3$, applying Lemma 4.3 to $V_k(\lambda)$, we have

$$V_k(\lambda) = \int_{X_k}^{2X_k} e(u^k\lambda)\mathrm{d}u + O(1) = \frac{1}{k}\int_{X_k^k}^{(2X_k)^k} v^{\frac{1}{k}-1}e(v\lambda)\mathrm{d}v + O(1) = \frac{1}{k}\sum_{X_k^k<m\leqslant(2X_k)^k} m^{\frac{1}{k}-1}e(m\lambda) + O(1). \tag{4.5}$$

Putting equation (4.5) into I_1, we see that

$$\begin{aligned} I_1 = {} & \frac{1}{54}\sum_{q\leqslant P}\frac{B(n,q)}{\varphi^4(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left(\sum_{X_2^2<m\leqslant(2X_2)^2} m^{-\frac{1}{2}}e(m\lambda)\right)\left(\sum_{X_3^3<m\leqslant(2X_3)^3} m^{-\frac{2}{3}}e(m\lambda)\right)^3 e(-n\lambda)\mathrm{d}\lambda \\ & + O\left(\sum_{q\leqslant P}\frac{|B(n,q)|}{\varphi^4(q)}\int_{-\frac{1}{qQ}}^{\frac{1}{qQ}}\left|\sum_{X_2^2<m\leqslant(2X_2)^2} m^{-\frac{1}{2}}e(m\lambda)\right|\left|\sum_{X_3^3<m\leqslant(2X_3)^3} m^{-\frac{2}{3}}e(m\lambda)\right|^2 \mathrm{d}\lambda\right). \end{aligned} \tag{4.6}$$

By using the elementary estimate

$$\sum_{X_k^k<m\leqslant(2X_k)^k} m^{\frac{1}{k}-1}e(m\lambda) \ll N^{\frac{1}{k}-1}\min\left(N, \frac{1}{|\lambda|}\right), \tag{4.7}$$

and Lemma 4.4 with $r_0 = 1$, the O-term in equation (4.6) can be estimated as follows:

$$\ll \sum_{q\leqslant P}\frac{|B(n,q)|}{\varphi^4(q)}\left(\int_0^{\frac{1}{N}} N^{\frac{7}{6}}\mathrm{d}\lambda + \int_{\frac{1}{N}}^{\infty} N^{-\frac{11}{6}}\cdot\frac{1}{\lambda^3}\mathrm{d}\lambda\right) \ll N^{\frac{1}{6}}L^{65} \ll N^{\frac{1}{2}}L^{-A}.$$

If the interval of the integral in the main term of equation (4.6) is extended to $[-1/2, 1/2]$, then from equation (3.1), we can see that the resulting error is

$$\ll L^{65}\int_{\frac{1}{qQ}}^{\frac{1}{2}} N^{-\frac{5}{2}}\cdot\frac{\mathrm{d}\lambda}{\lambda^4} \ll N^{-\frac{5}{2}}q^3Q^3L^{65} \ll N^{-\frac{5}{2}}(PQ)^3L^{65} \ll N^{\frac{1}{2}-\varpi}$$

for some $\varpi > 0$. Therefore, by Lemma 4.2, equation (4.6) becomes

$$I_1 = \frac{1}{54}\mathfrak{S}(n)\mathfrak{J}(n) + O\left(N^{\frac{1}{2}}L^{-A}\right), \tag{4.8}$$

where

$$\mathfrak{J}(n) := \sum_{\substack{m_1+m_2+m_3+m_4=n \\ X_2^2<m_1\leqslant(2X_2)^2 \\ X_3^3<m_i\leqslant(2X_3)^3 \\ i=2,3,4}} m_1^{-\frac{1}{2}}(m_2m_3m_4)^{-\frac{2}{3}} \asymp N^{\frac{1}{2}}. \tag{4.9}$$

In order to estimate the contribution of I_j for $j = 2, 3, \ldots, 8$, we shall need the following three preliminary lemmas, i.e. Lemmas 4.5–4.7, whose proofs are exactly the same as Lemmas 3.5–3.7 in Zhang and Li [10], so we omit the details herein. In view of this, for $k \in \{2, 3\}$, we recall the definition of $W_k(\chi, \lambda)$ in equation (4.3) and write

$$J_k(g) = \sum_{r\leqslant P}[g, r]^{-1+\varepsilon}\sum_{\chi \bmod r}^{*}\max_{|\lambda|\leqslant\frac{1}{rQ}}|W_k(\chi, \lambda)|,$$

and

$$K_k(g) = \sum_{r\leqslant P}[g, r]^{-1+\varepsilon}\sum_{\chi \bmod r}^{*}\left(\int_{-\frac{1}{rQ}}^{\frac{1}{rQ}}|W_k(\chi, \lambda)|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}.$$

Here and below, Σ^* indicates that the summation is taken over all primitive characters.

Lemma 4.5. *Let P and Q be defined as in equation* (3.1). *Then, we have*

$$K_3(g) \ll g^{-1+\varepsilon} N^{-\frac{1}{6}} L^c.$$

Lemma 4.6. *Let P and Q be defined as in equation* (3.1). *Then, we have*

$$J_3(g) \ll g^{-1+\varepsilon} N^{\frac{1}{3}} L^c.$$

Lemma 4.7. *Let P and Q be defined as in equation* (3.1). *Then, for any $A > 0$, we have*

$$J_2(1) \ll N^{\frac{1}{2}} L^{-A}.$$

Now, we concentrate on estimating the terms I_j for $j = 2, 3, \dots, 8$. We begin with the term I_8, which is the most complicated one. Reducing the Dirichlet characters in I_8 into primitive characters, we have

$$\begin{aligned} |I_8| &= \left| \sum_{q \leqslant P} \frac{1}{\varphi^4(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} \left(\sum_{\chi_2 \bmod q} C_2(\chi_2, a) W_2(\chi_2, \lambda) \right) \right. \\ &\quad \left. \times \left(\sum_{\chi_3 \bmod q} C_3(\chi_3, a) W_3(\chi_3, \lambda) \right)^3 e(-n\lambda) \mathrm{d}\lambda \right| \\ &= \left| \sum_{q \leqslant P} \sum_{\chi_2 \bmod q} \sum_{\chi_3^{(1)} \bmod q} \sum_{\chi_3^{(2)} \bmod q} \sum_{\chi_3^{(3)} \bmod q} \frac{1}{\varphi^4(q)} \cdot B(n, q, \chi_2, \chi_3^{(1)}, \chi_3^{(2)}, \chi_3^{(3)}) \right. \\ &\quad \left. \times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} W_2(\chi_2, \lambda) W_3(\chi_3^{(1)}, \lambda) W_3(\chi_3^{(2)}, \lambda) W_3(\chi_3^{(3)}, \lambda) e(-n\lambda) \mathrm{d}\lambda \right| \\ &\leqslant \sum_{r_2 \leqslant P} \sum_{r_3^{(1)} \leqslant P} \sum_{r_3^{(2)} \leqslant P} \sum_{r_3^{(3)} \leqslant P} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3^{(1)} \bmod r_3^{(1)}} \sideset{}{^*}\sum_{\chi_3^{(2)} \bmod r_3^{(2)}} \sideset{}{^*}\sum_{\chi_3^{(3)} \bmod r_3^{(3)}} \\ &\quad \times \sum_{\substack{q \leqslant P \\ r_0 | q}} \frac{|B(n, q, \chi_2\chi^0, \chi_3^{(1)}\chi^0, \chi_3^{(2)}\chi^0, \chi_3^{(3)}\chi^0)|}{\varphi^4(q)} \\ &\quad \times \int_{-\frac{1}{qQ}}^{\frac{1}{qQ}} |W_2(\chi_2\chi^0, \lambda) W_3(\chi_3^{(1)}\chi^0, \lambda) W_3(\chi_3^{(2)}\chi^0, \lambda) W_3(\chi_3^{(3)}\chi^0, \lambda)| \mathrm{d}\lambda, \end{aligned}$$

where χ^0 is the principal character modulo q and $r_0 = [r_2, r_3^{(1)}, r_3^{(2)}, r_3^{(3)}]$. For $q \leqslant P$ and $X_k < p \leqslant 2X_k$ with $k \in \{2, 3\}$, we have $(q, p) = 1$. From this and the definition of $W_k(\chi, \lambda)$, we obtain $W_2(\chi_2\chi^0, \lambda) = W_2(\chi_2, \lambda)$ and $W_3(\chi_3^{(i)}\chi^0, \lambda) = W_3(\chi_3^{(i)}, \lambda)$ for primitive characters χ_2 and $\chi_3^{(i)}$ with $i = 1, 2, 3$. Therefore, by Lemma 4.4, we obtain

$$\begin{aligned} |I_8| &\leqslant \sum_{r_2 \leqslant P} \sum_{r_3^{(1)} \leqslant P} \sum_{r_3^{(2)} \leqslant P} \sum_{r_3^{(3)} \leqslant P} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3^{(1)} \bmod r_3^{(1)}} \sideset{}{^*}\sum_{\chi_3^{(2)} \bmod r_3^{(2)}} \sideset{}{^*}\sum_{\chi_3^{(3)} \bmod r_3^{(3)}} \\ &\quad \times \int_{-\frac{1}{r_0Q}}^{\frac{1}{r_0Q}} |W_2(\chi_2, \lambda) W_3(\chi_3^{(1)}, \lambda) W_3(\chi_3^{(2)}, \lambda) W_3(\chi_3^{(3)}, \lambda)| \mathrm{d}\lambda \\ &\quad \times \sum_{\substack{q \leqslant P \\ r_0 | q}} \frac{|B(n, q, \chi_2\chi^0, \chi_3^{(1)}\chi^0, \chi_3^{(2)}\chi^0, \chi_3^{(3)}\chi^0)|}{\varphi^4(q)} \\ &\ll L^{65} \cdot \sum_{r_2 \leqslant P} \sum_{r_3^{(1)} \leqslant P} \sum_{r_3^{(2)} \leqslant P} \sum_{r_3^{(3)} \leqslant P} r_0^{-1+\varepsilon} \sideset{}{^*}\sum_{\chi_2 \bmod r_2} \sideset{}{^*}\sum_{\chi_3^{(1)} \bmod r_3^{(1)}} \sideset{}{^*}\sum_{\chi_3^{(2)} \bmod r_3^{(2)}} \sideset{}{^*}\sum_{\chi_3^{(3)} \bmod r_3^{(3)}} \\ &\quad \times \int_{-\frac{1}{r_0Q}}^{\frac{1}{r_0Q}} |W_2(\chi_2, \lambda) W_3(\chi_3^{(1)}, \lambda) W_3(\chi_3^{(2)}, \lambda) W_3(\chi_3^{(3)}, \lambda)| \mathrm{d}\lambda. \end{aligned}$$

In the last integral, we pick out $|W_2(\chi_2,\lambda)|$ and $|W_3(\chi_3^{(1)},\lambda)|$ and then use Cauchy's inequality to derive that

$$\begin{aligned}|I_8| \ll L^{65}&\left(\sum_{r_2\leqslant P}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\max_{|\lambda|\leqslant\frac{1}{r_2Q}}|W_2(\chi_2,\lambda)|\right)\left(\sum_{r_3^{(1)}\leqslant P}\ \sideset{}{^*}\sum_{\chi_3^{(1)} \bmod r_3^{(1)}}\max_{|\lambda|\leqslant\frac{1}{r_3^{(1)}Q}}|W_3(\chi_3^{(1)},\lambda)|\right)\\ &\times\sum_{r_3^{(2)}\leqslant P}\ \sideset{}{^*}\sum_{\chi_3^{(2)} \bmod r_3^{(2)}}\left(\int_{-\frac{1}{r_3^{(2)}Q}}^{\frac{1}{r_3^{(2)}Q}}|W_3(\chi_3^{(2)},\lambda)|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}\\ &\times\sum_{r_3^{(3)}\leqslant P} r_0^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_3^{(3)} \bmod r_3^{(3)}}\left(\int_{-\frac{1}{r_3^{(3)}Q}}^{\frac{1}{r_3^{(3)}Q}}|W_3(\chi_3^{(3)},\lambda)|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}.\end{aligned}\tag{4.10}$$

Now we introduce the iterative procedure to bound the sums over $r_3^{(3)}, r_3^{(2)}, r_3^{(1)}$, and r_2, consecutively and respectively. We first estimate the above sum over $r_3^{(3)}$ in equation (4.10) via Lemma 4.5. Since

$$r_0=[r_2,r_3^{(1)},r_3^{(2)},r_3^{(3)}]=[[r_2,r_3^{(1)},r_3^{(2)}],r_3^{(3)}],$$

the sum over $r_3^{(3)}$ is

$$\begin{aligned}&=\sum_{r_3^{(3)}\leqslant P}[[r_2,r_3^{(1)},r_3^{(2)}],r_3^{(3)}]^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_3^{(3)} \bmod r_3^{(3)}}\left(\int_{-\frac{1}{r_3^{(3)}Q}}^{\frac{1}{r_3^{(3)}Q}}|W_3(\chi_3^{(3)},\lambda)|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}\\ &= K_3([r_2,r_3^{(1)},r_3^{(2)}])\ll[r_2,r_3^{(1)},r_3^{(2)}]^{-1+\varepsilon}N^{-\frac{1}{6}}L^c.\end{aligned}\tag{4.11}$$

By Lemma 4.5 again, the contribution of the quantity on the right-hand side of equation (4.11) to the sum over $r_3^{(2)}$ in equation (4.10) is

$$\begin{aligned}&\ll N^{-\frac{1}{6}}L^c\cdot\sum_{r_3^{(2)}\leqslant P}[[r_2,r_3^{(1)}],r_3^{(2)}]^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_3^{(2)} \bmod r_3^{(2)}}\left(\int_{-\frac{1}{r_3^{(2)}Q}}^{\frac{1}{r_3^{(2)}Q}}|W_3(\chi_3^{(2)},\lambda)|^2\mathrm{d}\lambda\right)^{\frac{1}{2}}\\ &= N^{-\frac{1}{6}}L^c\cdot K_3([r_2,r_3^{(1)}])\ll[r_2,r_3^{(1)}]^{-1+\varepsilon}N^{-\frac{1}{3}}L^c.\end{aligned}\tag{4.12}$$

By Lemma 4.6, the contribution of the quantity on the right-hand side of equation (4.12) to the sum over $r_3^{(1)}$ in equation (4.10) is

$$\begin{aligned}&\ll N^{-\frac{1}{3}}L^c\cdot\sum_{r_3^{(1)}\leqslant P}[r_2,r_3^{(1)}]^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_3^{(1)} \bmod r_3^{(1)}}\max_{|\lambda|\leqslant\frac{1}{r_3^{(1)}Q}}|W_3(\chi_3^{(1)},\lambda)|\\ &= N^{-\frac{1}{3}}L^c\cdot J_3(r_2)\ll r_2^{-1+\varepsilon}L^c.\end{aligned}\tag{4.13}$$

Finally, from Lemma 4.7, inserting the bound on the right-hand side of equation (4.13) to the sum over r_2 in equation (4.10), we obtain

$$\begin{aligned}|I_8|&\ll L^c\cdot\sum_{r_2\leqslant P}[1,r_2]^{-1+\varepsilon}\ \sideset{}{^*}\sum_{\chi_2 \bmod r_2}\max_{|\lambda|\leqslant\frac{1}{r_2Q}}|W_2(\chi_2,\lambda)|\\ &= L^c\cdot J_2(1)\ll N^{\frac{1}{2}}L^{-A}.\end{aligned}\tag{4.14}$$

For the estimation of the terms $I_2, I_3,\ldots, I_7$, by noting equations (4.5) and (4.7), we obtain

$$\left(\int_{-\frac{1}{Q}}^{\frac{1}{Q}} |V_k(\lambda)|^2 \mathrm{d}\lambda\right)^{\frac{1}{2}} \ll \left(\int_{-\frac{1}{Q}}^{\frac{1}{Q}} N^{\frac{2}{k}-2} \min\left(N, \frac{1}{|\lambda|}\right)^2 \mathrm{d}\lambda + \frac{1}{Q}\right)^{\frac{1}{2}}$$
$$\ll N^{\frac{1}{k}-1}\left(\int_0^{\frac{1}{N}} N^2 \mathrm{d}\lambda + \int_{\frac{1}{N}}^{\frac{1}{Q}} \frac{\mathrm{d}\lambda}{\lambda^2}\right)^{\frac{1}{2}} + \frac{1}{Q^{1/2}} \ll N^{\frac{1}{k}-\frac{1}{2}}.$$

Using this estimate and the upper bound of $V_k(\lambda)$, which derives from equations (4.5) and (4.7), that $V_k(\lambda) \ll N^{\frac{1}{k}}$, we can argue similarly to the treatment of I_8 and obtain

$$\sum_{j=2}^{7} I_j \ll N^{\frac{1}{2}} L^{-A}. \tag{4.15}$$

Combining equations (4.4), (4.8), (4.14), and (4.15), we can derive the conclusion of Proposition 3.1.

5 The singular series

In this section, we shall investigate the properties of the singular series that appear in Proposition 3.1.

Lemma 5.1. *Let p be a prime and $p^\alpha \| k$. For $(a, p) = 1$, if $\ell \geqslant \gamma(p)$, we have $C_k(p^\ell, a) = 0$, where*

$$\gamma(p) = \begin{cases} \alpha + 2, & \text{if } p \neq 2 \text{ or } p = 2, \alpha = 0; \\ \alpha + 3, & \text{if } p = 2, \alpha > 0. \end{cases}$$

Proof. See Lemma 8.3 of Hua [11]. □

For $k \geqslant 1$, we define

$$S_k(q, a) = \sum_{m=1}^{q} e\left(\frac{am^k}{q}\right).$$

Lemma 5.2. *Suppose that $(p, a) = 1$. Then,*

$$S_k(p, a) = \sum_{\chi \in \mathscr{A}_k} \overline{\chi(a)} \tau(\chi),$$

where $\mathscr{A}_k$ denotes the set of non-principal characters χ modulo p for which χ^k is principal, and $\tau(\chi)$ denotes the Gauss sum

$$\sum_{m=1}^{p} \chi(m) e\left(\frac{m}{p}\right).$$

In addition, there hold $|\tau(\chi)| = p^{1/2}$ and $|\mathscr{A}_k| = (k, p-1) - 1$.

Proof. See Lemma 4.3 of Vaughan [7]. □

Lemma 5.3. *For $(p, n) = 1$, we have*

$$\left| \sum_{a=1}^{p-1} \frac{S_2(p, a) S_3^3(p, a)}{p^4} e\left(-\frac{an}{p}\right) \right| \leqslant 8p^{-\frac{3}{2}}. \tag{5.1}$$

Proof. We denote by Σ the left-hand side of equation (5.1). By Lemma 5.2, we have

$$\Sigma = \frac{1}{p^4}\sum_{a=1}^{p-1}\left(\sum_{\chi_2\in\mathscr{A}_2}\overline{\chi_2(a)}\tau(\chi_2)\right)\left(\sum_{\chi_3\in\mathscr{A}_3}\overline{\chi_3(a)}\tau(\chi_3)\right)^3 e\left(-\frac{an}{p}\right).$$

If $|\mathscr{A}_k| = 0$ for some $k \in \{2, 3\}$, then $\Sigma = 0$. If this is not the case, then

$$\Sigma = \frac{1}{p^4}\sum_{\chi_2\in\mathscr{A}_2}\sum_{\chi_3^{(1)}\in\mathscr{A}_3}\sum_{\chi_3^{(2)}\in\mathscr{A}_3}\sum_{\chi_3^{(3)}\in\mathscr{A}_3}\tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})$$
$$\times\sum_{a=1}^{p-1}\overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)}\,e\left(-\frac{an}{p}\right).$$

From Lemma 5.2, the quadruple outer sums have not more than eight terms. In each of these terms, we have

$$|\tau(\chi_2)\tau(\chi_3^{(1)})\tau(\chi_3^{(2)})\tau(\chi_3^{(3)})| = p^2.$$

Since in any one of these terms $\overline{\chi_2(a)\chi_3^{(1)}(a)\chi_3^{(2)}(a)\chi_3^{(3)}(a)}$ is a Dirichlet character $\chi \pmod p$, the inner sum is

$$\sum_{a=1}^{p-1}\chi(a)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\sum_{a=1}^{p-1}\chi(-an)e\left(-\frac{an}{p}\right) = \overline{\chi(-n)}\tau(\chi).$$

From the fact that $\tau(\chi^0) = -1$ for principal character $\chi^0 \bmod p$, we have

$$|\overline{\chi(-n)}\tau(\chi)| \leqslant p^{\frac{1}{2}}.$$

By the above arguments, we obtain

$$|\Sigma| \leqslant \frac{1}{p^4}\cdot 8\cdot p^2\cdot p^{\frac{1}{2}} = 8p^{-\frac{3}{2}}.$$

This completes the proof of Lemma 5.3. □

Lemma 5.4. *Let $\mathscr{L}(p, n)$ denote the number of solutions to the following congruence*:

$$x_1^2 + x_2^3 + x_3^3 + x_4^3 \equiv n \pmod p, \quad 1 \leqslant x_1, x_2, x_3, x_4 \leqslant p-1.$$

Then, we have $\mathscr{L}(p, n) > 0$ for $n \equiv 0 \pmod 2$.

Proof. We have

$$p\cdot\mathscr{L}(p, n) = \sum_{a=1}^{p} C_2(p, a)C_3^3(p, a)e\left(-\frac{an}{p}\right) = (p-1)^4 + E_p,$$

where

$$E_p = \sum_{a=1}^{p-1} C_2(p, a)C_3^3(p, a)e\left(-\frac{an}{p}\right).$$

By Lemma 5.2, we obtain

$$|E_p| \leqslant (p-1)(\sqrt{p}+1)(2\sqrt{p}+1)^3.$$

It is easy to check that $|E_p| < (p-1)^4$ for $p \geqslant 17$. Therefore, we obtain $\mathscr{L}(p, n) > 0$ for $p \geqslant 17$. For $p = 2, 3, 5, 7, 11, 13$, we can check $\mathscr{L}(p, n) > 0$ directly provided that $n \equiv 0 \pmod 2$. This completes the proof of Lemma 5.4. □

Lemma 5.5. *$A(n, q)$ is multiplicative in q.*

Proof. By the definition of $A(n,q)$ in equation (4.1), we only need to show that $B(n,q)$ is multiplicative in q. Suppose that $q=q_1q_2$ with $(q_1,q_2)=1$. Then, we have

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a=1\\(a,q_1q_2)=1}}^{q_1q_2} C_2(q_1q_2,a)C_3^3(q_1q_2,a)e\left(-\frac{an}{q_1q_2}\right) \\ &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1}\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_1q_2,a_1q_2+a_2q_1)C_3^3(q_1q_2,a_1q_2+a_2q_1)e\left(-\frac{a_1n}{q_1}\right)e\left(-\frac{a_2n}{q_2}\right). \end{aligned} \tag{5.2}$$

For $(q_1,q_2)=1$ and $k\in\{2,3\}$, there holds

$$\begin{aligned} C_k(q_1q_2,a_1q_2+a_2q_1) &= \sum_{\substack{m=1\\(m,q_1q_2)=1}}^{q_1q_2} e\left(\frac{(a_1q_2+a_2q_1)m^k}{q_1q_2}\right) \\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1}\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\left(\frac{(a_1q_2+a_2q_1)(m_1q_2+m_2q_1)^k}{q_1q_2}\right) \\ &= \sum_{\substack{m_1=1\\(m_1,q_1)=1}}^{q_1} e\left(\frac{a_1(m_1q_2)^k}{q_1}\right)\sum_{\substack{m_2=1\\(m_2,q_2)=1}}^{q_2} e\left(\frac{a_2(m_2q_1)^k}{q_2}\right) \\ &= C_k(q_1,a_1)C_k(q_2,a_2). \end{aligned} \tag{5.3}$$

Putting equation (5.3) into equation (5.2), we deduce that

$$\begin{aligned} B(n,q_1q_2) &= \sum_{\substack{a_1=1\\(a_1,q_1)=1}}^{q_1} C_2(q_1,a_1)C_3^3(q_1,a_1)e\left(-\frac{a_1n}{q_1}\right)\sum_{\substack{a_2=1\\(a_2,q_2)=1}}^{q_2} C_2(q_2,a_2)C_3^3(q_2,a_2)e\left(-\frac{a_2n}{q_2}\right) \\ &= B(n,q_1)B(n,q_2). \end{aligned}$$

This completes the proof of Lemma 5.5. □

Lemma 5.6. *Let* $A(n,q)$ *be defined as in equation* (4.1). *Then,*

(*i*) *we have*

$$\sum_{q>Z}|A(n,q)| \ll Z^{-\frac{1}{2}+\varepsilon}d(n),$$

and thus the singular series $\mathfrak{S}(n)$ *is absolutely convergent and satisfies* $\mathfrak{S}(n)\ll d(n)$;

(*ii*) *there exists an absolute positive constant* $c^*>0$ *such that*

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}$$

for any integer n *satisfying* $n\equiv 0 \pmod 2$.

Proof. From Lemma 5.5, we know that $B(n,q)$ is multiplicative in q. Therefore, there holds

$$B(n,q)=\prod_{p^t\|q}B(n,p^t)=\prod_{p^t\|q}\sum_{\substack{a=1\\(a,p)=1}}^{p^t} C_2(p^t,a)C_3^3(p^t,a)e\left(-\frac{an}{p^t}\right). \tag{5.4}$$

From equation (5.4) and Lemma 5.1, we deduce that $B(n,q)=\prod_{p\|q}B(n,p)$ or 0 according to whether q is square-free or not. Thus, we have

$$\sum_{q=1}^{\infty} A(n,q) = \sum_{\substack{q=1 \\ q \text{ square-free}}}^{\infty} A(n,q). \tag{5.5}$$

Write

$$\mathscr{V}(p,a) := C_2(p,a)C_3^3(p,a) - S_2(p,a)S_3^3(p,a).$$

Then,

$$A(n,p) = \frac{1}{(p-1)^4}\sum_{a=1}^{p-1} S_2(p,a)S_3^3(p,a)e\left(-\frac{an}{p}\right) + \frac{1}{(p-1)^4}\sum_{a=1}^{p-1}\mathscr{V}(p,a)e\left(-\frac{an}{p}\right). \tag{5.6}$$

Applying Lemma 4.1 and noting that $S_k(p,a) = C_k(p,a) + 1$, we obtain $S_k(p,a) \ll p^{\frac{1}{2}}$, and thus $\mathscr{V}(p,a) \ll p^{\frac{3}{2}}$. Therefore, the second term in equation (5.6) is $\leqslant c_1 p^{-\frac{3}{2}}$. On the other hand, from Lemma 5.3, we can see that the first term in equation (5.6) is $\leqslant 2^4 \cdot 8p^{-\frac{3}{2}} = 128p^{-\frac{3}{2}}$. Let $c_2 = c_1 + 128$. Then, we have proved that, for $p \nmid n$, there holds

$$|A(n,p)| \leqslant c_2 p^{-\frac{3}{2}}. \tag{5.7}$$

Moreover, if we use Lemma 4.1 directly, it follows that

$$|B(n,p)| = \left|\sum_{a=1}^{p-1} C_2(p,a)C_3^3(p,a)e\left(-\frac{an}{p}\right)\right| \leqslant \sum_{a=1}^{p-1}|C_2(p,a)||C_3(p,a)|^3 \leqslant (p-1)\cdot 2^4\cdot p^2\cdot 54 = 864p^2(p-1),$$

and therefore,

$$|A(n,p)| = \frac{|B(n,p)|}{\varphi^4(p)} \leqslant \frac{864p^2}{(p-1)^3} \leqslant \frac{2^3\cdot 864p^2}{p^3} = \frac{6912}{p}. \tag{5.8}$$

Let $c_3 = \max(c_2, 6912)$. Then, for square-free q, we have

$$\begin{aligned}|A(n,q)| &= \left(\prod_{\substack{p\mid q\\ p\nmid n}}|A(n,p)|\right)\left(\prod_{\substack{p\mid q\\ p\mid n}}|A(n,p)|\right) \leqslant \left(\prod_{\substack{p\mid q\\ p\nmid n}}\left(c_3p^{-\frac{3}{2}}\right)\right)\left(\prod_{\substack{p\mid q\\ p\mid n}}(c_3p^{-1})\right)\\ &= c_3^{\omega(q)}\left(\prod_{p\mid q}p^{-\frac{3}{2}}\right)\left(\prod_{p\mid(n,q)}p^{\frac{1}{2}}\right) \ll q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}}.\end{aligned}$$

Hence, by equation (5.5), we obtain

$$\begin{aligned}\sum_{q>Z}|A(n,q)| &\ll \sum_{q>Z}q^{-\frac{3}{2}+\varepsilon}(n,q)^{\frac{1}{2}} = \sum_{d\mid n}\sum_{q>\frac{Z}{d}}(dq)^{-\frac{3}{2}+\varepsilon}d^{\frac{1}{2}} = \sum_{d\mid n}d^{-1+\varepsilon}\sum_{q>\frac{Z}{d}}q^{-\frac{3}{2}+\varepsilon}\\ &\ll \sum_{d\mid n}d^{-1+\varepsilon}\left(\frac{Z}{d}\right)^{-\frac{1}{2}+\varepsilon} = Z^{-\frac{1}{2}+\varepsilon}\sum_{d\mid n}d^{-\frac{1}{2}+\varepsilon} \ll Z^{-\frac{1}{2}+\varepsilon}d(n).\end{aligned}$$

This proves (i) of Lemma 5.6.

To prove (ii) of Lemma 5.6, by Lemma 5.5, we first note that

$$\begin{aligned}\mathfrak{S}(n) &= \prod_p\left(1+\sum_{t=1}^{\infty}A(n,p^t)\right) = \prod_p(1+A(n,p))\\ &= \left(\prod_{p\leqslant c_3}(1+A(n,p))\right)\left(\prod_{\substack{p>c_3\\ p\nmid n}}(1+A(n,p))\right)\left(\prod_{\substack{p>c_3\\ p\mid n}}(1+A(n,p))\right).\end{aligned} \tag{5.9}$$

From equation (5.7), we have

$$\prod_{\substack{p>c_3 \\ p\nmid n}} (1 + A(n,p)) \geqslant \prod_{p>c_3} \left(1 - \frac{c_3}{p^{3/2}}\right) \geqslant c_4 > 0. \tag{5.10}$$

By equation (5.8), we know that there exists $c_5 > 0$ such that

$$\prod_{\substack{p>c_3 \\ p\mid n}} (1 + A(n,p)) \geqslant \prod_{\substack{p>c_3 \\ p\mid n}} \left(1 - \frac{c_3}{p}\right) \geqslant \prod_{p\mid n} \left(1 - \frac{c_3}{p}\right) \gg (\log\log n)^{-c_5}. \tag{5.11}$$

On the other hand, it is easy to see that

$$1 + A(n,p) = \frac{p \cdot \mathscr{L}(p,n)}{\varphi^4(p)}.$$

By Lemma 5.4, we know that $\mathscr{L}(p,n) > 0$ for all p with $n \equiv 0 \pmod 2$, and thus $1 + A(n,p) > 0$. Therefore, there holds

$$\prod_{p\leqslant c_3} (1 + A(n,p)) \geqslant c_6 > 0. \tag{5.12}$$

Combining the estimates (5.9)–(5.12) and taking $c^* = c_5 > 0$, we derive that

$$\mathfrak{S}(n) \gg (\log\log n)^{-c^*}.$$

This completes the proof of Lemma 5.6. □

6 Proof of Proposition 3.2

In this section, we first present some lemmas that will be used to prove Proposition 3.2.

Lemma 6.1. *Suppose that $\alpha \in \mathbb{R}$ and that there exist integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying*

$$1 \leqslant q \leqslant N^{\frac{3}{4}}, \quad (a,q) = 1, \quad q|\beta| \leqslant N^{-\frac{3}{4}},$$

where $\beta = \alpha - a/q$. Then, for any $\varepsilon > 0$, we have

$$f_2(\alpha) \ll X_2^{1-\frac{1}{8}+\varepsilon} + \frac{X_2^{1+\varepsilon}}{\sqrt{q(1 + X_2^2|\beta|)}}.$$

Proof. See Theorem 3 of Kumchev [12]. □

Lemma 6.2. *Suppose that α is a real number, and that $|\alpha - a/q| \leqslant q^{-2}$ with $(a,q) = 1$. Let $\beta = \alpha - a/q$. Then, we have*

$$f_k(\alpha) \ll d^{\delta_k}(q)(\log X_k)^c \left(X_k^{1/2}\sqrt{q(1 + X_k^k|\beta|)} + X_k^{4/5} + \frac{X_k}{\sqrt{q(1 + X_k^k|\beta|)}} \right),$$

where $\delta_k = \frac{1}{2} + \frac{\log k}{\log 2}$ and c is an absolute constant.

Proof. See Theorem 1 of Ren [13]. □

Lemma 6.3. *Suppose that α is a real number, and that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with*

$$(a,q) = 1, \quad 1 \leqslant q \leqslant Q \quad \textit{and} \quad q|\beta| \leqslant Q^{-1},$$

where $\beta = \alpha - a/q$. *If* $W^{1/2} \leqslant Q \leqslant W^{5/2}$, *then we have*

$$\sum_{W<p\leqslant 2W} e(p^3\alpha) \ll W^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}}W^{1+\varepsilon}}{(1+W^3|\beta|)^{1/2}}.$$

Proof. See Lemma 8.5 of Zhao [14]. □

Lemma 6.4. *Suppose that* α *is a real number, and that there exist integers* $a \in \mathbb{Z}$ *and* $q \in \mathbb{N}$ *satisfying*

$$(a,q) = 1, \quad 1 \leqslant q \leqslant Q \quad \text{and} \quad q|\beta| \leqslant Q^{-1},$$

where $\beta = \alpha - a/q$. *If* $X_3^{1/2} \leqslant Q \leqslant X_3^{5/2}$, *then we have*

$$f_3(\alpha) \ll X_3^{1-\frac{1}{12}+\varepsilon} + \frac{X_3^{1+\varepsilon}}{\sqrt{q(1+X_3^3|\beta|)}}.$$

Proof. For any fixed Q satisfying $X_3^{1/2} \leqslant Q \leqslant X_3^{5/2}$, by Dirichlet's lemma on rational approximation (for instance, see Lemma 2.1 of Vaughan [7]), there exist integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that

$$\alpha = \frac{a}{q} + \beta, \quad |\beta| \leqslant \frac{1}{qQ}$$

with $1 \leqslant q \leqslant Q$ and $(a,q) = 1$. Next, we shall discuss the upper bound of $f_3(\alpha)$ according to the size of q and $|\beta|$.

Case 1 If $q \leqslant X_3^{1/2}$ and $|\beta| \leqslant q^{-1}X_3^{-5/2}$, we have

$$X_3^{1/2}\sqrt{q(1+X_3^3|\beta|)} \leqslant \frac{X_3}{\sqrt{q(1+X_3^3|\beta|)}}.$$

Then, by Lemma 6.2, we obtain

$$f_3(\alpha) \ll X_3^{1-\frac{1}{5}+\varepsilon} + \frac{X_3^{1+\varepsilon}}{\sqrt{q(1+X_3^3|\beta|)}},$$

which is acceptable.

Case 2 If $q > X_3^{1/2}$, it follows from Lemma 6.3 that

$$f_3(\alpha) \ll X_3^{1-\frac{1}{12}+\varepsilon} + q^{-\frac{1}{6}}X_3^{1+\varepsilon} \ll X_3^{1-\frac{1}{12}+\varepsilon}.$$

Case 3 If $q \leqslant X_3^{1/2}$ and $q^{-1}X_3^{-5/2} \leqslant |\beta| \leqslant q^{-1}Q^{-1}$, we have $X_3^3|\beta| \geqslant X_3^{1/2}q^{-1} \geqslant 1$, which combined with Lemma 6.3 yields that

$$\begin{aligned} f_3(\alpha) &\ll X_3^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}}X_3^{1+\varepsilon}}{\sqrt{1+X_3^3|\beta|}} \ll X_3^{1-\frac{1}{12}+\varepsilon} + \frac{q^{-\frac{1}{6}}X_3^{1+\varepsilon}}{\sqrt{X_3^{1/2}q^{-1}}} \\ &\ll X_3^{1-\frac{1}{12}+\varepsilon} + q^{\frac{1}{3}}X_3^{\frac{3}{4}+\varepsilon} \ll X_3^{1-\frac{1}{12}+\varepsilon} + (X_2^{\frac{1}{2}})^{\frac{1}{3}}X_3^{\frac{3}{4}+\varepsilon} \ll X_3^{1-\frac{1}{12}+\varepsilon}. \end{aligned}$$

Combining the above three cases, we derive the desired conclusion of Lemma 6.4. □

Lemma 6.5. *Let* $f_k(\alpha)$ *be defined as above. Then, we have*

$$\int_0^1 |f_2^2(\alpha)f_3^4(\alpha)|\mathrm{d}\alpha \ll N^{\frac{4}{3}+\varepsilon}.$$

Proof. Trivially, the conclusion can be deduced by counting the number of solutions to the underlying Diophantine equation

$$x_1^2 - x_2^2 = y_1^3 + y_2^3 - y_3^3 - y_4^3$$

with $X_2 < x_1, x_2 \leqslant 2X_2$ and $X_3 < y_i \leqslant 2X_3$ for $i = 1, 2, 3, 4$. If $x_1 \neq x_2$, the contribution is bounded by $X_3^{4+\varepsilon}$. If $x_1 = x_2$, the contribution is bounded by

$$\ll X_2 \cdot \int_0^1 |f_3(\alpha)|^4 \mathrm{d}\alpha.$$

By Lemma 2.5 of Vaughan [7], we have

$$\int_0^1 |f_3(\alpha)|^4 \mathrm{d}\alpha \ll X_3^{2+\varepsilon},$$

and thus the contribution with $x_1 = x_2$ is $\ll X_2 \cdot X_3^{2+\varepsilon} \ll N^{\frac{7}{6}+\varepsilon}$. Combining the above two cases, we deduce that

$$\int_0^1 |f_2^2(\alpha) f_3^4(\alpha)| \mathrm{d}\alpha \ll X_3^{4+\varepsilon} + X_2 \cdot X_3^{2+\varepsilon} \ll N^{\frac{4}{3}+\varepsilon}.$$

This completes the proof of Lemma 6.5. □

Define the multiplicative function $w_3(q)$ by

$$w_3(p^{3u+v}) = \begin{cases} 3p^{-u-\frac{1}{2}}, & \text{if } u \geqslant 0,\ v = 1; \\ p^{-u-1}, & \text{if } u \geqslant 0,\ 2 \leqslant v \leqslant 3. \end{cases}$$

Lemma 6.6. *For $\gamma \in \mathbb{R}$, we define*

$$\Gamma(\gamma) \coloneqq \sum_{q \leqslant X_3} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{|\alpha - \frac{a}{q}| \leqslant X_3} \frac{w_3^2(q) d^c(q) |\sum_{X_3 < p \leqslant 2X_3} e(p^3(\alpha + \gamma))|^2}{1 + X_3^3 |\alpha - a/q|} \mathrm{d}\alpha$$

One has uniformly for $\gamma \in \mathbb{R}$ that

$$\Gamma(\gamma) \ll X_3^2 N^{-1+\varepsilon}.$$

Proof. See Lemma 2.2 of Zhao [14]. □

For $\mathscr{A} \subseteq (X_3, 2X_3] \cap \mathbb{N}$, we define

$$g(\alpha) = g_{\mathscr{A}}(\alpha) = \sum_{n \in \mathscr{A}} (\log n) e(n^3 \alpha). \tag{6.1}$$

Lemma 6.7. *Let $\mathscr{M}$ be the union of the intervals $\mathscr{M}(q, a)$ for $1 \leqslant a \leqslant q \leqslant X_3^{3/4}$ and $(a, q) = 1$, where*

$$\mathscr{M}(q, a) = \{\alpha : |q\alpha - a| \leqslant X_3^{-9/4}\}.$$

Suppose that $G(\alpha)$ and $h(\alpha)$ are integrable functions of period one. Let $g(\alpha) = g_{\mathscr{A}}(\alpha)$ be given in equation (6.1), *and let $\mathfrak{m} \subseteq [0, 1)$ be a measurable set. Then, we have*

$$\int_{\mathfrak{m}} g(\alpha) G(\alpha) h(\alpha) \mathrm{d}\alpha \ll N^{\frac{1}{3}} \mathcal{J}_0^{\frac{1}{4}} \left(\int_{\mathfrak{m}} |G(\alpha)|^2 \mathrm{d}\alpha \right)^{\frac{1}{4}} \mathscr{J}^{1/2}(\mathfrak{m}) + N^{\frac{7}{24}+\varepsilon} \mathscr{J}(\mathfrak{m}),$$

where

$$\mathscr{J}(\mathfrak{m}) = \int_{\mathfrak{m}} |G(\alpha) h(\alpha)| \mathrm{d}\alpha, \quad \mathcal{J}_0 = \sup_{\beta \in [0,1)} \int_{\mathscr{M}} \frac{w_3^2(q) |h(\alpha + \beta)|^2}{(1 + X_3^3 |\alpha - a/q|)^2} \mathrm{d}\alpha.$$

Proof. See Lemma 3.1 of Zhao [14]. □

For the proof of Proposition 3.2, we define a general Hardy-Littlewood dissection employed in our application of the circle method. When X is a positive number with $X \leqslant \sqrt{N}$, we take $\mathfrak{N}(X)$ to be the union of the intervals

$$\mathfrak{N}(q, a, X) = \{\alpha : |q\alpha - a| \leqslant XN^{-1}\},$$

with $1 \leqslant a \leqslant q \leqslant X$ and $(a, q) = 1$. In addition, when $X \leqslant \sqrt{N}/2$, we put $\mathfrak{K}(X) = \mathfrak{N}(2X)\backslash\mathfrak{N}(X)$. Finally, we take

$$\mathfrak{m}_1 = \mathfrak{m} \cap \mathfrak{N}\left(N^{\frac{1}{8}}\right), \quad \mathfrak{m}_2 = \mathfrak{m}\backslash\mathfrak{N}\left(N^{\frac{1}{8}}\right).$$

For $\alpha \in \mathfrak{m}_2$, by Dirichlet's lemma on rational approximation (e.g. see Lemma 2.1 of Vaughan [7]), there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ satisfying

$$1 \leqslant q \leqslant N^{\frac{3}{4}}, \quad |q\alpha - a| \leqslant N^{-\frac{3}{4}}, \quad (a, q) = 1.$$

Since $\alpha \in \mathfrak{m}_2$, we know that either $q > N^{\frac{1}{8}}$ or $N|q\alpha - a| > N^{\frac{1}{8}}$. Therefore, by Lemmas 6.1 and 6.4, it is easy to obtain

$$\sup_{\alpha\in\mathfrak{m}_2}|f_2(\alpha)| \ll X_2^{1-\frac{1}{8}+\varepsilon} + \frac{X_2^{1+\varepsilon}}{\sqrt{N^{1/8}}} \ll N^{\frac{7}{16}+\varepsilon}, \tag{6.2}$$

and

$$\sup_{\alpha\in\mathfrak{m}_2}|f_3(\alpha)| \ll X_3^{1-\frac{1}{12}+\varepsilon} + \frac{X_3^{1+\varepsilon}}{\sqrt{N^{1/8}}} \ll N^{\frac{11}{36}+\varepsilon}. \tag{6.3}$$

Define

$$\mathcal{I}(t) \coloneqq \int_{\mathfrak{m}_2} |f_2^2(\alpha)f_3^t(\alpha)|\mathrm{d}\alpha, \quad t \geqslant 1.$$

Taking

$$g(\alpha) = f_3(\alpha), \quad h(\alpha) = f_3(-\alpha), \quad G(\alpha) = |f_2(\alpha)|^2|f_3(\alpha)|^4$$

in Lemma 6.7, we obtain

$$\mathcal{I}(6) \ll N^{\frac{1}{3}}\mathcal{J}_0^{\frac{1}{4}}\left(\int_{\mathfrak{m}_2}|f_2^4(\alpha)f_3^8(\alpha)|\mathrm{d}\alpha\right)^{\frac{1}{4}}(\mathcal{I}(5))^{\frac{1}{2}} + N^{\frac{7}{24}+\varepsilon}\cdot\mathcal{I}(5), \tag{6.4}$$

where

$$\mathcal{J}_0 = \sup_{\beta\in[0,1)} \sum_{q\leqslant X_3^{3/4}} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathscr{M}(q,a)} \frac{w_3^2(q)|h(\alpha+\beta)|^2}{(1 + X_3^3|\alpha - a/q|)^2}\mathrm{d}\alpha$$

with

$$\mathscr{M}(q, a) = \{\alpha : |q\alpha - a| \leqslant X_3^{-9/4}\}.$$

By Lemma 6.6, we obtain

$$\mathcal{J}_0 \ll \Gamma(\gamma) \ll N^{-\frac{1}{3}+\varepsilon}. \tag{6.5}$$

From equations (6.2) and (6.3), we have

$$\begin{aligned}\int_{\mathfrak{m}_2} |f_2^4(\alpha)f_3^8(\alpha)|\mathrm{d}\alpha &\ll (\sup_{\alpha\in\mathfrak{m}_2} |f_2(\alpha)|^2)(\sup_{\alpha\in\mathfrak{m}_2} |f_3(\alpha)|^2)\cdot \mathcal{I}(6) \\ &\ll \left(N^{\frac{7}{16}+\varepsilon}\right)^2 \cdot \left(N^{\frac{11}{36}+\varepsilon}\right)^2 \cdot \mathcal{I}(6) \ll N^{\frac{107}{72}+\varepsilon}\cdot \mathcal{I}(6).\end{aligned} \tag{6.6}$$

Putting equations (6.5) and equation (6.6) into (6.4), we derive that

$$\mathcal{I}(6) \ll N^{\frac{179}{288}+\varepsilon}(\mathcal{I}(6))^{\frac{1}{4}}(\mathcal{I}(5))^{\frac{1}{2}} + N^{\frac{7}{24}+\varepsilon}\cdot \mathcal{I}(5). \tag{6.7}$$

It follows from Cauchy's inequality and Lemma 6.5 that

$$\mathcal{I}(5) \ll \left(\int_0^1 |f_2^2(\alpha)f_3^4(\alpha)|\mathrm{d}\alpha\right)^{\frac{1}{2}}\left(\int_{\mathfrak{m}_2} |f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha\right)^{\frac{1}{2}} \ll \left(N^{\frac{4}{3}+\varepsilon}\right)^{\frac{1}{2}}(I(6))^{\frac{1}{2}} \ll N^{\frac{2}{3}+\varepsilon}\cdot(I(6))^{\frac{1}{2}}. \tag{6.8}$$

Inserting equation (6.8) into equation (6.7), we have

$$\mathcal{I}(6) \ll N^{\frac{275}{288}+\varepsilon}(\mathcal{I}(6))^{\frac{1}{2}} + N^{\frac{23}{24}+\varepsilon}(\mathcal{I}(6))^{\frac{1}{2}} \ll N^{\frac{23}{24}+\varepsilon}(\mathcal{I}(6))^{\frac{1}{2}},$$

which implies

$$\mathcal{I}(6) \ll N^{\frac{23}{12}+\varepsilon} = N^{2-\frac{1}{12}+\varepsilon}. \tag{6.9}$$

We define the function $\Xi : [0,1] \to [0,1]$ by putting $\Xi(\alpha) = 0$ for $\alpha \in [0,1]\backslash\mathfrak{N}\left(N^{\frac{1}{8}}\right)$, and when $\alpha \in \mathfrak{N}\left(N^{\frac{1}{8}}\right) \cap \mathfrak{N}\left(q, a, N^{\frac{1}{8}}\right)$, by writing

$$\Xi(\alpha) = (q + qN|\alpha - a/q|)^{-1}.$$

Define

$$\mathfrak{m}_3 = \mathfrak{m}_1 \cap \mathfrak{N}\left(N^{\frac{1}{18}}\right), \quad \mathfrak{m}_4 = \mathfrak{m}_1\backslash\mathfrak{N}\left(N^{\frac{1}{18}}\right).$$

By noting the fact that $\mathfrak{m}_4 \subseteq \mathfrak{N}\left(N^{\frac{1}{8}}\right)\backslash\mathfrak{N}\left(N^{\frac{1}{18}}\right)$, hence for $\alpha \in \mathfrak{m}_4$, it follows from Lemmas 6.1 and 6.4 that

$$|f_2(\alpha)|^2 \ll N^{1+\varepsilon}\Xi(\alpha) \quad \text{and} \quad |f_3(\alpha)|^4 \ll N^{\frac{11}{9}+\varepsilon},$$

which yield the estimate

$$\begin{aligned}\int_{\mathfrak{m}_4} |f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha &\ll (\sup_{\alpha\in\mathfrak{m}_4} |f_3(\alpha)|^4)\cdot N^{1+\varepsilon}\cdot \int_{\mathfrak{N}\left(N^{\frac{1}{8}}\right)} \Xi(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha \\ &\ll N^{\frac{20}{9}+\varepsilon}\cdot \int_{\mathfrak{N}\left(N^{\frac{1}{8}}\right)} \Xi(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha.\end{aligned} \tag{6.10}$$

By Lemma 2 of Brüdern [15], we obtain

$$\int_{\mathfrak{N}\left(N^{\frac{1}{8}}\right)} \Xi(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha \ll N^{-\frac{1}{3}+\varepsilon},$$

from which, using equation (6.10), we conclude that

$$\int_{\mathfrak{m}_4} |f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha \ll N^{2-\frac{1}{9}+\varepsilon}. \tag{6.11}$$

For $\alpha \in \mathfrak{m}_3$, by Lemmas 6.1 and 6.4, we obtain

$$|f_2(\alpha)|^2 \ll N^{1+\varepsilon}\Xi(\alpha) \quad \text{and} \quad |f_3(\alpha)|^2 \ll N^{\frac{2}{3}+\varepsilon}\Xi(\alpha).$$

Hence, for $\alpha \in \mathfrak{m}_3$, there holds

$$|f_2^2(\alpha)f_3^2(\alpha)| \ll N^{\frac{5}{3}+\varepsilon}\Xi^2(\alpha),$$

which combined with the trivial estimate $f_3(\alpha) \ll N^{\frac{1}{3}+\varepsilon}$ yields

$$\int_{\mathfrak{m}_3} |f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha \ll N^{\frac{5}{3}+\varepsilon} \cdot N^{\frac{2}{3}+\varepsilon} \cdot \int_{\mathfrak{m}\cap\mathfrak{N}\left(N^{\frac{1}{18}}\right)} \Xi^2(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha. \tag{6.12}$$

This leaves the set $\mathfrak{m} \cap \mathfrak{N}\left(N^{\frac{1}{18}}\right)$ for treatment, and this set is covered by the union of sets $\mathfrak{R}(Y) = \mathfrak{N}(2Y)\backslash\mathfrak{N}(Y)$ as Y runs over the sequence $2^{-j}N^{\frac{1}{18}}$ with $P \ll Y \leqslant N^{\frac{1}{18}}/2$. Note that $\Xi(\alpha) \ll Y^{-1}$ for $\alpha \notin \mathfrak{N}(Y)$. Moreover, Lemma 2 of Brü dern [15] supplies the following upper bound:

$$\int_{\mathfrak{N}(2Y)} \Xi(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha \ll YN^{-\frac{2}{3}+\varepsilon} + N^{-\frac{1}{3}+\varepsilon},$$

which implies that

$$\int_{\mathfrak{R}(Y)} \Xi^2(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha \ll N^{-\frac{2}{3}+\varepsilon} + N^{-\frac{1}{3}+\varepsilon}Y^{-1} \ll N^{-\frac{2}{3}+\varepsilon} + N^{-\frac{1}{3}+\varepsilon}P^{-1} \ll N^{-\frac{17}{36}+\varepsilon}. \tag{6.13}$$

By a splitting argument, from equations (6.12) and (6.13), we derive that

$$\begin{aligned}\int_{\mathfrak{m}_3} |f_2^2(\alpha)f_3^6(\alpha)|\mathrm{d}\alpha &\ll N^{\frac{7}{3}+\varepsilon} \times \max_{P\ll Y\leqslant N^{\frac{1}{18}}/2} \int_{\mathfrak{R}(Y)} \Xi^2(\alpha)|f_3(\alpha)|^2\mathrm{d}\alpha \\ &\ll N^{\frac{7}{3}+\varepsilon} \cdot N^{-\frac{17}{36}+\varepsilon} \ll N^{2-\frac{5}{36}+\varepsilon}.\end{aligned} \tag{6.14}$$

Combining equations (6.9), (6.11), and (6.14), we obtain the conclusion of Proposition 3.2.

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