Research Article

Shiping Zhong*, Zehui Zhao, and Xinjie Wan

Solitons for the coupled matrix nonlinear Schrödinger-type equations and the related Schrödinger flow

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Abstract: In this article, the coupled matrix nonlinear Schrödinger (NLS) type equations are gauge equivalent to the equation of Schrödinger flow from $\mathbb{R}^1$ to complex Grassmannian manifold $\widetilde{G}_{n,k} = \mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(n-k, \mathbb{C})$, which generalizes the correspondence between Schrödinger flow to the complex 2-sphere $\mathbb{CS}^2(1) \hookrightarrow \mathbb{C}^3$ and the coupled Landau-Lifshitz (CLL) equation. This gives a geometric interpretation of the matrix generalization of the coupled NLS equation (i.e., CLL equation) via Schrödinger flow to the complex Grassmannian manifold $\widetilde{G}_{n,k}$. Finally, we explicit soliton solutions of the Schrödinger flow to the complex Grassmannian manifold $\widetilde{G}_{2,1}$.

Keywords: coupled matrix nonlinear Schrödinger-type equations, complex Grassmannian manifold, Schrödinger flow, gauge equivalence, soliton solutions

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1 Introduction

In this article, the matrix generalization of the second Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy (i.e., the coupled matrix nonlinear Schrödinger [NLS] type equations) (e.g., see [1–3]):

$$\begin{cases} \varepsilon\Psi_{1t} - \Psi_{1xx} + 2\Psi_1\Psi_2\Psi_1 = 0, \\ \varepsilon\Psi_{2t} + \Psi_{2xx} - 2\Psi_2\Psi_1\Psi_2 = 0, \end{cases} \tag{1}$$

where $\Psi_1 = \Psi_1(x,t), \Psi_2 = \Psi_2(x,t)$ are $k \times (n-k)$ and $(n-k) \times k$ complex matrix-valued functions and $\varepsilon^2 = \pm 1$. Note that the matrix form of the coupled matrix NLS equations (1) is

$$u_t = \frac{1}{\varepsilon^2}[\sigma_3, u_{xx}] - \frac{1}{2\varepsilon^2}[u, [u, [\sigma_3, u]]], \tag{2}$$

where

$$u = u(x,t) = \begin{pmatrix} 0 & \Psi_1 \\ \Psi_2 & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbf{m} \text{ (see Secton 2)}, \sigma_3 = \frac{\varepsilon}{2}\begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix},$$

*** Corresponding author: Shiping Zhong,** School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, P. R. China, e-mail: zhongshiping@gnnu.edu.cn
Zehui Zhao: School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, P. R. China, e-mail: zhaozehui@gnnu.edu.cn
Xinjie Wan: School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, P. R. China, e-mail: xinjiewan2023@163.com

which is called the $\widetilde{G}_{n,k}$-NLS equation (see Section 2), where $[\cdot,\cdot]$ denotes Lie bracket. These symmetry reductions and special orders of Ψ_1 and Ψ_2 lead to several relevant cases (e.g., see [1–16] for more details on this fact) in the following:

Matrix Case 1 (Matrix-Nonlocal-types): The matrix nonlocal NLS equation (e.g., see [2]):

$$\sqrt{-1}Q_t(x,t) - Q_{xx}(x,t) + 2\zeta^2 Q(x,t)Q^*(-x,t)Q(x,t) = 0, \quad x,t \in \mathbb{R}, \tag{3}$$

is a reduction of equation (1) by $\Psi_1(x,t) = Q(x,t)$ and $\Psi_2(x,t) = \zeta^2 Q^*(-x,t)$, $n = 2k$, $\zeta^2 = \pm 1$, $\varepsilon = \sqrt{-1}$, where the dagger indicates the complex conjugate transpose. In the special case where Q is a column vector, then equation (3) reduces to the vector nonlocal NLS equation (Vector-Nonlocal-types) (see [2]):

$$Q^{(n-1)}_{-1,1} : \sqrt{-1}Q_t(x,t) - Q_{xx}(x,t) + 2\zeta^2 Q(x,t)Q^*(-x,t)Q(x,t) = 0, \quad x,t \in \mathbb{R}, \tag{4}$$

namely, equation (4) is a reduction of equation (1) by $\Psi_1(x,t) = Q(x,t) = (q_1(x,t), \ldots, q_{n-1}(x,t))$, $\Psi_2(x,t) = \zeta^2 Q^*(-x,t)$, $k = 1$, $\zeta^2 = \pm 1$, $\varepsilon = \sqrt{-1}$. And the natural generalization of equation (4) is a hierarchy of two-family-parameter $\{(\varepsilon_{x_{\vec{j}}}, \varepsilon_{t_{\vec{j}}}) | \varepsilon_{x_{\vec{j}}}, \varepsilon_{t_{\vec{j}}} = \pm 1, j = 1, 2, \ldots, n-1\}$ equation (called $Q^{(n-1)}_{\varepsilon_{x_{\vec{j}}},\varepsilon_{t_{\vec{j}}}}$ hierarchy) (see [15]):

$$Q^{(n-1)}_{\varepsilon_{x_{\vec{j}}},\varepsilon_{t_{\vec{j}}}} : \sqrt{-1}Q_t(x,t) - Q_{xx}(x,t) + 2\zeta^2 Q(x,t)Q^*(\varepsilon_{x_{\vec{j}}}x, \varepsilon_{t_{\vec{j}}}t)Q(x,t) = 0, \quad x,t \in \mathbb{R}, \tag{5}$$

which is a reduction of equation (1) by $\Psi_1(x,t) = Q(x,t) = (q_1(x,t), \ldots, q_{n-1}(x,t))$, $\Psi_2(x,t) = \zeta^2 Q^*(\varepsilon_{x_{\vec{j}}}x, \varepsilon_{t_{\vec{j}}}t) = (q_1(\varepsilon_{x_{\vec{1}}}x, \varepsilon_{t_{\vec{1}}}t), \ldots, q_{n-1}(\varepsilon_{x_{\overrightarrow{n-1}}}x, \varepsilon_{t_{\overrightarrow{n-1}}}t))$, $k = 1$, $\zeta^2 = \pm 1$, $\varepsilon = \sqrt{-1}$.

Matrix Case 2 (Matrix-Local-types). The matrix NLS equation by Fordy and Kulish in [9] (or see [8,12,13]):

$$\sqrt{-1}Q_t(x,t) - Q_{xx}(x,t) + 2\zeta^2 Q(x,t)Q^*(x,t)Q(x,t) = 0, \quad x,t \in \mathbb{R}, \tag{6}$$

which is a reduction of equation (1) by $\Psi_1(x,t) = Q(x,t)$, $\Psi_2(x,t) = \zeta^2 Q^*(x,t)$, $n = 2k$, $\zeta^2 = \pm 1$, $\varepsilon = \sqrt{-1}$. In the special case where Q is a column vector, then equation (6) reduces to the vector local NLS equation (Vector-Local-types) (see [6,10,16]):

$$Q^{(n-1)}_{1,1} : \sqrt{-1}Q_t(x,t) - Q_{xx}(x,t) + 2\zeta^2 Q(x,t)Q^*(x,t)Q(x,t) = 0, \tag{7}$$

i.e., equation (7) is a reduction of equation (1) by $\Psi_1(x,t) = Q(x,t) = (q_1(x,t), \ldots, q_{n-1}(x,t))$, $\Psi_2(x,t) = \zeta^2 Q^*(x,t)$, $k = 1$, $\zeta^2 = \pm 1$, $\varepsilon = \sqrt{-1}$. Though equations (1)–(7) have some dynamical properties, e.g., solitons, general N-solitons, multi-soliton solutions, reflectionless solutions, and rogue wave solutions. Hence, a quite relevant question arises: does there exist a unified geometric interpretation of equations (3)–(7) or equation (1)?

This question is motivated by the work of geometric characterization of equation (6); it is proved by Terng and Uhlenbeck [13] that equation (6) is gauge equivalent to the complex compact Grassmannian manifolds $U(n)/U(k) \times U(n-k)$, and by Terng and Thorbergsson in [17] for the other three classical Hermitian symmetric spaces. Chen [18] generalized their result to the cases of $u(k, n-k)$ and $\mathcal{G}l(n, \mathbb{R})$. This gives a unified geometric interpretation of the three typical second-order matrix NLS equations in the second matrix-AKNS hierarchies via Schrödinger flow. Along this route, in [19–22], the authors gave a complete description of the theory of vortex filament on symmetric algebras up to the second-order and third-order approximation from a purely geometric way. Recently, the first author showed that the coupled NLS equations, which are a reduction of equation (1) by $\Psi_1 = \varphi_1(x,t)$, $\Psi_2 = \varphi_2(x,t)$, $k = 1$, $n = 2$, $\varepsilon = \sqrt{-1}$, are geometrically interpreted as the equation of Schrödinger flow from $\mathbb{R}^1$ to the complex 2-sphere $\mathbb{CS}^2(1) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 1\} \hookrightarrow \mathbb{R}^{3,3} \cong \mathbb{C}^3$ with the standard holomorphic metric, and is also gauge equivalent to the unconventional system of the coupled Landau-Lifshitz (CLL) equation (see [23,24]). CLL reads as the following evolution equation:

$$S_t = -\frac{\sqrt{-1}}{2}[S, S_{xx}],$$

where $S = S(x,t)$ is a 2×2 complex-matrix with $S^2 = I_{2\times 2}$ ($I_{2\times 2}$ stands for the 2×2 unit matrix) and $\mathrm{tr}S = 0$. Some physical and geometrical properties of CLL are also discussed in [23,24].

On the other hand, by the loop group factorization method, Terng and Uhlenbeck [25] first constructed Bäcklund transformations for the Zakharov-Shabat (ZS)-AKNS $\mathrm{sl}(n,\mathbb{C})$-hierarchy. Afterward, using this method, many hierarchies (such as SU(1,1)-hierarchy, vmKdV-hierarchy, $A(1)^{n-1}$-KdV-hierarchy, Gelfand-Dickey-hierarchy, $\hat{B}_n^{(1)}$-hierarchy and $\hat{A}_{2n}^{(2)}$-KdV-hierarchy) of Bäcklund and Darboux transformations are obtained (see [26–32]).

This article is organized as follows. Section 2 gives preliminary about the symmetric Lie algebras $\mathcal{G}l(n,\mathbb{C})$ and the second-order matrix-AKNS hierarchy. In Section 3, we show that the coupled matrix NLS-type equations are gauge equivalent to the equation of Schrödinger flow from $\mathbb{R}^1$ to complex Grassmannian manifold $\widetilde{G}_{n,k}$, which generalizes the correspondence between Schrödinger flow to the complex 2-sphere $\mathbb{C}\mathbb{S}^2(1) \hookrightarrow \mathbb{C}^3$ and the CLL equations. Section 4 explicits the soliton solutions of the Schrödinger flow to the complex Grassmannian manifold $\widetilde{G}_{2,1}$.

2 Symmetric Lie algebras and the second-order matrix-AKNS hierarchy

In this section, we recall some fundamental facts about the complex general linear Lie algebra $\mathcal{G}l(n,\mathbb{C})$ $(n \geq 2)$ with index k $(1 \leq k < n)$: the space of all $n \times n$ complex matrices and the second-order matrix-AKNS hierarchy.

First, we recall the concept of symmetric Lie algebra $\mathbf{g}$. The so-called symmetric Lie algebra $\mathbf{g}$ is a Lie algebra that has a decomposition as a vector space sum: $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$ satisfying the (bracket) symmetric conditions: $[\mathbf{k},\mathbf{k}] \subset \mathbf{k}$, $[\mathbf{m},\mathbf{m}] \subset \mathbf{k}$ and $[\mathbf{k},\mathbf{m}] \subset \mathbf{m}$ (see [9,33,34]). In such a symmetric Lie algebra, there is an element denoted by σ_3 in $\mathbf{k}$ such that $\mathbf{k} = \mathrm{Kernel}\left(\mathrm{ad}_{\sigma_3}\right) = \{\chi \in \mathbf{g} | [\chi, \sigma_3] = 0\}$. A homogeneous space is a manifold M with a transitive action of a Lie group G. Equivalently, it is a manifold of the form G/K, where G is a Lie group and K is a closed subgroup of G.

Now, we recall some results of the complex general linear Lie algebra $\mathcal{G}l(n,\mathbb{C})$ $(n \geq 2)$ with index k $(1 \leq k < n)$.

Lemma 1. *The complex general linear Lie algebra $\mathcal{G}l(n,\mathbb{C}) = \mathbf{k} \oplus \mathbf{m}$ is a symmetric Lie algebra, where*

$$\mathbf{k} = \mathrm{Kernel}\left(\mathrm{ad}_{\sigma_3}\right) = \left\{\begin{pmatrix} A_{k\times k} & 0 \\ 0 & B_{(n-k)\times(n-k)} \end{pmatrix} \in \mathcal{G}l(n,\mathbb{C})\right\}$$

and

$$\mathbf{m} = \left\{\begin{pmatrix} 0 & U_{k\times(n-k)} \\ V_{(n-k)\times k} & 0 \end{pmatrix} \in \mathcal{G}l(n,\mathbb{C}) \;\middle|\; \forall U_{k\times(n-k)} \quad \text{and} \quad V_{(n-k)\times k}\right\}.$$

And the adjoint obit space

$$\widetilde{G}_{n,k} = \{E^{-1}\sigma_3 E | \ \forall E \in \mathrm{GL}(n,\mathbb{C})\}$$

is a homogeneous symmetric space, where

$$\sigma_3 = \frac{\varepsilon}{2}\begin{pmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{pmatrix}, \varepsilon^2 = \pm 1. \tag{8}$$

Proof. Let us define a (left) operation of the complex general linear Lie group $\mathrm{GL}(n,\mathbb{C})$ on $\widetilde{G}_{n,k}$ by

$$\Phi : \mathrm{GL}(n,\mathbb{C}) \times \widetilde{G}_{n,k} \to \widetilde{G}_{n,k}, (X,\gamma) \mapsto \Phi(X,\gamma) = X \circ \gamma = X\gamma X^{-1},$$

since

$$\Phi(X,\gamma) = X \circ \gamma = X\gamma X^{-1} = XE^{-1}\sigma_3 E X^{-1} = XE^{-1}\sigma_3(XE^{-1})^{-1}.$$

It is obvious to see that an operation satisfies the following:

$$I_{n\times n}\circ\gamma=I_{n\times n}\gamma I_{n\times n}^{-1}=\gamma,\quad \forall\gamma\in\widetilde{G}_{n,k},$$
$$(XY)\circ\gamma=(XY)\gamma(XY)^{-1}=X(Y\gamma Y^{-1})X^{-1}=X\circ(Y\circ\gamma),\quad \forall X,Y\in \mathrm{GL}(n,\mathbb{C}),$$

and the action is transitive. In fact, $\forall\gamma_1=E_1^{-1}\sigma_3E_1$, $\gamma_2=E_2^{-1}\sigma_3E_2\in\widetilde{G}_{n,k}$, then $\exists\, X=E_2^{-1}E_1\in\mathrm{GL}(n,\mathbb{C})$, s.t.,

$$X\circ\gamma_1=E_2^{-1}E_1\gamma_1(E_2^{-1}E_1)^{-1}=E_2^{-1}E_1E_1^{-1}\sigma_3E_1E_1^{-1}E_2=E_2^{-1}\sigma_3E_2=\gamma_2.$$

Moreover, the isotropy group at the point $\sigma_3\in\widetilde{G}_{n,k}$ is

$$G_{\sigma_3}=\{X\in\mathrm{GL}(n,\mathbb{C})|X\circ\sigma_3=\sigma_3\}=\{X\in\mathrm{GL}(n,\mathbb{C})|X\sigma_3=\sigma_3X\}$$
$$=\left\{\begin{pmatrix}A_{k\times k} & 0\\ 0 & B_{(n-k)\times(n-k)}\end{pmatrix}\in\ \mathrm{GL}(n,\mathbb{C})\right\}:=K.$$

Hence, $\widetilde{G}_{n,k}$ is a homogeneous space of the group $\mathrm{GL}(n,\mathbb{C})$; in fact, the map

$$G/K\to\widetilde{G}_{n,k},\,[x]\mapsto X\circ\sigma_3$$

is diffeomorphism and the K-principle bundle $K\to\mathrm{GL}(n,\mathbb{C})\to\widetilde{G}_{n,k}$. The Lie algebra $\mathcal{G}l(n,\mathbb{C})$ $(n\geq 2)$ with index k $(1\leq k<n)$ of $\mathrm{GL}(n,\mathbb{C})$ decomposes as $\mathcal{G}l(n,\mathbb{C})=\mathbf{k}\oplus\mathbf{m}$, where

$$\mathbf{k}=\left\{\begin{pmatrix}A_{k\times k} & 0\\ 0 & B_{(n-k)\times(n-k)}\end{pmatrix}\in\mathcal{G}l(n,\mathbb{C})\right\}$$

is the Lie algebra of K with the property: $[\mathbf{k},\mathbf{k}]\subset\mathbf{k}$ and

$$\mathbf{m}=\left\{\begin{pmatrix}0 & U_{k\times(n-k)}\\ V_{(n-k)\times k} & 0\end{pmatrix}\in\mathcal{G}l(n,\mathbb{C})\,\middle|\,\forall U_{k\times(n-k)}\,\text{and}\,V_{(n-k)\times k}\right\}.$$

It is easy to verify that the symmetric connections fulfill

$$[\mathbf{k},\mathbf{m}]\subset\mathbf{m},\ \ [\mathbf{m},\mathbf{m}]\subset\mathbf{k}.$$

Therefore, $\mathcal{G}l(n,\mathbb{C})=\mathbf{k}\oplus\mathbf{m}$ is a symmetric algebra, and hence, $\widetilde{G}_{n,k}$ is a homogeneous symmetric space. □

Now, we recall the concept of $(J^2=\pm1)$-Kähler manifold (M,J,g) (such as see [35]). A manifold will be called to have a $J^2=\pm1$-structure if an almost complex $(J^2=-1)$ or almost product $(J^2=1)$ structure J is an isometry and $\nabla J=0$, where ∇ denotes the Levi-Civita connection of g. It is also said that (M,J,g) is a $(J^2=\pm1)$-Kähler manifold.

Lemma 2. *The adjoint obit space $\widetilde{G}_{n,k}$ is a $J^2=\pm1$-Kähler manifold with tensor $J_\gamma=[\gamma,\cdot]$ at a point $\gamma\in\widetilde{G}_{n,k}$.*

Proof. First, we consider the tangent space of a point $\gamma=E^{-1}\sigma_3E$ on $\widetilde{G}_{n,k}$. We can decompose the element P of $\mathcal{G}l(n,\mathbb{C})$ of the form $P=\mathrm{diag(P)}+\widetilde{P}$, where $\widetilde{P}=\text{off-diag(P)}\in\mathbf{m}$ is defined to be the off-diagonal part of P with respect to the decomposition $\mathcal{G}l(n,\mathbb{C})=\mathbf{k}\oplus\mathbf{m}$. So,

$$\sigma(t)=\exp(-t\,\mathrm{diag(P)}-t\widetilde{P})\sigma_3\exp(t\,\mathrm{diag(P)}+t\widetilde{P})$$

is a curve on $\widetilde{G}_{n,k}$ passing the point σ_3. By taking its derivation, we can obtain the tangent space of the point σ_3 of $\widetilde{G}_{n,k}$:

$$\sigma_3\widetilde{P}-\widetilde{P}\sigma_3+\sigma_3\,\mathrm{diag(P)}-\mathrm{diag(P)}\sigma_3=[\sigma_3,\widetilde{P}],$$

which is a matrix of form $\widetilde{P}$, because $\mathrm{diag(P)}$ is communicative with σ_3. So the tangent space $T_\gamma\widetilde{G}_{n,k}$ at γ consisted of

$$T_\gamma\widetilde{G}_{n,k}=\{E^{-1}[\sigma_3,P]E|\ \forall P\in\mathcal{G}l(n,\mathbb{C})\}=\{E^{-1}[\sigma_3,\widetilde{P}]E|\ \forall\widetilde{P}\in\mathbf{m}\}.$$

Hence, $\forall X = E^{-1}[\sigma_3, P]E \in T_\gamma \widetilde{G}_{n,k}, P \in \mathbf{m}$. Let us define an operation J_γ at γ by

$$J_\gamma = [\gamma, \cdot] : T_\gamma \widetilde{G}_{n,k} \to T_\gamma \widetilde{G}_{n,k}, X \mapsto J_\gamma(X) = [\gamma, X]. \tag{9}$$

Therefore, $\forall X = E^{-1}[\sigma_3, \widetilde{P}]E \in T_\gamma \widetilde{G}_{n,k}$; it is a direct verification that

$$\begin{aligned}
J_\gamma(X) = [\gamma, X] = \gamma X - X\gamma &= E^{-1}\sigma_3 E E^{-1}[\sigma_3, \widetilde{P}]E - E^{-1}[\sigma_3, \widetilde{P}]EE^{-1}\sigma_3 E \\
&= E^{-1}\sigma_3[\sigma_3, \widetilde{P}]E - E^{-1}[\sigma_3, \widetilde{P}]\sigma_3 E \\
&= \frac{\varepsilon^2}{2}E^{-1}\widetilde{P}E - \frac{\varepsilon^2}{2}E^{-1}(-\widetilde{P})E \\
&= \varepsilon^2 E^{-1}\widetilde{P}E, \\
J_\gamma^2(X) = [\gamma, [\gamma, X]] = [\gamma, \varepsilon^2 E^{-1}\widetilde{P}E] &= E^{-1}\sigma_3 E\varepsilon^2 E^{-1}\widetilde{P}E - \varepsilon^2 E^{-1}\widetilde{P}EE^{-1}\sigma_3 E \\
&= \varepsilon^2 E^{-1}\sigma_3\widetilde{P}E - \varepsilon^2 E^{-1}\widetilde{P}\sigma_3 E \\
&= \varepsilon^2 E^{-1}[\sigma_3, \widetilde{P}]E \\
&= \varepsilon^2 X = \pm X.
\end{aligned}$$

This shows that the $J^2 = \pm 1$-Kähler structure of $\widetilde{G}_{n,k}$ is given by (9). In fact, we can define bi-invariant metric on $\widetilde{G}_{n,k}$:

$$\langle \cdot, \cdot \rangle_\gamma : T_\gamma \widetilde{G}_{n,k} \to T_\gamma \widetilde{G}_{n,k}, \quad (X, Y) \mapsto \langle X, Y \rangle_\gamma = -\operatorname{tr}(XY).$$

It is a direct verification that $\forall X, Y, Z \in T_\gamma \widetilde{G}_{n,k}$, we have

$$\begin{aligned}
d\omega_\gamma(X, Y, Z) &= X(\omega_\gamma(Y, Z)) - Y(\omega_\gamma(X, Z)) + Z(\omega_\gamma(X, Y)) - \omega_\gamma([X, Y], Z) + \omega_\gamma([X, Z], Y) - \omega_\gamma([Y, Z], X) \\
&= \langle \nabla_X(J_\gamma Y), Z \rangle_\gamma + \langle J_\gamma Y, \nabla_X Z \rangle_\gamma - \langle \nabla_Y(J_\gamma X), Z \rangle_\gamma - \langle J_\gamma X, \nabla_Y Z \rangle_\gamma + \langle \nabla_Z(J_\gamma X), Y \rangle_\gamma + \langle J_\gamma X, \nabla_Z Y \rangle_\gamma \\
&\quad - \langle J_\gamma[X, Y], Z \rangle_\gamma + \langle J_\gamma[X, Z], Y \rangle_\gamma - \langle J_\gamma[Y, Z], X \rangle_\gamma \\
&= 0.
\end{aligned}$$

□

From Lemma 1, the symmetric space $\widetilde{G}_{n,k}$ is simply written as $\widetilde{G}_{n,k} = \mathrm{GL}(n, \mathbb{C})/\mathrm{GL}(k, \mathbb{C}) \times \mathrm{GL}(n-k, \mathbb{C})$ and is called the complex Grassmannian manifold.

Next, let us recall the three typical classes of the Hermitian symmetric Lie algebras $\mathcal{Gl}(n, \mathbb{C})$ with index k ($1 \le k < n$) having three types.

The first subclass of symmetric Lie algebras $\mathcal{Gl}(n, \mathbb{C})$ consists of Hermitian symmetric Lie algebras $u(n) \subset \mathcal{Gl}(n, \mathbb{C})$ ($n \ge 2$) with index k ($1 \le k < n$) of compact type. In fact, for any given $1 \le k < n$, $u(n)$ is decomposable as $u(n) = \mathbf{k_1} \oplus \mathbf{m_1}$ satisfy the symmetric conditions, where

$$\mathbf{k_1} = \operatorname{Kernel}\left(\operatorname{ad}_{\sigma_3}\right) = \left\{\begin{pmatrix} A_{k\times k} & 0 \\ 0 & B_{(n-k)\times(n-k)} \end{pmatrix} \in u(n)\right\}$$

and

$$\mathbf{m_1} = \left\{\begin{pmatrix} 0 & U_{k\times(n-k)} \\ -U^*_{(n-k)\times k} & 0 \end{pmatrix} \in u(n)\right\},$$

where $U^*_{(n-k)\times k}$ stands for the transposed conjugate matrix of $U_{k\times(n-k)}$, σ_3 is given by (8), and $\varepsilon^2 = -1$.

The second subclass of symmetric Lie algebras $\mathcal{Gl}(n, \mathbb{C})$ consists of Hermitian symmetric Lie algebras $u(k, n-k) \subset \mathcal{Gl}(n, \mathbb{C})$ with index k ($1 \le k < n$) of noncompact type. In this case, we see that $u(k, n-k)$ is decomposable as $u(k, n-k) = \mathbf{k_2} \oplus \mathbf{m_2}$ satisfy the symmetric conditions, where

$$\mathbf{k_2} = \operatorname{Kernel}(\operatorname{ad}_{\sigma_3}) = \left\{\begin{pmatrix} A_{k\times k} & 0 \\ 0 & B_{(n-k)\times(n-k)} \end{pmatrix} \in u(k, n-k)\right\}$$

and

$$\mathbf{m_2} = \left\{\begin{pmatrix} 0 & U_{k\times(n-k)} \\ U^*_{(n-k)\times k} & 0 \end{pmatrix} \in u(k, n-k)\right\},$$

where σ_3 is given by (8), and $\varepsilon^2 = -1$.

The third subclass of symmetric Lie algebras $\mathcal{G}l(n,\mathbb{C})$ consists of para-Hermitian symmetric Lie algebras $\mathcal{G}l(n,\mathbb{R}) \subset \mathcal{G}l(n,\mathbb{C})$ $(n \geq 2)$ with index k $(1 \leq k < n)$. In this case, for any given $1 \leq k < n$, we see that $\mathcal{G}l(n,\mathbb{R}) = \mathbf{k}_3 \oplus \mathbf{m}_3$ is a symmetric Lie algebra, where

$$\mathbf{k}_3 = \mathrm{Kernel}\big(\mathrm{ad}_{\sigma_3}\big) = \left\{\begin{pmatrix} A_{k\times k} & 0 \\ 0 & B_{(n-k)\times(n-k)} \end{pmatrix} \in \mathcal{G}l(n,\mathbb{R})\right\}$$

and

$$\mathbf{m}_3 = \left\{\begin{pmatrix} 0 & U^+_{k\times(n-k)} \\ U^-_{(n-k)\times k} & 0 \end{pmatrix} \in \mathcal{G}l(n,\mathbb{R}) \,\middle|\, \forall U^+_{k\times(n-k)} \quad \text{and} \quad U^-_{(n-k)\times k}\right\},$$

where σ_3 is given by (8), and $\varepsilon^2 = 1$.

Finally, we briefly review the second matrix-AKNS hierarchy on a symmetric Lie algebra. It is well known that the coupled matrix NLS equations are equivalent to the compatibility condition of a Lax pair for the potential matrix

$$u = \begin{pmatrix} 0 & \Psi_1(x,t) \\ \Psi_2(x,t) & 0 \end{pmatrix} \in \mathbf{m},$$

where the first equation in the Lax pair is the so-called matrix ZS or AKNS system (see [1]). Specifically, the coupled matrix NLS equations (1) admit the Lax pair

$$E_x = (-\sigma_3\lambda + u)E,\, E_t = (\sigma_3\lambda^2 - u\lambda + \varepsilon^2 P_{-1}(u))E, \tag{10}$$

where $E : \mathbb{R}^2 \times \mathbb{C} \to \mathrm{GL}(n,\mathbb{C})$ and

$$P_{-1}(u) = 2\sigma_3(u_x - u^2) = \varepsilon\begin{pmatrix} -\Psi_1(x,t)\Psi_2(x,t) & \Psi_{1x}(x,t) \\ -\Psi_{2x}(x,t) & \Psi_2(x,t)\Psi_1(x,t) \end{pmatrix}.$$

We call $E : \mathbb{R}^2 \to \mathrm{GL}(n,\mathbb{C})$ a frame of the solution u of the $\widetilde{G}_{n,k}$-NLS equation (2) with $E(0,0,\lambda) = I_{n\times n}$. Hence, rewrite equation (10) as:

$$\varepsilon^2 u_t = P_{-1}(u)_x + [P_{-1}(u), u]. \tag{11}$$

3 Schrödinger flow into the complex Grassmannian manifold $\widetilde{G}_{n,k}$

It is well known that the equation of Schrödinger flow from a Riemannian manifold (M,g) to a $J^2 = \pm 1$-Kähler manifold (N,J,h), where J satisfies $J^2 = \pm 1$ and compatible to the metric h, is given by the following Hamiltonian gradient flow:

$$u_t = J_u\tau(u), \tag{12}$$

where $\tau(u)$ is the tension field of map $u : M \to N$ ([13,36–38]).

Theorem 1. *The equation* (12) *of Schrödinger flow from* $\mathbb{R}^1$ *to complex Grassmannian manifold* $\widetilde{G}_{n,k}$ *is*

$$\gamma_t = \varepsilon^2[\gamma, \gamma_{xx}]. \tag{13}$$

Proof. Let $\gamma = E^{-1}\sigma_3 E \in \widetilde{G}_{n,k}$ be the equation of Schrödinger flow from $\mathbb{R}^1$ to complex Grassmannian manifold $\widetilde{G}_{n,k}$, namely, γ solves [13]

$$J_\gamma \gamma_t = \nabla_{\gamma_x}\gamma_x,$$

where $E = E(x,t) \in \mathrm{GL}(n,\mathbb{C})$, J_γ is the $J^2 = \pm 1$-Kähler structure of $\widetilde{G}_{n,k}$ at the point γ, ∇_x the covariant derivative $\nabla_{\frac{\partial}{\partial x}}$ on the pull-back bundle $\gamma^{-1}T\widetilde{G}_{n,k}$ induced from the Levi-Civita connection on $\widetilde{G}_{n,k}$, and by γ_x the $\nabla_x \gamma$.

Let $\gamma = E^{-1}(t,x)\sigma_3 E(t,x)$ be a map from the line $\mathbb{R}$ to $\widetilde{G}_{n,k}$, where σ_3 is given by (8). Without loss of generality, we may assume that E satisfies: $E_x = PE$ for some $P \in \mathbf{m}$, where $\mathbf{m}$ fits with $\mathcal{Gl}(n,\mathbb{C}) = \mathbf{k} \oplus \mathbf{m}$. In fact, if E does not meet the requirement, we may make a transform:

$$E \to \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} E,$$

(this is because the form $E^{-1}\sigma_3 E$ is invariant up to the transform) such that by suitably choosing A and B (through solving a linear differential system of A and B), P can be modified so that the new P satisfies $P \in \mathbf{m}$. Based on this fact, it is shown that

$$\begin{aligned} \nabla_{\gamma_x}\gamma &= \gamma_x = E^{-1}[\sigma_3, P]E \ \text{ (taking the tangent part)}, \\ \nabla_{\gamma_x}\gamma_x &= \gamma_{xx} + E^{-1}[P,[\sigma_3,P]]E \ \text{ (taking the tangent part of) } (\nabla_x\gamma)_x. \end{aligned}$$

Hence, we have

$$\begin{aligned} \gamma_t = \varepsilon^2 J_\gamma \nabla_{\gamma_x}\gamma_x &= \varepsilon^2[\gamma, \nabla_{\gamma_x}\gamma_x] \\ &= \varepsilon^2[\gamma,\gamma_{xx}] + [\gamma, E^{-1}[P,[\sigma_3,P]]E] \\ &= \varepsilon^2[\gamma,\gamma_{xx}] + [E^{-1}\sigma_3 E, E^{-1}[P,[\sigma_3,P]]E] \\ &= \varepsilon^2[\gamma,\gamma_{xx}], \ \text{(since } P \in \mathbf{m}). \end{aligned}$$

□

Hence, the Lax pair of equation (13) is

$$\delta_x = -\gamma\lambda\delta, \ \delta_t = (\gamma\lambda^2 - \varepsilon^2[\gamma,\gamma_x]\lambda)\delta. \tag{14}$$

Let $\mathbb{CS}^{2,\varepsilon}(1) = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : z_1^2 - \varepsilon^2 z_2^2 + z_3^2 = 1\} \hookrightarrow \mathbb{C}^3$ with the metric (i.e., $\mathrm{d}z^2 = \mathrm{d}z_1^2 - \varepsilon^2\mathrm{d}z_2^2 + \mathrm{d}z_3^2$). Now, for vectors $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3)$ in $\mathbb{C}^3$, we define the cross-product of u and v by:

$$u \times_\varepsilon v = (u_2v_3 - u_3v_2, -\varepsilon^2(u_3v_1 - u_1v_3), u_1v_2 - u_2v_1).$$

Corollary 1. *The equation of Schrödinger flow from $\mathbb{R}^1$ to complex Grassmannian manifold $\widetilde{G}_{2,1}$ is*

$$\mathbf{s}_t = \varepsilon^2 \mathbf{s} \times_\varepsilon \mathbf{s}_{xx}. \tag{15}$$

Proof. $\forall E = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$, then

$$\begin{aligned} \tau : \widetilde{G}_{2,1} \to \mathbb{CS}^{2,\varepsilon}(1), \gamma = E^{-1}\sigma_3 E &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} \frac{\varepsilon}{2} & 0 \\ 0 & -\frac{\varepsilon}{2} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{\varepsilon}{2(ad-bc)}\begin{pmatrix} ad+bc & 2bd \\ -2ac & -ad-bc \end{pmatrix} \in \widetilde{G}_{2,1} \\ &\mapsto \begin{pmatrix} s_3 & s_1 - \varepsilon s_2 \\ s_1 + \varepsilon s_2 & -s_3 \end{pmatrix} := S \\ &\leftrightarrow \mathbf{s} = (s_1,s_2,s_3), \end{aligned}$$

where $(s_1,s_2,s_3) = \left(\frac{bd-ac}{ad-bc}, \frac{\varepsilon(bd+ac)}{ad-bc}, \frac{ad+bc}{ad-bc}\right) \in \mathbb{C}^3$ satisfying $s_1^2 - \varepsilon^2 s_2^2 + s_3^2 = 1$. It is a direct computation that $\gamma \times_\varepsilon \gamma_{xx}$ is just $[\tau^{-1}\gamma, \tau_*^{-1}\gamma_{xx}]$, where τ_* is the tangent map induced from τ. Hence, the equation of Schrödinger flow from $\mathbb{R}^1$ to $\widetilde{G}_{2,1}$ returns to CLL equations (ε-CLL)

$$S_t = \frac{\varepsilon}{2}[S, S_{xx}], \tag{16}$$

where $S = S(x,t)$ is a 2×2 matrix of the form

$$S(x,t)=\begin{pmatrix} s_3(x,t) & s_1(x,t)-\varepsilon s_2(x,t)\\ s_1(x,t)+\varepsilon s_2(x,t) & -s_3(x,t)\end{pmatrix}, \tag{17}$$

where $s_1^2-\varepsilon^2 s_2^2+s_3^2=1$ by the requirement $S^2=I$, S is identified with a vector $\mathbf{s}=(s_1,s_2,s_3)\in\mathbb{CS}^{2,\varepsilon}(1)\hookrightarrow\mathbb{C}^3$, and the ε-CLL equation (16) becomes equation (15). □

Note that when $\varepsilon=\sqrt{-1}$, $\mathbb{CS}^{2,\varepsilon}(1)$ becomes the complex 2-sphere $\mathbb{CS}^2(1)=\{(z_1,z_2,z_3)\in\mathbb{C}^3: z_1^2+z_2^2+z_3^2=1\}$. From Corollary 1, the equation of Schrödinger flow the complex 2-sphere $\mathbb{CS}^2(1)$ with the standard holomorphic metric (i.e., $\mathrm{d}z^2=\mathrm{d}z_1^2+\mathrm{d}z_2^2+\mathrm{d}z_3^2$) is

$$\mathbf{s}_t=-\mathbf{s}\times\mathbf{s}_{xx},$$

where $\times$ denotes the cross-product of $\mathbb{C}^3$, which also refers to [23]. By Corollary 1, we can interpret $\mathbb{CS}^{2,\varepsilon}(1)$ as a symmetric space $\widetilde{G}_{2,1}=\mathrm{GL}(2,\mathbb{C})/\mathrm{GL}(1,\mathbb{C})\times\mathrm{GL}(1,\mathbb{C})$.

Theorem 2. *Equations* (11) *and* (13) *are gauge equivalent to each other.*

Proof. Let $u:\mathbb{R}^2\to\mathbf{m}$ be a smooth solution of equation (11), we let $G:\mathbb{R}^2\to\mathrm{GL}(n,\mathbb{C})$ satisfy the following linear system:

$$G_x=-Gu,\quad G_t=-\varepsilon^2 GP_{-1}(u),\quad (\text{since } \varepsilon^2u_t=P_{-1}(u)_x+[P_{-1}(u),u], G \text{ exist}).$$

By using the gauge transformation:

$$\delta=G\varphi,\ \ \gamma=G\sigma_3G^{-1},$$

where φ satisfies equation (10), we see that

$$\delta_x=(G\varphi)_x=G_x\varphi+G\varphi_x=-Gu\varphi+G(-\sigma_3\lambda+u)\varphi=-G\sigma_3\lambda\varphi=-\gamma\lambda\delta$$

and

$$\begin{aligned}\delta_t=(G\varphi)_t=G_t\varphi+G\varphi_t&=-\varepsilon^2GP_{-1}(u)\varphi+G(\sigma_3\lambda^2-u\lambda+\varepsilon^2P_{-1}(u))\varphi\\&=(G\sigma_3\lambda^2-Gu\lambda)\varphi\\&=(\gamma\lambda^2-GuG^{-1}\lambda)\delta\\&=(\gamma\lambda^2-\varepsilon^2[\gamma,\gamma_x]\lambda)\delta,\ \text{since}\,[\gamma,\gamma_x]=\varepsilon^2GuG^{-1}.\end{aligned}$$

Namely, $\gamma=\gamma(x,t)=G(x,t)\sigma_3G(x,t)^{-1}$ is a solution of the Schrödinger flow (13) on $\widetilde{G}_{n,k}$. Hence, we have a gauge transform from equation (11) to equation (13).

Next, if $\gamma=E^{-1}(t,x)\sigma_3E(t,x)\in\widetilde{G}_{n,k}$ is a solution of the Schrödinger flow equation (13). Without loss of generality, we may assume that E satisfies (see [39]):

$$E_xE^{-1}=\begin{pmatrix}0 & R_1\\ R_2 & 0\end{pmatrix}\coloneqq P_1\in\mathbf{m}.$$

By $\gamma_t=\varepsilon^2[\gamma,\gamma_{xx}]$, it is a direct verification that

$$[\sigma_3,E_tE^{-1}]=P_{1x},\ \ \text{since}\ J_\gamma(\gamma_{xx})=\varepsilon^2E^{-1}P_{1x}E,\ \gamma_t=E^{-1}[\sigma_3,E_tE^{-1}]E.$$

Hence, we have

$$E_tE^{-1}=\begin{pmatrix}\xi_{11} & \dfrac{R_{1x}}{\varepsilon}\\ -\dfrac{R_{2x}}{\varepsilon} & \xi_{22}\end{pmatrix}\coloneqq P_2.$$

Moreover, from the integrability condition: $E_{xt} = E_{tx}$, i.e., $P_{1t} = P_{2x} + [P_2, P_1]$, we can obtain

$$\xi_{11} = -\frac{1}{\varepsilon}R_1R_2 + C_1(t), \quad \xi_{22} = \frac{1}{\varepsilon}R_2R_1 + C_2(t),$$

where $C(t) = \begin{pmatrix} C_1(t) & 0 \\ 0 & C_2(t) \end{pmatrix}$ depend only on t, and

$$E_tE^{-1} = \frac{1}{\varepsilon}\begin{pmatrix} -R_1R_2 & R_{1x} \\ -R_{2x} & R_2R_1 \end{pmatrix} + \begin{pmatrix} C_1(t) & 0 \\ 0 & C_2(t) \end{pmatrix} = \frac{2}{\varepsilon^2}\sigma_3(P_{1x} - P_1^2) + C(t).$$

In order to vanish C_1 and C_2, we take

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix},$$

where $\rho_1 = \rho_1(t), \rho_2 = \rho_2(t)$ depend only on t, satisfying

$$\rho_{1t} = -\rho_1C_1, \rho_{2t} = -\rho_2C_2.$$

Let $g = \rho E$; hence,

$$g_xg^{-1} = \rho E_xE^{-1}\rho^{-1} = \rho P_1\rho^{-1} = \rho\begin{pmatrix} 0 & R_1 \\ R_2 & 0 \end{pmatrix}\rho^{-1} := \begin{pmatrix} 0 & \Psi_1 \\ \Psi_2 & 0 \end{pmatrix} := u, \tag{18}$$

and moreover,

$$\begin{aligned} g_tg^{-1} &= (\rho_tE + \rho E_t)E^{-1}\rho^{-1} \\ &= \left(\rho_tE + \rho\left(\frac{2}{\varepsilon^2}\sigma_3(P_{1x} - P_1^2)E + C(t)E\right)\right)E^{-1}\rho^{-1} \\ &= \frac{2}{\varepsilon^2}\rho\sigma_3(P_{1x} - P_1^2)\rho^{-1} \\ &= \frac{2}{\varepsilon^2}\sigma_3(\rho P_{1x}\rho^{-1} - \rho P_1^2\rho^{-1}) \\ &= \frac{2}{\varepsilon^2}\sigma_3(u_x - u^2) := \frac{1}{\varepsilon^2}P_{-1}(u), \end{aligned} \tag{19}$$

where $P_{-1}(u) = 2\sigma_3(u_x - u^2)$. Let $G = g^{-1}$, $\varphi = G^{-1}\delta$, where δ satisfies equation (14). From equations (18) and (19), we obtain

$$G_x = -Gu, G_t = -\frac{1}{\varepsilon^2}GP_{-1}(u) = -\varepsilon^2GP_{-1}(u).$$

Moreover, we have

$$\begin{aligned} \varphi_x &= -G^{-1}G_xG^{-1}\delta + G^{-1}\delta_x = (-\sigma_3\lambda + u)\varphi, \\ \varphi_t &= -G^{-1}G_tG^{-1}\delta + G^{-1}\delta_t = (\sigma_3\lambda^2 - u\lambda + \varepsilon^2P_{-1}(u))\varphi, \end{aligned}$$

i.e., φ satisfies equation (10). Hence, we have proved that equation (13) is gauge equivalent to equation (10). □

From equations (10), (11), (13), (14), and Theorem 2, we have the following:

Theorem 3. *The coupled matrix NLS equations* (1) *on* $\mathcal{G}l(n, \mathbb{C}) = \mathbf{k} \oplus \mathbf{m}$ *are gauge equivalent to the equation* (13) *of Schrödinger flow from* $\mathbb{R}^1$ *to complex Grassmannian manifold* $\widetilde{G}_{n,k}$.

Corollary 2. *The matrix NLS equation* (6) *on the first subclass* $u(n) = \mathbf{k}_1 \oplus \mathbf{m}_1$ *or the second subclass* $u(k, n-k) = \mathbf{k}_2 \oplus \mathbf{m}_2$ *is gauge equivalent to the equation* (13) *of Schrödinger flow from* $\mathbb{R}^1$ *to complex Grassmannian manifold* $U(n)/U(k) \times U(n-k)$ *or* $U(k, n-k)/U(k) \times U(n-k)$ *(see* [13,18]).

Corollary 3. *The matrix version of the nonlinear heat equations*

$$\begin{cases}\Psi_{1t} - \Psi_{1xx} + 2\Psi_1\Psi_2\Psi_1 = 0,\\ \Psi_{2t} + \Psi_{2xx} - 2\Psi_2\Psi_1\Psi_2 = 0,\end{cases}$$

which are a reduction of equation (1) *by* $\varepsilon = 1$, *and* $\Psi_1 = \Psi_1(x,t)$, $\Psi_2 = \Psi_2(x,t)$ *are* $k \times (n-k)$ *and* $(n-k)\times k$ *real matrix-valued function, on the third subclass* $\mathcal{G}l(n,\mathbb{R}) = \mathbf{k}_3 \oplus \mathbf{m}_3$ *is gauge equivalent to equation* (13) *of Schrödinger flow from* $\mathbb{R}^1$ *to manifold* $\mathrm{GL}(n,\mathbb{R})/\mathrm{GL}(k,\mathbb{R})\times\mathrm{GL}(n-k,\mathbb{R})$ (*see* [18]).

4 Darboux transformations and explicit soliton solutions

In this section, we use the loop factorization method given in [25] to construct Darboux transformation for $\widetilde{G}_{n,k}$-NLS, and we apply Darboux transformation (Theorems 4 and 5) to:

(1) the trivial solution $u = 0$ of the coupled NLS equations

$$\begin{cases}\varepsilon\Psi_{1t} - \Psi_{1xx} + 2\Psi_1\Psi_2\Psi_1 = 0,\\ \varepsilon\Psi_{2t} + \Psi_{2xx} - 2\Psi_2\Psi_1\Psi_2 = 0,\end{cases} \tag{20}$$

where $\Psi_1 = \Psi_1(x,t)$, $\Psi_2 = \Psi_2(x,t)$ are the complex valued function and $\varepsilon^2 = \pm 1$, to obtain soliton solutions;

(2) the constant map solution $\gamma(x,t) = a$ and to the solutions of the Schrödinger flow (13) on $\widetilde{G}_{2,1}$.

Let $\alpha_1, \alpha_2 \in \mathbb{C}$, $\{v_1, v_2\}$ a basis of $\mathbb{C}^2$, π the projection of $\mathbb{C}^2$ onto $\mathbb{C}v_1$ along $\mathbb{C}v_2$ and

$$f_{\alpha_1,\alpha_2,\pi}(\lambda) = I_{2\times 2} + \frac{\alpha_1-\alpha_2}{\lambda-\alpha_1}(I_{2\times 2} - \pi).$$

Theorem 4. (*Darboux transformation for the* $\widetilde{G}_{2,1}$*-NLS equation* (2))

Let $u = \begin{pmatrix}0 & \Psi_1\\ \Psi_2 & 0\end{pmatrix}$ *be a solution of the* $\widetilde{G}_{2,1}$*-NLS equation* (2), *and* $E(x,t,\lambda)$ *a frame of* u. *Let* $v_1 = \begin{pmatrix}c_1\\ c_2\end{pmatrix} \in \mathbb{C}^2$, $v_2 = \begin{pmatrix}d_1\\ d_2\end{pmatrix} \in \mathbb{C}^2$, $c_i, d_i, \alpha_1, \alpha_2 \in \mathbb{C}$, $i = 1,2$, *and* $\alpha_1 \neq \alpha_2$, $\det(v_1, v_2) \neq 0$, and $\pi = \frac{1}{c_1d_2 - c_2d_1}\begin{pmatrix}c_1d_2 & -c_1d_1\\ c_2d_2 & -c_2d_1\end{pmatrix}$. *If*

$$\tilde{v}_1 = (\tilde{c}_1, \tilde{c}_2)^T \coloneqq E(x,t,\alpha_1)^{-1}(v_1),\ \tilde{v}_2 = (\tilde{d}_1, \tilde{d}_2)^T \coloneqq E(x,t,\alpha_2)^{-1}(v_2),$$

and suppose $\tilde{v}_1, \tilde{v}_2$ *are linearly independent. Then, we have the following.*

(*i*) $\tilde{u} = u + (\alpha_1 - \alpha_2)[\sigma_3, \tilde{\pi}]$ *is a solution of the* $\widetilde{G}_{2,1}$*-NLS equation* (2), *and*

$$\tilde{E}(x,t,\lambda) = f_{\alpha_1,\alpha_2,\tilde{\pi}}(\lambda)E(x,t,\lambda)f_{\alpha_1,\alpha_2,\tilde{\pi}}(x,t)(\lambda)^{-1}$$

is a frame for $\tilde{u}$, *where*

$$\tilde{\pi} = \frac{1}{\tilde{c}_1\tilde{d}_2 - \tilde{c}_2\tilde{d}_1}\begin{pmatrix}\tilde{c}_1\tilde{d}_2 & -\tilde{c}_1\tilde{d}_1\\ \tilde{c}_2\tilde{d}_2 & -\tilde{c}_2\tilde{d}_1\end{pmatrix}.$$

(*ii*) $\tilde{u}$ *satisfies* $d\tilde{u} = \theta(\cdot,\cdot,\alpha)\tilde{u}$, *where*

$$\theta(x,t,\alpha) = (-\sigma_3\alpha + u)\mathrm{d}x(\sigma_3\alpha^2 - u\alpha + \varepsilon^2 P_{-1}(u))\mathrm{d}t,$$

$$\sigma_3 = \frac{\varepsilon}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ u = \begin{pmatrix}0 & \Psi_1\\ \Psi_2 & 0\end{pmatrix},\ P_{-1}(u) = \varepsilon\begin{pmatrix}-\Psi_1\Psi_2 & \Psi_{1x}\\ -\Psi_{2x} & \Psi_2\Psi_1\end{pmatrix},\ i.e.,$$

$$\begin{cases}\tilde{\mathcal{V}}_x = \begin{pmatrix}-\dfrac{\alpha}{2}\varepsilon & \Psi_1\\ \Psi_2 & \dfrac{\alpha}{2}\varepsilon\end{pmatrix}\tilde{\mathcal{V}},\\[2ex] \tilde{\mathcal{V}}_t = \begin{pmatrix}\dfrac{\alpha^2}{2}\varepsilon - \varepsilon^3\Psi_1\Psi_2 & -\Psi_1\alpha + \varepsilon^3\Psi_{1x}\\ -\alpha\Psi_2 - \varepsilon^3\Psi_{2x} & -\dfrac{\alpha^2}{2}\varepsilon + \varepsilon^3\Psi_2\Psi_1\end{pmatrix}\tilde{\mathcal{V}}.\end{cases} \tag{21}$$

Moreover, if $\alpha_1 \neq \alpha_2$, *and* $\tilde{v}_1(x,t) = (\tilde{c}_1(x,t), \tilde{c}_2(x,t))^T$, $\tilde{v}_2(x,t) = (\tilde{d}_1(x,t), \tilde{d}_2(x,t))^T$ *are solutions of equation* (21) *with* $\alpha = \alpha_1$ *and* $\alpha = \alpha_2$, *respectively. Suppose* $\tilde{v}_1, \tilde{v}_2$ *are linearly independent, then the formula of* $\tilde{u}$ *given in (i) is a solution of the* $\widetilde{G}_{2,1}$*-NLS equation* (2).

Theorem 5. (*Darboux transformation for the Schrödinger flow on* $\widetilde{G}_{2,1}$). *Let* γ *be a solutions of Schrödinger flow* (13) *on* $\widetilde{G}_{2,1}$, $g(x,t)$ *a Schrödinger frame of* γ, *and* $u := g^{-1}g_x$ *the solution of the* $\widetilde{G}_{2,1}$*-NLS equation* (2). *Let* $E(x,t,\lambda) = g(0,0)$ *for all* $\lambda \in \mathbb{C}$. *Let* $\alpha_1, \alpha_2 \in \mathbb{C}\backslash\mathbb{R}$ *be a constant,* π *a constant Hermitian projection onto* $V_1 \in \mathbb{C}^2$ *along* $V_2 \in \mathbb{C}^2$, *and* $\tilde{\pi}(x,t)$ *the Hermitian projection onto* $\tilde{V}_1(x,t) := E(x,t,\alpha_1)^{-1}(V_1)$ *along* $\tilde{V}_2(x,t) := E(x,t,\alpha_1)^{-1}(V_2)$. *Then,*

$$\gamma(x,t) = g(x,t) f_{\alpha_1,\alpha_2,\tilde{\pi}}(0)(x,t)^{-1}\sigma_3 f_{\alpha_1,\alpha_2,\tilde{\pi}}(0)(x,t) g(x,t)^{-1}$$

is a new solution of (13).

Example 4.1. (1-Soliton solutions for the coupled NLS equations (20))

Note that

$$E(x,t,\lambda) = \exp(\sigma_3(-\lambda x + \lambda^2 t)) = \begin{pmatrix} e^{-B(x,t,\lambda)} & 0 \\ 0 & e^{B(x,t,\lambda)} \end{pmatrix},$$

where $B(x,t,\lambda) = \frac{\varepsilon}{2}(-\lambda x + \lambda^2 t)$, is the frame of the solution $u = 0$ of the $\widetilde{G}_{n,k}$-NLS equation (2) with $E(0,0,\lambda) = I$. Let π be a constant Hermitian projection onto $V_1 \in \mathbb{C}^2$ along $V_2 \in \mathbb{C}^2$, where $v_1(x,t) = (c_1(x,t), c_2(x,t))^T$, $v_2(x,t) = (d_1(x,t), d_2(x,t))^T$, $c_i = c_i(x,t)$, $d_i = d_i(x,t) \in \mathbb{C}$, $i = 1,2$, we have

$$\pi = \frac{1}{c_1 d_2 - c_2 d_1}\begin{pmatrix} c_1 d_2 & -c_1 d_1 \\ c_2 d_2 & -c_2 d_1 \end{pmatrix}.$$

Then,

$$\tilde{v}_1 = E(x,t,\alpha)^{-1}(v_1) = (e^{B(x,t,\alpha_1)}c_1, e^{-B(x,t,\alpha_1)}c_2)^T := (\tilde{c}_1, \tilde{c}_2)^T,$$
$$\tilde{v}_2 = E(x,t,\alpha)^{-1}(v_2) = (e^{B(x,t,\alpha_2)}d_1, e^{-B(x,t,\alpha_2)}d_2)^T := (\tilde{d}_1, \tilde{d}_2)^T,$$

and

$$\tilde{\pi} = \frac{1}{\tilde{c}_1\tilde{d}_2 - \tilde{c}_2\tilde{d}_1}\begin{pmatrix} \tilde{c}_1\tilde{d}_2 & -\tilde{c}_1\tilde{d}_1 \\ \tilde{c}_2\tilde{d}_2 & -\tilde{c}_2\tilde{d}_1 \end{pmatrix} = \frac{1}{e^{\eta}c_1 d_2 - e^{-\eta}c_2 d_1}\begin{pmatrix} e^{\eta}c_1 d_2 & -e^{\xi}c_1 d_1 \\ e^{-\xi}c_2 d_2 & -e^{-\eta}c_2 d_1 \end{pmatrix},$$

namely,

$$\tilde{u} = u + (\alpha_1 - \alpha_2)[\sigma_3, \tilde{\pi}(x,t)] = \frac{\varepsilon(\alpha_1 - \alpha_2)}{e^{-\eta}c_2 d_1 - e^{\eta}c_1 d_2}\begin{pmatrix} 0 & e^{\xi}c_1 d_1 \\ e^{-\xi}c_2 d_2 & 0 \end{pmatrix}$$

is a solution of the $\widetilde{G}_{n,k}$-NLS equation (2), where $\xi = \frac{\varepsilon}{2}(-(\alpha_1 + \alpha_2)x + (\alpha_1^2 + \alpha_2^2)t)$ and $\eta = \frac{\varepsilon}{2}(-(\alpha_1 - \alpha_2)x + (\alpha_1^2 - \alpha_2^2)t)$. Hence, $\tilde{\Psi}_1 = \frac{\varepsilon(\alpha_1 - \alpha_2)}{e^{-\eta}c_2 d_1 - e^{\eta}c_1 d_1}e^{\xi}c_1 d_1$ and $\tilde{\Psi}_2 = \frac{\varepsilon(\alpha_1 - \alpha_2)}{e^{-\eta}c_2 d_1 - e^{\eta}c_1 d_2}e^{-\xi}c_2 d_2$ are a solution of the coupled NLS equations (20).

If $c_1 = c_2 = d_1 = d_2 = 1$, $\alpha_1 = r + is$, and $\alpha_2 = r - is$, then

$$\tilde{\pi}(x,t) = \frac{1}{e^{i\varepsilon(-sx+2rst)} - e^{-i\varepsilon(-sx+2rst)}}\begin{pmatrix} e^{i\varepsilon(-sx+2rst)} & -e^{\varepsilon(-rx+(r^2-s^2)t)} \\ e^{-\varepsilon(-rx+(r^2-s^2)t)} & -e^{-i\varepsilon(-sx+2rst)} \end{pmatrix},$$

i.e.,

$$\tilde{\Psi}_1(x,t) = \frac{2is\varepsilon e^{\varepsilon(-rx+(r^2-s^2)t)}}{e^{-i\varepsilon(-sx+2rst)} - e^{i\varepsilon(-sx+2rst)}}, \quad \tilde{\Psi}_2(x,t) = \frac{2is\varepsilon e^{\varepsilon(-(-rx+(r^2-s^2)t))}}{e^{-i\varepsilon(-sx+2rst)} - e^{i\varepsilon(-sx+2rst)}}$$

is a soliton solution of the coupled NLS equations (20).

Example 4.2. (1-Soliton solutions for Schrödinger flow on $\widetilde{G}_{2,1}$)

The constant map $\gamma(x,t)=\sigma_3$ is a solution of the Schrödinger flow (3.1) on $\widetilde{G}_{2,1}$, $g(x,t)=I_{2\times 2}$ is a Schrödinger frame for γ, $u=0$ is the corresponding solution of the $\widetilde{G}_{n,k}$-NLS equation (2), and

$$E(x,t,\lambda)=\exp(-\sigma_3\lambda x+\sigma_3\lambda^2 t)$$

is the frame of $u=0$, with $E(0,0,\lambda)=I_{2\times 2}$.

By Theorem 5 to $\gamma(x,t)=\sigma_3$, $g(x,t)=I_{2\times 2}$, and use $\tilde{\pi}(x,t)$ given in Example 4.1 to obtain soliton solution for Schrödinger flow on $\widetilde{G}_{2,1}$. We choose $\alpha_1=is$, $\alpha_2=-is$, and $c_i=d_i=1$, $i=1,2$; then, we have

$$f_{is,-is,\tilde{\pi}}(0)(x,t)=\begin{pmatrix} \dfrac{2e^{-\varepsilon sxi}}{e^{-\varepsilon sxi}-e^{\varepsilon sxi}}-1 & -\dfrac{2e^{-\varepsilon s^2 t}}{e^{-\varepsilon sxi}-e^{\varepsilon sxi}} \\ \dfrac{2e^{\varepsilon s^2 t}}{e^{-\varepsilon sxi}-e^{\varepsilon sxi}} & -\dfrac{2e^{\varepsilon sxi}}{e^{-\varepsilon sxi}-e^{\varepsilon sxi}}-1 \end{pmatrix}.$$

Hence,

$$\begin{aligned}\tilde{\gamma}(x,t)&=g(x,t)f_{is,-is,\tilde{\pi}}(0)(x,t)^{-1}\sigma_3 f_{is,-is,\tilde{\pi}}(0)(x,t)g(x,t)^{-1}\\ &=\begin{pmatrix} \dfrac{\varepsilon(6e^{2\varepsilon sxi}+e^{4\varepsilon sxi}+1)}{2(e^{2\varepsilon sxi}-1)^2} & \dfrac{-4\varepsilon cos(\varepsilon sx)e^{2\varepsilon sxi}}{e^{\varepsilon s^2 t}+e^{\varepsilon s(4xi+st)}-2e^{\varepsilon s(2xi+st)}} \\ \dfrac{2\varepsilon(e^{\varepsilon s(xi+st)}+e^{\varepsilon s(3xi+st)})}{(e^{2\varepsilon sxi}-1)^2} & -\dfrac{\varepsilon(6e^{2\varepsilon sxi}+e^{4\varepsilon sxi}+1)}{2(e^{2\varepsilon sxi}-1)^2} \end{pmatrix}\end{aligned}$$

is a soliton solution for Schrödinger flow on $\widetilde{G}_{2,1}$ (CLL equations (16)). From equation (17), we have

$$\begin{aligned}\mathbf{s}=(s_1,s_2,s_3)=\Bigg(&\frac{-2\varepsilon\cos(\varepsilon sx)e^{2\varepsilon sxi}}{e^{\varepsilon s^2 t}+e^{\varepsilon s(4xi+st)}-2e^{\varepsilon s(2xi+st)}}+\frac{\varepsilon(e^{\varepsilon s(xi+st)}+e^{\varepsilon s(3xi+st)})}{(e^{2\varepsilon sxi}-1)^2},\\ &\frac{2\cos(\varepsilon sx)e^{2\varepsilon sxi}}{e^{\varepsilon s^2 t}+e^{\varepsilon s(4xi+st)}-2e^{\varepsilon s(2xi+st)}}+\frac{e^{\varepsilon s(xi+st)}+e^{\varepsilon s(3xi+st)}}{(e^{2\varepsilon sxi}-1)^2},\ \frac{\varepsilon(6e^{2\varepsilon sxi}+e^{4\varepsilon sxi}+1)}{2(e^{2\varepsilon sxi}-1)^2}\Bigg)\end{aligned}$$

is a soliton solution of equation (15).

Theorem 3 gives a geometric interpretation of the matrix generalization of the coupled NLS equation (i.e., CLL equation) via Schrödinger flow to the complex Grassmannian manifold $\mathrm{GL}(n,\mathbb{C})/\mathrm{GL}(k,\mathbb{C})\times\mathrm{GL}(n-k,\mathbb{C})$. And Example 4.2 shows that 1-soliton solution for Schrödinger flow on $\mathbb{C}\mathbb{S}^{2,\varepsilon}(1)$ is given. There are many questions unclear in this aspect. For example, it is also of interest to consider N-soliton solutions for Schrödinger flow on $\mathbb{C}\mathbb{S}^{2,\varepsilon}(1)$. As you know, there are many models for NLS equation such as the coupled NLS-type equations (see [40]), the extended coupled nonlinear Schrödinger equations (see [41]). Is it possible to give a geometric interpretation of this systems via Schrödinger flow? These questions deserve study in the future.

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