

Research Article

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Weak solution of non-Newtonian polytropic variational inequality in fresh agricultural product supply chain problem

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Abstract: In this article, we study a class of variational inequality problems with non-Newtonian polytropic parabolic operators. We introduce a mapping with an adjustable parameter to control the polytropic term, which exactly meets the conditions of Leray-Schauder fixed point theory. At the same time, we construct a penalty function to transform the variational inequality into a regular parabolic initial boundary value problem. Thus, the existence is treated with a Leray-Schauder fixed point theory as well as a suitable version of Aubin-Lions lemma. Then, the uniqueness and stability of the solution are analyzed.

Keywords: variational inequality problem, generalized solution, non-Newtonian polytropic operator, existence, uniqueness and stability

MSC 2020: 35K99, 97M30

1 Introduction

Let $p \geq 2$, $m \in (0, p - 1]$ and assume that $\Omega \subset \mathbb{R}_N$ ($N \geq 2$) be a bounded domain and $\Omega_T = \Omega \times (0, T]$ with $T > 0$. The author of this study considered a kind of variational inequality problem

$$\min\{Lu, u - u_0\} = 0, \quad (x, t) \in \Omega_T, \quad (1)$$

with the non-Newtonian polytropic operator

$$Lu = \partial_t u - \nabla(|\nabla u^m|^{p-2} \nabla u^m) + \gamma u, \gamma \geq 0 \quad (2)$$

and the Dirichlet initial-boundary value condition

$$u(0, x) = u_0(x), \quad x \in \Omega; \quad u(t, x) = 0, \quad (x, t) \in \partial\Omega \times (0, T). \quad (3)$$

Variational inequality has a good application in the value analysis of financial products with early implementation clauses, for details see [1,2]. Recent years, much attention has been paid to the study of variational inequality with linear, quasi-linear, and degenerate parabolic operator [3–5]. In [3], Li and Bi considered two-dimensional variational inequality systems

$$\begin{cases} \min\{L_i u_i - f_i(x, t, u_1, u_2), u_i - u_{i,0}\} = 0, & (x, t) \in \Omega_T, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u(t, x) = 0, & (x, t) \in \partial\Omega \times (0, T) \end{cases}$$

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with degenerate parabolic operator

$$L_i u_i = \partial_t u_i - \operatorname{div}(|\nabla u_i|^{p_i-2} \nabla u_i), \quad i = 1, 2.$$

The existence of weak solution is studied by a limit process. In [4], Sun and Wu considered a kind of variational inequality problem with a double degenerate operator

$$Lu = \partial_t u - u \operatorname{div}(a(u)|\nabla u|^{p(x)-2} \nabla u) - \gamma u^{p(x)} - f(x, t).$$

Similar methods to those presented in [3] have been used, and the existence and uniqueness of the solutions in the weak sense are proved.

The structure of parabolic initial boundary value problem is simpler than that of variational inequality. When $m = 0$, initial boundary value problems with parabolic operator Lu have been extensively studied in the last few years, see for details [6–13]. Some articles are focused on the existence of generalized solutions related to this article [6–8]. Some results for uniqueness of generalized solutions can be found in [9–11]. There are other arguments worth studying, such as stability of boundary output feedback [12,13].

In this article, we extend the corresponding results in [3,6,9] to study a class of variational inequality problems with non-Newtonian polytropic parabolic operators with the Dirichlet initial-boundary value condition. Since m and p are coupled in Lu and Lu is degenerate, we plan to solve this problem with Leray-Schauder fixed point theory by constructing a map. In order to overcome the difficulty of establishing generalized solutions on variational inequalities, we turn the variational inequalities into regular problems through penalty functions. Some estimates of regular problems and continuity, boundedness and compactness of Leray-Schauder map are given by the inequality technique as well as a suitable version of Aubin-Lions lemma. In what follows, we prove the existence, uniqueness, and stability of the solution under the proper setting of the parameters in (1).

2 Statement of the problem and the main results

Our consideration in this article is motivated by an application model about fresh agricultural product supply chain. Here we consider a fresh agricultural product supply chain formed by a supplier and a retailer in which retailers face uncertain market demand. Assuming that the current time is 0, the time agreed in the contract for retailers to purchase agricultural products is T , and the retail price of agricultural products agreed in the contract meets

$$dP(t) = \mu P(t)dt + \sigma P(t)dW_t, \quad P(0) = P_0,$$

where μ and σ denote the expected rate and the volatility of return on the retail price of agricultural products, respectively. $\{W_t, t \geq 0\}$ is a Wiener process, which drives the random noise of the market. P_0 represents the market price at time 0.

Since fresh agricultural products are easy to deteriorate and decay, retailers will have no residual value for their remaining agricultural products. Therefore, the order of agricultural products must be placed before the sales season T . Retailers can buy a call option contract in which they have the right to purchase a certain amount of fresh agricultural products at the agreed price of K from 0 to T . Of course, retailers need to pay a certain premium of C to obtain such rights. If the retailer finds a more suitable source of goods, it will give up the option contract and lose the option premium C . This means that retailers can decide whether to exercise or hold options based on their own earnings. According to the literature [1–3], the value of options meets

$$\begin{cases} \min\{LC, C - \max(P - K, 0)\} = 0, & (P, t) \in [0, B] \times [0, T], \\ C(0, P) = \max(P - K, 0), & P \in [0, B], \\ C(t, B) = P - \exp\{-rt\}, \\ C(t, 0) = 0 \end{cases} \quad (4)$$

where B represents the artificially set price ceiling for agricultural products,

$$LC = \frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 C}{\partial P^2} + rP \frac{\partial C}{\partial P} - rC.$$

In addition, if the loss of agricultural products during transportation is considered, then the expression of LC is more complicated that σ in LC is a function of C and $\frac{\partial C}{\partial P}$, for details see [14].

More complex case than (5) is considered in this article. In doing so, we give a class of maximal monotone maps defined in [11,13]

$$G(x) = \begin{cases} 0, & x > 0, \\ -M_0, & x = 0, \end{cases} \quad (5)$$

where M_0 is a positive constant which can be chosen later. Since m and p are coupled in Lu and Lu is degenerate, problem (1) do not have a classical solution. So we consider its generalized case as follows.

Definition 1. A pair (u, ξ) is said to be a generalized solution of variational inequality (1), if (u, ξ) satisfies $u \in L^\infty(0, T; W^{1,p}(\Omega))$, $\partial_t u \in L^2(\Omega_T)$,

- (a) $u(x, t) \geq u_0(x)$, $u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,
- (b) $\xi \in G(u - u_0)$ for any $(x, t) \in \Omega_T$,
- (c) for every test function $\varphi \in C^\infty(\Omega_T)$, there admits the equality

$$\int_{\Omega_T} \partial_t u \cdot \varphi + |\nabla u^m|^{p-2} \nabla u^m \nabla \varphi dx dt + \gamma \int_{\Omega_T} u \varphi dx dt = \int_{\Omega_T} \xi \cdot \varphi dx dt.$$

Because of coupling in Lu , we cannot prove the existence of solution of problem (1) by using a common limit method. Here, we plan to use the Leray-Schauder fixed point theory as well as a suitable version of Aubin-Lions lemma based on the map

$$M : L^\infty(0, T; W_0^{1,p}(\Omega)) \times [0, 1] \rightarrow L^\infty(0, T; W_0^{1,p}(\Omega)), \quad (6)$$

such that for every function $\omega \in L^\infty(0, T; W_0^{1,p}(\Omega))$ and $\theta \in [0, 1]$, $u_\varepsilon = M(\omega, \theta)$ is a solution of the equation

$$\begin{cases} L_\varepsilon^{\theta, \omega} u_\varepsilon = -\theta \beta_\varepsilon(u_\varepsilon - u_0), & (x, t) \in \Omega_T, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x) = u_0 + \varepsilon, & x \in \Omega, \\ u_\varepsilon(x, t) = \varepsilon, & (x, t) \in \partial\Omega_T, \end{cases} \quad (7)$$

with an operator

$$L_\varepsilon^{\theta, \omega} u_\varepsilon = \partial_t u_\varepsilon - \operatorname{div} \left((|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) \right) + \gamma \omega. \quad (8)$$

Here $u_\varepsilon^m \in L^\infty(0, T; W_0^{1,p}(\Omega))$, $A_\theta(u_\varepsilon) = \theta u_\varepsilon^m + (1 - \theta)u_\varepsilon$, the penalty map $\beta_\varepsilon : \mathbb{R}_+ \rightarrow \mathbb{R}_-$ satisfies

$$\begin{aligned} \varepsilon \in (0, 1), \quad \beta_\varepsilon(\cdot) \in C^2(\mathbb{R}), \quad \beta_\varepsilon(x) \leq 0, \quad \beta'_\varepsilon(x) \geq 0, \quad \beta''_\varepsilon(x) \leq 0, \\ \beta_\varepsilon(x) = \begin{cases} 0 & x \geq \varepsilon, \\ -M_0 & x = 0, \end{cases} \quad \lim_{\varepsilon \rightarrow 0^+} \beta(x) = \begin{cases} 0, & x > 0, \\ -M_0, & x = 0. \end{cases} \end{aligned} \quad (9)$$

Then the existence of problem (1) is equivalent to $u_\varepsilon = M(u_\varepsilon, 1)$. With a similar method to that in [6,7], we may prove that the regularized problem (7) admits a weak solution as follows:

Definition 2. A function u_ε is said to be a solution of problem (7) if u_ε satisfies $u_\varepsilon^m \in L(0, T; W^{1,p}(\Omega))$,

$$\int_{\Omega} \partial_t u_\varepsilon \cdot \varphi + (|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) \nabla \varphi dx + \gamma \int_{\Omega} \omega \varphi dx = \int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) \cdot \varphi dx, \quad \forall t \in (0, T) \quad (10)$$

for any $\varphi \in C^\infty(\Omega_T)$.

Because of the denseness of $C^\infty(\Omega_T)$ in $L(0, T; W^{1,p}(\Omega))$, one can assert that the identities in Definition 2.1 and Definition 2.2 hold for any $L(0, T; W^{1,p}(\Omega))$. Furthermore, one can obtain the following inequalities [6,7]:

$$u_0 \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, \quad u_{\varepsilon_1} \leq u_{\varepsilon_2} \text{ for } \varepsilon_1 \leq \varepsilon_2. \quad (11)$$

It is clear that the constructed generalized solution in this article is non-negative. Then we give the restriction in (6) that $\omega \geq 0$.

3 Some preliminaries

To discuss the existence of weak solution of equation (1), we give certain useful estimates.

Lemma 1. For any fixed $t \in [0, T]$ and $\gamma \geq 0$,

$$\|u_\varepsilon\|_{L^p(\Omega)}^p \leq C, \quad (12)$$

$$\int_{\Omega} (|\theta \nabla u_\varepsilon^m + (1 - \theta) \nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\theta \nabla u_\varepsilon^m + (1 - \theta) \nabla u_\varepsilon|^2 dx \leq C, \quad (13)$$

$$\int_{\Omega} |\theta \nabla u_\varepsilon^m + (1 - \theta) \nabla u_\varepsilon|^p dx \leq C. \quad (14)$$

Proof. Since $u_\varepsilon^m \in L(0, T; W^{1,p}(\Omega))$, combining with (11) gives $\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon \in L(0, T; W^{1,p}(\Omega))$. So choosing a test function φ as $\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon$ in (10), it can be easily verified that

$$\begin{aligned} \int_{\Omega_T} \partial_\tau u_\varepsilon \cdot (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt + \Pi_1 &= -\theta \int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt \\ &\quad - \gamma \int_{\Omega} \omega (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt, \end{aligned} \quad (15)$$

where

$$\Pi_1 = \int_{\Omega} (|\theta \nabla u_\varepsilon^m + (1 - \theta) \nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} |\theta \nabla u_\varepsilon^m + (1 - \theta) \nabla u_\varepsilon|^2 dx dt.$$

From the first part of (11), $u_\varepsilon - u_0 \geq 0$ for any $(x, t) \in \Omega_T$. Combining with (9) leads to $|\beta_\varepsilon(u_\varepsilon - u_0)| \leq M_0$, such that $\int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt$ can be estimated as

$$-\theta \int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt \leq M_0 T (\max\{|u_{0,\varepsilon}|_\infty, |u_{0,\varepsilon}^m|_\infty\} + 1). \quad (16)$$

Since $\omega \geq 0$, $\theta \in (0, 1)$, it is easy from the first part of (12) to see that

$$\int_{\Omega} \omega \cdot (\theta u_\varepsilon^m + (1 - \theta) u_\varepsilon) dx dt \geq 0. \quad (17)$$

Using differential transformation method gives (recall that $m \in (0, p - 1]$)

$$\begin{aligned}
& \int_{\Omega} \int \partial_{\tau} u_{\varepsilon} \cdot (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon}) dx dt \\
&= \int_{\Omega} \int \frac{\theta}{m+1} \partial_{\tau} u_{\varepsilon}^{m+1} + \frac{1}{2} (1 - \theta) \partial_{\tau} u_{\varepsilon}^2 dx dt \\
&= \frac{\theta}{m+1} (\|u_{\varepsilon}\|_{L^{m+1}(\Omega)}^{m+1} - \|u_{0,\varepsilon}\|_{L^{m+1}(\Omega)}^{m+1}) + \frac{1}{2} (1 - \theta) (\|u_{\varepsilon}\|_{L^2(\Omega)}^2 - \|u_{0,\varepsilon}\|_{L^2(\Omega)}^2).
\end{aligned} \tag{18}$$

Since $A_1 \geq 0$, we drop Π_1 to arrive at

$$\frac{\theta}{m+1} \|u_{\varepsilon}\|_{L^p(\Omega)}^p + \frac{1}{2} (1 - \theta) \|u_{\varepsilon}\|_{L^2(\Omega)}^2 \leq \frac{\theta}{m+1} \|u_{0,\varepsilon}\|_{L^{m+1}(\Omega)}^{m+1} + \frac{1}{2} (1 - \theta) \|u_{0,\varepsilon}\|_{L^2(\Omega)}^2. \tag{19}$$

This implies (12) follows.

On the contrary, if we drop non-negative term $\int_{\Omega} \omega (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon}) dx dt$ and non-positive term $-\theta \int_{\Omega} \beta_{\varepsilon} (u_{\varepsilon} - u_0) (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon}) dx dt$ in (14),

$$\Pi_1 \leq \frac{2}{m+1} \|u_{0,\varepsilon}\|_{L^p(\Omega)}^p + \|u_{0,\varepsilon}\|_{L^2(\Omega)}^2. \tag{20}$$

Thus (13) follows. If $\varepsilon = 0$,

$$\int_{\Omega} \int |\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}|^p dx dt \leq \int_{\Omega} \int (|\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}} |\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}|^2 dx dt,$$

such that estimate (14) is an immediate result of (13). $\square$

Lemma 2. Assume that $\gamma \geq 0$, $m > 0$, and $p \geq 2$. Then there exists a constant C , independent of ε , such that

$$\int_{\Omega} |\partial_t (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon})|^2 dx \leq C(p, M_0, |\Omega|, |u_{0,\varepsilon}|_{\infty}), \tag{21}$$

$$\|\partial_t u_{\varepsilon}^m\|_{L^2(0,T;\Omega)} \leq C(p, T, |\Omega|), \tag{22}$$

$$\|\partial_t u_{\varepsilon}\|_{L^2(0,T;\Omega)} \leq C(p, T, |\Omega|). \tag{23}$$

Proof. Since $\theta \in (0, 1)$. It follows from the first part of (11) that

$$\partial_t u_{\varepsilon} \partial_t u_{\varepsilon}^m \leq m(|u_0|_{\infty}^{m-1} + 1) |\partial_t u_{\varepsilon}|^2. \tag{24}$$

Multiplying the first line of (7) by $\partial_t (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon})$ and integrating both sides of the equality over Ω_T , we have

$$\rho \int_{\Omega} \int |\partial_t (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon})|^2 dx dt \leq -\Pi_2 - \Pi_3 + \Pi_4, \tag{25}$$

where the constant π depends only upon m and $|u_0|_{\infty}$,

$$\Pi_2 = \int_{\Omega} \int (|\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}} (\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}) \partial_t (\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}) dx dt,$$

$$\Pi_3 = \gamma \int_{\Omega} \int \omega \cdot \partial_t (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon}) dx dt, \quad \Pi_4 = -\theta \int_{\Omega} \int \beta_{\varepsilon} (u_{\varepsilon} - u_0) \cdot \partial_t (\theta u_{\varepsilon}^m + (1 - \theta) u_{\varepsilon}) dx dt.$$

Using some differential transformation techniques,

$$-\Pi_2 = -\frac{1}{p} \int_0^t \int_{\Omega} \partial_t (|\theta \nabla u_{\varepsilon}^m + (1 - \theta) \nabla u_{\varepsilon}|^2 + \varepsilon)^{\frac{p}{2}} dx d\tau \leq \frac{1}{p} \int_{\Omega} (|\theta \nabla u_{\varepsilon}^m(\cdot, 0) + (1 - \theta) \nabla u_{\varepsilon}(\cdot, 0)|^2 + \varepsilon)^{\frac{p}{2}} dx. \tag{26}$$

It follows by Holder and Cauchy inequalities that

$$|\Pi_3| \leq 2\pi^{-1}\gamma^2 \int_{\Omega} \omega^2 dx + \frac{1}{8}\pi \int_{\Omega} |\partial_t(\theta u_{\varepsilon}^m + (1-\theta)u_{\varepsilon})|^2 dx, \quad (27)$$

$$|\Pi_4| = 2\pi^{-1}M_0^2|\Omega| + \frac{1}{8}\pi \int_{\Omega} |\partial_t(\theta u_{\varepsilon}^m + (1-\theta)u_{\varepsilon})|^2 dx. \quad (28)$$

Since $\omega \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$, combining (25), (26), (27), and (28), it is easy to verify that

$$\frac{3}{4}\rho \int_{\Omega} |\partial_t(\theta u_{\varepsilon}^m + (1-\theta)u_{\varepsilon})|^2 dx \leq C(p, M_0, |\Omega|, |u_{0,\varepsilon}|_{\infty}). \quad (29)$$

This implies that (21) follows. Using (24) again,

$$\theta^2 \int_{\Omega} |\partial_t u_{\varepsilon}^m|^2 dx + \rho \int_{\Omega} |\partial_t u_{\varepsilon}|^2 dx \leq \int_{\Omega} |\partial_t(\theta u_{\varepsilon}^m + (1-\theta)u_{\varepsilon})|^2 dx. \quad (30)$$

(22) and (23) are immediate results of (29) and (30). $\square$

Next, we analyze the continuity of map M . To this end, assume that $\theta_k \rightarrow \theta$ and $f_k \rightarrow f$ as $k \rightarrow \infty$ and define $u_{\varepsilon,k} = M(\theta_k, f_k)$, such that $u_{\varepsilon,k}$ is the solution of (9) with degenerate parabolic operator

$$L_{\varepsilon}^{\theta_k, \omega_k} u_{\varepsilon,k} = \partial_t u_{\varepsilon,k} - \operatorname{div} \left((|\nabla A_{\theta_k}(u_{\varepsilon,k})|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta_k}(u_{\varepsilon,k}) \right) + \gamma \omega_k. \quad (31)$$

Lemma 3. *The map $M : L^{\infty}(0, T; W_0^{1,p}(\Omega)) \times [0, 1] \rightarrow L^{\infty}(0, T; W_0^{1,p}(\Omega))$ is continuous for any $\theta \in (0, 1)$.*

Proof. From the first part of (11), it can be seen that $\{u_{\varepsilon,k}, k = 1, 2, 3, \dots\}$ is bounded, which together with the uniform estimates in k allow one to extract from the sequence $\{u_{\varepsilon,k}, k = 1, 2, 3, \dots\}$ a subsequence (for the sake of simplicity, we assume that it merely coincides with the whole of the sequence) and a function u_{ε} such that

$$u_{\varepsilon,k} \rightarrow u_{\varepsilon} \text{ a.e. } \Omega_T \text{ as } k \rightarrow \infty. \quad (32)$$

So that we remain to prove that $u_{\varepsilon} = M(\theta, f)$. From (12) and (13), one can infer that for any fix $\varepsilon \in (0, 1)$,

$$(|\nabla A_{\theta_k}(u_{\varepsilon,k})|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta_k}(u_{\varepsilon,k}) \overrightarrow{w} \chi_{\varepsilon} \text{ in } W_0^{1, \frac{p}{p-1}}(\Omega) \text{ as } k \rightarrow \infty, \quad (33)$$

such that combining (10) and (33) gives

$$\int_{\Omega} \partial_t u_{\varepsilon} \cdot \varphi + \chi_{\varepsilon} \nabla \varphi dx + \gamma \int_{\Omega} \omega \varphi dx = - \int_{\Omega} \beta_{\varepsilon}(u_{\varepsilon} - u_0) \cdot \varphi dx. \quad (34)$$

Hence, it remains to prove that

$$\int_{\Omega} \chi_{\varepsilon} \nabla \varphi dx = \int_{\Omega} \xi \cdot (|\nabla A_{\theta}(u_{\varepsilon})|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta}(u_{\varepsilon}) dx. \quad (35)$$

Choosing $\varphi = u_{\varepsilon,k} - u_{\varepsilon}$, multiplying $u_{\varepsilon,k} = M(\theta_k, f_k)$ or $L_{\varepsilon}^{\theta_k, \omega_k} u_{\varepsilon,k} = \beta_{\varepsilon}(u_{\varepsilon} - u_0)$ by φ , and integrating over Ω , we have that

$$\int_{\Omega} \partial_t u_{\varepsilon,k} \cdot \varphi + (|\nabla A_{\theta,k}(u_{\varepsilon,k})|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta,k}(u_{\varepsilon,k}) \nabla \varphi dx + \gamma \int_{\Omega} \omega_k \varphi dx = - \int_{\Omega} \beta_{\varepsilon}(u_{\varepsilon,k} - u_0) \cdot \varphi dx. \quad (36)$$

Combining (34) and (36), and integrating over $[0, T]$, one can have that

$$\begin{aligned} \int_{\Omega_T} (\partial_t u_{k,\varepsilon} - \partial_t u_{k,\varepsilon}) \cdot \varphi + \left[(|\nabla A_{\theta,k}(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta,k}(u_\varepsilon) - \chi_\varepsilon \right] \nabla \varphi dx dt \\ + \gamma \int_{\Omega_T} (\omega_k - \omega) \varphi dx dt = - \int_0^t \int_{\Omega} [\beta_\varepsilon(u_{k,\varepsilon}) - \beta_\varepsilon(u_\varepsilon)] \cdot \varphi dx dt. \end{aligned} \quad (37)$$

From (32), we infer that

$$\lim_{k \rightarrow \infty} \int_{\Omega_T} (\omega_k - \omega) \varphi dx dt = \lim_{k \rightarrow \infty} \int_0^t \int_{\Omega} [\beta_\varepsilon(u_{k,\varepsilon}) - \beta_\varepsilon(u_\varepsilon)] \cdot \varphi dx dt = 0. \quad (38)$$

Since $u_{k,\varepsilon}(x, 0) = u_\varepsilon(x, 0)$ for any $x \in \Omega$,

$$\int_{\Omega_T} (\partial_t u_{k,\varepsilon} - \partial_t u_\varepsilon) \cdot \varphi dx dt = \frac{1}{2} \int_{\Omega} (u_{k,\varepsilon} - u_\varepsilon)^2 dx \geq 0,$$

so we drop the non-negative term on the left-hand side and pass the limit $k \rightarrow \infty$ to arrive at

$$\lim_{k \rightarrow \infty} \int_{\Omega_T} \left[(|\nabla A_{\theta,k}(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta,k}(u_\varepsilon) - \chi_\varepsilon \right] \nabla \varphi dx dt \leq 0, \quad (39)$$

$$\left[(|\nabla A_{\theta_k}(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta_k}(u_\varepsilon) - (|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) \right] (\nabla A_{\theta_k}(u_\varepsilon) - \nabla A_\theta(u_\varepsilon)) \geq 0.$$

Since $\nabla A_{\theta_k}(u_\varepsilon) - \nabla A_\theta(u_\varepsilon)$ and $\nabla \varphi$ have the same sign,

$$\left[(|\nabla A_{\theta_k}(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_{\theta_k}(u_\varepsilon) - (|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) \right] \nabla \varphi \geq 0. \quad (40)$$

Subtracting (39) and (40), we infer that

$$\lim_{k \rightarrow \infty} \int_{\Omega_T} \left[(|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) - \chi_\varepsilon \right] \nabla \varphi dx dt \leq 0. \quad (41)$$

Obviously, if we swap $u_{\varepsilon,k}$ and u_ε , it is easy to obtain another inequality

$$\lim_{k \rightarrow \infty} \int_{\Omega_T} \left[\chi_\varepsilon - (|\nabla A_\theta(u_\varepsilon)|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla A_\theta(u_\varepsilon) \right] \nabla \varphi dx dt \leq 0. \quad (42)$$

(41) and (42) imply that Lemma 3.3 follows. $\square$

By virtue of Lemmas 3.1 and 3.2, and the Ascoli-Arzelà lemma [7], M maps any bounded set $[0, 1] \times Q$ into a compact set of $Q \subset L^\infty(0, T; W_0^{1,p}(\Omega))$. So we have the following result.

Lemma 4. *The map $M : L^\infty(0, T; W_0^{1,p}(\Omega)) \times [0, 1] \rightarrow L^\infty(0, T; W_0^{1,p}(\Omega))$ is compact.*

Let $y \in L^\infty(0, T; W_0^{1,p}(\Omega))$. By classical results (see, e.g., [8]), problem $u_\varepsilon = M(0, f) + y$, or

$$\partial_t u_\varepsilon - \nabla \left((|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon \right) + \gamma u_\varepsilon + y = 0 \quad (43)$$

with the same initial boundary condition in (7), admits a unique solution in $L^\infty(0, T; W_0^{1,p}(\Omega))$. Combining Lemmas 3.1–3.4 and the results related to (43), the following initial boundary problem

$$\begin{cases} Lu_\varepsilon = -\beta_\varepsilon(u_\varepsilon - u_0), & (x, t) \in Q_T, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \\ u_\varepsilon(x, t) = \varepsilon, & (x, t) \in \partial Q_T, \end{cases} \quad (44)$$

by using the properties of the Leray-Schauder degree, has a solution $u_\varepsilon \in L^\infty(0, T; W^{1,p}(\Omega))$ that satisfies $\partial_t u_\varepsilon \in L^\infty(0, T; L^2(\Omega))$,

$$\int_{\Omega} (\partial_t u_\varepsilon \cdot \varphi + (|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m \nabla \varphi + \gamma u_\varepsilon \varphi) dx = - \int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) \varphi dx \quad (45)$$

with $\varphi \in C^1(\bar{\Omega}_T)$. And, the solution of (44) satisfies (11), that is,

$$u_{0\varepsilon} \leq u_\varepsilon \leq |u_0|_\infty + \varepsilon, \quad u_{\varepsilon_1} \leq u_{\varepsilon_2} \text{ for } \varepsilon_1 \leq \varepsilon_2. \quad (46)$$

Based on Lemmas 3.1 and 3.2, we conclude that

$$\|u_\varepsilon\|_{L^p(\Omega)}^p \leq C, \quad (47)$$

$$\int_{\Omega} (|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} |\nabla u_\varepsilon^m|^2 dx \leq C, \quad (48)$$

$$\|\nabla u_\varepsilon^m\|_{L^p(\Omega)}^p \leq C, \quad (49)$$

$$\|\partial_t u_\varepsilon^m\|_{L^2(0,T;\Omega)} \leq C(p, T, |\Omega|), \quad (50)$$

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;\Omega)} \leq C(p, T, |\Omega|). \quad (51)$$

Indeed, choosing u_ε^m as a test function in (45) and repeating the proof of Lemma 3.1, (47)–(49) follow. As for (50) and (51), it is a consequence of taking $\partial_t u_\varepsilon^m$ as a test function.

4 Existence

In this section, we consider the existence of generalized solution to (1). Using what we mentioned in (46)–(51), the sequence $\{u_\varepsilon, \varepsilon \geq 0\}$ contains a subsequence denoted again by itself weakly convergent to a function $u \in L^\infty(0, T; W_0^{1,p}(\Omega))$, that is,

$$u_\varepsilon \rightarrow u \text{ a.e. weakly in } \Omega_T \text{ as } \varepsilon \rightarrow 0, \quad (52)$$

$$u_\varepsilon^m \xrightarrow{*} u^m \text{ weakly } * \text{ in } L^\infty(0, T; W_0^{1,p}(\Omega)) \text{ as } \varepsilon \rightarrow 0, \quad (53)$$

$$\partial_t u_\varepsilon \rightarrow \partial_t u \text{ weakly in } L^2(\Omega_T) \text{ as } \varepsilon \rightarrow 0. \quad (54)$$

From (46), one can infer that $u_\varepsilon \leq u, \forall (x, t) \in \Omega_T$.

Lemma 5. Assume that $\gamma \geq 0, m > 0$, and $p \geq 2$. Then for any fixed $t \in (0, T)$,

$$\int_{\Omega} |u_\varepsilon^m - u^m|^p dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

Proof. Taking a test function $(u_\varepsilon^m - \varepsilon - u^m)$ in (45), we infer that

$$\begin{aligned} & \int_{\Omega} \partial_t u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx + \int_{\Omega} (|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m \nabla (u_\varepsilon^m - \varepsilon - u^m) dx \\ &= -\gamma \int_{\Omega} u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx - \int_{\Omega} \beta_\varepsilon(u_\varepsilon - u_0) \cdot (u_\varepsilon^m - \varepsilon - u^m) dx. \end{aligned} \quad (55)$$

From (9) and (46), $u_\varepsilon^m - \varepsilon - u^m \leq 0$, such that $-\int_\Omega \beta_\varepsilon(u_\varepsilon - u_0) \cdot (u_\varepsilon^m - \varepsilon - u^m) dx$ is non-positive and we drop it to arrive at

$$\begin{aligned} & \int_\Omega \partial_t u_\varepsilon \cdot (u_\varepsilon - \varepsilon - u) dx + \gamma \int_\Omega u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx \\ & \leq - \int_\Omega (|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m \nabla (u_\varepsilon^m - u^m) dx. \end{aligned} \quad (56)$$

This by adding $\int_\Omega \left((|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m \right) \nabla (u_\varepsilon^m - u^m) dx$ both hand sides of (50), leads to

$$\begin{aligned} & \int_\Omega \partial_t u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx + \gamma \int_\Omega u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx \\ & + \int_\Omega \left((|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m \right) \nabla (u_\varepsilon^m - u^m) dx \\ & \leq - \int_\Omega |\nabla u^m|^{p-2} \nabla u^m \nabla (u_\varepsilon^m - u^m) dx. \end{aligned} \quad (57)$$

Applying Holder's inequality and joining with Lemmas 3.1 and 3.2, (52), and (53), one can pass to the limit $\varepsilon \rightarrow 0$ to arrive at

$$\left| \int_\Omega \partial_t u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx \right| \leq C \sqrt{\int_\Omega |\partial_t u_\varepsilon|^2 dx} \cdot \int_\Omega (u_\varepsilon^m - \varepsilon - u^m)^2 dx \rightarrow 0, \quad (58)$$

$$\int_\Omega u_\varepsilon \cdot (u_\varepsilon^m - \varepsilon - u^m) dx \rightarrow 0, \quad (59)$$

$$\int_\Omega |\nabla u^m|^{p-2} \nabla u^m \nabla (u_\varepsilon^m - u^m) dx \rightarrow 0. \quad (60)$$

Combining (57), (58), (59), and (60) and passing to the limit, one can infer that

$$\lim_{\varepsilon \rightarrow 0} \int_\Omega \left((|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m \right) \nabla (u_\varepsilon^m - u^m) dx \leq 0. \quad (61)$$

Since $0 \leq \varepsilon \leq 1$, it follows from [6] that

$$\begin{aligned} & \left((|\nabla u_\varepsilon^m|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m \right) \nabla (u_\varepsilon^m - u^m) \geq (|\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m) \nabla (u_\varepsilon^m - u^m) \\ & \geq C(p) |\nabla u_\varepsilon^m - \nabla u^m|^p \geq 0. \end{aligned} \quad (62)$$

Combining (61) and (62), the proof of Lemma 4.1 is completed. $\square$

Using Trigonometric inequality,

$$\int_\Omega ||\nabla u_\varepsilon^m|^{p-2} \nabla u_\varepsilon^m - |\nabla u^m|^{p-2} \nabla u^m| dx \leq \int_\Omega ||\nabla u_\varepsilon^m|^{p-2} - |\nabla u^m|^{p-2}| \cdot |\nabla u_\varepsilon^m| dx + \int_\Omega |\nabla u^m|^{p-2} |\nabla u_\varepsilon^m - \nabla u^m| dx.$$

If $p \in [2, 3]$, applying $|a^r - b^r| \leq |a - b|^r$, $a > 0$, $b > 0$, $r \in [0, 1]$, Holder's inequality as well as Lemma 4.1,

$$\int_\Omega ||\nabla u_\varepsilon^m|^{p-2} - |\nabla u^m|^{p-2}| |\nabla u_\varepsilon^m| dx \leq \int_\Omega |\nabla u_\varepsilon^m - \nabla u^m|^{p-2} |\nabla u_\varepsilon^m| dx \rightarrow 0.$$

If $p > 3$, one from Lemma 4.1 can obtain

$$\int_{\Omega} ||\nabla u_{\varepsilon}^m|^{p-2} - |\nabla u^m|^{p-2}| |\nabla u| dx \leq \int_{\Omega} (|\nabla u_{\varepsilon}^m| + |\nabla u^m|)^{p-2} |\nabla u_{\varepsilon}^m - \nabla u^m| dx \rightarrow 0.$$

Besides that using Lemma 4.1 also gives

$$\int_{\Omega} |\nabla u_{\varepsilon}^m|^{p-2} |\nabla u_{\varepsilon}^m - \nabla u^m| dx \rightarrow 0.$$

Combining all the above, we have the following result.

Lemma 6. As $\varepsilon \rightarrow 0$, $|\nabla u_{\varepsilon}^m|^{p-2} \nabla u_{\varepsilon}^m$ converges to $|\nabla u^m|^{p-2} \nabla u^m$ with norm $L^1(\Omega)$.

Furthermore, we focus on the limit of $\beta_{\varepsilon}(u_{\varepsilon} - u_0)$. Since $\beta_{\varepsilon}(\cdot) \in C^2(\mathbb{R})$ in (9), (46), and (47), for all $(x, t) \in \Omega_T$, give

$$\beta_{\varepsilon}(u_{\varepsilon} - u_0) \rightarrow \xi \text{ as } \varepsilon \rightarrow 0.$$

We consider ξ in two cases: $u_{\varepsilon} \geq u_0 + \varepsilon$ and $0 < u_{\varepsilon} < u_0 + \varepsilon$. If $u_{\varepsilon} \geq u_0 + \varepsilon$, $\beta_{\varepsilon}(u_{\varepsilon} - u_0) = 0$, such that

$$\xi(x, t) = 0 \Leftrightarrow u > u_0. \quad (63)$$

If $u_0 \leq u_{\varepsilon} < u_0 + \varepsilon$, $\beta_{\varepsilon}(0) = -M_0$, and $\beta_{\varepsilon}(\cdot) \in C^2(\mathbb{R})$ imply that

$$\xi(x, t) = -M_0 \Leftrightarrow u = u_0. \quad (64)$$

Combining (63) and (64), it can be easily verified that

$$-\beta_{\varepsilon}(u_{\varepsilon} - u_0) \rightarrow \xi \in G(u - u_0) \text{ as } \varepsilon \rightarrow 0 \quad (65)$$

for all $(x, t) \in \Omega_T$. Furthermore, passing the limit $\varepsilon \rightarrow 0$ in the second line of (44) and the second part of (46),

$$u(x, 0) = u_0(x) \quad \text{in } \Omega, \quad u(x, t) \geq u_0(x) \quad \text{in } \Omega_T. \quad (66)$$

Combining Lemmas 4.1, 4.2, (65), and (66), we infer that (u, ξ) satisfies the conditions of Definition 2.1, such that (u, ξ) is a generalized solution of (1).

Theorem 1. Assume that $u_0 \in W^{1,p}(\Omega)$, $\gamma \geq 0$. Then (1) admits a solution within the class of Definition 2.1.

5 Stability and uniqueness

In this section, we study the stability and uniqueness of generalized solution for all $\gamma \geq 0$, $m > 0$, and $p \geq 2$. Consider generalized solutions (u_i, ξ_i) of (1) with initial conditions

$$u(0, x) = u_{0,i}(x), \quad x \in \Omega, \quad i = 1, 2. \quad (67)$$

Since $u_1, u_2 \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$, $u_1 - u_2 \in L^{\infty}(0, T; W_0^{1,p}(\Omega))$. So choosing $\varphi = u_1 - u_2$ in Definition 2.1, one can infer that

$$\int_{\Omega} \int_0^T \partial_t \varphi \cdot \varphi + (|\nabla u_1^m|^{p-2} \nabla u_1^m - |\nabla u_2^m|^{p-2} \nabla u_2^m) \nabla \varphi dx dt + \gamma \int_{\Omega} \int_0^T \varphi^2 dx dt = \int_{\Omega} \int_0^T (\xi_1 - \xi_2) \cdot \varphi dx dt. \quad (68)$$

As what we mentioned in (62),

$$(|\nabla u_1^m|^{p-2} \nabla u_1^m - |\nabla u_2^m|^{p-2} \nabla u_2^m) (\nabla u_1^m - \nabla u_2^m) \geq 0.$$

$f(u) = u$ and $g(u) = u^m$ are both increasing functions with u , $\text{sgn}(\nabla u_1^m - \nabla u_2^m) = \text{sgn}(\nabla u_1 - \nabla u_2)$, so

$$(|\nabla u_1^m|^{p-2} \nabla u_1^m - |\nabla u_2^m|^{p-2} \nabla u_2^m) \nabla \varphi \geq 0. \quad (69)$$

As for $\int_{\Omega} (\xi_1 - \xi_2) \cdot \varphi dx dt$, we first give the estimate

$$\int_{\Omega} (\xi_1 - \xi_2) \cdot \varphi dx dt \leq 0, \quad \forall t \in [0, T]. \quad (70)$$

Indeed, if $u_1(x, t) > u_2(x, t)$, then $u_1(x, t) > u_{1,0}(x)$. (5) and (65) imply $\xi_1 = 0 \leq \xi_2$, such that

$$(\xi_1 - \xi_2) \cdot \varphi = (\xi_1 - \xi_2) \cdot (u_1 - u_2) \leq 0. \quad (71)$$

On the contrary, if $u_1(x, t) < u_2(x, t)$, we swap $u_1(x, t)$ and $u_2(x, t)$, it is obvious to obtain

$$(\xi_1 - \xi_2) \cdot \varphi = (\xi_2 - \xi_1) \cdot (u_2 - u_1) \leq 0. \quad (72)$$

Furthermore, we remove the non-negative term (69) and $\gamma \int_{\Omega} \varphi^2 dx dt$ and also drop the non-positive term (70) in (68) to arrive at

$$\int_{\Omega} \partial_t \varphi \cdot \varphi dx dt \leq 0 \Leftrightarrow \int_{\Omega} |u_1 - u_2|^2 dx \leq \int_{\Omega} |u_{1,0} - u_{2,0}|^2 dx. \quad (73)$$

Uniqueness of solution can be found easily if $u_{1,0} = u_{2,0}$. Thus, we state our result as follows.

Theorem 2. Let (u_i, ξ_i) be a generalized solution of (1) with

$$u(0, x) = u_{0,i}(x), \quad x \in \Omega, \quad i = 1, 2.$$

Then

$$\|u_1 - u_2\|_{L^\infty(0,T; L^2(\Omega))} \leq \|u_{0,1}(\cdot) - u_{0,2}(\cdot)\|_{L^2(\Omega)}.$$

Moreover, problem (1) has a unique solution in the sense of Definition 2.1 with $\gamma \geq 0$, $m > 0$, and $p \geq 2$.

6 Conclusion

In this work, the existence, uniqueness, and stability of generalized solution to a kind of variational inequality problem with non-Newtonian polytropic parabolic operators

$$Lu = \partial_t u - \nabla(|\nabla u^m|^{p-2} \nabla u^m) + \gamma u, \quad \gamma \geq 0.$$

Since m and p are coupled in Lu and Lu is degenerate, we construct a map (6) based on the Leray-Schauder lemma's conditions and a penalty map function used to deal with inequality constraint that regular problem (7) is established. Some estimates of problem (7) and continuity, boundedness and compactness of map (6) are proved a Leray-Schauder fixed point theory as well as a suitable version of Aubin-Lions lemma. The existence, uniqueness, and stability of the solution are analyzed.

In this article, we limit that $\gamma \geq 0$. In the proof of Lemma 3.1, formula (17) holds only when γ is greater than 0. The Holder inequality is used in Lemma 3.2, where p needs to satisfy $p \geq 2$. In addition, if parameter p depends on x or t , (26) in Lemma 3.2 does not work, Lemma 3.2 is difficult to be proved in this case. We will study those focuses in the future.

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