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Research Article

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On H^2-solutions for a Camassa-Holm type equation

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Abstract: Camassa-Holm type equations arise as models for the unidirectional propagation of shallow water waves over a flat bottom. They also describe finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. Under appropriate assumption on the initial data, on the time T, and on the coefficients of such equation, we prove the well-posedness of the classical solutions for the Cauchy problem.

Keywords: existence, uniqueness, stability, Camassa-Holm type equation, Cauchy problem

MSC 2020: 35G25, 35K55

1 Introduction

In this article, we investigate the well-posedness of the following Cauchy problem:

$$\begin{cases} \partial_t u + \delta\partial_x u + 2\kappa\partial_x u^2 + a\partial_x^3 u - \beta^2\partial_t\partial_x^2 u + \alpha\partial_x u\partial_x^2 u + \gamma u\partial_x^3 u = 0, & 0 < t < T, \quad x \in \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with $\delta, \kappa, a, \beta, \alpha, \gamma \in \mathbb{R}$ are constant, such that

$$\beta \neq 0. \tag{1.2}$$

On the initial datum, we assume

$$u_0 \in H^2(\mathbb{R}), \quad u_0 \not\equiv 0, \tag{1.3}$$

and one of the following:

$$\beta^6 - 8(|\kappa + \alpha + 2\gamma| + |2\alpha - \gamma|)^2[A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0]T^2 > 0, \tag{1.4}$$

$$|\kappa + \alpha + 2\gamma| + |2\alpha - \gamma| < \frac{|\beta|^3}{2\sqrt{2}T\sqrt{A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0}}, \tag{1.5}$$

$$|\beta|^3 \log(A_2(\beta)) > \frac{(|\kappa + \alpha + 2\gamma| + |2\alpha - \gamma|)T}{\sqrt{2}}, \tag{1.6}$$

$$\kappa = 0, \quad \gamma = 2\alpha, \tag{1.7}$$

$$\alpha = 2\gamma, \quad \gamma \neq 0, \quad \kappa \neq 0, \quad T < \frac{(\pi - 2\arctan(\sqrt{B_0 + \beta^2 E_0}))|\beta|^{\frac{5}{2}}}{\tau_3^2(\beta^2)}, \tag{1.8}$$

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where

$$\begin{aligned} A_0 &= \|u_0\|^2_{L^2(\mathbb{R})}, \quad B_0 = \|\partial_x u_0\|^2_{L^2(\mathbb{R})}, \quad E_0 = \|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}, \\ A_1 &:= 8(|\kappa + \alpha + 2\gamma| + |2\alpha - \gamma|)^2, \qquad \tau_3^2(\beta^2) := \max\{|\kappa|(A_0 + \beta^2 B_0), 3|\gamma|\}, \\ A_2(\beta) &:= \frac{A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0 + 1}{A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0}. \end{aligned} \tag{1.9}$$

Observe that (1.4) is satisfied if one of the following holds

$$|\beta| = T, \qquad T > \sqrt{A_1(B_0 + E_0) + \sqrt{A_1^2(B_0 + E_0)^2 + 4A_1(A_0 + B_0)}}\,, \tag{1.10}$$

$$T = |\beta|^3, \quad A_1 < \frac{1}{A_0 + (\sqrt[3]{T^2} + 1)B_0 + \sqrt[3]{2}\,TE_0}. \tag{1.11}$$

Equation (1.1) arises as a model for the unidirectional propagation of shallow water waves over a flat bottom, where $u(t, x)$ represents the water-free surface in nondimensional variables. It was first obtained in [1] as an abstract bi-Hamiltonian equation with infinitely many conservation laws [2–4], and subsequently from physical principles [5–8].

Equation (1.1) is also deduced in [9–11] as an equation describing finite length, small amplitude radial deformation waves in cylindrical compressible hyperelastic rods. Moreover, (1.1) is a re-expression of the geodesic flow in the group of compressible diffeomorphisms of the circle [12], just like the Euler equation that is an expression of the geodesic flow in the group of incompressible diffeomorphisms of the torus [13]. This geometric interpretation leads to a proof that equation (1.1) satisfies the least action principle [14]: a state of the system is transformed to another nearby state through a uniquely determined flow that minimizes the energy (see also [15]).

If (1.8) holds, (1.1) is known as the Camassa-Holm equation. Local well-posedness results are proved in [16–19]. It is also known that there exist global solutions for a certain class of initial data and solutions that blow up in finite time for a large class of initial data [16,20,21]. Existence and uniqueness results for global weak solutions are proven in [21–33]. The convergence of finite difference schemes is proved in [34,35], existence of the traveling wave solutions in [36], and the well-posedness of periodic solutions for the Cauchy problem in [37,38]. In [39–42], using a compensated compactness argument in the L^p setting [43–46], the convergence of the solution of (1.1) to the unique entropy one of the Burgers equation is proven, and in [47], it was proved using a kinetic approach [48].

Equation (1.1) is also studied in [49,50]. By using the method of asymptotic integrability, the authors found that only three equations from this family were asymptotically integrable up to third order: the Camassa-Holm equation, the Korteweg-deVries equation, and one with

$$\kappa = -\frac{2\gamma}{\beta^2}, \qquad \alpha = \gamma. \tag{1.12}$$

The Korteweg-deVries equation ($\beta = \alpha = \gamma = 0$) models weakly nonlinear unidirectional long waves and arises in various physical contexts. For example, it models surface waves of small amplitude and long wavelength in shallow water. In this context, $u(t, x)$ represents the wave height above a flat bottom, with x being proportional to distance in the propagation direction and t being proportional to the elapsed time. The Korteweg-deVries equation is completely integrable and possesses solitary wave solutions that are solitons. The Cauchy problem for the Korteweg-deVries equation is studied in [51,52], in [53], and the references cited therein. In particular, in [51,52], the authors prove that the well-posedness of the Korteweg-deVries equation, under Assumption (1.3) for any T.

Observe that, by using (1.12) in (1.1), and properly scaling, we gain

$$\partial_t u + \partial_x u + 3\partial_x u^2 + \partial_x^3 u - \beta^2 \partial_t \partial_x^2 u - \frac{9\beta^2}{2}\partial_x u \partial_x^2 u - \frac{3\beta^2}{2} u \partial_x^3 u = 0. \tag{1.13}$$

By rescaling, shifting the dependent variable, and finally applying a Galilean boost, Equation (1.13) can be transformed into the following form [49,54]:

$$\partial_t u - \partial_t \partial_x^2 u + 4u\partial_x u = 3\partial_x u \partial_x^2 u + u\partial_x^3 u. \tag{1.14}$$

From a mathematical point of view, the local and global well-posedness of (1.14) in energy spaces is proven in [55–58] and the references cited therein. In [59–61], the well-posedness of the entropy solution is proven, while, in [62], the well-posedness of the homogeneous initial boundary value problem is studied. In [63], the well-posedness of the periodic solution of (1.14) is analyzed, while, in [64], the convergence of some numerical schemes is proven. Possible estimates on the blow-up time T for (1.14) are given in [18,65,66] and the references cited therein.

In this article, we obtained the following estimates on the blow-up time T of the H^2 norm for (1.1). We can obtain some additional ones.

- Assuming (1.5) and choosing $\kappa = 1$, $\beta = 1$, $\alpha = 3$, and $\gamma = 1$ in (1.1), we obtain (1.14) and we can estimate T as follows:

$$T \geq \frac{1}{18\sqrt{2}\sqrt{(A_0 + 2B_0 + E_0)}}. \tag{1.15}$$

- By assuming (1.12) in (1.1), we have the following equation:

$$\partial_t u + \delta\partial_x u - \frac{2\gamma}{\beta^2}\partial_x u^2 + a\partial_x^3 u - \beta^2\partial_t\partial_x^2 u + \gamma\partial_x u\partial_x^2 u + \gamma u\partial_x^3 u = 0, \tag{1.16}$$

  and by assuming (1.5), we obtain

$$|\gamma|(|3\beta^2 - 2| + \beta^2) < \frac{|\beta|^5}{2\sqrt{2}T\sqrt{A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0}}. \tag{1.17}$$

  Therefore, fixed T and u_0, we have that (1.16) admits an unique classical solution, when

$$|\gamma| < \frac{|\beta|^5}{2\sqrt{2}T(|3\beta^2 - 2| + \beta^2)\sqrt{A_0 + (\beta^2 + 1)B_0 + \beta^2 E_0}}. \tag{1.18}$$

- By choosing $\beta = 1$ and $\gamma \neq 0$ and assuming (1.17), we can estimate the blow-up time T for (1.16) as follows:

$$T \geq \frac{1}{4\sqrt{2}|\gamma|\sqrt{(A_0 + 2B_0 + E_0)}}. \tag{1.19}$$

One more interesting equation belonging to equation (1.1) is as follows:

$$\partial_t u + \delta\partial_x u + 3u\partial_x u + a\partial_x^3 u - \beta^2\partial_t\partial_x^2 u - 2\beta^2\partial_x u\partial_x^2 u - \beta^2 u\partial_x^3 u = 0, \tag{1.20}$$

which was deduced in [67], in the context of the integrable shallow water wave equation with linear and nonlinear dispersions. Here, we obtain the following on the blow-up time T for (1.20):

$$T \geq \frac{|\beta|^3}{\sqrt{2}(|3 - 8\beta^2| + 6\beta^2)\sqrt{A_0 + (\beta^2 + 1)B_0 + \beta^2 E}}. \tag{1.21}$$

Taking $\kappa = \frac{b+1}{2}$, $\alpha = b\beta^2$, and $\gamma = \beta^2$, (1.1) reads:

$$\partial_t u + \delta\partial_x u + (b + 1)u\partial_x u + a\partial_x^3 u - \beta^2\partial_t\partial_x^2 u + b\beta^2\partial_x u\partial_x^2 u + \beta^2 u\partial_x^3 u, \tag{1.22}$$

which is known as the b-equation and was deduced in [68–70] as the family of asymptotically equivalent shallow water wave equations that emerges at quadratic order accuracy, for each $b \neq -1$, by an appropriate Kodama transformation. If $b = -1$, [68,69] show that the corresponding Kodama transformation is singular and the asymptotic ordering is violated.

The solution of (1.22) is studied numerically for various values of b in [70,71], under assumption $\delta = a = 0$. Using the energy space technique, the local and global well-posedness of the Cauchy problem for (1.22) is proven in [72,73]. Moreover, [72] gives an estimate on the blow-up time T. Here, by assuming (1.5), (1.2), and (1.3), we prove that (1.22) admits an unique classical solution, if the following inequality is verified:

$$(\beta^2+1)|b+1|+\beta^2|b|<\frac{|\beta|^3}{2\sqrt{2}T\sqrt{A_0+(\beta^2+1)B_0+\beta^2E_0}}, \tag{1.23}$$

which gives an estimate on the blow-up time T

$$T\geq\frac{|\beta|^3}{2\sqrt{2}((\beta^2+1)|b+1|+\beta^2|b|)\sqrt{A_0+(\beta^2+1)B_0+\beta^2E_0}}. \tag{1.24}$$

Choosing

$$\delta=0,\quad \kappa=\frac{1}{2},\quad a=0,\quad \alpha=\theta-1,\quad \gamma=-\theta,\quad 0<\theta<1, \tag{1.25}$$

(1.1) becomes

$$\partial_t u+u\partial_x u-\beta^2\partial_t\partial_x^2 u+(\theta-1)\partial_x u\partial_x^2 u-\theta u\partial_x^3 u=0. \tag{1.26}$$

It models equations of some dispersive schemes [74]. In [75], by using the energy spaces technique, the local and global well-posedness of the Cauchy problem for (1.26) is proven. They provide an estimate of the blow-up time T. Here, by using (1.25), we show that (1.26) admits an unique classical solution, if

$$|1-2\theta|+2|3\theta-2|<\frac{|\beta|^3}{\sqrt{2}T\sqrt{A_0+(\beta^2+1)B_0+\beta^2E_0}}, \tag{1.27}$$

that gives the following estimate on the blow-up time T

$$T\geq\frac{|\beta|^3}{\sqrt{2}(|1-2\theta|+2|3\theta-2|)\sqrt{A_0+(\beta^2+1)B_0+\beta^2E_0}}. \tag{1.28}$$

Finally, observe that if $\delta=a=\alpha=\gamma=0$, (1.1) gives the following equation:

$$\partial_t u+\kappa\partial_x u^2-\beta^2\partial_t\partial_x^2 u=0, \tag{1.29}$$

which is known as the Benjamin-Bona-Mahony equation. In [76], the well-posedness of the classical solution of (1.29) is proven, for each choose of β and T, and under assumption,

$$u_0\in H^1(\mathbb{R}). \tag{1.30}$$

Here, by assuming (1.5), we prove that (1.29) admits an H^2 solution, if

$$|\kappa|<\frac{|\beta|^3}{2\sqrt{2}T\sqrt{A_0+(\beta^2+1)B_0+\beta^2E_0}} \tag{1.31}$$

holds.

The main result of this article is the following theorem.

Theorem 1.1. *Assuming* (1.2), (1.3), *and one within* (1.4), (1.10), (1.11), (1.5), (1.6), (1.7), *and* (1.8) *hold, there exists a unique solution u of* (1.1), *such that*

$$\begin{aligned}&u\in H^1((0,T)\times\mathbb{R})\cap L^\infty(0,T;H^2(\mathbb{R}))\cap W^{1,\infty}((0,T)\times\mathbb{R}),\\&\partial_t\partial_x u\in L^\infty(0,T;L^2(\mathbb{R})).\end{aligned} \tag{1.32}$$

Moreover, if u_1 *and* u_2 *are two solutions of* (1.1) *in correspondence of the initial data* $u_{1,0}$ *and* $u_{2,0}$, *we have that*

$$\|u_1(t,\cdot)-u_2(t,\cdot)\|^2_{H^1(\mathbb{R})}\leq\frac{\tau_2^2e^{Ct}}{\tau_3^2}\|u_{1,0}-u_{2,0}\|^2_{H^1(\mathbb{R})}, \tag{1.33}$$

for some suitable $C>0$, *and every* $0\leq t\leq T$, *where*

$$\tau_1^2=\min\{1,\beta^2\},\quad \tau_2^2=\max\{1,\beta^2\}. \tag{1.34}$$

Theorem 1.1 gives some conditions on κ, β, α, γ, T, and u_0, to have classical solutions for (1.1). Moreover, it says that the solutions of (1.1) are classical, for each choice of κ, β, α, γ, T, and u_0, if (1.7) holds (see Lemma 2.3), or, under the condition (Corollary 2.2):

$$\kappa + \alpha + 2\gamma = 0, \quad 2\alpha - \gamma = 0. \tag{1.35}$$

Observe that, if $\kappa = \alpha^2$, with $\alpha \neq 0$, then (1.35) is equivalent

$$\alpha^2 + 5\alpha = 0, \quad \gamma = 2\alpha.$$

Therefore, Theorem 1.1 holds also in the case

$$(\kappa, \alpha, \gamma) = (25, -5, -10). \tag{1.36}$$

In general, thanks to [77, Lemma 2.2] [78], Theorem 1.1 holds also in the following cases:

$$\kappa = \alpha^{2n}, \quad \alpha = -5^{\frac{1}{2n-1}}, \quad n \neq \frac{1}{2}, \quad \kappa = (\alpha + q)^{2n}, \quad q > \frac{1}{5}\left(-\frac{5}{2n}\right)^{\frac{2n}{2n-1}} + \left(\frac{5}{2n}\right)^{2n-1}, \quad n \neq 0. \tag{1.37}$$

Note that (1.36) and (1.37) do not imply (1.36).

This article is organized as follows. In Section 2, we prove several *a priori* estimates on a vanishing viscosity approximation of (1.1). Those play a key role in the proof of our main result, which is given in Section 3.

2 Vanishing viscosity approximation

Our existence argument is based on passing to the limit in a vanishing viscosity approximation of (1.1).

Fix a small number $0 < \varepsilon < 1$ and let $u_\varepsilon = u_\varepsilon(t, x)$ be the unique classical solution of the following problem [79–82]:

$$\begin{cases} \partial_t u_\varepsilon + \delta\partial_x u_\varepsilon + 2\kappa u_\varepsilon \partial_x u_\varepsilon + a\partial_x^3 u_\varepsilon - \beta^2 \partial_t \partial_x^2 u_\varepsilon + \alpha \partial_x u_\varepsilon \partial_x^2 u_\varepsilon + \gamma u_\varepsilon \partial_x^3 u_\varepsilon = -\varepsilon \partial_x^4 u_\varepsilon, & 0 < t < T, \quad x \in \mathbb{R}, \\ u_\varepsilon(0, x) = u_{\varepsilon,0}(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

where $u_{\varepsilon,0}$ is a C^∞ approximation of u_0, such that

$$\|u_{\varepsilon,0}\|_{H^2(\mathbb{R})} \leq \|u_0\|_{H^2(\mathbb{R})}, \quad \sqrt{\varepsilon}\,\|\partial_x^3 u_{\varepsilon,0}\|_{L^2(\mathbb{R})} \leq C_0, \tag{2.2}$$

where C_0 is a positive constant independent of ε.

Let us prove some *a priori* estimates on u_ε.

Following [81, Lemma 1], we prove the following result.

Lemma 2.1. (H^2 estimate) *We have that*

$$\begin{aligned} &\frac{1}{\sqrt{A_\varepsilon(t) + (\beta^2 + 1)B_\varepsilon(t) + \beta^2 E_\varepsilon(t)}} + \frac{2\sqrt{2}(|\kappa + \alpha + 2\gamma| + |2\alpha - \gamma|)t}{|\beta|^3} \\ &\qquad \geq \frac{1}{\sqrt{A_\varepsilon(0) + (\beta^2 + 1)B_\varepsilon(0) + \beta^2 E_\varepsilon(0)}} \end{aligned} \tag{2.3}$$

for every $0 \leq t \leq T$, *where*

$$A_\varepsilon(t) := \|u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})}, \quad B_\varepsilon(t) := \|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})}, \quad E_\varepsilon(t) := \|\partial_x^2 u(t, \cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.4}$$

In particular, under Assumption (1.3), *there exists a constant* $C > 0$, *dependent on* β, u_0, *and* T, *but not* ε, *such that*

$$\|u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2 + 1)\|\partial_x u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t, \cdot)\|^2_{L^2(\mathbb{R})} \leq C, \tag{2.5}$$

$$\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \leq C, \tag{2.6}$$

$$\varepsilon\int_{0}^{t}\|\partial_x^2u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le C, \tag{2.7}$$

$$\varepsilon\int_{0}^{t}\|\partial_x^3u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le C, \tag{2.8}$$

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\le C, \tag{2.9}$$

for every $0\le t\le T$.

Proof. By multiplying (2.1) by $2u_\varepsilon-2\partial_x^2u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+(\beta^2+1)\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)\\
&\quad=2\int_{\mathbb{R}}u\partial_tu\mathrm{d}x-2\beta^2\int_{\mathbb{R}}u_\varepsilon\partial_t\partial_x^2u_\varepsilon\mathrm{d}x-2\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_tu_\varepsilon\mathrm{d}x+2\beta^2\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_t\partial_x^2u_\varepsilon\mathrm{d}x\\
&\quad=-2\kappa\int_{\mathbb{R}}u_\varepsilon^2\partial_xu_\varepsilon\mathrm{d}x+2\kappa\int_{\mathbb{R}}u_\varepsilon\partial_xu_\varepsilon\partial_x^2u_\varepsilon\mathrm{d}x-2\alpha\int_{\mathbb{R}}u_\varepsilon\partial_xu_\varepsilon\partial_x^2u_\varepsilon\mathrm{d}x\\
&\qquad+2\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2\mathrm{d}x-2\gamma\int_{\mathbb{R}}u_\varepsilon^2\partial_x^3u_\varepsilon\mathrm{d}x+2\gamma\int_{\mathbb{R}}u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon\mathrm{d}x\\
&\qquad-2\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x^4u_\varepsilon\mathrm{d}x+2\varepsilon\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^4u_\varepsilon\mathrm{d}x-2\delta\int_{\mathbb{R}}u_\varepsilon\partial_xu_\varepsilon\mathrm{d}x\\
&\qquad-2a\int_{\mathbb{R}}u_\varepsilon\partial_x^3u_\varepsilon\mathrm{d}x+2\delta\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^2u_\varepsilon\mathrm{d}x+2a\int_{\mathbb{R}}\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon\mathrm{d}x\\
&\quad=-(\kappa+\alpha)\int_{\mathbb{R}}(\partial_xu_\varepsilon)^3\mathrm{d}x+(2\alpha-\gamma)\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2\mathrm{d}x+4\gamma\int_{\mathbb{R}}u_\varepsilon\partial_xu_\varepsilon\partial_x^2u_\varepsilon\mathrm{d}x\\
&\qquad+2\varepsilon\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^3u_\varepsilon\mathrm{d}x-2\varepsilon\|\partial_x^3u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2a\int_{\mathbb{R}}\partial_xu_\varepsilon\partial_x^2u_\varepsilon\mathrm{d}x\\
&\quad=-(\kappa+\alpha+2\gamma)\int_{\mathbb{R}}(\partial_xu_\varepsilon)^3\mathrm{d}x+(2\alpha-\gamma)\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2\mathrm{d}x\\
&\qquad-2\varepsilon\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}-2\varepsilon\|\partial_x^3u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, we have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+(\beta^2+1)\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\big)+2\varepsilon\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\qquad+2\varepsilon\|\partial_x^3u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad=-(\kappa+\alpha+2\gamma)\int_{\mathbb{R}}(\partial_xu_\varepsilon)^3\mathrm{d}x+(2\alpha-\gamma)\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2\mathrm{d}x\\
&\quad\le|\kappa+\alpha+2\gamma|\|\partial_xu_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+|2\alpha-\gamma|\|\partial_xu_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned} \tag{2.10}$$

We define

$$\begin{aligned}
X_\varepsilon(t)&\coloneqq\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+(\beta^2+1)\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
\ell_1&\coloneqq|\kappa+\alpha+2\gamma|,\quad \ell_2\coloneqq|2\alpha-\gamma|.
\end{aligned} \tag{2.11}$$

Observe that, thanks to (2.11),

$$\begin{aligned}\ell_1\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= \frac{\ell_1}{\beta^2+1}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}(\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq \frac{\ell_1}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}(\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq \frac{\ell_1}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}X_\varepsilon(t), \qquad (2.12)\\ \ell_2\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &= \frac{\ell_2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\leq \frac{\ell_2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}X_\varepsilon(t).\end{aligned}$$

Therefore, by (2.10)–(2.12),

$$\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} \leq \frac{\ell_1+\ell_2}{\beta^2}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}X_\varepsilon(t). \qquad (2.13)$$

Thanks to (2.11) and the Hölder inequality,

$$\begin{aligned}(\partial_x u_\varepsilon(t,\cdot))^2 &= 2\int_{-\infty}^{x}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon \mathrm{d}y = 2\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon|\mathrm{d}x\\ &\leq 2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &= \frac{2}{\beta^2}|\beta|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}|\beta|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\ &= \frac{2}{\beta^2}\sqrt{\beta^2\ \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\sqrt{\beta^2\ \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\\ &\leq \frac{2}{\beta^2}\sqrt{(\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\sqrt{\beta^2\ \|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}} \leq \frac{2}{\beta^2}X_\varepsilon(t).\end{aligned}$$

Hence,

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{\sqrt{2}}{|\beta|}X_\varepsilon^{\frac{1}{2}}(t). \qquad (2.14)$$

By (2.11), (2.12), and (2.14), we obtain

$$\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} \leq \frac{\sqrt{2}(\ell_1+\ell_2)}{|\beta|^3}X_\varepsilon^{\frac{3}{2}}(t), \qquad (2.15)$$

that is,

$$X_\varepsilon^{-\frac{3}{2}}(t)\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} \leq \frac{\sqrt{2}(\ell_1+\ell_2)}{|\beta|^3}.$$

By integrating on $(0,t)$, we have that

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \geq \frac{1}{\sqrt{X_\varepsilon(0)}} - \frac{2\sqrt{2}(\ell_1+\ell_2)t}{|\beta|^3}. \qquad (2.16)$$

By using (2.11) in (2.16), thank to (2.4), we have (2.3).

Assume (1.4) and we prove (2.5). Thanks to (1.9), we can define

$$X_0 := \|u_0\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})} = A_0 + (\beta^2+1)B_0 + \beta^2 E_0. \qquad (2.17)$$

Consequently, by (2.2) and (2.11),

$$X_\varepsilon(0) \leq X_0 \ \Rightarrow\ \frac{1}{\sqrt{X_\varepsilon(0)}} \geq \frac{1}{\sqrt{X_0}}. \qquad (2.18)$$

Moreover,

$$\frac{2\sqrt{2}(\ell_1+\ell_2)t}{|\beta|^3} \le \frac{2\sqrt{2}(\ell_1+\ell_2)T}{|\beta|^3} \Rightarrow -\frac{2\sqrt{2}(\ell_1+\ell_2)t}{|\beta|^3} \ge -\frac{2\sqrt{2}(\ell_1+\ell_2)T}{|\beta|^3}. \tag{2.19}$$

It follows from (2.3), (2.11), (2.17), (2.18), and (2.19) that

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \ge \frac{1}{\sqrt{X_0}} - \frac{2\sqrt{2}(\ell_1+\ell_2)T}{|\beta|^3},$$

which gives

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \ge \frac{|\beta|^3 - 2\sqrt{2}(\ell_1+\ell_2)\sqrt{X_0}\,T}{|\beta|^3\sqrt{X_0}}.$$

Thanks to (1.4), (1.9), (2.11), and (2.17), there exists a constant $C > 0$, dependent on β, u_0, and T, but not ε, such that

$$\frac{1}{\sqrt{X_\varepsilon(t)}} \ge \frac{C}{|\beta|^3\sqrt{X_0}}.$$

Therefore, by (2.17),

$$\sqrt{X_\varepsilon(t)} \le \frac{|\beta|^3\sqrt{X_0}}{C} \le C \Rightarrow X_\varepsilon(t) \le C. \tag{2.20}$$

By using (2.11) in (2.20), we have (2.5).

We prove (2.6). Thanks to (2.5), (2.11), and (2.14), we obtain

$$\|\partial_x u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (2.6).

Now, we prove (2.7) and (2.8). We begin by observing that, by (2.5), (2.6), and (2.10),

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})\\ &\quad + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C.\end{aligned}$$

By integrating on $(0,t)$, by (2.2), (2.11), and (2.17), we obtain

$$\begin{aligned}&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + (\beta^2+1)\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\quad + 2\varepsilon\int_0^t \|\partial_x^2 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s + 2\varepsilon\int_0^t \|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le X_0 + Ct \le C,\end{aligned}$$

which gives (2.7) and (2.8).

Finally, we prove (2.9). Thanks to (2.5) and the Hölder inequality,

$$u_\varepsilon^2(t,x) = 2\int_{-\infty}^x u_\varepsilon\partial_x u_\varepsilon \mathrm{d}y \le 2\int_{\mathbb{R}} |u_\varepsilon||\partial_x^2 u_\varepsilon|\mathrm{d}x \le 2\|u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C.$$

Hence,

$$\|u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (2.9). □

Since (1.10) and (1.11) and (1.5) imply (1.4), we have the following corollary.

Corollary 2.1. *If* (1.10) *or* (1.11) *or* (1.5) *hold, then we have* (2.5)–(2.9).

Corollary 2.2. *Fix* $T > 0$ *and assume* (1.35). *Then,* (2.5)–(2.8) *hold, for each choose of* β, T, *and* u_0.

Proof. Let $0 \leq t \leq T$. We begin by observing that, thanks to (1.35), (1.4) reads

$$|\beta| > 0,$$

which is verified by Assumption (1.2). Therefore, arguing as in Lemma 2.1, we have (2.5)–(2.8), for each choose of β, T, and u_0. ☐

Lemma 2.2. *We have that*

$$\frac{e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}}{A_\varepsilon(t) + (\beta^2+1)B_\varepsilon(t) + \beta^2 E_\varepsilon(t)} + e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}} - 1 \geq \frac{1}{A_{0,\varepsilon} + (\beta^2+1)B_{0,\varepsilon} + \beta^2 E_{0,\varepsilon}}, \tag{2.21}$$

for every $0 \leq t \leq T$, *where* $A_\varepsilon(t)$, $B_\varepsilon(t)$, $E_\varepsilon(t)$, ℓ_1, *and* ℓ_2 *are defined in* (2.4) *and* (2.11), *respectively. In particular, under Assumption* (1.6), *we have* (2.5). *Moreover,* (2.6)–(2.9) *hold.*

Proof. We begin observing that, arguing as in Lemma 2.1, we have (2.15). Thanks to the Young inequality,

$$X_\varepsilon^{\frac{3}{2}}(t) \leq \frac{1}{2}(X_\varepsilon + X_\varepsilon^2). \tag{2.22}$$

It follows from (2.15) and (2.22) that

$$\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} \leq \frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3}(X_\varepsilon + X_\varepsilon^2),$$

that is,

$$\frac{1}{X_\varepsilon^2(t)}\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} \leq \frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3 X_\varepsilon(t)} + \frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3}. \tag{2.23}$$

Observe that

$$\frac{1}{X_\varepsilon^2(t)}\frac{\mathrm{d}X_\varepsilon(t)}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{X_\varepsilon(t)}\right).$$

Consequently by (2.23), we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{X_\varepsilon(t)}\right) + \frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3 X_\varepsilon(t)} \geq -\frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3}. \tag{2.24}$$

By multiplying (2.24) by $e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}}{X_\varepsilon(t)}\right) \geq -\frac{(\ell_1+\ell_2)}{\sqrt{2}|\beta|^3} e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}.$$

It follows from an integration on $(0, t)$ that

$$\frac{e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}}{X_\varepsilon(t)} - \frac{1}{X_{0,\varepsilon}} \geq -\left(e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}} - 1\right),$$

that is,

$$\frac{e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}}{X_\varepsilon(t)} + e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}} - 1 \geq \frac{1}{X_{0,\varepsilon}}. \tag{2.25}$$

By using (2.3) and (2.11), in (2.25), we have (2.21).

Assume (1.6) and we prove (2.5). We begin by observing that (2.4), (2.11), (2.21), and (2.25),

$$\frac{e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}+\left(e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}-1\right)X_\varepsilon(t)}{X_\varepsilon(t)}\geq\frac{1}{X_{0,\varepsilon}}.$$

Hence, by (2.2), (2.11), (2.17), and (2.18),

$$X_\varepsilon(t)\leq X_{0,\varepsilon}e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}+X_{0,\varepsilon}\left(e^{\frac{(\ell_1+\ell_2)t}{\sqrt{2}|\beta|^3}}-1\right)X_\varepsilon(t)\leq X_0e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}+X_0\left(e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}-1\right)X_\varepsilon(t).$$

Therefore,

$$\left[1-X_0\left(e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}-1\right)\right]X_\varepsilon(t)\leq X_0e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}.$$

Thanks to (1.6), (1.9), and (2.11), there exists an constant $C>0$, dependent on β, u_0, and T, but not ε, such that

$$\frac{X_\varepsilon(t)}{C}\leq X_0e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}.$$

Hence,

$$X_\varepsilon(t)\leq CX_0e^{\frac{(\ell_1+\ell_2)T}{\sqrt{2}|\beta|^3}}\leq C. \tag{2.26}$$

By using (2.11) in (2.26), we have (2.5).

Finally, arguing as in Lemma 2.1, we have (2.6)–(2.9). □

Lemma 2.3. *Fix* $T>0$ *and assume* (1.7). *Then,* (2.5)–(2.9) *hold, for each choose of* β, T, *and* u_0.

Proof. Let $0\leq t\leq T$. We begin by observing that, by (1.7), (2.1) reads

$$\partial_tu_\varepsilon+\delta\partial_xu_\varepsilon-\beta^2\partial_t\partial_x^2u_\varepsilon+\gamma\partial_x^3u_\varepsilon+\alpha\partial_xu_\varepsilon\partial_x^2u_\varepsilon+2\alpha u_\varepsilon\partial_x^3u_\varepsilon=-\varepsilon\partial_x^4u_\varepsilon. \tag{2.27}$$

We prove that

$$\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon\int_{\mathbb{R}}\|\partial_x^3u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds\leq\|\partial_xu_0\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_0\|^2_{L^2(\mathbb{R})}. \tag{2.28}$$

Observe that

$$-2\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2dx-4\alpha\int_{\mathbb{R}}u_\varepsilon\partial_x^2u_\varepsilon\partial_x^3u_\varepsilon dx=-2\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2dx-2\alpha\int_{\mathbb{R}}\partial_xu_\varepsilon(\partial_x^2u_\varepsilon)^2dx=0.$$

Consequently, multiplying (2.28) by $-2\partial_x^2u_\varepsilon$ and arguing as in Lemma 2.1, and integration on $\mathbb{R}$ gives

$$\frac{d}{dt}(\|\partial_xu_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})+2\varepsilon\int_0^t\|\partial_x^3u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}ds=0.$$

By integrating on $(0,t)$, by (2.2), we have (2.28).

We prove that

$$\|\partial_xu_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\leq\sqrt{2}\sqrt{\|\partial_xu_0\|^2_{L^2(\mathbb{R})}+\beta^2\|\partial_x^2u_0\|^2_{L^2(\mathbb{R})}}. \tag{2.29}$$

Thanks to (2.28) and the Hölder inequality,

$$
\begin{aligned}
(\partial_x u_\varepsilon(t,x))^2 &= 2\int\limits_{-\infty}^{x} \partial_x u_\varepsilon \partial_x^2 u_\varepsilon \mathrm{d}y = 2\int\limits_{\mathbb{R}} |\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| \mathrm{d}x \\
&\le 2\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \\
&\le 2(\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}).
\end{aligned}
$$

Hence,

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^\infty(\mathbb{R})} \le 2(\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}),$$

which gives (2.29).

Now, we prove that

$$
\begin{aligned}
&\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\,\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int\limits_0^t \|\partial_x^4 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \\
&\le \|u_0\|^2_{H^2(\mathbb{R})} + 5\sqrt{2}|\alpha|(\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}T.
\end{aligned}
\tag{2.30}
$$

We begin by observing that

$$2\alpha\int\limits_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon + 4\alpha\int\limits_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon\mathrm{d}x = -\alpha\int\limits_{\mathbb{R}}(\partial_x u_\varepsilon)^3\mathrm{d}x - 8\alpha\int\limits_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\mathrm{d}x = 5\alpha\int\limits_{\mathbb{R}}(\partial_x u_\varepsilon)^3\mathrm{d}x.$$

Therefore, multiplying (2.27) by $2u$ and arguing as in Lemma 2.1, integration on $\mathbb{R}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} = -5\alpha\int\limits_{\mathbb{R}}(\partial_x u_\varepsilon)^3\mathrm{d}x. \tag{2.31}$$

Thanks to (2.28) and (2.29),

$$5|\alpha|\int\limits_{\mathbb{R}}|\partial_x u_\varepsilon|^3\mathrm{d}x \le |\alpha|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 5\sqrt{2}|\alpha|(\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}.$$

It follows from (2.31) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + 2\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 5\sqrt{2}|\alpha|(\|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})})^{\frac{3}{2}}.$$

By integrating on $(0,t)$, by (2.2), we have (2.28).

Finally, thanks to (2.30) and the Hölder inequality, we have that

$$\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})} \le C(T). \tag{2.32}$$

Therefore, the proof is concluded. □

Lemma 2.4. *Assume* (1.8). *Then, we have* (2.5)–(2.9).

Proof. We begin by observing that, thanks to (1.8), (2.1) reads

$$\partial_t u_\varepsilon + \delta\partial_x u_\varepsilon - 2\kappa u_\varepsilon\partial_x u_\varepsilon + a\partial_x^3 u_\varepsilon - \beta^2\partial_t\partial_x^2 u_\varepsilon + 2\gamma\partial_x u_\varepsilon\partial_x^2 u_\varepsilon + \gamma u_\varepsilon\partial_x^3 u_\varepsilon = -\varepsilon\partial_x^4 u_\varepsilon. \tag{2.33}$$

We prove that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon\int\limits_0^t\|\partial_x^2 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le \|u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_0\|^2_{L^2(\mathbb{R})}. \tag{2.34}$$

Observe that,

$$4\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx + 2\gamma\int_{\mathbb{R}} u_\varepsilon^2\partial_x^3 u_\varepsilon dx = 4\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx - 4\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_x^2 u_\varepsilon dx = 0.$$

Therefore, by multiplying (2.33) by u_ε and arguing as in Lemma 2.1, and integrating on $\mathbb{R}$, we obtain (2.34).

Moreover, thanks to (2.34) and the Hölder inequality,

$$\|u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \sqrt{2}(\|u_0\|_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_0\|^2_{L^2(\mathbb{R})}).$$

We prove that

$$\arctan(\sqrt{B_\varepsilon(t) + \beta^2 E_\varepsilon(t)}) \le \arctan(\sqrt{B_0 + \beta^2 E_0}) + \frac{\tau_3^2(\beta^2)t}{2|\beta|^{\frac{5}{2}}}, \tag{2.35}$$

where B_0, E_0, $\tau_3^2(\beta^2)$, $B_\varepsilon(t)$, and $E_\varepsilon(t)$ are defined in (1.9) and (2.4). Multiplying (2.33) by $-2\partial_x^2 u_\varepsilon$ and arguing as in Lemma 2.1, an integration on $\mathbb{R}$ gives

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_x u_\varepsilon(t,\cdot)\|^2 + \beta^2\,\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad = \kappa\int_{\mathbb{R}}(\partial_x u_\varepsilon)^3 dx + 3\gamma\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 dx \\
&\quad \le |\kappa|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 3|\gamma|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}\tag{2.36}$$

Observe that, by (2.34),

$$|\kappa|\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \frac{|\kappa|(\|u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_0\|^2_{L^2(\mathbb{R})})}{\beta^2}\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}.$$

Therefore, by (2.36), we have

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_x u_\varepsilon(t,\cdot)\|^2 + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}) + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad \le \frac{|\kappa|(\|u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_0\|^2_{L^2(\mathbb{R})})}{\beta^2}\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}\\
&\qquad + \frac{3|\gamma|}{\beta^2}\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})}(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}).
\end{aligned}\tag{2.37}$$

Due to the Young inequality,

$$\begin{aligned}
\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &\le 2\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon| dx\\
&\le 2\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\
&= \frac{2}{|\beta|}\,\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}|\beta|\|\partial_x^2 u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\
&= \frac{2}{|\beta|}\sqrt{\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\sqrt{\beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}}\\
&\le \frac{2}{|\beta|}(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}).
\end{aligned}$$

Hence,

$$\|\partial_x u_\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{\sqrt{2}}{\sqrt{|\beta|}}\sqrt{(\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})}. \tag{2.38}$$

We define

$$Y_\varepsilon(t) := \|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}. \tag{2.39}$$

It follows from (2.37), (2.38), and (2.39) that

$$\begin{aligned}\frac{\mathrm{d}Y_\varepsilon(t)}{\mathrm{d}t} + 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} &\le \frac{|\kappa|(\|u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x u_0\|^2_{L^2(\mathbb{R})})}{|\beta|^{\frac{5}{2}}} Y_\varepsilon^{\frac{1}{2}}(t) + \frac{3|\gamma|}{|\beta|^{\frac{5}{2}}} Y_\varepsilon^{\frac{3}{2}}(t)\\ &\le \frac{\tau_3^2(\beta^2)}{|\beta|^{\frac{5}{2}}} Y_\varepsilon^{\frac{1}{2}}(t)(1+Y_\varepsilon(t)),\end{aligned} \tag{2.40}$$

where $\tau_3^2(\beta^2)$ is defined in (1.9). Consequently, we have that

$$\frac{1}{Y_\varepsilon^{\frac{1}{2}}(t)(1+Y_\varepsilon(t))}\frac{\mathrm{d}Y_\varepsilon(t)}{\mathrm{d}t} \le \frac{\tau_3^2(\beta^2)}{|\beta|^{\frac{5}{2}}}.$$

By integrating on $(0, t)$, by (2.2) and (2.39), we obtain that

$$\arctan\left(Y_\varepsilon^{\frac{1}{2}}(t)\right) \le \arctan\left(Y_{0,\varepsilon}^{\frac{1}{2}}\right) + \frac{\tau_3^2(\beta^2)t}{2|\beta|^{\frac{5}{2}}} \le \arctan\left(Y_0^{\frac{1}{2}}\right) + \frac{\tau_3^2(\beta^2)t}{2|\beta|^{\frac{5}{2}}}, \tag{2.41}$$

where $Y_0 := \|\partial_x u_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_0\|^2_{L^2(\mathbb{R})}$.

By using (1.9) and (2.4) in (2.41), we have (2.35).

We prove that under Assumption (1.8), there exists a constant $C > 0$, dependent on β, u_0 and T, but not ε, such that

$$\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C, \tag{2.42}$$

for every $0 \le t \le T$.

We begin by observing that, by (1.9), (2.4), and (2.35), we have that

$$\arctan(\sqrt{B_\varepsilon(t) + \beta^2 E_\varepsilon(t)}) \le \arctan(\sqrt{B_0 + \beta^2 E_0}) + \frac{\tau_3^2(\beta^2)T}{2|\beta|^{\frac{5}{2}}}. \tag{2.43}$$

Moreover, by (1.8), we have that

$$\arctan(\sqrt{B_0 + \beta^2 E_0}) + \frac{\tau_3^2(\beta^2)T}{2|\beta|^{\frac{5}{2}}} < \frac{\pi}{2}. \tag{2.44}$$

Consequently, by (2.43) and (2.44), we obtain

$$\arctan(\sqrt{B_\varepsilon(t) + \beta^2 E_\varepsilon(t)}) \le C, \quad C \in \left(0, \frac{\pi}{2}\right). \tag{2.45}$$

(2.42) follows from (2.4) and (2.45).

Finally, arguing as in Lemma 2.1, we have (2.6) and (2.8). Therefore, the proof is concluded. □

Lemma 2.5. *Assume that one within* (1.4), (1.10), (1.11), *and* (1.5)–(1.8). *There exists a constant* $C > 0$, *dependent on* β, T, *and* u_0, *but not* ε, *such that*

$$\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\varepsilon^2\int_0^t \|\partial_x^4 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s \le C, \tag{2.46}$$

for every $0 \le t \le T$.

Proof. By multiplying (2.1) by $2\varepsilon\partial_x^4 u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^2\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})\\
&=-2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x-2\beta^2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t\partial_x^2 u_\varepsilon \mathrm{d}x\\
&=-2\kappa\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x-2\alpha\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x-2\gamma\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x\\
&\quad-2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}-2\delta\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x-2a\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_x^4 u_\varepsilon \mathrm{d}x\\
&=2\kappa\varepsilon\int_{\mathbb{R}}(\partial_x u_\varepsilon)^2\partial_x^3 u_\varepsilon \mathrm{d}x+2\kappa\varepsilon\int_{\mathbb{R}}u_\varepsilon\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon \mathrm{d}x+(2\alpha-\gamma)\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x\\
&\quad-2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\delta\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_x^3 u_\varepsilon \mathrm{d}x\\
&=-3\kappa\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 \mathrm{d}x+(2\alpha-\gamma)\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x-2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

Therefore, (2.6), we have that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^2\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})})+2\varepsilon^2\|\partial_x^4 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&=\ -3\kappa\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^2 u_\varepsilon)^2 \mathrm{d}x+(2\alpha-\gamma)\varepsilon\int_{\mathbb{R}}\partial_x u_\varepsilon(\partial_x^3 u_\varepsilon)^2 \mathrm{d}x\\
&\le\ 3|\kappa|\varepsilon\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+|\alpha-\gamma|\varepsilon\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le\ C\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+C\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}
$$

By integrating on $(0,t)$, by (2.2), (2.7), and (2.8), we obtain

$$
\begin{aligned}
&\varepsilon\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+\varepsilon\beta^2\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2\varepsilon^2\int_0^t\|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s\\
&\le\ C_0+C\varepsilon\int_0^t\|\partial_x^2 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s+C\varepsilon\int_0^t\|\partial_x^3 u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R})}\mathrm{d}s\le C,
\end{aligned}
$$

which gives (2.46). □

Lemma 2.6. *Assume that one within* (1.4), (1.10), (1.11), *and* (1.5)–(1.8). *There exists a constant* $C>0$, *dependent on* β, T, *and* u_0, *but not* ε, *such that*

$$\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}+\beta^2\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\le C, \tag{2.47}$$

$$\|\partial_t u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\le C, \tag{2.48}$$

for every $0\le t\le T$.

Proof. By multiplying (2.1) by $2\partial_t u_\varepsilon$, an integration on $\mathbb{R}$ gives

$$
\begin{aligned}
2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}&=-2\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x+2\beta^2\int_{\mathbb{R}}\partial_t u_\varepsilon\partial_x^2 u_\varepsilon \mathrm{d}x-2\alpha\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x\\
&\quad-2\gamma\int_{\mathbb{R}}u_\varepsilon\partial_x^3 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x-2\varepsilon\int_{\mathbb{R}}\partial_x^4 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x-2\delta\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x-2a\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x\\
&=-2\kappa\int_{\mathbb{R}}u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x-2\beta^2\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}+2(\gamma-\alpha)\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x\\
&\quad+2\gamma\int_{\mathbb{R}}u_\varepsilon\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon \mathrm{d}x+2\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t\partial_x u_\varepsilon \mathrm{d}x-2\delta\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_t u_\varepsilon \mathrm{d}x+2a\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon \mathrm{d}x.
\end{aligned}
$$

Therefore, we have that

$$\begin{aligned}
&2\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\quad= -2\kappa\int_{\mathbb{R}} u_\varepsilon\partial_x u_\varepsilon\partial_t u_\varepsilon dx + 2(\gamma-\alpha)\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_x^2 u_\varepsilon\partial_t u_\varepsilon dx + 2\gamma\int_{\mathbb{R}} u_\varepsilon\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon dx\\
&\qquad+2\varepsilon\int_{\mathbb{R}}\partial_x^3 u_\varepsilon\partial_t\partial_x u_\varepsilon dx - 2\delta\int_{\mathbb{R}}\partial_x u_\varepsilon\partial_t u_\varepsilon dx + 2a\int_{\mathbb{R}}\partial_x^2 u_\varepsilon\partial_t\partial_x u_\varepsilon dx.
\end{aligned}\tag{2.49}$$

Since, $0<\varepsilon<1$, due to (2.5), (2.6), (2.9), Lemma 2.5 and the Young inequality,

$$\begin{aligned}
2|\kappa|\int_{\mathbb{R}}|u_\varepsilon||\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx &\le 2|\kappa|\|u\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx\\
&\le C\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx = \int_{\mathbb{R}}|C\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx\\
&\le C\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|\gamma-\alpha|\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_x^2 u_\varepsilon||\partial_t u_\varepsilon|dx &\le 2|\gamma-\alpha|\|\partial_x u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_t u_\varepsilon|dx\\
&\le C\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_t u_\varepsilon|dx = \int_{\mathbb{R}}|C\partial_x^2 u_\varepsilon||\partial_t u_\varepsilon|dx\\
&\le C\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|\gamma|\int_{\mathbb{R}}|u_\varepsilon||\partial_x^2 u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx &\le 2|\gamma|\|u_\varepsilon\|_{L^\infty((0,T)\times\mathbb{R})}\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_t\partial_x|dx\\
&\le C\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx = \int_{\mathbb{R}}\left|\frac{C\partial_x^2 u_\varepsilon}{\beta}\right||\beta\partial_t\partial_x u_\varepsilon|dx\\
&\le C\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\\
&\le C + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})},\\
2\varepsilon\int_{\mathbb{R}}|\partial_x^3 u_\varepsilon||\partial_t u_\varepsilon|dx &= \int_{\mathbb{R}}\left|\frac{2\varepsilon\partial_x^3 u_\varepsilon}{\beta}\right||\beta\partial_t\partial_x u_\varepsilon|dx\\
&\le 2\varepsilon^2\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le 2\varepsilon\|\partial_x^3 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C_0 + +\frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|\delta|\int_{\mathbb{R}}|\partial_x u_\varepsilon||\partial_t u_\varepsilon|dx &\le 4\delta^2\|\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C + \frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})},\\
2|a|\int_{\mathbb{R}}|\partial_x^2 u_\varepsilon||\partial_t\partial_x u_\varepsilon|dx &= \int_{\mathbb{R}}\left|\frac{2a\partial_x^2 u_\varepsilon}{\beta}\right||\beta\partial_t\partial_x u_\varepsilon|dx\\
&\le \frac{2a^2}{\beta^2}\|\partial_x^2 u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}\\
&\le C + \frac{\beta^2}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})}.
\end{aligned}$$

Consequently, by (2.49), we obtain

$$\frac{1}{2}\|\partial_t u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\partial_t\partial_x u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C,$$

which gives (2.47).

Finally, we prove (2.48). Thanks to (2.47) and the Hölder inequality,

$$(\partial_t u_\varepsilon(t,x))^2 = 2\int_{-\infty}^{x} \partial_t u_\varepsilon \partial_t\partial_x u_\varepsilon dy \le 2\int_{\mathbb{R}} |\partial_t u_\varepsilon||\partial_t\partial_x u_\varepsilon| dx \le 2\|\partial_t u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x u_\varepsilon(t,\cdot)\|_{L^2(\mathbb{R})} \le C.$$

Hence,

$$\|\partial_t u_\varepsilon\|^2_{L^\infty((0,T)\times\mathbb{R})} \le C,$$

which gives (2.48). □

3 Proof of Theorem 1.1

This section is devoted to the proof of theorem.

By using the Sobolev immersion theorem, we prove the following result.

Lemma 3.1. *Assume that one within* (1.4)–(1.8) *and* (1.10) *holds. There exist a subsequence* $\{u_{\varepsilon_k}\}_{k\in\mathbb{N}}$ *of* $\{u_\varepsilon\}_{\varepsilon>0}$ *and a limit function u, which satisfies* (1.32) *such that*

$$u_{\varepsilon_k} \to u \text{ a.e. and in } L^p_{\mathrm{loc}}((0,T)\times\mathbb{R}), \quad 1 \le p < \infty. \tag{3.1}$$

Moreover, u is the solution of (1.1).

Proof. Thanks to Lemma 2.1, Corollaries 2.1 *and* 2.2, Lemmas 2.2 *and* 2.3, (2.4), and, Lemma 2.6,

$$\{u_\varepsilon\}_{\varepsilon>0} \text{ is uniformly bounded in } H^1((0,T)\times\mathbb{R}), \tag{3.2}$$

which gives (3.1).

Observe that, thanks to Lemma 2.1, Corollaries 2.1 and 2.2, and Lemmas 2.2 and 2.3

$$u \in L^\infty(0,T;H^2(\mathbb{R})),$$

while, by Lemma 2.6,

$$u \in W^{1,\infty}((0,T)\times\mathbb{R}).$$

Moreover, by Lemma 2.6, we have that

$$\partial_t\partial_x u \in L^2((0,T)\times\mathbb{R}).$$

Therefore, (1.32) holds and u is the solution of (1.1). □

Following [83, Theorem 1.1], we prove Theorem 1.1.

Proof of Theorem 1.1. Lemma 3.1 gives the existence of a solution u of (1.1) such that (1.32) holds.

We prove (1.33). Let u_1 and u_2 be two solutions of (1.1), which satisfy (1.32), that is,

$$\begin{cases} \partial_t u_1 + \delta\partial_x u_1 + \kappa\partial_x u_1^2 - \beta^2\partial_t\partial_x^2 u_1 + a\partial_x^3 u_1 + \dfrac{\alpha-\gamma}{2}\partial_x((\partial_x u_1)^2) - \gamma\partial_x(u_1\partial_x^2 u_1), & t>0, x\in\mathbb{R}, \\ u_1(0,x) = u_{1,0}(x), & x\in\mathbb{R}, \end{cases}$$

$$\begin{cases} \partial_t u_2 + \delta\partial_x u_2 + \kappa\partial_x u_2^2 - \beta^2\partial_t\partial_x^2 u_2 + a\partial_x^3 u_2 + \dfrac{\alpha-\gamma}{2}\partial_x((\partial_x u_2)^2) - \gamma\partial_x(u_2\partial_x^2 u_2), & t > 0, x \in \mathbb{R}, \\ u_2(0, x) = u_{2,0}(x), & x \in \mathbb{R}. \end{cases}$$

Then, the function

$$\omega = u_1 - u_2 \tag{3.3}$$

is the solution of the following Cauchy problem:

$$\begin{cases} \partial_t\omega + \nu\partial_x\omega - \beta^2\partial_t\partial_x^2\omega + \kappa\partial_x(u_1^2 - u_2^2) + a\partial_x^3\omega + \dfrac{\alpha-\gamma}{2}\partial_x((\partial_x u_1)^2 - (\partial_x u_1)^2) & \\ \qquad - \gamma\partial_x(u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2) = 0, & t > 0, x \in \mathbb{R}, \\ \omega_0(x) = u_{1,0}(x) - u_{2,0}(x), & x \in \mathbb{R}. \end{cases} \tag{3.4}$$

Observe that, thanks to (3.3),

$$\begin{aligned} \partial_x(u_1^2 - u_2^2) &= \partial_x((u_1 + u_2)(u_1 - u_2)) = \partial_x((u_1 + u_2)\omega), \\ \partial_x((\partial_x u_1)^2 - (\partial_x u_1)^2) &= \partial_x((\partial_x u_1 + \partial_x u_2)(\partial_x u_1 - \partial_x u_2)) = \partial_x((\partial_x u_1 + \partial_x u_2)\omega). \end{aligned} \tag{3.5}$$

Moreover, again by (3.3),

$$\begin{aligned} \partial_x(u_1\partial_x^2 u_1 - u_2\partial_x^2 u_2) &= \partial_x(u_1\partial_x^2 u_1 - u_1\partial_x^2 u_2 + u_1\partial_x^2 u_2 - u_2\partial_x^2 u_2) \\ &= \partial_x(u_1\partial_x^2\omega - \partial_x^2 u_2\omega) + \partial_x(\partial_x^2 u_2\omega). \end{aligned} \tag{3.6}$$

Consequently, by (3.5) and (3.6), equation (3.4) reads

$$\begin{aligned} &\partial_t\omega + \delta\partial_x\omega - \beta^2\partial_t\partial_x^2\omega + \kappa\partial_x((u_1 + u_2)\omega) + a\partial_x^3\omega \\ &\qquad + \frac{\alpha-\gamma}{2}\partial_x((\partial_x u_1 + \partial_x u_2)\omega) - \gamma\partial_x(u_1\partial_x^2\omega - \partial_x^2 u_2\omega) = 0. \end{aligned} \tag{3.7}$$

Since $u_1, u_2 \in L^\infty(0, T; H^2(\mathbb{R}))$, there exists a constant $C > 0$, such that

$$\begin{aligned} &\|u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C, \\ &\|\partial_x u_1\|_{L^\infty((0,T)\times\mathbb{R})}, \|\partial_x u_2\|_{L^\infty((0,T)\times\mathbb{R})} \le C, \\ &\|\partial_x^2 u_2(t, \cdot)\|_{L^2(\mathbb{R})} \le C. \end{aligned} \tag{3.8}$$

Moreover, by (3.8), we have that

$$|u_1 + u_2| \le |u_1| + |u_2| \le C, \quad |\partial_x u_1 + \partial_x u_2| \le |\partial_x u_1 + \partial_x u_2| \le C. \tag{3.9}$$

Observe that

$$\begin{aligned} 2\int_{\mathbb{R}} \omega(\partial_t\omega - \beta^2\partial_t\partial_x^2\omega)dx &= \frac{d}{dt}(\|\omega(t, \cdot)\| + \beta^2 \|\partial_x\omega(t, \cdot)\|^2_{L^2(\mathbb{R})}), \\ 2\delta\int_{\mathbb{R}} \omega\partial_x\omega dx &= 0, \\ 2\kappa\int_{\mathbb{R}} \omega\partial_x((u_1 + u_2)\omega)dx &= -2\kappa\int_{\mathbb{R}} (u_1 + u_2)\omega\partial_x\omega dx, \\ (\alpha - \gamma)\int_{\mathbb{R}} \omega\partial_x((\partial_x u_1 + \partial_x u_2)\omega)dx &= -(\alpha - \gamma)\int_{\mathbb{R}} (\partial_x u_1 + \partial_x u_2)\omega\partial_x\omega dx, \\ 2a\int_{\mathbb{R}} \omega\partial_x^3\omega dx &= -2a\int_{\mathbb{R}} \partial_x\omega\partial_x^2\omega dx = 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} -2\gamma\int_{\mathbb{R}} \omega\partial_x(u_1\partial_x^2\omega - \partial_x^2 u_2\omega)dx &= 2\gamma\int_{\mathbb{R}} u_1\partial_x\omega\partial_x^2\omega dx - 2\gamma\int_{\mathbb{R}} \partial_x^2 u_2\omega\partial_x\omega dx \\ &= -\gamma\int_{\mathbb{R}} \partial_x u_1(\partial_x\omega)^2 dx - 2\gamma\int_{\mathbb{R}} \partial_x^2 u_2\omega\partial_x\omega dx. \end{aligned} \tag{3.11}$$

By multiplying (3.7) by 2ω, thanks to (3.10), we have that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t,\cdot)\| + \beta^2\,\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})})\\ &\quad = 2\kappa\int_{\mathbb{R}}(u_1+u_2)\omega\partial_x\omega\mathrm{d}x + (\alpha-\gamma)\int_{\mathbb{R}}(\partial_x u_1+\partial_x u_2)\omega\partial_x\omega\mathrm{d}x + \gamma\int_{\mathbb{R}}\partial_x u_1(\partial_x\omega)^2\mathrm{d}x + 2\gamma\int_{\mathbb{R}}\partial_x^2 u_2\omega\partial_x\omega\mathrm{d}x.\end{aligned}\tag{3.12}$$

Due to (3.8), (3.9), and the Young inequality,

$$\begin{aligned}2|\kappa|\int_{\mathbb{R}}|u_1+u_2||\omega||\partial_x\omega|\mathrm{d}x &\le C\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x \le C\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |\alpha-\gamma|\int_{\mathbb{R}}|\partial_x u_1+\partial_x u_2||\omega||\partial_x\omega|\mathrm{d}x &\le C\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x \le C\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ |\gamma|\int_{\mathbb{R}}|\partial_x u_1|(\partial_x\omega)^2\mathrm{d}x &\le C\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})},\\ 2|\gamma|\int_{\mathbb{R}}|\partial_x^2 u_2\omega||\partial_x\omega|\mathrm{d}x &\le \gamma^2\int_{\mathbb{R}}\omega^2(\partial_x^2 u_2)^2\mathrm{d}x + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le \gamma^2\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\|\partial_x^2 u_2(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}\\ &\le C\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})} + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

It follows from (3.12) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t,\cdot)\| + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}) \le C\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + C\|\omega(t,\cdot)\|^2_{L^\infty(\mathbb{R})}.\tag{3.13}$$

Thanks to the Hölder inequality,

$$\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le 2\int_{\mathbb{R}}|\omega||\partial_x\omega|\mathrm{d}x \le 2\|\omega(t,\cdot)\|_{L^2(\mathbb{R})}\|\partial_x\omega(t,\cdot)\|_{L^2(\mathbb{R})}.$$

Therefore, by the Young inequality,

$$\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le \|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}.$$

It follows from (3.13) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\omega(t,\cdot)\| + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}) \le C\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le C(\|\omega(t,\cdot)\| + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})}).$$

The Gronwall Lemma and (3.4) give

$$\|\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega(t,\cdot)\|^2_{L^2(\mathbb{R})} \le e^{C(T)t}(\|\omega_0\|^2_{L^2(\mathbb{R})} + \beta^2\|\partial_x\omega_0\|^2_{L^2(\mathbb{R})}).$$

By (1.34), we have that

$$\tau_1^2\,\|\omega\|^2_{H^1(\mathbb{R})} \le \tau_2^2 e^{C(T)t}\|\omega_0\|^2_{H^1(\mathbb{R})}.\tag{3.14}$$

Therefore, (1.33) follows from (3.4) and (3.14). □

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