



Research Article

Khaled Alhazmy*, Fuad Ali Ahmed Almahti, El Mehdi Bouba, and Mohammed Tamekkante

On $2r$ -ideals in commutative rings with zero-divisors

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Abstract: In this article, we are interested in uniformly pr -ideals with order ≤ 2 (which we call $2r$ -ideals) introduced by Rabia Üregen in [On uniformly pr -ideals in commutative rings, Turkish J. Math. **43** (2019), no. 4, 18781886]. Several characterizations and properties of these ideals are given. Moreover, the comparison between the (nonzero) $2r$ -ideals and certain classes of classical ideals gives rise to characterizations of certain rings based only on the properties of the ideals consisting only of zero-divisors. Namely, among other things, we compare the class of (nonzero) $2r$ -ideals with the class of (minimal) prime ideals, the class of minimal prime ideals and their squares, and the class of primary ideals. The study of $2r$ -ideal in polynomial rings allows us to give a new characterization of the rings satisfying the famous A -property.

Keywords: uniformly pr -ideals, zero-divisors, A -property

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1 Introduction

Throughout, all rings considered are *commutative* with nonzero unity. Let R be a ring, I be an ideal of R , and S be a subset of R . Set $S^* := S \setminus \{0\}$ and $(I : S) := \{x \in R \mid xS \subseteq I\}$. The set of zero-divisors of R and the set of regular elements of R are denoted by $Z(R)$ and $\text{Reg}(R)$, respectively. The ideal I is said to be proper if $I \neq R$. The radical of I is denoted by $\sqrt{I} := \{x \in R \mid x^n \in I \text{ for some integer } n \geq 1\}$ and the nil-radical of R is denoted by $\text{nil}(R) := \sqrt{(0)}$. The total ring of fractions of R is denoted $Q(R) := \left\{ \frac{a}{b} \mid a \in R \text{ and } b \in \text{Reg}(R) \right\}$. The ring R is said to be a total quotient ring if $R = Q(R)$, or equivalently, every element in R is either a zero-divisor or a unit.

It is known that there are so many important rings with zero divisors that have interesting properties whose counterparts for the integral domains become trivial. Recently, there has been a lot of attention to the ideal theory of these rings (see [1–4]). For a ring R , the properties of $Q(R)$ provided by its ideals come from the ideals of R consisting entirely of zero divisors. An example of such ideals are the z^0 -ideals (studied under the name d -ideals in [4]).

* **Corresponding author: Khaled Alhazmy**, Department of Mathematics, Faculty of Sciences, King Khalid University, P. O. Box. 9004, Abha, Saudi Arabia, e-mail: khazmy@kku.edu.sa

Fuad Ali Ahmed Almahti: Department of Mathematics, Faculty of Sciences, King Khalid University, P. O. Box. 9004, Abha, Saudi Arabia, e-mail: fuadalialmahdy@hotmail.com

El Mehdi Bouba: Department of Mathematics and Computer, Faculty Multidisciplinary of Nador, University Mohammed First, Nador, Morocco, e-mail: mehdi8bouba@hotmail.fr

Mohammed Tamekkante: Department of Mathematics, Faculty of Science, University Moulay Ismail Meknes, Box 11201, Zitoune, Morocco, e-mail: tamekkante@yahoo.fr

Definition 1.1. [1] A proper ideal I of a ring R is said to be a z^0 -ideal if, for each $a \in I$, we have $P_a \subseteq I$, where P_a is the intersection of all minimal prime ideals containing a and, by convention, $P_a = R$ if a is not contained in any minimal prime ideal.

If R is reduced, then $Q(R)$ is Von Neumann regular if and only if every proper (principal) ideal is a z^0 -ideal [1, Corollary 1.14]. In [5], Mohamadian generalized the notion of ideals z^0 by introducing the concepts of r -ideals.

Definition 1.2. [5] A proper ideal I of a ring R is called an r -ideal if, whenever, $x, y \in R$ with $xy \in I$, we have $x \in I$ or $y \in Z(R)$, which are also consisting entirely of zero divisors.

Let $C(X)$ be the ring of real valued continuous functions on a Tychonoff space X . It is proved in [5, Proposition 5.4] that, over $C(X)$, every r -ideal is a z^0 -ideal if and only if X is a ∂ -space (a space in which the boundary of any zeroset is contained in a zeroset with empty interior). Several recently introduced notions are related to the notion of r -ideals (see, for example, [6,7]).

In [8], Urengin introduced the concept of uniformly pr -ideals as follows:

Definition 1.3. [8] A proper ideal I of a ring R is said to be a uniformly pr -ideal if there exists a positive integer n such that, whenever $x, y \in R$ with $xy \in I$, then $x^n \in I$ or $y \in Z(R)$. The order of I is the smallest positive integer for which the aforementioned property holds.

Hence, r -ideals are just uniformly pr -ideals with order 1. So, the order of an uniformly a pr -ideal I measures how far away I is from being an r -ideal.

In this article, we are interested to uniformly pr -ideals with order ≤ 2 , which we call $2r$ -ideals. Section 2 gives several characterizations and properties of $2r$ -ideals. It is also proved that every r -ideal is a $2r$ -ideal and that minimal prime ideals and their squares are $2r$ -ideals. Thus, several results in Section 2 study the rings in which every $2r$ -ideal is an r -ideal, the rings in which the only (nonzero) $2r$ -ideals are the minimal prime ideals (resp. and their squares), and the rings in which every (nonzero) $2r$ -ideals are primary. These comparisons give rise to characterizations of several rings using the properties of ideals consisting of zero-divisors. Section 3, among other things, studies some types of $2r$ -ideals in the polynomial rings. In addition, the $2r$ -ideals are used to characterize the rings satisfying the A -property.

2 On $2r$ -ideals in commutative rings

In this section, we introduce the class of $2r$ -ideals and study some of their properties. We also compare the class of $2r$ -ideals with some other classes of ideals.

Definition 2.1. Let R be a ring. A proper ideal I of R is said to be a $2r$ -ideal if, whenever $a, b \in R$ with $ab \in I$, then $a^2 \in I$ or $b \in Z(R)$.

By definition, the r -ideals are $2r$ -ideals and the $2r$ -ideals are uniformly pr -ideals. However, these concepts are different, as the following example shows.

Example 2.2. Consider the ring $R = k[x, y]/(xy)$, where k is a field. It is clear that R is reduced and noetherian with two minimal prime ideals; $P_1 = (\bar{x})$ and $P_2 = (\bar{y})$. Hence, $Z(R) = P_1 \cup P_2$. One can easily verify that $\bar{x} + \bar{y}$ and $\bar{x}^2 + \bar{y}$ are regular elements of R . It is also easy to see that $(P_1)^2$ is a $2r$ -ideal. Now, $\bar{x}(\bar{x} + \bar{y}) = \bar{x}^2 \in (P_1)^2$, $\bar{x} \notin (P_1)^2$, and $\bar{x} + \bar{y} \notin Z(R)$. Accordingly, $(P_1)^2$ is not an r -ideal. The ideal $(P_1)^3$ is a uniformly pr -ideal (with order 3). On the other hand, $\bar{x}(\bar{x}^2 + \bar{y}) = \bar{x}^3 \in (P_1)^3$, $\bar{x}^2 \notin (P_1)^3$, and $\bar{x}^2 + \bar{y} \notin Z(R)$. Accordingly, $(P_1)^3$ is not a $2r$ -ideal.

Recall that a proper ideal I of a ring R is said to be strongly quasi primary if, whenever, $a, b \in R$ with $ab \in I$, then either $a^2 \in I$ or $b \in \sqrt{I}$ [9]. Strongly quasi primary ideals contain primary ideals, and they are used to characterize divided domains [9, Theorem 2.2].

The first result in this section, while an immediate consequence of the definition of $2r$ -ideals, is an important fact since it emphasizes that $2r$ -ideals are entirely consisting of zero-divisors.

Proposition 2.3. *Let R be a ring and I be a proper ideal of R . If I is a $2r$ -ideal, then $I \subseteq Z(R)$. The equivalence holds if I is strongly quasi primary.*

Proof. For each $a \in I$, we have 1. $a \in I$ and $I^2 \notin I$. Then, $a \in Z(R)$, and so $I \subseteq Z(R)$. Suppose now that I is a strongly quasi primary ideal with $I \subseteq Z(R)$. Let $a, b \in R$ with $ab \in I$ and $a^2 \notin I$. Then, $b \in \sqrt{I} \subseteq Z(R)$ (since $I \subseteq Z(R)$). Hence, I is a $2r$ -ideal. \square

Recall that, for a proper ideal I of a ring R , the ideal generated by the squares of elements of I (i.e., $\{a^2 | a \in I\}$) is denoted by $[I]_2$ [10]. If 2 is a unit in R , then $[I]_2 = I^2$ [10, Theorem 5]. Let $f: R \rightarrow Q(R)$ be the natural homomorphism defined by $f(x) = \frac{x}{1}$, and let J^c denote the contraction of J in R , i.e., $J^c := f^{-1}(J) = \{x \in R | \frac{x}{1} \in J\}$.

The next result provides some useful characterizations of $2r$ -ideals.

Theorem 2.4. *Let R be a ring and I be a proper ideal of R . The following statements are equivalent.*

1. I is a $2r$ -ideal of R .
2. For every $a \in R$, either $a^2 \in I$ or $(I : a) \subseteq Z(R)$.
3. For any ideals J and K of R with $JK \subseteq I$, either $[J]_2 \subseteq I$ or $K \subseteq Z(R)$.
4. $[(I : r)]_2 \subseteq I$ for each $r \in \text{Reg}(R)$.
5. $[J^c]_2 \subseteq I \subseteq J^c$ for some ideal J of $Q(R)$.

Proof.

- (1) \Rightarrow (2) Let $a \in R$ such that $a^2 \notin I$. For each $b \in (I : a)$, we have $ab \in I$, and so $b \in Z(R)$. Hence, $(I : a) \subseteq Z(R)$.
- (2) \Rightarrow (3) Let J and K be two ideals of R with $JK \subseteq I$. Suppose that $K \not\subseteq Z(R)$ and consider $k \in K \setminus Z(R)$. For each $j \in J$, we have $k \in (I : j)$, and so $(I : j) \not\subseteq Z(R)$. Then, $j^2 \in I$. Consequently, $[J]_2 \subseteq I$.
- (3) \Rightarrow (4) Let $r \in \text{Reg}(R)$ and set $J = (I : r)$ and $K = (r)$. Then, $JK = (I : r)(r) \subseteq I$ and $K \not\subseteq Z(R)$. Then, $[(I : r)]_2 = [J]_2 \subseteq I$.
- (4) \Rightarrow (5) Set $J = (\text{Reg}(R))^{-1}I$. We have $I \subseteq J^c$. Now, let $x \in J^c$. Then, $\frac{x}{1} \in J$. Therefore, $xr \in I$ for some $r \in \text{Reg}(R)$. Hence, $x \in (I : r)$. So, $x^2 \in [(I : r)]_2 \subseteq I$. Accordingly, $[J^c]_2 \subseteq I$.
- (5) \Rightarrow (1) Let $a, b \in R$ such that $ab \in I$ and $b \in \text{Reg}(R)$. Since $I \subseteq J^c$, we have $\frac{ab}{1} \in J$. Thus, $\frac{a}{1} = \frac{ab}{1} \cdot \frac{1}{b} \in J$. So, $a \in J^c$, and then $a^2 \in [J^c]_2 \subseteq I$. Accordingly, I is a $2r$ -ideal of R . \square

Following [5, Proposition 2.2], the r -ideals of a ring R are exactly the contraction of the proper ideals of $Q(R)$. Similarly, the aforementioned theorem shows that the $2r$ -ideals are the ideals lying between J^c and $[J^c]_2$, where J ranges over all the proper ideals of $Q(R)$.

The following theorem gathers some useful facts about $2r$ -ideals. These facts will be used in the sequel without explicit mention.

Theorem 2.5. *Let R be a ring. Then, the following hold.*

1. If I is a $2r$ -ideal of R , then \sqrt{I} is an r -ideal.
2. If I is an r -ideal of R and J is an ideal of R such that $[I]_2 \subseteq J \subseteq I$, then J is a $2r$ -ideal. In particular, I^2 and $[I]_2$ are $2r$ -ideals.
3. If I is an r -ideal of R and $I \subseteq J$ is an ideal of R , then IJ is a $2r$ -ideal.
4. If P is a minimal prime ideal of R , then P is an r -ideal and so P^2 and $[P]_2$ are $2r$ -ideals.
5. If I and J are $2r$ -ideals (resp. r -ideals) of R , then $I \cap J$ is a $2r$ -ideal (resp. an r -ideal).

6. If I and J are r -ideals of R , then IJ is a $2r$ -ideals.
7. Let I and J be coprime ideals (I and J are proper and $I + J = R$). If $IJ = I \cap J$ is a $2r$ -ideal, then so are I and J .
8. Every maximal $2r$ -ideal of R is prime.
9. If I is a $2r$ -ideal of R and P is a minimal prime ideal over I , then P is an r -ideal.

Proof.

- (1) Clear.
- (2) Follows from Theorem 2.4(1) \Leftrightarrow (5) and [5, Proposition 2.2].
- (3) Follows from (2) since $[I]_2 \subseteq I^2 \subseteq IJ \subseteq I$.
- (4) Since $P \subseteq Z(R)$, it is clear that P is an r -ideal. Now, by (2), P^2 and $[P]_2$ are $2r$ -ideals.
- (5) Clear.
- (6) Let $a, b \in R$ with $ab \in IJ$ and $b \notin Z(R)$. We have $ab \in I$ and $ab \in J$. Then, $a \in I$ and $a \in J$. Therefore, $a^2 \in IJ \subseteq I \cap J$. Hence, IJ is a $2r$ -ideal.
- (7) Since I and J are coprime, then $IJ = I \cap J$. Let $x \in I$ and $y \in J$ such that $x + y = 1$. Let $a, b \in R$ with $ab \in I$ and $b \notin Z(R)$. Then, $ayb \in IJ$. Therefore, $a^2y^2 \in IJ \subseteq I$. Thus, $a^2 = a^2x^2 + 2a^2xy + a^2y^2 \in I$. Hence, I is a $2r$ -ideal. Similarly, J is a $2r$ -ideal.
- (8) Let P be a maximal $2r$ -ideal of R , and let $a, b \in R$ with $ab \in P$ and $a \notin P$. We have to show that $b \in P$. By (1), \sqrt{P} is an r -ideal, and so a $2r$ -ideal. By the maximality of P , we obtain $P = \sqrt{P}$. Thus, P is an r -ideal. It is easy to see that $(P : a)$ is proper and is an r -ideal, which contains of course P . Again, by the maximality of P , we obtain that $P = (P : a)$. Hence, $b \in (P : a) = P$.
- (9) Let $a, b \in R$ such that $ab \in P$ and $b \in \text{Reg}(R)$. By [11, Theorem 2.1], there exists $x \notin P$ and a positive integer $n \geq 1$ such that $xa^n b^n \in I$. Since $b^n \in \text{Reg}(R)$, we obtain $x^2 a^{2n} \in I \subseteq P$. Then, $a \in P$. \square

Example 2.2 shows that the $2r$ -ideals need not be r -ideals. The next theorem characterizes the reduced noetherian rings in which the notions of $2r$ -ideals and r -ideals coincide. This result can be used readily to provide examples discriminating between the two notions. For a ring R , $\text{Min}(R)$ denotes the set of minimal prime ideals of R .

Theorem 2.6. *Let R be a noetherian reduced ring. Then, every $2r$ -ideal of R is an r -ideal if and only if R is isomorphic to a finite direct product of domains.*

Proof. (\Rightarrow) Since R is reduced and noetherian, R admits a finite number of minimal prime ideals; $\text{Min}(R) = \{P_1, \dots, P_n\}$, and $Z(R) = \bigcup_{i=1}^n P_i$. If $n = 1$, then R is a domain, as desired. So, suppose that $n \geq 2$. Let $j \in \{1, \dots, n\}$ and set $Q_j = \bigcap_{i \neq j} P_i$. By [11, Theorem 2.5], there exists $y \in P_j$ such that $y \notin \bigcup_{i \neq j} P_i$. Let $x \in Q_j^*$.

We have $xy \in \bigcap_{i=1}^n P_i = (0)$. If $x + y \in P_j$, then $x \in P_j$. So, $x \in \bigcap_{i=1}^n P_i = (0)$, a contradiction. Now, if $x + y \in P_i$ for some $j \neq i \in \{1, \dots, n\}$, then $y \in P_i$, a contradiction. Hence, $x + y$ does not belong to any minimal prime ideal, and then $x + y$ is regular. Let J be an ideal of R containing Q_j . Since Q_j is an r -ideal (as an intersection of r -ideals), $Q_j J$ is a $2r$ -ideal (by Theorem 2.5(3)), and so it is an r -ideal. Since $x(x + y) = x^2 \in Q_j J$, $x \in Q_j J$. Hence, $Q_j \subseteq Q_j J$, which means that $Q_j = Q_j J$. In particular, $Q_j^2 = Q_j$. Since R is noetherian, Q_j is generated by an idempotent element e ; $Q_j = (e)$. Suppose that $P_j + Q_j$ is contained in a maximal ideal M . Hence, $Q_j = Q_j M$, and so $e = em$ for some $m \in M$. Thus, $e(1 - m) = 0$. Hence, since $e \notin P_j$ (otherwise $e = 0$), $1 - m \in P_j \subseteq M$, a contradiction. Thus, $P_j + Q_j$ is not contained in any maximal ideal, and so $P_j + Q_j = R$. Thus, $P_j + P_i = R$ for each $j \neq i \in \{1, \dots, n\}$. By using the generalized Chinese Remainder Theorem, we found that $R \cong R/P_1 \times \dots \times R/P_n$, as desired.

(\Leftarrow) Set $R = \prod_{i=1}^n D_i$ for some domains D_i . Let J be a $2r$ -ideal of R . Set $J = \prod_{i=1}^n I_i$, where I_i is an ideal of D_i for each i . Since J must be proper, there exists some ideals I_i , which are proper (in D_i). Suppose, for example, that $I_1 \neq R_1$. Let $x \in I_1$. We have $(x, 1, \dots, 1)(1, 0, \dots, 0) = (x, 0, \dots, 0) \in J$. Since $(1, 0, \dots, 0)^2 \notin J$, we obtain $(x, 1, \dots, 1) \in Z(R)$. Thus, $x \in Z(D_1) = (0)$. Hence, $I_1 = (0)$. Consequently, each I_i in J is either (0) or equal to D_i . Thus, $\sqrt{J} = J$. Hence, by Theorem 2.5 (1), J is an r -ideal, as desired. \square

Of course, a ring need not be a finite direct product of domains to have the property that every $2r$ -ideal is an r -ideal. In fact, we can consider any total quotient ring, and so in such a ring, any ideal is an r -ideal [5].

The following result shows that if every ideal is a $2r$ -ideal, then R is also a total quotient ring.

Theorem 2.7. *Let R be a ring. Then, the followings are equivalent:*

1. R is a total quotient ring.
2. Every proper ideal is an r -ideal.
3. Every (nonzero) proper principal ideal of R is a $2r$ -ideal.
4. Every (nonzero) proper ideal of R is a $2r$ -ideal.
5. Every (nonzero) strongly quasi primary ideal of R is a $2r$ -ideal.
6. Every (nonzero) primary ideal of R is a $2r$ -ideal.
7. Every (nonzero) prime ideal of R is a $2r$ -ideal.
8. Every (nonzero) maximal ideal of R is a $2r$ -ideal.

Proof. (1) \Rightarrow (2) Let I be a proper ideal of R . Let $a, b \in R$ with $ab \in I$ and $b \notin Z(R)$. Since R is a total quotient ring, b is a unit element. Hence, $a \in I$. So, I is an r -ideal.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (4) Let I be a proper ideal of R . Let $a, b \in R$ with $ab \in I$ and $b \notin Z(R)$. If $ab = 0$, then $a = a^2 = 0 \in I$. So, suppose that $ab \neq 0$. Since (ab) is a $2r$ -ideal, we obtain $a^2 \in (ab) \subseteq I$. Thus, I is a $2r$ -ideal.

The implications (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8) are clear.

(8) \Rightarrow (1) Let x be a nonunit element of R . Since (x) is a proper ideal of R , it is contained in a maximal ideal M . But M is a $2r$ -ideal, and so $M \subseteq Z(R)$. Thus, $x \in Z(R)$. Accordingly, R is a total quotient ring. \square

Recall that a proper ideal I of a ring R is said to be semi-primary if \sqrt{I} is prime. Recall also that a ring is said to be decomposable if it admits a nontrivial idempotent.

Lemma 2.8. *Let R be a ring such that every nonzero $2r$ -ideal is semi-primary. Then,*

1. either $\text{nil}(R) \neq (0)$ and $Z(R)$ are prime ideals (and so R is indecomposable),
2. or R is reduced with at most two minimal prime ideals.

Proof. Assume that R is not reduced. Let P and Q be two prime ideals contained in $Z(R)$. Since R is not reduced, $P \cap Q \neq (0)$. Since $P \cap Q$ is an r -ideal, we obtain that $P \cap Q = \sqrt{P \cap Q}$ is prime. Hence, P and Q are comparable. In particular, minimal prime ideals are comparable. Hence, $\text{nil}(R) = P$ is prime. On the other hand, $Z(R)$ is a union of some prime ideals (which are necessarily contained in $Z(R)$, and so they are comparable). Hence, $Z(R)$ is an ideal, and so prime. Suppose that R is decomposable, and let e be a nontrivial idempotent of R . Then, $e \in Z(R)$ and $1 - e \in Z(R)$. So, $1 \in Z(R)$, a contradiction.

Suppose now that R is reduced but not a domain. Let P_1 and P_2 be two different minimal prime ideals. If $P_1 \cap P_2 \neq (0)$, then, as earlier, P_1 and P_2 are comparable, a contradiction. Then, $P_1 \cap P_2 = (0)$, and so R admits exactly two minimal prime ideals. \square

Lemma 2.9. *Let R be a ring such that every $2r$ -ideal is semi-primary. Then,*

1. either $\text{nil}(R) \neq (0)$ is prime and $Z(R)$ is an ideal (and so R is indecomposable),
2. or R is a domain.

Proof. The zero ideal is a $2r$ -ideal. Hence, $\sqrt{0} = \text{nil}(R)$ is prime. Now, the result follows from Lemma 2.8. \square

When a prime ideal consists entirely of zero-divisors, then it must be an r -ideal (and so a $2r$ -ideal). The next proposition shows that the $2r$ -ideals are not necessarily all prime, except in a domain.

Proposition 2.10. *Let R be a ring. Then, the followings are equivalent:*

1. Every $2r$ -ideal of R is a (minimal) prime ideal.
2. R is a domain.
3. (0) is the only $2r$ -ideal of R .

Proof. (1) \Rightarrow (2) It follows from the fact that (0) is a $2r$ -ideal.

(2) \Rightarrow (3) Since a $2r$ -ideal must be contained in $Z(R) = (0)$, then (0) is the only $2r$ -ideal of R .

(3) \Rightarrow (1) Let $z \in R^*$. It is easy to verify that $\text{ann}(z)$ is an r -ideal, and so a $2r$ -ideal. Thus, $\text{ann}(z) = (0)$. Hence, z is a regular element, and so R is a domain. Thus, every $2r$ -ideal of R (here (0) is the only $2r$ -ideal) is a (minimal) prime. \square

The fact that (0) is an r -ideal forces the ring in Proposition 2.10 to be a domain. To avoid this, we will now focus on the noetherian rings in which only the nonzero $2r$ -ideals must be (minimal) prime.

Recall that a minimal ideal of a ring R is a nonzero ideal that does not contain any other nonzero ideal of R .

Theorem 2.11. *Let R be a noetherian ring. Then, the following are equivalent:*

1. Every nonzero $2r$ -ideal is a minimal prime ideal.
2. Every nonzero $2r$ -ideal is a prime ideal.
3. One of the following holds:
 - (a) R is a domain.
 - (b) R is isomorphic to a product of two domains.
 - (c) $Z(R)$ is a minimal ideal of R .

Proof. (1) \Rightarrow (2) Clear.

(2) \Rightarrow (3) Assume that R is not reduced. By Lemma 2.8, $\text{nil}(R) \neq (0)$ and $Z(R)$ are prime ideals and R is indecomposable. If $Z(R)^2$ is prime, then $Z(R)^2 = Z(R)$, and so $Z(R) = (e)$ is generated by an idempotent e . Hence, $Z(R) = (0)$, a contradiction. Accordingly, $Z(R)^2$ is not prime. Since $Z(R)^2$ is a $2r$ -ideal, we found that $Z(R)^2 = (0)$. Hence, $\text{nil}(R) = Z(R)$ and $Z(R)^2 = (0)$. Let I be a nonzero subideal of $Z(R)$. We have $Z(R)^2 = (0) \subseteq I \subseteq Z(R)$. Hence, by Theorem 2.5(2), I is a $2r$ -ideal, and then prime. Since $\text{nil}(R) = Z(R)$, $I = Z(R)$. Thus, $Z(R)$ is a minimal ideal.

Assume now that R is reduced. Since every nonzero $2r$ -ideal is a prime ideal contained in $Z(R)$, we conclude that every (nonzero) $2r$ -ideal is an r -ideal. Hence, from Theorem 2.6, R is a finite direct product of domains. On the other hand, by Lemma 2.8, R admits at most two minimal prime ideals. Thus, R is a domain or is isomorphic to a product of two domains.

(3) \Rightarrow (1) If R is a domain, the conclusion is obvious. Now, suppose that $R \cong D_1 \times D_2$ for some domains D_1 and D_2 . As in the proof of Theorem 2.6, the only nonzero $2r$ -ideals of $D_1 \times D_2$ are $(0) \times D_2$ and $D_1 \times (0)$, which are clearly minimal primes of $D_1 \times D_2$. Finally, suppose that $Z(R)$ is a minimal ideal of R . Thus, $Z(R)$ is a minimal prime ideal. Let I be a nonzero $2r$ -ideal of R . Then, since $I \subseteq Z(R)$, we obtain that $I = Z(R)$, and so it is a minimal prime ideal. \square

It is clear that minimal prime ideals and their squares are $2r$ -ideals. Next, we investigate when the set of $2r$ -ideals of a ring R coincides with the set $\{P, P^2 \mid P \in \text{Min}(R)\}$.

Theorem 2.12. *Let R be a ring. Then, the following are equivalent:*

1. The only $2r$ -ideals are P and P^2 with $P \in \text{Min}(R)$.
2. One of the following holds:
 - (a) R is a domain,
 - (b) $Z(R)$ is a minimal ideal.

Proof. (1) \Rightarrow (2) Suppose that R is not a domain. Since (0) is a $2r$ -ideal and (0) is not prime, $(0) = P^2$ for some minimal prime ideal P . So, $(0) \neq \text{nil}(R) = P$ is the unique minimal prime ideal of R . By Lemma 2.9, $Z(R)$ is an ideal, and so a nonzero r -ideal. Hence, $Z(R) = P$. Let I be a nonzero subideal of $Z(R)$. We have $P^2 = (0) \subseteq I \subseteq P$. Hence, I is a $2r$ -ideal, and then $I = P = Z(R)$. Thus, $Z(R)$ is a minimal ideal.

(2) \Rightarrow (1) If R is a domain, then the result is clear. So, suppose that $Z(R)$ is a minimal ideal. Then, $Z(R)$ is the unique minimal prime ideal of R (since (0) is not prime). Every nonzero $2r$ -ideal I is a subideal of $Z(R)$, and so $I = Z(R)$. Hence, it suffices to show that (0) is a square of $Z(R)$. Let $a \in Z(R)^*$. By the minimality of $Z(R)$, we obtain that $Z(R) = (a)$. Assume that $Z(R)^2 = Z(R)$. Then, $a \in Z(R)^2 = (a^2)$. Thus, $a = a^2r$ for some $r \in R$. Since $a \in Z(R) = \text{nil}(R)$, we obtain that $a = 0$, a contradiction. So, again by the minimality of $Z(R)$, $Z(R)^2 = (0)$. \square

The next three results investigate when, in noetherian rings and decomposable rings, the only nonzero $2r$ -ideals are the minimal prime ideals and their squares.

Theorem 2.13. *Let R be a decomposable ring. Then, the following are equivalent:*

1. *The only nonzero $2r$ -ideals are P and P^2 with $P \in \text{Min}(R)$.*
2. *Every nonzero $2r$ -ideal is primary.*
3. *Every nonzero $2r$ -ideal is semi-primary.*
4. *R is isomorphic to a product of two domains.*
5. *Every nonzero $2r$ -ideal is a minimal prime.*

Proof. (1) \Rightarrow (3) and (2) \Rightarrow (3) are clear.

(3) \Rightarrow (4) By Lemma 2.8, since R is decomposable, R must be reduced and it admits exactly two minimal prime ideals P_1 and P_2 (since a domain is indecomposable). Let e be a nontrivial idempotent of R . Either $e \in P_1$ or $e \in P_2$. Suppose, for example, that $e \in P_1$. Then, $1 - e \in P_2$. Hence, $P_1 + P_2 = R$, and so $R \cong R/P_1 \times R/P_2$ is a product of two domains.

(4) \Rightarrow (5) Without the loss of generality, set $R = D_1 \times D_2$ for some domains D_1 and D_2 . As in the proof of Theorem 2.6, the only nonzero $2r$ -ideals of R are $(0) \times D_2$ and $D_1 \times (0)$, which are clearly minimal prime ideals of R .

(5) \Rightarrow (1) & (2) Clear. \square

Theorem 2.14. *Let R be a reduced noetherian ring. Then, the following are equivalent:*

1. *The only nonzero $2r$ -ideals are P and P^2 with $P \in \text{Min}(R)$.*
2. *R admits two minimal prime ideals P_1 and P_2 , and for each $i \in \{1, 2\}$, there is no ideal strictly between P_i and $[P_i]_2$.*

Proof. (1) \Rightarrow (2) By the hypothesis, if P is a minimal prime ideal, then it must be a nonzero $2r$ -ideal. Thus, R cannot be a domain. Since R is reduced, by Lemma 2.8, R admits exactly two minimal prime ideals P_1 and P_2 . Let $i \in \{1, 2\}$ and consider an ideal I such that $[P_i]_2 \subseteq I \subseteq P_i$. Then, I is a $2r$ -ideal. Clearly, $I \neq (0)$. Otherwise $[P_i]_2 = (0)$ and so, since R is reduced, $P_i = (0)$, a contradiction. Hence, since I is a nonzero $2r$ -ideal, we obtain that $I = P_i$ or $I = P_i^2$. In particular, $[P_i]_2 = P_i$ or $[P_i]_2 = P_i^2$. In the first case, we obtain that $P_i = P_i^2 = [P_i]_2$ (since $[P_i]_2 \subseteq P_i^2 \subseteq P_i$). Hence, we have always $P_i^2 = [P_i]_2$. Then, $I = P_i$ or $I = [P_i]_2$, as desired.

(2) \Rightarrow (1) Since R is reduced, $Z(R) = P_1 \cup P_2$. Let I be a nonzero $2r$ -ideal of R . Then, $I \subseteq Z(R) = P_1 \cup P_2$. Suppose that there exists $x \in I \setminus P_1$ and $y \in I \setminus P_2$. Then, $x \in P_2$ and $y \in P_1$. Moreover, $x + y \in I \subseteq Z(R)$. Thus, $x + y \in P_1$ or $x + y \in P_2$. So, $x \in P_1$ or $y \in P_2$, a contradiction. Hence, either $I \subseteq P_1$ or $I \subseteq P_2$. Suppose, for example, that $I \subseteq P_1$. Let $x \in I^*$ and $p_2 \in P_2^*$. If $x + p_2 \in P_1$, then $p_2 \in P_1 \cap P_2 = (0)$, a contradiction. Thus, $x + p_2 \notin P_1$. Similarly, $x + p_2 \notin P_2$. Thus, $r = x + p_2 \notin Z(R)$. Let p_1 be an arbitrary element of P_1 . We have, $p_1r = p_1(x + p_2) = p_1x \in I$. Since I is a $2r$ -ideal, we obtain $p_1^2 \in I$. Thus, $[P_1]_2 \subseteq I$. But there is no ideal strictly between P_1 and $[P_1]_2$. Hence, $I = P_1$ or $I = [P_1]_2$. On the other hand, $[P_1]_2 \subseteq P_1^2 \subseteq P_1$. Thus, $P_1^2 = P_1$ or $P_1^2 = [P_1]_2$. If $P_1^2 = P_1$, then, since R is noetherian, $P_1 = (e)$ for some idempotent e . Hence, $[P_1]_2 = P_1^2$. So, we have always $P_1^2 = [P_1]_2$. Consequently, $I = P_1$ or $I = P_1^2$, as desired. \square

Theorem 2.15. *Let R be a nonreduced noetherian ring. Then, the following are equivalent:*

1. *The only nonzero $2r$ -ideals are P and P^2 with $P \in \text{Min}(R)$.*
2. *One of the following holds:*
 - (a) *$Z(R)$ is a minimal ideal of R .*
 - (b) *$\text{nil}(R) = Z(R)$, $Z(R)^2$ is a minimal ideal, and there is no ideal strictly between $Z(R)$ and $Z(R)^2$.*

Proof. (1) \Rightarrow (2) By Lemma 2.8, $P = \text{nil}(R) \neq (0)$ is the unique minimal prime ideal and $Z(R)$ is an ideal. Since $Z(R)$ is an r -ideal, $Z(R) = P$ or $Z(R) = P^2$. But $P = \text{nil}(R) \subseteq Z(R)$. Thus, $Z(R) = \text{nil}(R) = P$. If $P^2 = (0)$, then every nonzero subideal of P is a $2r$ -ideal (by Theorem 2.5(2)), and so equal to P . Thus, $P = Z(R)$ is a minimal ideal. Now, assume that $P^2 \neq (0)$. Suppose that P^2 is not an r -ideal. Hence, there exists $x \notin P^2$ and $r \notin Z(R) = P$ such that $xr \in P^2$. Clearly, $x \in P$. Now, we have $P^2 \subseteq P^2 + (x) \subseteq P$. Thus, $P^2 + (x)$ is a $2r$ -ideal, and so $P^2 + (x) = P$ (since $x \notin P^2$). Thus, for each $p \in P$, $p = xa + b$ for some $a \in R$ and $b \in P^2$. Hence, $pr = xra + br \in P^2$. So, $rP \subseteq P^2$. Since R is noetherian, there exists an integer $n \geq 3$ such that $P^n = (0)$. Let n be the smallest one with this property. We have $rP^{n-1} \subseteq P^n = (0)$. So, $P^{n-1} = (0)$ (since r is regular), a contradiction. Consequently, P^2 is a nonzero r -ideal of R . Then, by Theorem 2.5(3), P^3 is a $2r$ -ideal. Thus, $P^3 = (0)$ or $P^3 = P$ or $P^3 = P^2$. If $P^3 = P$ (resp. $P^3 = P^2$), then, using the Nakayama's Lemma, $P = (0)$ (resp. $P^2 = (0)$), a contradiction. Then, $P^3 = (0)$. Let $0 \neq I \subseteq P^2$. Since P^2 is an r -ideal of R and $(0) = P^4 = (P^2)^2 \subseteq I \subseteq P^2$, I is a $2r$ -ideal. So, $I = P$ or $I = P^2$. If $I = P$ then $P = P^2$, and so $P = (0)$, a contradiction. Thus, $I = P^2$, and so P^2 is a minimal ideal. Moreover, each ideal between P and P^2 is a $2r$ -ideal, and then equal to P or P^2 . Thus, there are no ideals strictly between P and P^2 .

(2) \Rightarrow (1) It is clear that, in the both cases, $P = \text{nil}(R) = Z(R)$ must be the unique minimal prime ideal of R . Suppose that $Z(R)$ is a minimal ideal and let I be a nonzero $2r$ -ideal of R . Then, $(0) \neq I \subseteq Z(R)$, and then, $I = Z(R) = P$ is a minimal prime ideal.

Suppose now that (b) holds. Then, since $P^2 = \text{nil}(R)^2$ is minimal, $P^3 = P^2$ or $P^3 = (0)$. Since R is noetherian, the first case leads to $P^2 = (0)$, a contradiction. Then, $P^3 = (0)$. Let $(0) \neq I$ be a $2r$ -ideal of R . Hence, $I \subseteq P$. If $I \subseteq P^2$, then $I = P^2$. So, suppose that $I \not\subseteq P^2$. Then, $P^2 \subsetneq P^2 + I \subseteq P$. Hence, $P^2 + I = P$. Thus, $P^2 = P^3 + IP = IP \subseteq I$. Hence, $P^2 \subsetneq I \subseteq P$, and so $I = P$. \square

It is easy to verify that every primary ideal consisting of zero divisors is an r -ideal, and so a $2r$ -ideal. In the rest of this section, we are interested to rings in which every (nonzero) $2r$ -ideal of R is a primary.

Recall from [12] that a ring R is said to be a UN -ring if every nonunit element a of R is a product of a unit and nilpotent elements. Following [6, Proposition 2.25], R is a UN -ring if and only if every element of R is either nilpotent or unit if and only if $\text{nil}(R)$ is a maximal ideal of R .

Theorem 2.16. *Let R be a ring. Then, the followings are equivalent:*

1. *Every $2r$ -ideal of R is a primary ideal.*
2. *R is a domain or R is a UN -ring.*

Proof. (1) \Rightarrow (2) Since (0) is a $2r$ -ideal, then by hypothesis (0) is a primary ideal of R , and so $\text{nil}(R) = Z(R)$. Suppose that R is neither a UN -ring nor a domain. Let M be a maximal ideal of R , then there is $x \in M \setminus \text{nil}(R)$. Consider $0 \neq y \in \text{nil}(R)$, and let n be the smallest integer such that $y^n = 0$. For each $a, b \in R$ such that $ab \in (xy^{n-1})$ and $b \notin Z(R)$, we have $a^2b^2 = 0$. Thus, $a^2 = 0 \in (xy^{n-1})$ since $b^2 \notin \text{nil}(R)$. Thus, (xy^{n-1}) is a $2r$ -ideal, and so it is a primary ideal with $\sqrt{(xy^{n-1})} = \text{nil}(R)$. Since $xy^{n-1} \in (xy^{n-1})$ and $x \notin \text{nil}(R)$, then $y^{n-1} \in (xy^{n-1})$. Then, $y^{n-1}(1 - rx) = 0$ for some $r \in R$. Thus, $1 - rx \in Z(R) = \text{nil}(R) \subseteq M$, a contradiction. So, R must be either domain or a UN -ring.

(2) \Rightarrow (1) If R is a domain, then (0) is the only $2r$ -ideal of R , which is primary. If R is a UN -ring. Then, every proper ideal of R is primary, in particular every $2r$ -ideal of R is primary. \square

Theorem 2.17. *Let R be a noetherian ring. Then, the following are equivalent:*

1. *Every nonzero $2r$ -ideal of R is primary.*
2. *One of the following holds:*
 - (a) *R is a domain.*
 - (b) *$R \cong D_1 \times D_2$, where D_1 and D_2 are two domains.*
 - (c) *R is a UN-ring.*
 - (d) *$\text{nil}(R)$ is prime and minimal, $Z(R)$ is a maximal ideal, and there are no prime ideals strictly between $\text{nil}(R)$ and $Z(R)$.*

Proof. (\Rightarrow) Suppose that R is reduced. Let I be a nonzero $2r$ -ideal. Then, $I \subseteq Z(R)$, and so $\sqrt{I} \subseteq Z(R)$. Let $a, b \in R$ such that $ab \in I$ and $b \notin Z(R)$. Then, $b \notin \sqrt{I}$. Since I is primary, we obtain that $a \in I$. Hence, I is an r -ideal. Thus, every (nonzero) $2r$ -ideal is an r -ideal. By using Theorem 2.6, we conclude that R is isomorphic to a finite product of domains. by Lemma 2.8, R admits at most two minimal prime ideals. Thus, R is a domain or is isomorphic to a product of two domains.

Suppose that R is not reduced. By Lemma 2.8, $P = \text{nil}(R) \neq (0)$ is a prime ideal, $Z(R)$ is an ideal, and R is indecomposable. If $\text{nil}(R) = Z(R)$, then (0) is primary. Hence, by Theorem 2.16, R is a UN-ring.

Now, suppose that $P = \text{nil}(R) \neq Z(R)$. Let $Q \subsetneq Z(R)$ be a prime ideal of R . We claim that $Q^2 = (0)$. Suppose that $Q^2 \neq (0)$. Then, Q^2 is primary (since it is a $2r$ -ideal). If $Q^2 = Q$, then $Q = (e)$ for some idempotent e . Since R is indecomposable, $Q = (0)$. Thus, $P \subseteq Q = (0)$, a contradiction. Hence, $Q^2 \neq Q$. Let $z \in Z(R) \setminus Q$ and $x \in Q \setminus Q^2$. Since $Q^2 + (xz)$ is a $2r$ -ideal, $Q^2 + (xz)$ is primary. Moreover, $xz \in Q^2 + (xz)$ and $z \notin Q = \sqrt{Q^2 + (xz)}$. Thus, $x \in Q^2 + (xz)$. Hence, $x(1 - zr) \in Q^2$ for some $r \in R$. But $1 - zr \notin \sqrt{Q^2 + (xz)} = Q \subseteq Z(R)$, otherwise $1 \in Z(R)$. Then, $x \in Q^2$, a contradiction. Consequently, $Q^2 = (0)$. Thus, $P = Q$ and $P^2 = (0)$. So, $P^2 = (0)$, and there is no ideal strictly between P and $Z(R)$. Let $P^2 = (0) \neq I \subseteq P$. Then, I is a $2r$ -ideal, and so primary. Let $x \in P$ and $z \in Z(R) \setminus P$. We have that $J = I + (xz) \subseteq P$, and it is also primary. We have $xz \in I + (xz)$, and $z \notin P = \sqrt{I + (xz)}$. Thus, $x \in I + (xz)$, and so $x(1 - zr) \in I$ for some $r \in R$. Thus, since $1 - zr \notin P \subseteq Z(R)$, we obtain that $x \in I$. Thus, $I = P$. So, P is a minimal ideal of R .

Consider a maximal ideal M such that $Z(R) \subseteq M$. We have $MP = (0)$ or $MP = P$ (Since P is minimal). On the other hand, by minimality of P , we obtain that it is principal. So set $P = xR$. If $MP = P$, then $xM = (x)$, and so $x = xm$ for some $m \in M$. Hence, $x(1 - m) = 0$. Thus, $1 - m \in Z(R) \subseteq M$, a contradiction. Hence, $MP = 0$, and so $M \subseteq Z(R)$. Thus, $Z(R) = M$ is maximal.

(\Leftarrow) If (a) or (b) or (c) holds the result follows from Theorems 2.11 and 2.16. Now, suppose that (d) holds. Let I be a nonzero $2r$ -ideal of R . Then, $I \subseteq Z(R)$. Since there are no prime ideals strictly between $\text{nil}(R)$ and $Z(R)$, the only possible minimal prime ideals over I are $\text{nil}(R)$ and $Z(R)$. Hence, $\sqrt{I} = \text{nil}(R)$ or $\sqrt{I} = Z(R)$. In the first case, $I \subseteq \text{nil}(R)$, and so $I = \text{nil}(R)$, which is prime. In the second case, $\sqrt{I} = Z(R)$, I is clearly primary since $Z(R)$ is maximal. \square

Corollary 2.18. *Let (R, M) be a local noetherian ring. Then, the followings are equivalent:*

1. *Every nonzero $2r$ -ideal of R is primary.*
2. *One of the following holds:*
 - (a) *R is a domain.*
 - (b) *R is a UN-ring.*
 - (c) *$\text{Spec}(R) = \{\text{nil}(R), M\}$ such that $\text{nil}(R)$ is a minimal ideal of R .*

Proof. (1) \Rightarrow (2) Follows from Theorem 2.17.

(2) \Rightarrow (1) It suffices to prove that if (c) holds then every nonzero $2r$ -ideals of R is primary. To do so, it suffices by Theorem 2.17, to prove that $Z(R) = M$. Since $P = \text{nil}(R)$ is minimal, $PM = P$ or $PM = (0)$. Since R is noetherian, $PM = P$ leads to $P = (0)$, a contradiction. Thus, $PM = (0)$, and so $M \subseteq Z(R)$. Hence, $Z(R) = M$, as desired. \square

3 $2r$ -Ideals in polynomial rings

We begin this section with following easy fact.

Proposition 3.1. *Let R be a ring and I be a proper ideal of R . If $I[X]$ is a $2r$ -ideal of $R[X]$, then I is a $2r$ -ideal of R .*

Proof. Let $a, b \in R$ such that $ab \in I$ and $a^2 \notin I$. Then, $ab \in I[X]$ and $a^2 \notin I[X]$. Thus, since $I[X]$ is a $2r$ -ideal of $R[X]$, $b \in Z(R[X])$. So, $b \in Z(R)$. \square

Remark 3.2. Note that the converse implication in Proposition 3.1 is not always true as we will see in Theorem 3.4.

Let R be a ring and let $f \in R[x]$ be a polynomial in one variable over R . The content of f , denoted by $c(f)$, is the ideal of R generated by the coefficients of f . The content of a polynomial, $c(f)$, satisfies a number of multiplicative properties. For example, the Dedekind-Mertens Lemma (see, for example, [13, Theorem 1]) asserts that for every two polynomials f and g in $R[x]$:

$$c(f)c(g)^{k+1} = c(g)^k c(fg), \quad \text{where } k = \deg(f).$$

The next result characterizes when $I[X]$ is a $2r$ -ideal of $R[X]$.

Theorem 3.3. *Let R be a ring and I be a proper ideal of R . Then, the following are equivalent.*

1. $I[X]$ is a $2r$ -ideal of $R[X]$.
2. For each ideal A of R and each finitely generated ideal B of R , $AB \subseteq I$ implies that $[A]_2 \subseteq I$ or $\text{ann}_R(B) \neq (0)$.
3. For each finitely generated ideals A and B of R , $AB \subseteq I$ implies that $[A]_2 \subseteq I$ or $\text{ann}_R(B) \neq (0)$.

Proof. (1) \Rightarrow (2) Let A be an ideal of R and B a finitely generated ideal of R with $AB \subseteq I$ and $[A]_2 \not\subseteq I$. There exists $a \in A$ such that $a^2 \notin I$. Set $B = \sum_{i=0}^n Rb_i$ and $g = \sum_{i=0}^n b_i X^i$. Hence, $B = c(g)$. We have $aB \subseteq I$. Then, $ag \in I[X]$. Since $a^2 \notin I[X]$, we obtain that $\text{ann}_{R[X]}(g) \neq (0)$. By using McCoy's Theorem [14], $dg = 0$ for some $d \in R^*$. This makes $dB = 0$, and hence, $\text{ann}_R(B) \neq (0)$, as desired.

(2) \Rightarrow (3) Clear.

(3) \Rightarrow (1) Let $f, g \in R[X]$ with $fg \in I[X]$ and $g \notin Z(R[X])$. We claim that $f^2 \in I[X]$. Assume that $f \neq 0$. Following [13, Theorem 1], we have $c(f)c(g)^{n+1} = c(g)^n c(fg)$, where $n = \deg(f)$. Hence, $c(f)c(g)^{n+1} \subseteq I$ since $c(fg) \subseteq I$. Moreover, $\text{ann}_{R[X]}(g) = (0)$ implies that $\text{ann}_{R[X]}(g^{n+1}) = (0)$. Thus, $\text{ann}_R(c(g^{n+1})) = (0)$. Since $c(g^{n+1}) \subseteq c(g)^{n+1}$, it follows that $\text{ann}_R(c(g)^{n+1}) \subseteq \text{ann}_R(c(g^{n+1}))$ and hence $\text{ann}_R(c(g)^{n+1}) = (0)$. Since $c(f)$ and $c(g)^{n+1}$ are finitely generated ideals of R , we obtain $[c(f)]_2 \subseteq I$. Set $f = \sum_{i=0}^n a_i X^i$. We have:

$$f^2 = \sum_{k=0}^n (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} 2a_i a_j X^{i+j} = \sum_{k=0}^n (a_k)^2 X^{2k} + \sum_{0 \leq i < j \leq n} [(a_i + a_j)^2 - a_i^2 - a_j^2] X^{i+j}.$$

Thus, $f^2 \in [c(f)]_2[X] \subseteq I[X]$. Consequently, $I[X]$ is a $2r$ -ideal of $R[X]$. \square

Recall that a ring R is said to be an A -ring if each finitely generated ideal contained in $Z(R)$ has a non-zero annihilator. It is well known that noetherian rings, zero-dimensional rings, and polynomial rings are A -rings. Moreover, a ring R is an A -rings if and only if so is $Q(R)$ (see [11,15,16]).

Next, we characterize the A -rings in terms of $2r$ -ideals.

Theorem 3.4. Let R be a ring. Then, the followings are equivalent:

1. R is an A -ring.
2. For each ideal $2r$ -ideal I of R , $I[X]$ is a $2r$ -ideal of $R[X]$.
3. For each proper ideal J of $Q(R)$, $J[X]$ is a $2r$ -ideal of $Q(R)[X]$.

Proof. (1) \Rightarrow (2) Let I be a $2r$ -ideal of R . Let A an ideal of R and B a finitely generated ideal of R with $AB \subseteq I$ and $[A]_2 \not\subseteq I$. By Theorem 2.4, $B \subseteq Z(R)$. Since R is an A -ring, we obtain that $\text{ann}_R(B) \neq (0)$. Accordingly, by Theorem 3.3, $I[X]$ is a $2r$ -ideal of $R[X]$.

(2) \Rightarrow (1) Following [17, Proposition 3.5 (c)], to prove that R is an A -ring, we have to show that, for $f \in R[X]$, $c(f) \subseteq Z(R)$ implies that $f \in Z(R[X])$. Set $I = c(f)$. The ideal $J = S^{-1}I$ of $Q(R)$ is proper since $c(f) \subseteq Z(R)$. Hence, J^c is a proper ideal of R , and so an r -ideal of R (by [5, Proposition 2.2]). Moreover, $I \subseteq J^c$. By hypothesis, $J^c[X]$ is a $2r$ -ideal, and so $f \in I[X] \subseteq J^c[X] \subseteq Z(R[X])$.

(1) \Leftrightarrow (3) Theorem 2.7 states that every proper ideal of $Q(R)$ is a $2r$ -ideal. Moreover, R is an A -ring if and only if so is $Q(R)$. Now, applying the equivalence (1) \Leftrightarrow (2) for $Q(R)$, we obtain that $Q(R)$ is an A -ring if and only if for each proper ideal J of $Q(R)$, $J[X]$ is a $2r$ -ideal of $Q(R)[X]$. \square

We now turn to characterize the $2r$ -ideals of the quotient rings.

Proposition 3.5. *Let $I \subseteq J$ be two ideal of a ring R . The followings are equivalent:*

1. J/I is a $2r$ -ideal of R/I .
2. For each $a, b \in R$ with $ab \in J$, we have $a^2 \in J$ or $(I : b) \neq I$.

In particular, if I is an r -ideal of R and J/I is a $2r$ -ideal of R/I , then J is a $2r$ -ideal of R .

Proof. (1) \Rightarrow (2) Let $a, b \in R$ with $ab \in J$ and $a^2 \notin J$. Then, $\bar{a}^2 \notin J/I$. Hence, $\bar{b} \in Z(R/I)$. Let $x \notin I$ such that $\bar{b}x = \bar{0}$. We have $bx \in I$, and so $x \in (I : b)$. Hence, since $x \notin I$, we obtain that $(I : b) \neq I$.

(2) \Rightarrow (1) Let $a, b \in R$ such that $\bar{a}\bar{b} \in J/I$ and $\bar{a}^2 \notin J/I$. Then, $ab \in J$, and $a^2 \notin J$. Thus, $(I : b) \neq I$. Since $I \subseteq (I : b)$, there exists $x \in (I : b)$ such that $x \notin I$. Hence, $\bar{b}x = \bar{0}$ and $\bar{x} \neq \bar{0}$. So, $\bar{b} \in Z(R/I)$. Thus, J/I is a $2r$ -ideal of R/I .

Now, suppose that I is an r -ideal. Following [5, Proposition 2.2], if $b \in R$ satisfies $(I : b) \neq I$, then $b \in I$ or $b \in Z(R)$. In both cases, $b \in Z(R)$ since $I \subseteq Z(R)$. Hence, the desired result follows from (1) \Rightarrow (2). \square

The next example shows that even if I is an r -ideal of R and J is a $2r$ -ideal of R , then J/I needs not to be a $2r$ -ideal of R/I .

Example 3.6. Consider the ring $A = \frac{k[x, y]}{(x^2, xy)}$ with k is a field. The ideal $P = \frac{(x, y)}{(x^2, xy)}$ is a prime ideal of A . So, set $R = A_P$. Then, R is a local ring with maximal ideal $M = (\bar{x}/\bar{1}, \bar{y}/\bar{1}) = Z(R)$. Suppose Q is a prime ideal of A_P that is not a maximal. Then, there exists a prime ideal q of A that is contained strictly in P such that $Q = qA_P$. Moreover, we can write q as $q = q_0/(x^2, xy)$, where q_0 is a prime ideal of $k[x, y]$ that satisfies $(x^2, xy) \subseteq q_0 \subseteq (x, y)$. Since $k[x, y]$ is of krull dimension 2 and $(x^2, xy) \subseteq q_0 \subseteq (x, y)$ is a chain of prime ideals, we conclude that $q_0 = (x)$. Therefore, $Q = (\bar{x}/\bar{1})$. So, R has exactly one nonmaximal prime ideal, which is $\text{nil}(R) = (\bar{x}/\bar{1})$. Then, both $\text{nil}(R)$ and M are r -ideals of R . However, $M/\text{nil}(R) \neq (0)$ is not a $2r$ -ideal of the domain R/P (by Proposition 2.10).

Proposition 3.7. *Let $f : A \rightarrow B$ be a ring homomorphism. The following are equivalent:*

1. $f(\text{Reg}(A)) \subseteq \text{Reg}(B)$.
2. For each $2r$ -ideal J of B , $f^{-1}(J)$ is a $2r$ -ideal of A .

Proof. (1) \Rightarrow (2) Let J be a $2r$ -ideal of B . Let $a, b \in A$ such that $ab \in f^{-1}(J)$ and $a \in \text{Reg}(A)$. Then $f(a)f(b) \in J$ and $f(a) \in \text{Reg}(B)$, and hence, $f(b)^2 \in J$. Therefore, $b^2 \in f^{-1}(J)$, and thus, $f^{-1}(J)$ is a $2r$ -ideal.

(2) \Rightarrow (1) Let $x \in \text{Reg}(A)$. Suppose that $f(x) \in Z(B)$. Then, $f(x)y = 0$ for some $y \in B^*$. If $J = \text{ann}_B(y)$, then J is an r -ideal (and hence $2r$ -ideal) of B . Therefore, $f^{-1}(J)$ is a $2r$ -ideal of A , and thus, $x \in f^{-1}(J) \subseteq Z(A)$, a contradiction. Hence, $f(\text{Reg}(A)) \subseteq \text{Reg}(B)$. \square

Corollary 3.8.

- Let $R \subseteq T$ be rings such that R is essential in T (i.e., $R \cap I \neq (0)$ for every nonzero ideal I of T). If I is a $2r$ -ideal of T , then $I \cap R$ is a $2r$ -ideal of R .
- Let S be a multiplicatively closed set of a ring R . If J is a $2r$ -ideal of $S^{-1}R$, then J^c is a $2r$ -ideal of R .

Proof. (1) Let $x \in \text{Reg}(R)$. Suppose that $x \in Z(T)$. Then, $\text{ann}_T(x) \neq (0)$. Hence, since R is essential in T , $R \cap \text{ann}_T(x) \neq (0)$, and so there exists $r \in R^*$ such that $rx = 0$, which means that $x \in Z(R)$, a contradiction. Consequently, $\text{Reg}(R) \subseteq \text{Reg}(T)$. Now, the result follows from Proposition 3.9.

(2) Let $f: R \rightarrow S^{-1}R$ be the natural homomorphism defined by $f(x) = \frac{x}{1}$. It is clear that $f(\text{Reg}(R)) \subseteq \text{Reg}(S^{-1}R)$. Hence, the result follows from Proposition 3.9. \square

Proposition 3.9. *Let $f: A \rightarrow B$ be a surjective ring homomorphism such that $f(Z(A)) \subseteq Z(B)$ and $\ker(f) \subseteq Z(A)$. If I is a $2r$ -ideal of A containing $\ker(f)$, then $f(I)$ is a $2r$ -ideal of B .*

Proof. Let I be a $2r$ -ideal of A containing $\ker(f)$ and let $f(a)f(b) \in f(I)$ for some $a, b \in A$. Since $\ker(f) \subseteq I$, it follows that $ab \in I$, and hence, $a \in Z(A)$ or $b^2 \in I$. Therefore, $f(a) \in Z(B)$ (as $f(Z(A)) \subseteq Z(B)$) or $f(b)^2 \in f(I)$. \square

Proposition 3.10. *Let S be a multiplicatively closed set of R such that $S \cap Z(R) = \emptyset$. If I is a $2r$ -ideal of R , then $S^{-1}I$ is a $2r$ -ideal of $S^{-1}R$.*

Proof. Let $\frac{a}{s}, \frac{b}{t} \in S^{-1}R$ such that $\frac{a}{s} \frac{b}{t} \in S^{-1}I$ and $\frac{b}{t} \notin Z(S^{-1}R)$. Then, $uab \in I$ for some $u \in S$. Let $x \in R$ such that $bx = 0$. Hence, $\frac{b}{t} \cdot \frac{x}{1} = \frac{0}{1}$. So, $\frac{x}{1} = \frac{0}{1}$. That is, $xs = 0$ for some $s \in S$. But $S \subseteq \text{Reg}(R)$, and so $x = 0$. Hence, $b \in \text{Reg}(R)$, and so $ub \in \text{Reg}(R)$. Hence, $a^2 \in I$, and then $\left(\frac{a}{s}\right)^2 \in S^{-1}I$. Consequently, $S^{-1}I$ is a $2r$ -ideal of $S^{-1}R$. \square

Proposition 3.11. *Let I_1 and I_2 be ideals of rings R_1 and R_2 , respectively. Then, the followings are equivalent:*

1. $I_1 \times I_2$ is a $2r$ -ideal of $R_1 \times R_2$.
2. “ I_1 is a $2r$ -ideal of R_1 and $I_2 = R_2$ ” or “ I_2 is a $2r$ -ideal of R_2 and $I_1 = R_1$ ” or “ I_1 and I_2 are $2r$ -ideals of R_1 and R_2 , respectively.”

Proof. (1) \Rightarrow (2) Suppose that $J = I_1 \times I_2$ is a $2r$ -ideal of $R_1 \times R_2$. Since J must be proper ideal of $R_1 \times R_2$, then $I_1 \neq R_1$ or $I_2 \neq R_2$. Suppose that $I_1 \neq R_1$. Let $a, b \in R_1$ such that $ab \in I_1$ and $b \notin Z(R_1)$. Then, $(a, 0)(b, 1) \in I_1 \times I_2$ and $(b, 1) \notin Z(R_1 \times R_2)$. So, $(a, 0)^2 = (a^2, 0) \in I_1 \times I_2$. Thus, $a^2 \in I_1$, and then I_1 is a $2r$ -ideal of R_1 . Similarly, if I_2 is proper, then I_2 is $2r$ -ideal of R_2 . Thus, $J = I_1 \times R_2$ or $J = R_1 \times I_2$ or $J = I_1 \times I_2$ for some $2r$ -ideals I_1 and I_2 of R_1 and R_2 , respectively.

(2) \Rightarrow (1) Suppose that I_1 and I_2 are $2r$ -ideals of R_1 and R_2 , respectively. Let $(a, b)(a', b') \in I_1 \times I_2$ and $(a', b') \notin Z(R_1 \times R_2)$. Then, $aa' \in I_1, bb' \in I_2, a' \notin Z(R_1)$ and $b' \notin Z(R_2)$. Hence, $a^2 \in I_1$ and $b^2 \in I_2$ since I_1 and I_2 are a $2r$ -ideals of R_1 and R_2 , respectively. Thus, $(a, b)^2 = (a^2, b^2) \in I_1 \times I_2$, and so $I_1 \times I_2$ is a $2r$ -ideal of $R_1 \times R_2$. The other cases can be proved similarly. \square

Let R be a ring and M an R -module. The trivial ring extension of R by M is the ring $R := R \ltimes M$, where the underlying group is $R \times M$, and the multiplication is defined by $(a, m)(b, m') = (ab, am' + bm)$. It is also called the (Nagata) idealization of M over R and is denoted by $R(+M)$. This construction was first introduced, in 1962, by Nagata [18] with the objective to emphasize the interaction between rings and their modules and, more importantly, to provide numerous families of examples of rings with zero-divisors.

Proposition 3.12. *Let R be a ring, I an ideal of R , and M an R -module. Then the following are equivalent:*

1. $I \ltimes M$ is a $2r$ -ideal of $R \ltimes M$.
2. For each $a, b \in R$ with $ab \in I$, we have $a^2 \in I$ or $b \in Z(R) \cup Z(M)$.

In particular, If I is a $2r$ -ideal of R , then $I \ltimes M$ is a $2r$ -ideal of $R \ltimes M$, and the converse holds when $Z(M) \subseteq Z(R)$.

Proof. Recall, from [19, Theorem 3.5], that $(x, m) \in Z(R \ltimes M)$ if and only if $x \in Z(R) \cup Z(M)$.

(1) \Rightarrow (2) Let $a, b \in R$ with $ab \in I$ and $a^2 \notin I$. Then, $(a, 0)(b, 0) = (ab, 0) \in I \times M$ and $(a, 0)^2 = (a^2, 0) \notin I \times M$. Therefore, $(b, 0) \in Z(R \times M)$, that is, $b \in Z(R) \cup Z(M)$.

(2) \Rightarrow (1) Let $(a, m), (b, m') \in R \times M$ such that $(a, m)(b, m') = (ab, am' + bm) \in I \times M$ and $(b, m') \notin Z(R \times M)$. Then, $ab \in I$ and $b \notin Z(R) \cup Z(M)$. Hence, $a^2 \in I$. Thus, $(a, m)^2 = (a^2, 2am) \in I \times M$, and so $I \times M$ is a $2r$ -ideal of $R \times M$. \square

The next example shows that the converse of the last assertion in the previous proposition is not valid if one drop the hypothesis that $Z(M) \subseteq Z(R)$.

Example 3.13. Set $R := \mathbb{Z}$ and $M := \mathbb{Z}/p\mathbb{Z}$, where p is a positive prime integer. By [19, Theorems 3.2 & 3.5], $p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \subseteq Z(\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z})$ is a maximal ideal of $\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, and so an r -ideal. However, $p\mathbb{Z}$ is not a $2r$ -ideal of the domain \mathbb{Z} .

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