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Research Article

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Multidimensional sampling-Kantorovich operators in *BV*-spaces

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Abstract: The main purpose of this article is to prove a result of convergence in variation for a family of multidimensional sampling-Kantorovich operators in the case of averaged-type kernels. The setting in which we work is that one of BV-spaces in the sense of Tonelli.

Keywords: sampling-Kantorovich operators, convergence in variation, multidimensional variation, absolute continuity, averaged-type kernels

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1 Introduction

In recent years, the study of the approximation properties of Kantorovich-type operators has been a challenging topic, and a wide literature has been devoted to the subject (see, e.g., [1–8]). As it is well known, the pioneering idea goes back to Kantorovich [9], who introduced the operators $(BK_n f)(x) := (n+1)\left(\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(u)du\right)p_{k,n}(x)$, $x \in [0,1]$, $n \in \mathbb{N}$, a modified version of the classical Bernstein polynomials $(B_n f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)p_{k,n}(x)$, $p_{k,n}(x) := \binom{n}{k}x^k(1-x)^{n-k}$, replacing the values of the function over the points $\left(\frac{k}{n}\right)$ by means of an integral mean. Such operators, linked to the Bernstein polynomials by the relation $(B_n f)'(x) = (BK_{n-1}f')(x)$ in case of an absolutely continuous function f, allowed to obtain, in the L^p-spaces, the analog of the classical Weierstrass approximation theorem in $C([0,1])$: indeed Lorentz [10] proved that $\|BK_n f - f\|_{L^p([0,1])} \to 0$, as $n \to +\infty$.

The idea to replace the values of the function by means of an integral mean was applied to a wide range of operators, including the following generalized sampling operators:

$$(S_w f)(x) = \sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\chi(wx-k), \quad x \in \mathbb{R},\ w > 0,$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded function and χ is a kernel (see Section 2). Such operators, introduced with the aim to provide a generalized version of the classical sampling theorem, have been widely studied in last 40 years, together with other related operators (see, e.g., [11–21]), also in view of their deep and natural connections to applicative problems of signal and image reconstruction. Their Kantorovich version was introduced in [22], where the authors obtain convergence results in the general setting of Orlicz spaces, and later on, the approximation properties of such operators were deeply investigated in several function

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spaces, as well as in multidimensional settings [23–26]. We point out that the multidimensional version of such operators

$$(K_w f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N}\left[w^N \int_{\frac{\mathtt{k}}{w}}^{\frac{\mathtt{k}+1}{w}} f(\mathtt{u})d\mathtt{u}\right]\chi(w\mathtt{t}-\mathtt{k}), \quad \mathtt{t}\in\mathbb{R}^N,\ w>0,$$

introduced in [24], proved to be very useful in order to solve some applicative problems of digital image reconstruction and processing (see, e.g., [27,28]). In this direction, among the several function spaces, the setting of the spaces of functions of bounded variation is of particular interest. Indeed, in [29], the problem of the estimate in variation for the operators $\{K_w f\}_{w>0}$ is studied and an applicative interpretation of the variation diminishing-type estimates is given. Besides estimates in variation, it is natural to face the problem of the convergence in variation, that is the natural notion of convergence in BV-spaces. Results about convergence in variation have been obtained in the one-dimensional case (see [30] for $\{BK_n\}_{n\in\mathbb{N}}$ and [31] for $\{K_w\}_{w>0}$), but the multidimensional case, much more delicate but nevertheless crucial for applications to digital images, is still an open problem.

In this article, we address this issue and obtain a result of convergence in variation (Theorem 3) in the general case of averaged-type kernels (see Section 2) for multidimensional sampling-Kantorovich operators $\{K_w f\}_{w>0}$.

We will work in the frame of BV-spaces in the sense of Tonelli. As it is well known, several generalizations of the Jordan variation to the multidimensional case have been proposed in the literature, including the distributional variation, the Vitali variation, the Cesari generalized variation, the Ascoli-Arzelà variation, and others. We refer to the monograph by Appell et al. [32] for an exhaustive presentation of the different notions of variation, also in the multidimensional frame. We choose to work with the variation introduced by Tonelli [33] for functions of two variables and later extended to the case of N-variables by Radó [34] and Vinti [35], since this concept seems to be very suitable in order to obtain approximation results for families of integral and discrete operators (see, e.g., [25,36–41]). Moreover, the natural geometrical aspects connected to the definition and construction of the Tonelli variation allow us to discuss the previously mentioned applicative issues about digital images [18,29].

In order to reach our goal, we will use an indirect approach. In particular, starting from a natural relation between the sampling-Kantorovich operators and the generalized sampling series applied to a singular integral (see (2) of Section 2), we will prove the main theorem (Theorem 3) using a convergence result for the singular integrals $\{I_{\psi_w} f\}_{w>0}$ (Theorem 2), together with an estimate in variation and a convergence result for the generalized sampling operators [18,19]. As it is natural, we have to work within a suitable subspace of $BV(\mathbb{R}^N)$ (see Section 2).

The article is organized as follows: after a preliminary section where the main notations and definitions are presented (Section 2), the main results are proved in Section 3 and examples of kernels to which the results can be applied are presented in Section 4.

2 Preliminaries

We now recall the definition of the space $BV(\mathbb{R}^N)$ in the sense of Tonelli [33–35].

For $\mathtt{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $f : \mathbb{R}^N \to \mathbb{R}$ and $I = \prod_{i=1}^N [a_i, b_i] \subset \mathbb{R}^N$, we will use the following notations:

- $\mathtt{x}'_j = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \in \mathbb{R}^{N-1}$, $\mathtt{x} = (\mathtt{x}'_j, x_j)$ and $f(\mathtt{x}) = f(\mathtt{x}'_j, x_j)$, $j = 1, \ldots, N$;
- $I'_j = [\mathtt{a}'_j, \mathtt{b}'_j] = \prod_{i\neq j}[a_i, b_i]$ and $I = [\mathtt{a}, \mathtt{b}] = [\mathtt{a}'_j, \mathtt{b}'_j] \times [a_j, b_j]$, $\quad j = 1, \ldots, N$;
- $\alpha\mathtt{x} = (\alpha x_1, \ldots, \alpha x_N)$, for $\alpha \in \mathbb{R}$, and, for $\alpha \neq 0$, $\frac{\mathtt{x}}{\alpha} = \left(\frac{x_1}{\alpha}, \ldots, \frac{x_N}{\alpha}\right)$.

By $V[f] := \sup_{[a,b]\subset\mathbb{R}} V_{[a,b]}[f]$, we denote the Jordan variation of f over $\mathbb{R}$, where $V_{[a,b]}[f] = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$, the supremum being taken over all the possible partitions $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$, is the Jordan variation of f on $[a, b]$.

For $I = \prod_{i=1}^{N}[a_i, b_i] \subset \mathbb{R}^N$ and $j = 1, \ldots, N$, we define the so-called Tonelli integrals, namely the $(N-1)$-dimensional integrals, as follows:

$$\Phi_j(f, I) := \int_{\mathtt{a}'_j}^{\mathtt{b}'_j} V_{[a_j,b_j]}[f(\mathtt{x}'_j, \cdot)]\, d\mathtt{x}'_j,$$

where $V_{[a_j,b_j]}[f(\mathtt{x}'_j, \cdot)]$ is the one-dimensional Jordan variation of the jth section of f, and their Euclidean norm $\Phi(f, I) := \left\{\sum_{j=1}^{N} \Phi_j^2(f, I)\right\}^{\frac{1}{2}}$. As usual, $\Phi(f, I) = +\infty$ if $\Phi_j(f, I) = +\infty$ for some $j = 1, \ldots, N$.

The variation of f on $I \subset \mathbb{R}^N$ is defined as follows:

$$V_I[f] := \sup \sum_{k=1}^{m} \Phi(f, J_k),$$

where the supremum is taken over all the finite families of N-dimensional intervals $\{J_1, \ldots, J_m\}$, which form partitions of I. Moreover,

$$V[f] := \sup_{I\subset\mathbb{R}^N} V_I[f],$$

where the supremum is taken over all the intervals $I \subset \mathbb{R}^N$, is the Tonelli variation of f on $\mathbb{R}^N$.

We will also use the notation

$$V^j[f](\mathtt{x}'_j) := V[f(\mathtt{x}'_j, \cdot)], \quad \mathtt{x}'_j \in \mathbb{R}^{N-1},$$

so that $V^j[f] : \mathbb{R}^{N-1} \to \mathbb{R}$, $j = 1, \ldots, N$, where $f(\mathtt{x}'_j, \cdot)$ are the jth sections of f.

Definition 1. A measurable and bounded function $f : \mathbb{R}^N \to \mathbb{R}$ ($f \in M(\mathbb{R}^N)$) is said to be *of bounded variation* on $\mathbb{R}^N$ ($f \in BV(\mathbb{R}^N)$) if $V[f] < +\infty$.

It is obvious that, for every $f \in BV(\mathbb{R}^N)$, ∇f exists a.e. in $\mathbb{R}^N$ and $\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^N)$, for every $j = 1, \ldots, N$.

We now recall the definition of local absolute continuity on $\mathbb{R}^N$.

Definition 2. A function $f : \mathbb{R}^N \to \mathbb{R}$ is locally absolutely continuous in the sense of Tonelli ($f \in AC_{\text{loc}}(\mathbb{R}^N)$) if, for every interval $I = \prod_{i=1}^{N}[a_i, b_i]$ and for every $j = 1, 2, \ldots, N$, the jth section of f, $f(\mathtt{x}'_j, \cdot) : [a_j, b_j] \to \mathbb{R}$, is absolutely continuous, for every $\mathtt{x}'_j \in [\mathtt{a}'_j, \mathtt{b}'_j]$.

We will denote by $AC(\mathbb{R}^N) := BV(\mathbb{R}^N) \cap AC_{\text{loc}}(\mathbb{R}^N)$ the space of the absolutely continuous functions on $\mathbb{R}^N$. We recall that, for every $f \in AC(\mathbb{R}^N)$, $V[f] = \int_{\mathbb{R}^N} |\nabla f(\mathtt{x})|\, d\mathtt{x}$ [34,35].

We will now introduce the family of sampling-Kantorovich operators, namely

$$(K_w f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w^N \int_{\frac{\mathtt{k}}{w}}^{\frac{\mathtt{k}+1}{w}} f(\mathtt{u})\, d\mathtt{u} \right] \chi(w\mathtt{t} - \mathtt{k}), \quad \mathtt{t} \in \mathbb{R}^N, \ w > 0,$$

where $f : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a bounded function and $\chi : \mathbb{R}^N \to \mathbb{R}$ is a *kernel*.

The operators $\{K_w f\}_{w>0}$ have been introduced in [24] as the Kantorovich version of the multivariate generalized sampling series

$$(S_w f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} f\left(\frac{\mathtt{k}}{w}\right) \chi(w\mathtt{t} - \mathtt{k}), \quad \mathtt{t} \in \mathbb{R}^N, \ w > 0.$$

In [29], the following estimate in variation was obtained for the sampling-Kantorovich operators $\{\bar{K}_w^m f\}_{w>0}$, with kernels $\bar{\chi}_m$ of averaged type (see definition (1) below), proving that such operators map $BV(\mathbb{R}^N)$ into itself.

Theorem 1. ([29], Theorem 1) *For every $f \in BV(\mathbb{R}^N)$, $m \in \mathbb{N}$, $w > 0$,*

$$V[\bar{K}_w^m f] \leq N \frac{m+1}{m} \prod_{i=1}^{N} \|\chi_i\|_1 V[f],$$

and hence $\bar{K}_w^m f \in BV(\mathbb{R}^N)$. Moreover, $\bar{K}_w^m f \in AC(\mathbb{R}^N)$.

In this article, we will go a step further and prove a result of convergence in variation for $\{\bar{K}_w^m f\}_{w>0}$. Obviously, the setting will be the same, namely we will consider sampling-Kantorovich operators with kernels $\bar{\chi}_m$ of averaged type, denoted by

$$(\bar{K}_w^m f)(\mathtt{t}) := \sum_{\mathtt{k}\in\mathbb{Z}^N} \left[w^N \int_{\frac{\mathtt{k}}{w}}^{\frac{\mathtt{k}+1}{w}} f(\mathtt{u})d\mathtt{u} \right] \bar{\chi}_m(w\mathtt{t} - \mathtt{k}), \quad \mathtt{t} \in \mathbb{R}^N, \; w > 0. \tag{1}$$

We will similarly denote by $\bar{S}_w^m f$ the multivariate generalized sampling series of f with averaged kernel $\bar{\chi}_m$.

Saying that the kernels are of averaged-type means that

$$\bar{\chi}_m(\mathtt{t}) := \prod_{i=1}^{N} \bar{\chi}_{i,m}(t_i),$$

where

$$\bar{\chi}_{i,m}(t) := \frac{1}{m} \int_{-\frac{m}{2}}^{\frac{m}{2}} \chi_i(t+v)dv, \quad t \in \mathbb{R},$$

for some $m \in \mathbb{N}$, and $\chi_i : \mathbb{R} \longrightarrow \mathbb{R}$ is a one-dimensional kernel for every $i = 1, \ldots, N$, i.e., it satisfies the following conditions:

(χ_1) $\chi_i \in L^1(\mathbb{R})$ is such that $\sum_{\mathrm{k}\in\mathbb{Z}} \chi_i(\mathrm{u} - \mathrm{k}) = 1$ for every $\mathrm{u} \in \mathbb{R}$;

(χ_2) $A_{\chi_i} := \sup_{\mathrm{u}\in\mathbb{R}} \sum_{\mathrm{k}\in\mathbb{Z}} |\chi_i(\mathrm{u} - \mathrm{k})| < +\infty$, where the convergence of the series is uniform on the compact sets of $\mathbb{R}$.

One can immediately verify that $\bar{\chi}_{i,m}$ is a kernel itself and that

$$\|\bar{\chi}_{i,m}\|_{L^1(\mathbb{R})} \leq \|\chi_i\|_{L^1(\mathbb{R})}$$

for every $i = 1, \ldots, N$.

Moreover, $\bar{\chi}_m$ turns out to be a multidimensional kernel, namely it satisfies the multidimensional versions of (χ_1) and (χ_2), that is,

(χ_1^N) $\bar{\chi}_m \in L^1(\mathbb{R}^N)$ is such that $\sum_{\mathtt{k}\in\mathbb{Z}^N} \bar{\chi}_m(\mathtt{u} - \mathtt{k}) = 1$ for every $\mathtt{u} \in \mathbb{R}^N$;

(χ_2^N) $A_{\bar{\chi}_m} := \sup_{\mathtt{u}\in\mathbb{R}^N} \sum_{\mathtt{k}\in\mathbb{Z}^N} |\bar{\chi}_m(\mathtt{u} - \mathtt{k})| < +\infty$, where the convergence of the series is uniform on the compact sets of $\mathbb{R}^N$.

We refer to Section 4 for examples of kernels that fulfill all the above assumptions.

It is immediate to see that, if χ is a multidimensional kernel, then both the family of operators $\{K_w f\}_{w>0}$ and $\{S_w f\}_{w>0}$ are well-defined, for instance, for every bounded function, and therefore, for every $f \in BV(\mathbb{R}^N)$: indeed, if $|f(\mathtt{t})| \leq C$ for some $C > 0$, by (χ_2), $|(K_w f)(\mathtt{t})| \leq C \sum_{\mathtt{k}\in\mathbb{Z}^N} |\chi(w\mathtt{t} - \mathtt{k})| \leq C A_\chi$ for every $\mathtt{t} \in \mathbb{R}^N$.

In order to prove the main result about the convergence in variation for $\{\bar{K}_w^m f\}_{w>0}$, we will use an indirect approach. In particular, it is well known that, in the one-dimensional case, it is possible to write the sampling-Kantorovich operators as the generalized sampling series of a singular integral (see, e.g., [31]). Here, we will make use of the analogous identity in the multidimensional case. In particular, let us denote by

$$I_{\varphi_w} f(\mathtt{t}) := (f * \varphi_w)(\mathtt{t}) = \int_{\mathbb{R}^N} f(\mathtt{u})\varphi_w(\mathtt{t}-\mathtt{u})d\mathtt{u},$$

$\mathtt{t} \in \mathbb{R}^N$, $w > 0$, the singular integral of f with kernel $\{\varphi_w\}_{w>0}$. The family φ_w is, as usual, an approximate identity, namely it satisfies the following assumptions:

(A1) $\varphi_w \in L^1(\mathbb{R}^N)$, $\|\varphi_w\|_{L^1(\mathbb{R}^N)} \leq A$, for some constant $A > 0$ and $\int_{\mathbb{R}^N} \varphi_w(\mathtt{u})d\mathtt{u} = 1$, for every $w > 0$;

(A2) for every fixed $\delta > 0$, $\lim_{w\to+\infty} \int_{|\mathtt{u}|>\delta} |\varphi_w(\mathtt{u})| d\mathtt{u} = 0$.

Let us now consider $\psi_w(\mathtt{t}) := w^N \chi_{[-1,0]^N}(w\mathtt{t})$, where $\chi_{[-1,0]^N}(\mathtt{t}) = \begin{cases} 1, & \mathtt{t} \in [-1,0]^N, \\ 0, & \text{otherwise}, \end{cases}$ is the characteristic function of $[-1,0]^N$. Then, it is immediate to see that $\{\psi_w\}_{w>0}$ is an approximate identity (with $A = 1$) and

$$(K_w f)(\mathtt{t}) = \left(S_w\left(I_{\psi_w} f\right)\right)(\mathtt{t}), \tag{2}$$

for every $w > 0$, $\mathtt{t} \in \mathbb{R}^N$.

Using such relation, we will prove the main convergence result by means of an estimate in variation [19], a convergence result for the generalized sampling operators [18] and for the singular integrals $\{I_{\psi_w} f\}_{w>0}$ (Theorem 2 of Section 3). In order to establish it, we have to introduce a subspace of $L^1(\mathbb{R}^N)$ (see [16]) and, of course, some notations.

We recall that an *admissible partition* over the i-th axis is a partition $\Sigma_i := (x_{i,j_i})_{j_i \in \mathbb{Z}}$ such that

$$0 < \underline{\Delta} := \min_{i=1,\ldots,N} \inf_{j_i \in \mathbb{Z}} \left(x_{i,j_i} - x_{i,j_i-1}\right) \leq \max_{i=1,\ldots,N} \sup_{j_i \in \mathbb{Z}} \left(x_{i,j_i} - x_{i,j_i-1}\right) =: \overline{\Delta} < +\infty.$$

A sequence $\Sigma = (\mathtt{x}_\mathtt{j})_{\mathtt{j}\in\mathbb{Z}^N} \subset \mathbb{R}^N$, $\mathtt{x}_\mathtt{j} = (x_{1,j_1}, \ldots, x_{N,j_N})$, and $\mathtt{j} = (j_1, \ldots, j_N) \in \mathbb{Z}^N$, is said to be an *admissible sequence* if it is the cartesian product of admissible partitions $\Sigma_i = (x_{i,j_i})_{j_i\in\mathbb{Z}}$. For a fixed admissible sequence Σ, the $l^p(\Sigma)$-norm of $f : \mathbb{R}^N \to \mathbb{R}$ is defined as follows:

$$\|f\|_{l^p(\Sigma)} := \left\{ \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} |f(\mathtt{x})|^p \Delta_\mathtt{j} \right\}^{\frac{1}{p}}, \quad 1 \leq p < +\infty,$$

where $Q_\mathtt{j} = \prod_{i=1}^N [x_{i,j_i-1}, x_{i,j_i}[$ and $\Delta_\mathtt{j} := \prod_{i=1}^N (x_{i,j_i} - x_{i,j_i-1})$ denotes the volume of $Q_\mathtt{j}$.

The sampling grid, that is, the cartesian product of $\left(\frac{k_i}{w}\right)_{k_i\in\mathbb{Z}}$, $i = 1, \ldots, N$, is indeed an admissible sequence: for its importance, it will be denoted by Σ_w^N. Similarly, by Σ_w^{N-1}, we will denote the cartesian product of $\left(\frac{k_i}{w}\right)_{k_i\in\mathbb{Z}}$, $i \neq j$, that is, the sampling grid on $\mathbb{R}^{N-1}$, excluding the jth coordinate.

Then, the subspace $\Lambda^p(\mathbb{R}^N)$, $p \geq 1$, is defined as follows:

$$\Lambda^p(\mathbb{R}^N) := \left\{ f \in M(\mathbb{R}^N) : \|f\|_{l^p(\Sigma)} < +\infty, \quad \text{for every admissible sequence } \Sigma \right\}.$$

We recall that $\Lambda^p(\mathbb{R}^N)$ is a proper linear subspace of $L^p(\mathbb{R}^N)$ that contains, among others, all the measurable functions with compact support: this one and other properties of $\Lambda^p(\mathbb{R}^N)$ can be found in [16] and [42].

We will also use the following notation introduced in [19]: by $BV_\Lambda(\mathbb{R}^N)$, we denote the space of functions $f \in M(\mathbb{R}^N)$ such that the jth sections $f(\mathtt{x}'_j, \cdot)$ are of bounded variation on $\mathbb{R}$ for a.e. in $\mathtt{x}'_j \in \mathbb{R}^{N-1}$ and $V^j[f] \in \Lambda^1(\mathbb{R}^{N-1})$, for every $j = 1, \ldots, N$.

Of course, $BV_\Lambda(\mathbb{R}^N)$ is a subspace of $BV(\mathbb{R}^N)$ and, for example, it contains all the functions of bounded variation with compact support.

Finally, R_{loc} will denote the space of all the locally Riemann integrable functions $f : \mathbb{R}^N \to \mathbb{R}$.

3 Convergence in $BV(\mathbb{R}^N)$ by means of sampling-Kantorovich operators

In this section, we will prove the main result, that is, the convergence in variation by means of sampling-Kantorovich operators. In order to do this, we will provide an estimate (Proposition 1) and a convergence result (Theorem 2) for the singular integrals, which will be an intermediate step to reach the main result.

Results about convergence in L^p by means of singular integrals are well known (see, e.g., [43]). We will now study the operators $\{I_{\psi_w}\}_w$ in the subspace $\Lambda^p(\mathbb{R}^N)$. First, we will state an estimate in $BV_\Lambda(\mathbb{R}^N)$ for the singular integrals

$$\left(I_{\psi_w}g\right)(\mathtt{x}) = \int_{\mathbb{R}^N} \psi_w(\mathtt{t})g(\mathtt{x}-\mathtt{t})d\mathtt{t}, \quad \mathtt{x} \in \mathbb{R}^N.$$

Proposition 1. *If $g \in BV_\Lambda(\mathbb{R}^N)$, then $I_{\psi_w}g \in BV_\Lambda(\mathbb{R}^N)$ for every $w > 0$, and for every admissible sequence Σ^{N-1} (in $\mathbb{R}^{N-1}$) with lower mesh $\underline{\Delta} \geq \frac{1}{w}$, there is a constant $C > 0$ such that*

$$\|V^j\left[I_{\psi_w}g\right]\|_{\ell^1(\Sigma^{N-1})} \leq C\|V^j[g]\|_{\ell^1(\Sigma^{N-1})}. \tag{3}$$

Proof. Since $g \in BV_\Lambda(\mathbb{R}^N)$, in particular, $g(\mathtt{x}'_j, \cdot)$ is of bounded variation on $\mathbb{R}$ for a.e. $\mathtt{x}'_j \in \mathbb{R}^{N-1}$. Now, let $\Sigma^{N-1} = (\bar{\mathtt{x}}_{\mathtt{k}'_j})_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}}$ be an admissible partition (in $\mathbb{R}^{N-1}$), with associated intervals $Q_{\mathtt{k}'_j}$ of volume $\Delta_{\mathtt{k}'_j}$, and let $D = \{s_j^0 < s_j^0 < \ldots < s_j^\mu\}$ be an increasing sequence in $\mathbb{R}$. Then we have

$$\begin{aligned}
\|V^j\left[I_{\psi_w}g\right]\|_{\ell^1(\Sigma^{N-1})} &= \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \sup_D \sum_{\lambda=1}^{\mu} |I_{\psi_w}g(\mathtt{x}'_j, s_j^\lambda) - I_{\psi_w}g(\mathtt{x}'_j, s_j^{\lambda-1})|\Delta_{\mathtt{k}'_j} \\
&= \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \sup_D \sum_{\lambda=1}^{\mu} \left| \int_{\mathbb{R}^N} \psi_w(\mathtt{t})[g(\mathtt{x}'_j - \mathtt{t}'_j, s_j^\lambda - t_j) - g(\mathtt{x}'_j - \mathtt{t}'_j, s_j^{\lambda-1} - t_j)]d\mathtt{t} \right| \Delta_{\mathtt{k}'_j} \\
&\leq \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \int_{\mathbb{R}^N} |\psi_w(\mathtt{t})| V[g(\mathtt{x}'_j - \mathtt{t}'_j, \cdot)]d\mathtt{t}\Delta_{\mathtt{k}'_j} \\
&\leq \int_{\mathbb{R}^N} |\psi_w(\mathtt{t})| \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} V[g(\mathtt{x}'_j - \mathtt{t}'_j, \cdot)]\Delta_{\mathtt{k}'_j}d\mathtt{t} \\
&\leq \int_{\mathbb{R}^N} |\psi_w(\mathtt{t})| \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{u}'_j \in \bar{Q}_{\mathtt{k}'_j}} V[g(\mathtt{u}'_j, \cdot)]\Delta_{\mathtt{k}'_j}d\mathtt{t},
\end{aligned}$$

$w > 0$, $j = 1, \ldots, N$, where $\bar{Q}_{\mathtt{k}'_j} = \prod_{i=1}^{N-1}\left[\bar{x}_{i,j_i-1}, \bar{x}_{i,j_i} + \frac{1}{w}\right[$. Indeed, it is sufficient to note that $\psi_w(\mathtt{t}) = 0$ if $t_i \notin \left[-\frac{1}{w}, 0\right]$, for some $i = 1, \ldots, N$. Now, we have that $\bar{Q}_{\mathtt{k}'_j} = \prod_{i=1}^{N-1}\left([\bar{x}_{i,j_i-1}, \bar{x}_{i,j_i}[\cup \left[\bar{x}_{i,j_i}, \bar{x}_{i,j_i} + \frac{1}{w}\right[\right) \subseteq \bigcup_{n_1=0,1} \cdots \bigcup_{n_{N-1}=0,1} Q_{\mathtt{k}'_j+\mathtt{n}}$, with $\mathtt{n} = (n_1, \ldots, n_{N-1})$. Therefore,

$$\begin{aligned}
\|V^j\left[I_{\psi_w}g\right]\|_{\ell^1(\Sigma^{N-1})} &\leq \int_{\mathbb{R}^N} |\psi_w(\mathtt{t})| \sum_{n_1=0,1} \cdots \sum_{n_{N-1}=0,1} \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{u}'_j \in Q_{\mathtt{k}'_j+\mathtt{n}}} V[g(\mathtt{u}'_j, \cdot)]\Delta_{\mathtt{k}'_j}d\mathtt{t} \\
&= 2^{N-1} \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{u}'_j \in Q_{\mathtt{k}'_j}} V[g(\mathtt{u}'_j, \cdot)]\Delta_{\mathtt{k}'_j} = 2^{N-1}\|V^j[g]\|_{\ell^1(\Sigma^{N-1})} < +\infty,
\end{aligned}$$

since $V^j[g] \in \Lambda^1(\mathbb{R}^{N-1})$. This means, taking into account Lemma 3 of [16], that $V^j[I_{\psi_w}g] \in \Lambda^1(\mathbb{R}^{N-1})$, for every $j = 1, \ldots, N$, and thus $I_{\psi_w}g \in BV_\Lambda$. □

We will now prove a result of convergence in $\Lambda^p(\mathbb{R}^N)$ for the singular integrals $I_{\psi_w}g$, where $\psi_w(\mathtt{t}) = w^N\chi_{[-1,0]^N}(w\mathtt{t})$, $\mathtt{t} \in \mathbb{R}^N$, $w > 0$, which will be used in the proof of the main convergence result.

Theorem 2. *If* $g \in \Lambda^p(\mathbb{R}^N) \cap R_{\text{loc}}$, *then*

$$\lim_{w\to+\infty} \|I_{\psi_w} g - g\|_{\ell^p(\Sigma_w)} = 0$$

for every admissible sequence Σ_w *with upper mesh size* $\overline{\Delta} = \frac{1}{w}$.

Proof. Using assumption $(A1)$ and Jensen's inequality (recalling that $|u|^p$, $u \in \mathbb{R}$, $p \geq 1$, is a convex function and that $A = 1$ for ψ_w) we can write

$$\begin{aligned}|\left(I_{\psi_w} g\right)(\mathtt{x}) - g(\mathtt{x})|^p &= \left| \int_{\mathbb{R}^N} \psi_w(\mathtt{t}) g(\mathtt{x} - \mathtt{t}) d\mathtt{t} - g(\mathtt{x})\right|^p = \left| \int_{\mathbb{R}^N} \psi_w(\mathtt{t})[g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})] d\mathtt{t}\right|^p \\ &\leq \int_{\mathbb{R}^N} |\psi_w(\mathtt{t})||g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p d\mathtt{t} \\ &= \left\{ \int_{|\mathtt{t}|\leq\delta} |\psi_w(\mathtt{t})||g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p d\mathtt{t} + \int_{|\mathtt{t}|>\delta} |\psi_w(\mathtt{t})||g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p d\mathtt{t} \right\}\end{aligned}$$

for every $\delta > 0$, $\mathtt{x} \in \mathbb{R}^N$. Therefore,

$$\begin{aligned}\|I_{\psi_w} g - g\|^p_{\ell^p(\Sigma_w)} &= \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} |\left(I_{\psi_w} g\right)(\mathtt{x}) - g(\mathtt{x})|^p \Delta_\mathtt{j} \\ &\leq \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} \int_{|\mathtt{t}|\leq\delta} |\psi_w(\mathtt{t})||g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p d\mathtt{t}\Delta_\mathtt{j} \\ &\quad + \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} \int_{|\mathtt{t}|>\delta} |\psi_w(\mathtt{t})||g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p d\mathtt{t}\Delta_\mathtt{j} := (S_1 + S_2).\end{aligned}$$

We now estimate S_1. There holds

$$S_1 \leq \int_{|\mathtt{t}|\leq\delta} |\psi_w(\mathtt{t})| \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} |g(\mathtt{x} - \mathtt{t}) - g(\mathtt{x})|^p \Delta_\mathtt{j} d\mathtt{t} \leq \int_{|\mathtt{t}|\leq\delta} |\psi_w(\mathtt{t})| \sum_{\mathtt{j}\in\mathbb{Z}^N} \sup_{\mathtt{x}\in Q_\mathtt{j}} \omega_1(g, \mathtt{x}, 2\delta)\Delta_\mathtt{j} d\mathtt{t},$$

where $\omega_1(g, \mathtt{x}, \delta) := \sup\{|g(\mathtt{t} + \mathtt{h}) - g(\mathtt{t})| : \mathtt{t}, \mathtt{t} + \mathtt{h} \in \prod_{i=1}^N [x_i - \delta/2, x_i + \delta/2]\}$ is the modulus of smoothness of g. Denoting $\tau_1(g, \delta)_p := \|\omega_1(g, \cdot, \delta)\|_p$, by Proposition 10 of [16] (see also Proposition 22 of [42]), there exists $C > 0$ (independent by the admissible sequence) such that

$$S_1 \leq \|\omega_1(g, \cdot, 2\delta)\|^p_{\ell^p(\Sigma_w)} \int_{|\mathtt{t}|\leq\delta} |\psi_w(\mathtt{t})| d\mathtt{t} \leq \|\omega_1(g, \cdot, 2\delta)\|^p_{\ell^p(\Sigma_w)} \leq \left(C\tau_1\left(g, \delta + \frac{1}{w}\right)_p\right)^p.$$

Now, let us fix $\varepsilon > 0$, then Proposition 7 of [16], there exists $\bar{w} > 0$ such that, for every $w \geq \bar{w}$, $\tau_1(g, \frac{2}{w})_p < \varepsilon$. Therefore, if we consider $\delta = \frac{1}{\bar{w}}$,

$$S_1 \leq \left(C\tau_1\left(g, \frac{2}{\bar{w}}\right)_p\right)^p < (C\varepsilon)^p$$

for every $w \geq \bar{w}$. About S_2, it is sufficient to note that, for every $w \geq \sqrt{N}\bar{w}$, $\psi_w(\mathtt{t}) = 0$, if $|\mathtt{t}| > \delta = \frac{1}{\bar{w}}$ (indeed, $|\mathtt{t}| > \delta$ implies that there exists $j = 1, \ldots, N$ such that $|t_j| > \frac{\delta}{\sqrt{N}} = \frac{1}{\sqrt{N}\bar{w}}$ and so $w|t_j| \geq \sqrt{N}\bar{w}|t_j| > 1$). Therefore, $S_2 = 0$, for $w \geq \sqrt{N}\bar{w}$, and so we conclude that

$$\|I_{\psi_w} g - g\|_{\ell^p(\Sigma_w)} \leq C\varepsilon$$

for sufficiently large $w > 0$. □

We are now ready to prove the main convergence result.

Theorem 3. *Let $f \in AC(\mathbb{R}^N)$ be such that $\frac{\partial f}{\partial x_j} \in \Lambda^1(\mathbb{R}^N) \cap R_{\text{loc}}$, for every $j = 1, \dots, N$, and let $\bar{\chi}^m$ be an averaged kernel with compact support. Then,*

$$\lim_{w \to +\infty} V[\bar{K}_w^m f - f] = 0.$$

Proof. We first note that, with the above assumptions, $f \in BV_\Lambda(\mathbb{R}^N)$. Indeed, for every admissible sequence Σ^{N-1} in $\mathbb{R}^{N-1}$ (with associated intervals $Q_{\mathtt{k}'_j}$ of volume $\Delta_{\mathtt{k}'_j}$),

$$\|V^j[f]\|_{\ell^1(\Sigma^{N-1})} = \left\| \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_j}(\cdot, x_j) \right| dx_j \right\|_{\ell^1(\Sigma^{N-1})}$$

since $f \in AC(\mathbb{R}^N)$, for every $j = 1, \dots, N$. Now, let us consider the admissible sequence $\Sigma = (k_j)_{k_j \in \mathbb{Z}}$ in $\mathbb{R}$ associated to the intervals $Q_{k_j} := [k_j, k_j + 1[$, $k_j \in \mathbb{Z}$, of measure $\Delta_{k_j} = 1$. Obviously, the sequence $\Sigma^N = \Sigma^{N-1} \times \Sigma$, obtained as the cartesian product of Σ^{N-1} for the $N - 1$ components (other than j) and Σ for the j-th one, is an admissible sequence in $\mathbb{R}^N$. Moreover there holds

$$\begin{aligned}
\|V^j[f]\|_{\ell^1(\Sigma^{N-1})} &= \left\| \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_j}(\cdot, x_j) \right| dx_j \right\|_{\ell^1(\Sigma^{N-1})} \\
&= \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \int_{\mathbb{R}} \left| \frac{\partial f}{\partial x_i}(\mathtt{x}'_j, x_j) \right| dx_j \Delta_{\mathtt{k}'_j} \\
&= \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \sum_{k_j \in \mathbb{Z}} \int_{k_j}^{k_j+1} \left| \frac{\partial f}{\partial x_j}(\mathtt{x}'_j, x_j) \right| dx_j \Delta_{\mathtt{k}'_j} \\
&\leq \sum_{\mathtt{k}'_j \in \mathbb{Z}^{N-1}} \sup_{\mathtt{x}'_j \in Q_{\mathtt{k}'_j}} \sum_{k_j \in \mathbb{Z}} \sup_{u_{k_j} \in Q_{k_j}} \left| \frac{\partial f}{\partial x_j}\left(\mathtt{x}'_j, u_{k_j}\right) \right| \Delta_{k_j} \Delta_{\mathtt{k}'_i} \\
&\leq \left\| \frac{\partial f}{\partial x_j} \right\|_{\ell^1(\Sigma^N)} < +\infty,
\end{aligned} \tag{4}$$

since $\frac{\partial f}{\partial x_j} \in \Lambda^1(\mathbb{R}^N)$, by assumption.

Moreover, by Theorem 1, $\bar{K}_w^m f \in BV(\mathbb{R}^N)$.

Now we can write, for every $w > 0$, $m \in \mathbb{N}$,

$$V[\bar{K}_w^m f - f] \leq V[\bar{K}_w^m f - \bar{S}_w^m f] + V[\bar{S}_w^m f - f].$$

By (2), we have that $(\bar{K}_w^m f)(\mathtt{t}) = (\bar{S}_w^m (I_{\psi_w} f))(\mathtt{t})$, $\mathtt{t} \in \mathbb{R}^N$, and so, taking into account that, by Proposition 1, $I_{\psi_w} f \in BV_\Lambda(\mathbb{R}^N)$, by Theorem 3.1 of [19], we have the estimate

$$V[\bar{K}_w^m f - \bar{S}_w^m f] = V\left[\bar{S}_w^m \left(I_{\psi_w} f - f\right)\right] \leq \prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})} \sum_{j=1}^N \|V^j\left[I_{\psi_w} f - f\right]\|_{\ell^1(\Sigma_w^{N-1})},$$

where we recall that the admissible sequence Σ_w^{N-1} is the cartesian product of $\left(\frac{k_i}{w}\right)_{k_i \in \mathbb{Z}}$, $i \neq j$, namely the sampling grid.

Now, $I_{\psi_w} f \in AC(\mathbb{R}^N)$ because $f \in AC(\mathbb{R}^N)$ (see, e.g., Proposition 5 of [37] in the particular case of linear operators), therefore, similarly to (4), there holds

$$\begin{aligned}\|V^j\left[I_{\psi_w} f - f\right]\|_{\ell^1(\Sigma_w^{N-1})} &= \left\|\int_{\mathbb{R}}\left|\frac{\partial}{\partial x_j}(I_{\psi_w} f - f)(\cdot, x_j)\right| dx_j\right\|_{\ell^1(\Sigma_w^{N-1})}\\ &\leq \left\|\sum_{k_j\in\mathbb{Z}}\sup_{x_j\in\left[\frac{k_j}{w},\frac{k_j+1}{w}\right[}\left|\frac{\partial}{\partial x_j}(I_{\psi_w} f - f)(\cdot, x_j)\right|\frac{1}{w}\right\|_{\ell^1(\Sigma_w^{N-1})}\\ &\leq \left\|\frac{\partial}{\partial x_j}(I_{\psi_w} f - f)\right\|_{\ell^1(\Sigma_w^N)} = \left\|I_{\psi_w}\left(\frac{\partial f}{\partial x_j}\right) - \frac{\partial f}{\partial x_j}\right\|_{\ell^1(\Sigma_w^N)}.\end{aligned}$$

By assumption, $\frac{\partial f}{\partial x_j} \in \Lambda^1(\mathbb{R}^N) \cap R_{\text{loc}}$, for every $j = 1, \dots, N$, and therefore, by Theorem 2, there exists $\bar{w}_1 > 0$ such that, for every $w \geq \bar{w}_1$,

$$\left\|I_{\psi_w}\left(\frac{\partial f}{\partial x_j}\right) - \frac{\partial f}{\partial x_j}\right\|_{\ell^1(\Sigma_w^N)} < \frac{\varepsilon}{2N\prod_{i=1}^N \|\chi_i\|_{L^1(\mathbb{R})}}.$$

This implies that

$$V[\bar{K}_w^m f - \bar{S}_w^m f] < \frac{\varepsilon}{2}. \tag{5}$$

Finally, by Theorem 1 of [18], there exists $\bar{w}_2 > 0$ such that, for every $w \geq \bar{w}_2$,

$$V[\bar{S}_w^m f - f] < \frac{\varepsilon}{2}$$

which, together with (5), implies that

$$V[\bar{K}_w^m f - f] < \varepsilon$$

for every $w \geq \max\{\bar{w}_1, \bar{w}_2\}$. □

4 Examples of kernels

We will now give examples of families of product kernels of averaged type. First of all, we can consider the averaged kernels of Fejér-type, namely $\mathcal{F}_m(\mathtt{t}) := \prod_{i=1}^N \bar{F}_m(t_i)$, $\mathtt{t} \in \mathbb{R}^N$, where

$$\bar{F}_m(t) := \frac{1}{2m}\int_{-m/2}^{m/2} \operatorname{sinc}^2\left(\frac{t+v}{2}\right) dv, \quad t \in \mathbb{R},\ m \in \mathbb{N},$$

is the averaged version of the classical Fejér kernel $F(t) = \frac{1}{2}\operatorname{sinc}^2(\frac{x}{2})$, $x \in \mathbb{R}$. It is well known that $F(t)$ satisfies (χ_1) and (χ_2) (see, e.g., [24]), therefore $\mathcal{F}_m(\mathtt{t})$ is a kernel with the desired properties. Note that, as usual, the sinc-function is defined as $\operatorname{sinc}(x) := \begin{cases}\frac{\sin(\pi x)}{\pi x}, & x \neq 0,\\ 1, & x = 0.\end{cases}$

Another family of product kernels of averaged-type can be constructed as $\mathcal{J}_{m,n}(\mathtt{t}) := \prod_{i=1}^N \bar{J}_{m,n}(t_i)$, $\mathtt{t} \in \mathbb{R}^N$ starting from the averaged version of the Jackson kernel

$$\bar{J}_{m,n}(t) := \frac{c_n}{2m}\int_{-m/2}^{m/2} \operatorname{sinc}^{2n}\left(\frac{t+v}{2n\pi\alpha}\right) dv, \quad t \in \mathbb{R},\ m \in \mathbb{N},$$

where $c_n := \left[\int_{\mathbb{R}} \operatorname{sinc}^{2n}\left(\frac{u}{2n\pi\alpha}\right) du\right]^{-1}$, $n \in \mathbb{N}$, and $\alpha \geq 1$ (see, e.g., [29,43]).

The above kernels have unbounded support, thus, they fulfill all the conditions for Theorem 1, but not for Theorem 3, which holds for kernels with compact support. Nevertheless, it is also easy to provide examples of kernels of averaged-type with compact support: among them, there are the central B-splines of order $n \in \mathbb{N}$. Such kernels, well known in approximation theory (see, e.g., [43]), are defined as follows:

$$M_n(x) := \frac{1}{(n-1)!}\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left(\frac{n}{2}+x-i\right)_+^{n-1}, \quad x \in \mathbb{R},$$

where $(x)_+ := \max\{x, 0\}$ denotes "the positive part" of $x \in \mathbb{R}$, and satisfy conditions (χ_1) and (χ_2). Moreover, they are of averaged-type since

$$\bar{M}_{n,1}(t) = M_{n+1}(t), \quad t \in \mathbb{R},$$

for every $n \in \mathbb{N}$; in other words, the averaged kernel with $m = 1$ generated by a central B-spline of order n is a B-spline of order $n + 1$. Therefore, the product kernel $\mathcal{M}_1^n(\mathtt{t}) := \prod_{i=1}^{N}\bar{M}_{n,1}(t_i) = \prod_{i=1}^{N}M_{n+1}(t_i), \quad \mathtt{t} \in \mathbb{R}^N$, is an example of a kernel to which all our results can be applied.

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References

[1] H. Karsli, *On convergence of Chlodovsky and Chlodovsky-Kantorovich polynomials in the variation Seminorm*, Mediterr. J. Math. **10** (2013), 41–56.
[2] N. Deo, M. Dhamija, and D. Miclăuş, *Stancu-Kantorovich operators based on inverse Pólya-Eggenberger distribution*, Appl. Math. Comput. **273** (2016), 281–289.
[3] L. Coroianu and S. G. Gal, *L^p-approximation by truncated max-product sampling operators of Kantorovich-type based on Fejer kernel*, J. Integral Equations Appl. **29** (2017), no. 2, 349–364.
[4] D. Costarelli, A. R. Sambucini, and G. Vinti, *Convergence in Orlicz spaces by means of the multivariate max-product neural network operators of the Kantorovich type*, Neural Comput. Appl. **31** (2019), 5069–5078, DOI: https://doi.org/10.1007/s00521-018-03998-6.
[5] A. M. Acu, L. Hodis, and I. Rasa, *Multivariate weighted Kantorovich operators*, Math. Found. Comput. **3** (2020), 117–124.
[6] A. S. Kumar and B. Shivam, *Inverse approximation and GBS of bivariate Kantorovich type sampling series*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114** (2020), 82.
[7] S. Rahman, M. Mursaleen, and A. Khan, *A Kantorovich variant of Lupaş-Stancu operators based on Pólya distribution with error estimation*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **114** (2020), 75.

[8] A. M. Acu, A. Aral, and I. Rasa, *Generalized Bernstein Kantorovich operators: Voronovskaya type results, convergence in variation*, Carpathian J. Math. **38** (2022), 1–12.
[9] L. V. Kantorovich, *Sur certains développements suivant les polynomes de la forme de S. Bernstein. I*, C.R. Acad. Sc. URSS **20** (1930), 563–568 (in Russian).
[10] G. G. Lorentz, *Zur theorie der polynome von S. Bernstein*, Rec. Math. [Mat. Sbornik] N.S. **2** (1937), 543–556.
[11] P. L. Butzer and W. Splettstößer, *A sampling theorem for duration limited functions with error estimates*, Inform. Control **34** (1977), 55–65.
[12] P. L. Butzer, S. Ries, and R. L. Stens, *Approximation of continuous and discontinuous functions by generalized sampling series*, J. Approx. Theory **50** (1987), 25–39.
[13] P. L. Butzer, W. Splettstößer, and R. L. Stens, *The sampling theorem and linear prediction in signal analysis*, Jahresber. Deutsch. Math.-Verein. **90** (1988), 1–70.
[14] P. L. Butzer and R. L. Stens, Linear prediction by samples from the past, in: R. J. Marks (Ed.), *Advanced Topics in Shannon Sampling and Interpolation Theory, Springer Texts in Electrical Engineering*, Springer, New York, 1993, pp. 157–183.
[15] C. Bardaro, J. Musielak, and G. Vinti, *Nonlinear Integral Operators and Applications*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 9, New York, Berlin, 2003, DOI: https://doi.org/10.1515/9783110199277.
[16] C. Bardaro, I. Mantellini, R. Stens, J. Vautz, and G. Vinti, Generalized sampling approximation for multivariate discontinuous signals and application to image processing, in: A. Zayed, G. Schmeisser (Eds), *New Perspectives on Approximation and Sampling Theory – Festschrift in honor of Paul Butzer's 85th birthday*, Birkhäuser, Cham, 2014, pp. 87–114.
[17] A. Kivinukk and T. Metsmagi, *The variation detracting property of some Shannon sampling series and their derivatives*, Sampl. Theory Signal Image Process. **13** (2014), 189–206.
[18] L. Angeloni, D. Costarelli, and G. Vinti, *Convergence in variation for the multidimensional generalized sampling series and applications to smoothing for digital image processing*, Ann. Acad. Sci. Fenn. Math. **45** (2020), 751–770.
[19] L. Angeloni and G. Vinti, *Estimates in variation for multivariate sampling-type operators*, Dolomites Res. Notes Approx. **14** (2021), 1–9.
[20] A. Boccuto and A. R. Sambucini, *Some applications of modular convergence in vector lattice setting*, Sampl. Theory Signal Process. Data Anal. **20** (2022), 12.
[21] A. Boccuto and A. R. Sambucini, *Abstract integration with respect to measures and applications to modular convergence in vector lattice setting*, Results Math. **78** (2023), 4, DOI: https://doi.org/10.1007/s00025-022-01776-4.
[22] C. Bardaro, P. L. Butzer, R. L. Stens, and G. Vinti, *Kantorovich-type generalized sampling series in the setting of Orlicz spaces*, Sampl. Theory Signal Image Process. **6** (2007), 29–52.
[23] C. Bardaro and I. Mantellini, *Asymptotic formulae for multivariate Kantorovich type generalized sampling series*, Acta Math. Sin. **27** (2011), 1247–1258.
[24] D. Costarelli and G. Vinti, *Approximation by multivariate generalized sampling Kantorovich operators in the setting of Orlicz spaces*, Bollettino Unione Mat. Ital. **4** (2011), 445–468.
[25] O. Orlova and G. Tamberg, *On approximation properties of generalized Kantorovich-type sampling operators*, J. Approx. Theory **201** (2016), 73–86.
[26] D. Costarelli and G. Vinti, *Approximation properties of the sampling Kantorovich operators: regularization, saturation, inverse results and Favard classes in L^p-spaces*, J. Fourier Anal. Appl. **28** (2022), 49.
[27] F. Cluni, V. Gusella, and G. Vinti, *Masonry elastic characteristics assessment by thermographic images*, Meccanica **54** (2019), 1339–1349, DOI: https://doi.org/10.1007/s11012-019-00982-9.
[28] D. Costarelli, M. Seracini, and G. Vinti, *A segmentation procedure of the pervious area of the aorta artery from CT images without contrast medium*, Math. Methods Appl. Sci. **43** (2020), 114–133, DOI: https://doi.org/10.1002/mma.5838.
[29] L. Angeloni, D. Costarelli, M. Seracini, G. Vinti, and L. Zampogni, *Variation diminishing-type properties for multivariate sampling Kantorovich operators*, Boll. Unione Mat. Ital. **13** (2020), 595–605, DOI: https://doi.org/10.1007/s40574-020-00256-3.
[30] A. Kivinukk and T. Metsmagi, *Approximation in variation by the Kantorovich operators*, Proc. Estonian Acad. Sci. **60** (2011), 201–209.
[31] L. Angeloni, D. Costarelli, and G. Vinti, *A characterization of the absolute continuity in terms of convergence in variation for the sampling Kantorovich operators*, Mediterr. J. Math. **16** (2019), 44, DOI: https://doi.org/10.1007/s00009-019-1315-0.
[32] J. Appell, J. Banaś, and N. Merentes, *Bounded Variation and Around*, De Gruyter Series in Nonlinear Analysis and Applications, Vol. 17, De Gruyter, Berlin, Germany, 2014.
[33] L. Tonelli, *Su alcuni concetti dell'analisi moderna*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **11** (1942), no. 2, 107–118.
[34] T. Radó, *Length and Area*, American Mathematical Society Colloquium Publications, Vol. 30, American Mathematical Society, New York, 1948.
[35] C. Vinti, *Perimetro-variazione*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **18** (1964), 201–231.
[36] C. Bardaro, P. L. Butzer, R. L. Stens, and G. Vinti, *Convergence in variation and rates of approximation for Bernstein-type polynomials and singular convolution integrals*, Analysis **23** (2003), 299–340.
[37] L. Angeloni and G. Vinti, *Convergence in variation and rate of approximation for nonlinear integral operators of convolution type*, Results Math. **49** (2006), 1–23, Erratum: **57** (2010), 387–391.

[38] L. Angeloni and G. Vinti, *Approximation with respect to Goffman-Serrin variation by means of non-convolution type integral operators*, Numer. Funct. Anal. Optim. **31** (2010), 519–548.

[39] L. Angeloni and G. Vinti, Variation and approximation in multidimensional setting for Mellin operators, in: A. Zayed, G. Schmeisser (Eds), *New Perspectives on Approximation and Sampling Theory, Applied and Numerical Harmonic Analysis*, Birkhüser, Cham, 2014, pp. 299–317.

[40] L. Angeloni and G. Vinti, *A characterization of absolute continuity by means of Mellin integral operators*, Z. Anal. Anwend. **34** (2015), 343–356.

[41] L. Angeloni, *A new concept of multidimensional variation in the sense of Riesz and applications to integral operators*, Mediterr. J. Math. **14** (2017), 149.

[42] C. Bardaro, P. L. Butzer, R. L. Stens, and G. Vinti, *Approximation error of the Whittaker cardinal series in terms of an averaged modulus of smoothness covering discontinuous signals*, J. Math. Anal. Appl. **316** (2006), 269–306.

[43] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation, I*, Academic Press, New York-London, 1971.