



Research Article

Abdullah Alazemi, Milica Anđelić* and Darko Dimitrov

On the inverse Collatz-Sinogowitz irregularity problem

<https://doi.org/10.1515/math-2022-0572>

received January 30, 2023; accepted March 4, 2023

Abstract: The Collatz-Sinogowitz irregularity index is the oldest known numerical measure of graph irregularity. For a simple and connected graph G of order n and size m , it is defined as $CS(G) = \lambda_1 - 2m/n$, where λ_1 is the largest eigenvalue of the adjacency matrix of G , and $2m/n$ is the average vertex degree of G . Here, the Collatz-Sinogowitz inverse irregularity problem is studied. For every integer $i \geq 0$, it is shown that there exists a graph G such that $CS(G) = i$. Also, for every interval $I_i = (i, i + 1)$, it is shown that there are infinitely many graphs whose Collatz-Sinogowitz irregularity lies in I_i .

Keywords: irregularity measure, Collatz-Sinogowitz irregularity, inverse irregularity problem

MSC 2020: 05C50

1 Introduction

Let G be a simple graph, with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \dots, e_m\}$. The quantities n and m are called the order and size of G , respectively. The vertex degree $d_G(u)$ is the cardinality of the set of edges incident with u . A graph is *regular* if all its vertices have equal vertex degrees; contrarily, it is *irregular*. How much irregular a given graph is could be of high importance in solving many problems. It seems that the oldest numerical measure of graph irregularity was proposed by Collatz and Sinogowitz [1]. It is defined as:

$$CS(G) = \lambda_1 - \frac{2m}{n},$$

where λ_1 denotes the largest eigenvalue of the adjacency matrix of G , commonly called the *spectral radius* of G . Note that $2m/n$ is the average vertex degree of a graph G . It is well known that if a graph is regular, then $\lambda_1 = 2m/n$ and $\lambda_1 > 2m/n$ otherwise [1].

A more direct measure of irregularity was presented by Bell [2], who suggested that the variance $\text{Var}(G)$ of the vertex degrees

$$\text{Var}(G) = \frac{1}{n} \sum_{v \in V(G)} d_G(v)^2 - \left(\frac{1}{n} \sum_{v \in V(G)} d_G(v) \right)^2$$

serves for this purpose.

* **Corresponding author: Milica Anđelić**, Department of Mathematics, Kuwait University, Safat 13060, Kuwait, e-mail: milica.andelic@ku.edu.kw

Abdullah Alazemi: Department of Mathematics, Kuwait University, Safat 13060, Kuwait, e-mail: abdullah.alazemi@ku.edu.kw

Darko Dimitrov: Faculty of Information Studies, 8000 Novo Mesto, Slovenia, e-mail: darko.dimitrov11@gmail.com

Another well-established irregularity metric is the *Albertson irregularity* [3], defined as:

$$\text{irr}(G) = \sum_{uv \in E(G)} |d_G(u) - d_G(v)|.$$

The quantity $|d_G(u) - d_G(v)|$ in the aforementioned formula is called the *imbalance* of an edge $e = uv$. In order to evade the calculation of the absolute value, one inherently came up with the so-called σ -irregularity introduced in [4], which is defined as:

$$\sigma(G) = \sum_{uv \in E(G)} (d_G(u) - d_G(v))^2.$$

This irregularity measure not necessarily is the same for graphs with the same degree sequence. Recently in [5], the following variant of σ -irregularity was introduced:

$$\sigma_t(G) = \frac{1}{2} \sum_{(u,v) \in V^2(G)} (d_G(u) - d_G(v))^2,$$

where $V^2(G)$ is the Cartesian product of $V(G)$ with itself. It was named the *total σ -irregularity* and has the same value for all graphs with same degree sequence.

Before we proceed with the results on the inverse problem of Collatz-Sinogowitz irregularity, an additional necessary notation will be introduced. If the subgraph is obtained by removing at least one vertex or edge from the whole graph, then it is called a *proper subgraph*. A connected graph with equal number of vertices and edges is called a *unicyclic graph*. A graph G , with a vertex u labeled as root, is denoted by (G, u) . The *coalescence* $(G, u) * (H, v)$ of two disjoint rooted graphs (G, u) and (H, v) is a graph obtained after identifying u and v . The eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ of $(0, 1)$ -adjacency matrix A of a simple graph G comprise its spectrum, i.e., they are the roots of its characteristic polynomial $\phi(G, \lambda) = \det(\lambda I - A)$ (or simply of $\phi_G(\lambda)$).

Obtaining graphs with a previously determined value(s) of particular graph invariant(s) (known as an *inverse problem* regarding the considered graph invariant) could be very purposive and helpful for solving theoretical and practical problems. For example, generating chemical structure(s), which have the desired value of a given invariant, could significantly help in the research and production of chemical compounds with prescribed properties. For the Albertson irregularity, the Bell irregularity, the σ -irregularity, the σ_t -irregularity, and the total irregularity, some initial results with respect to the inverse problems can be found in [4–6].

In the sequel, we investigate the Collatz-Sinogowitz inverse irregularity problem.

2 The Collatz-Sinogowitz inverse irregularity problem

Here, we consider two variants of the inverse problem. First, we want to find a graph G , if there exists any, such that $\text{CS}(G) = k$, $k \in \mathbb{N}$. Second, we consider the real values of the Collatz-Sinogowitz index. Namely, we ask if for each interval $I_i = (i, i + 1)$, $i = 1, 2, \dots$ there are infinitely many graphs whose values of Collatz-Sinogowitz irregularity lie in I_i . We present solutions to both problems.

The following results will be used later in this section.

Theorem 2.1. [7] *Let H be a proper subgraph of a connected graph G . Then,*

$$\lambda_1(H) < \lambda_1(G).$$

Theorem 2.2. [8] *Let $(G, u) * (H, v)$ be the coalescence of two connected rooted graphs (G, u) and (H, v) . Then,*

$$\lambda_1((G, u) * (H, v)) \leq \sqrt{\lambda_1(G, u)^2 + \lambda_1(H, v)^2}.$$

*The equality holds if and only if $(G, u) * (H, v)$ is a star.*

Now, we present solutions to both variants of the inverse problem with respect to the Collatz-Sinogowitz irregularity. First, we find solutions within the class of unicyclic graphs. Later, we investigate the starlike graphs and the complete bipartite graphs.

2.1 Unicyclic graphs

Theorem 2.3. *For every non-negative integer i ,*

- (i) *there exists a unicyclic graph whose Collatz-Sinogowitz irregularity equals i ;*
- (ii) *there are infinitely many unicyclic graphs whose Collatz-Sinogowitz irregularities lie in the interval $(i, i + 1)$.*

Proof. (i) For $i = 0$, any cycle on n vertices, $n \geq 3$, fulfills the statement of the proposition. For $i \geq 1$, consider the unicyclic graph G depicted in Figure 1.

It enables the following equitable partition $\mathcal{D} : U_1 \cup U_2 \cup U_3$, where

- $U_1 = \{u \in V(G) : d(u) = 1\}$,
- $U_2 = \{u \in V(G) : d(u) = k + 2\}$,
- $U_3 = \{u \in V(G) : d(u) = 2\}$.

The corresponding quotient matrix is of the form:

$$D = \begin{pmatrix} 0 & 1 & 0 \\ k & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

It is well known that the largest eigenvalue of a graph is also an eigenvalue of any quotient matrix. (For this and other results on equitable partitions the reader is referred to [7].) Therefore, λ_1 is the largest root of

$$\phi_D(\lambda) = \lambda^3 - \lambda^2 - (k + 2)\lambda = \lambda(\lambda^2 - \lambda - (k + 2)),$$

i.e., $\lambda_1 = (1 + \sqrt{9 + 4k})/2$. We first observe that λ_1 is an integer if and only if $\sqrt{9 + 4k}$ is an odd integer, i.e., if and only if $9 + 4k = (2i + 3)^2$, $i = 1, 2, \dots$. Then, it follows that $k = i^2 + 3i$.

Observe that in such a way, the integer values of λ_1 , larger than 2, can be obtained.

For an illustration, in Table 1, the first smallest values of the parameters $i, k, \lambda_1, CS(G)$ are given. Recall that the average degree of any unicyclic graph is 2.

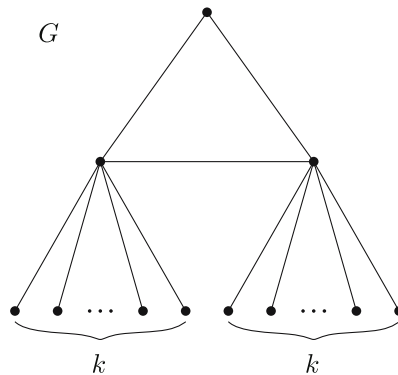


Figure 1: Unicyclic graph, which with certain values of the parameter k may have an arbitrary non-negative integer value of Collatz-Sinogowitz irregularity.

Table 1: Values of the parameters $i, k, \lambda_1, \text{CS}(G)$

i	1	2	3	4	5	6	7	8	9	10
k	4	10	18	28	40	54	70	88	108	130
λ_1	3	4	5	6	7	8	9	10	11	12
$\text{CS}(G)$	1	2	3	4	5	6	7	8	9	10

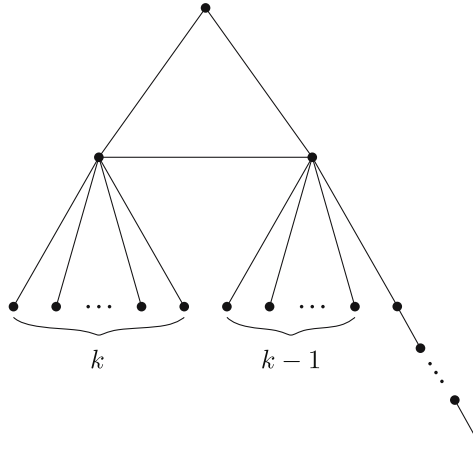


Figure 2: The coalescence of the graph G from Figure 1 and the path graph $P_n, n \geq 1$.

(ii) Consider the graph depicted in Figure 2. It is the coalescence of the graph G from Figure 1 and the path graph $P_n, n \geq 1$. We assume that G is rooted at one end vertex, denoted by v , and P_n is rooted at one end vertex, denoted by w . From Theorem 2.1, we have that

$$\lambda_1(G, v) < \lambda_1((G, v) * (P_n, w)),$$

and by Theorem 2.2, we have

$$\lambda_1((G, v) * (P_n, w)) \leq \sqrt{\lambda_1(G)^2 + \lambda_1(P_n)^2}.$$

We consider the graph G with $\lambda_1(G) = i, i \geq 3$ (i.e., $\text{CS}(G) = i - 2$). Also, by [9], $\lambda_1(P_n) < 2$. Thus, we have that

$$i < \lambda_1((G, v) * (P_n, w)) \leq \sqrt{\lambda_1(G)^2 + \lambda_1(P_n)^2} \leq \sqrt{i^2 + 4} < i + 1,$$

for $i \geq 2$. Therefore, it can be deduced that the spectral radius of $(G, v) * (P_n, w)$ lies in the interval $(i, i + 1), i \geq 3$, for any length n of P_n . Consequently, $\text{CS}((G, v) * (P_n, w))$ lies in the interval $(i - 2, i - 1), i \geq 3$.

It remains to prove that there are infinitely many unicyclic graphs whose CS index lies in the interval $(0, 1)$. Here, we consider the coalescence of a cycle of an arbitrary order $k \geq 3$ and a path of order $n \geq 2, (C_k, v) * (P_n, w)$. In this case,

$$2 < \lambda_1((C_k, v) * (P_n, w)) \leq \sqrt{\lambda_1(C_k)^2 + \lambda_1(P_n)^2} < \sqrt{4 + 4} < 3.$$

Thus, $\text{CS}((C_k, v) * (P_n, w)) \in (0, 1)$, for any $k \geq 3$ and any $n \geq 2$.

This completes the proof. □

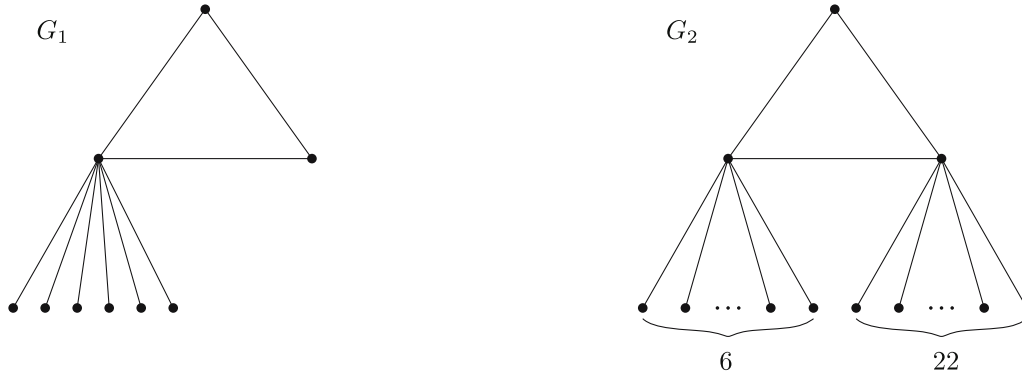


Figure 3: Unicyclic graphs, which are not isomorphic to those in Figure 1, but have non-negative integer values of their Collatz-Sinogowitz irregularity. It holds that $\lambda_1(G_1) = 3$ ($CS(G_1) = 1$) and $\lambda_1(G_2) = 5$ ($CS(G_2) = 3$).

Remark 2.1. There are unicyclic graphs, different from those considered in Theorem 2.3, whose Collatz-Sinogowitz irregularity is an integer. As examples, consider the graphs depicted in Figure 3. These graphs also enable equitable partitions, and their largest eigenvalues are the roots of the following polynomials.

- G_1 : $\lambda^4 - (k + 3)\lambda^2 - 2\lambda + k$. The remaining eigenvalues are 0 with multiplicity $k - 1$ and -1 with multiplicity 1, due to k vertices with the same neighborhoods and 2 vertices with the same closed neighborhood. If the number of pendant edges is 6, then $\lambda_1(G_1) = 3$ and $CS(G_1) = 1$.
- G_2 : $\lambda(\lambda^4 - (k + \ell + 3)\lambda^2 - 2\lambda + k\ell + k + \ell)$. The remaining eigenvalue is 0 with the multiplicity $k + \ell - 2$, assuming that the numbers of pendant edges are k and ℓ , respectively. For $k = 6$ and $\ell = 22$, $\lambda_1(G_2) = 5$ and $CS(G_2) = 3$.

Also, there exist unicyclic graphs with non-negative integer values of their Collatz-Sinogowitz irregularity and the length of the cycle larger than 3.

Proposition 2.1. *Let i be a positive integer. Then, there is a unicyclic graph, whose Collatz-Sinogowitz irregularity is i and the length of its cycle is larger than 3.*

Proof. In [10], one can find that the unicyclic graphs $UC(k, \dots, k)$ of arbitrary girth such that to each vertex of the cycle k , pendant edges are attached, have the largest eigenvalue equal to $\sqrt{k + 1} + 1$, no matter what the length of the cycle is. Hence,

$$CS(UC(k, \dots, k)) = \sqrt{k + 1} - 1.$$

By taking $k = (i + 1)^2 - 1$, $i \geq 1$, we obtain that $CS(UC(i^2 + 2i, \dots, i^2 + 2i)) = i$, independently of the graph's girth. □

By applying the same construction as in the proof of Theorem 2.3(ii), i.e., a coalescence of a graph G with an integer value of the Collatz-Sinogowitz irregularity and an arbitrarily long path, an infinite number of graphs with the Collatz-Sinogowitz irregularity within a given interval can be obtained. Alternatively, instead of using a coalescence of G and an arbitrarily long path, it is to expect that one can obtain the desired results by subdividing arbitrarily times an edge of a cycle of G .

Next, we extend our results in the class of bipartite graphs.

2.2 Bipartite graphs

In the class of bipartite graphs, we investigate the inverse Collatz-Sinogowitz irregularity problem for two subclasses, the starlike trees and complete bipartite graphs.

2.2.1 Starlikes trees

A tree that has exactly one vertex of degree at least three is called *starlike tree*. For a starlike tree, denoted as $S(n_1, n_2, \dots, n_k)$, with a vertex v of degree $k \geq 3$, it holds that

$$S(n_1, n_2, \dots, n_k) - v = P_{n_1} \cup P_{n_2} \cup \dots \cup P_{n_k},$$

where P_{n_i} , $i = 1, \dots, k$, is the path on n_i vertices. Up to isomorphism, the starlike tree is uniquely determined by parameters n_1, n_2, \dots, n_k . In 2001, Lepović and Gutman presented a sharp lower bound and a tight upper bound on the spectral radius of the starlike trees.

Theorem 2.4. [11] *Let λ_1 be the spectral radius of the starlike tree $S(n_1, n_2, \dots, n_k)$. Then for any positive integers, $n_1 \geq n_2 \geq \dots \geq n_k \geq 1$, $\sqrt{k} \leq \lambda_1(S(n_1, n_2, \dots, n_k)) < k / \sqrt{k-1}$.*

Using the aforementioned result, we are able to prove the following result for starlike graphs.

Theorem 2.5. *For every non-negative integer i , there are infinitely many starlike trees whose Collatz-Sinogowitz irregularity lies in the interval $(i, i + 1)$.*

Proof. Consider an arbitrary starlike graph with $k \geq 4$ branches, $S(n_1, n_2, \dots, n_k)$. By Theorem 2.4, it holds that $\sqrt{k} \leq \lambda_1(S(n_1, n_2, \dots, n_k)) < k / \sqrt{k-1}$. Since $k \geq 4$, the starlike tree must have at least five vertices. Therefore, for the average degree of $S(n_1, n_2, \dots, n_k)$, we have that $1.6 \leq \bar{d} < 2$. Consequently, it follows that $\sqrt{k} - 2 < \text{CS}(S(n_1, n_2, \dots, n_k)) < k / \sqrt{k-1} - 1.6$. Now, assume that $k = q^2$, where q is an integer greater than 1. Under these assumptions, $k / \sqrt{k-1} - 1.6 < q - 1$ holds, and hence, it can be deduced that $q - 2 < \text{CS}(S(n_1, n_2, \dots, n_k)) < q - 1$. By replacing $q - 2$ with i , we obtain that $i < \text{CS}(S(n_1, n_2, \dots, n_k)) < i + 1$, where i is a non-negative integer. The last relation is satisfied for any starlike tree with k arbitrary long branches ($k = (i + 2)^2$, $i = 0, 1, \dots$). \square

In the following proposition, we show that the intervals in Theorem 2.5 must be open.

Proposition 2.2. *The Collatz-Sinogowitz irregularity of a starlike graph cannot be an integer.*

Proof. The largest eigenvalue λ_1 of a starlike tree T is a root of its characteristic polynomial, which is a monic polynomial with integer coefficients. Therefore, if $\lambda_1 \in \mathbb{Q}$, then $\lambda_1 \in \mathbb{N}$. Otherwise, $\lambda_1 \in \mathbb{R} \setminus \mathbb{Q}$. Hence,

- If $\lambda_1 \in \mathbb{N}$, then $\text{CS}(T) = \lambda_1 - 2 + \frac{2}{n} \notin \mathbb{Z}$.
- If $\lambda_1 \in \mathbb{R} \setminus \mathbb{Q}$, then $\text{CS}(T) = \lambda_1 - 2 + \frac{2}{n} \in \mathbb{R} \setminus \mathbb{Z}$, and therefore, it is not an integer. \square

The argumentation in the proof of Proposition 2.2 leads to the following conclusion.

Corollary 2.1. *The Collatz-Sinogowitz irregularity of a graph is an integer if and only if both $\lambda_1(G)$ and $2m/n$ are integers.*

Finally, in the sequel, we consider the complete bipartite graphs.

2.2.2 Complete bipartite graphs

The spectral radius and the average degree of a complete bipartite graph $K_{q,r}$ are \sqrt{qr} and $2qr/n$, respectively [9, p. 8], and thus, Collatz-Sinogowitz irregularity of a complete bipartite graph is

$$\text{CS}(K_{q,r}) = \sqrt{qr} - \frac{2qr}{n}. \quad (1)$$

With the next two propositions, we show that the Collatz-Sinogowitz irregularity of a complete bipartite graph can be an even integer, but not an odd integer.

Proposition 2.3. *The Collatz-Sinogowitz irregularity of a complete bipartite graph cannot be an odd integer.*

Proof. Let G be a complete bipartite graph such that $\text{CS}(G)$ is an integer. Then, by Corollary 2.1, both $\lambda_1(G)$ and $2m/n$ are integers. Let $d = \gcd(q, r)$, $q = q'd$, $r = r'd$, and $\gcd(q', r') = 1$. Then, $\frac{2qr}{q+r} = \frac{2q'r'd}{q'+r'}$. If p is a prime factor of $q' + r'$, then either $p = 2$ or p is a factor of d . (If p is a factor of q' , then it is also a factor of r' , which contradicts the fact that $\gcd(q', r') = 1$.) If $q' + r'$ is odd, then any prime factor of $q' + r'$ is also a prime factor of d , i.e., $d = (q' + r')c$, for some $c \in \mathbb{N}$. Hence, in this case, $q = q'(q' + r')c$ and $r = r'(q' + r')c$, provided that $\gcd(q', r') = 1$ and $q' + r'$ is odd, i.e., exactly one of q', r' is even.

Then,

$$\text{CS}(G) = c(q' + r')\sqrt{q'r'} - 2cq'r',$$

with $\sqrt{q'r'}$ being an even integer ($q'r'$ is even). Consequently, $\text{CS}(G)$ is also even.

If $q' + r'$ is even, then $q' + r' = 2k$, for some $k \in \mathbb{N}$. From $\frac{2qr}{q+r} = \frac{2q'r'd}{q'+r'} = \frac{q'r'd}{k}$, similarly as in the previous case, we conclude that $d = kb$ for some $b \in \mathbb{N}$. Hence, $q = q'kb$, $r = r'kb$, $\gcd(q', r') = 1$, provided that $q' + r'$ is even, i.e., both q' and r' are odd.

Then,

$$\text{CS}(G) = kb\sqrt{q'r'} - q'r'b$$

is even as a difference of two odd numbers. □

Proposition 2.4. *For every positive even integer k , there exists an irregular complete bipartite graph whose Collatz-Sinogowitz irregularity is k .*

Proof. Consider the complete bipartite graph $K_{q,r}$. We may assume that $q < r$. Observe that when $q = r$, the graph is regular; then, $\lambda_1 = \bar{d}$ and $\text{CS}(K_{q,r}) = 0$.

Let set $q = 5i$ and $r = 20i$ ($n = 25i$), $i \geq 1$. Then,

$$\text{CS}(K_{q,r}) = \sqrt{qr} - \frac{2qr}{n} = 2i. \quad \square$$

We conclude the current section with the following result.

Theorem 2.6. *For every non-negative integer i , the number of complete bipartite graphs, whose Collatz-Sinogowitz irregularity lies in the interval $(i, i + 1)$, is infinitely large.*

Proof. We also assume that $q < r$. For $q = r$, we have $\text{CS}(K_{q,r}) = 0$. After an elementary analysis, applying the first- and second-derivative criteria, it can be obtained that $\text{CS}(K_{q,r})$ has its maximum either for $q_{\max} = \lfloor 0.087378r \rfloor$ or for $q_{\max} = \lceil 0.087378r \rceil$. The value of $\text{CS}(K_{q_{\max},r})$ is approximately $0.134884r$ (the exact value, due to the ceiling and flooring, can be obtained for particular r).

Let $q \geq q_{\max}$, and let $\text{CS}(K_{q,r}) \in [i - 1, i]$. We want to show that when q increases by one, then the value of CS lies in $[i - 1, i]$ or in $[i - 2, i - 1]$. This will follow if we prove that

$$f(q, r) = \text{CS}(K_{q,r}) - \text{CS}(K_{q+1,r}) = \frac{2r^2}{(q+r)(q+r+1)} + \sqrt{qr} - \sqrt{(q+1)r} < 1,$$

for $q \geq q_{\max}$. It can be easily checked that for any positive fixed $r \geq q$, $\partial f(q, r) / \partial q < 0$, which means that $f(q, r)$ is a decreasing function in q , and its maximum is obtained when $q = q_{\max}$. It follows that

$$f(0.1r, r) > f(q_{\max}, r) \geq f(q, r).$$

It is easy to verify that $f(0.1r, r)$ has one maximum, which is smaller than 0.25, and therefore also, $f(q, r) < 0.25$. Hence, we have shown that $\text{CS}(K_{q+1,r})$ lies in the same interval $[i - 1, i]$ as $\text{CS}(K_{q,r})$, or in the interval $[i - 2, i - 1]$.

Let r_f be a fixed value of r and let $\text{CS}(K_{q_{\max}, r_f}) \in [i - 1, i]$. Due to the facts that $\text{CS}(K_{q, r_f})$ decreases for $q \geq q_{\max}$, $r_f \geq q$, $\text{CS}(K_{r_f, r_f}) = 0$, and $\text{CS}(K_{q, r_f}) \leq \text{CS}(K_{q+1, r_f}) + 1$, it can be concluded that in each interval $[k - 1, k]$, $k = i, \dots, 1$, there is at least one value of $\text{CS}(q, r_f)$, for $q \in [q_{\max}, r_f]$ (actually, there are at least four values of $\text{CS}(q, r_f)$, because $f(q, r) < 0.25$). Recall that $\text{CS}(K_{q_{\max}, r_f}) = 0.134884r_f$. We can obtain an arbitrary large value for $\text{CS}(K_{q_{\max}, r_f})$ by choosing an appropriate value of r_f , or to turn around, we can obtain an arbitrarily large i such that $\text{CS}(K_{q_{\max}, r_f}) \in [i, i + 1]$, and with at least one value of $\text{CS}(q, r_f)$ in each interval $[k - 1, k]$, $k = i, \dots, 1$, for $q \in [q_{\max}, r_f]$. This can be satisfied for infinitely many instances r_f of r , since $r \rightarrow \infty$, and thus, we can obtain that in each interval $[i, i + 1]$, $i \geq 0$, there are infinitely many values of $\text{CS}(K_{q,r})$. \square

3 Conclusion

In this work, the inverse problem for the Collatz-Sinogowitz irregularity was considered. It was shown that for every interval $I_i = (i, i + 1)$, where i is a non-negative integer, there are infinitely many graphs whose Collatz-Sinogowitz irregularity lies in I_i . This was shown within the classes of unicyclic graphs, starlike graphs, and complete bipartite graphs. Also, it was shown that the Collatz-Sinogowitz irregularity of a starlike graph cannot be an integer, while the Collatz-Sinogowitz irregularity of a complete bipartite graph can be an even integer, but not an odd integer. On the other hand, it was proven that for any non-negative integer i , there exists a unicycle graph whose Collatz-Sinogowitz irregularity is i . In general, it was shown that the Collatz-Sinogowitz irregularity of a graph is an integer if and only if both, its spectral radius and its average degree, are integers.

Further investigation of other classes of graphs is of interest. We conclude with the following conjecture. Namely, we believe that Theorem 2.3 can be extended to c -cyclic graphs, for any $c \geq 2$.

Conjecture 3.1. For every positive integer i larger than a given constant and for every $c \geq 2$,

- (i) there is a connected c -cyclic graph of order n , whose Collatz-Sinogowitz irregularity equals i , provided that n divides $2(c - 1)$;
- (ii) there are infinitely many connected c -cyclic graphs whose Collatz-Sinogowitz irregularities lie in the interval $(i, i + 1)$.

Conflict of interest: The authors declare that there is no conflict of interest regarding the publication of this article.

References

- [1] L. Collatz and U. Sinogowitz, *Spektren endlicher Graphen*, Abh. Math. Semin. Univ. Hambg. **21** (1957), 63–77.
- [2] F. K. Bell, *A note on the irregularity of graphs*, Linear Algebra Appl. **161** (1992), 45–54.
- [3] M. O. Albertson, *The irregularity of a graph*, Ars Combin. **46** (1997), 219–225.
- [4] I. Gutman, M. Togan, A. Yurttas, A. S. Cevik, and I. N. Cangul, *Inverse problem for sigma index*, MATCH Commun. Math. Comput. Chem. **79** (2018), 491–508.
- [5] D. Dimitrov and D. Stevanović, *On the σ_t -irregularity and the inverse irregularity problem*, Appl. Math. Comput. **441** (2023), 127709.
- [6] M. Togan, A. Yurttas, U. Sanli, F. Celik, and I. N. Cangul, *Inverse problem for Bell index*, Filomat **34** (2020), 615–621.
- [7] D. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, London Mathematical Society Student Texts, Vol. 75, Cambridge University Press, Cambridge, 2009.
- [8] Z. W. Wang and J. M. Guo, *Some upper bounds on the spectral radius of a graph*, Linear Algebra Appl. **601** (2020), 101–112.
- [9] A. E. Brouwer and W. H. Haemers, *Spectra of Graphs*, Springer, New York, 2012.
- [10] F. Belardo, E. M. Li Marzi, and S. K. Simić, *Some results on the index of unicyclic graphs*, Linear Algebra Appl. **416** (2006), 1048–1059.
- [11] M. Lepović and I. Gutman, *Some spectral properties of starlike trees*, Bull. Cl. Sci. Math. Nat. Sci. Math. **26** (2001), 107–113.