



## Research Article

Abd-Semii Oluwatosin-Enitan Owolabi, Timilehin Opeyemi Alakoya, and Oluwatosin Temitope Mewomo\*

# A self-adaptive inertial extragradient method for a class of split pseudomonotone variational inequality problems

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**Abstract:** In this article, we study a class of pseudomonotone split variational inequality problems (VIPs) with non-Lipschitz operator. We propose a new inertial extragradient method with self-adaptive step sizes for finding the solution to the aforementioned problem in the framework of Hilbert spaces. Moreover, we prove a strong convergence result for the proposed algorithm without prior knowledge of the operator norm and under mild conditions on the control parameters. The main advantages of our algorithm are: the strong convergence result obtained without prior knowledge of the operator norm and without the Lipschitz continuity condition often assumed by authors; the minimized number of projections per iteration compared to related results in the literature; the inertial technique employed, which speeds up the rate of convergence; and unlike several of the existing results in the literature on VIPs with non-Lipschitz operators, our method does not require any linesearch technique for its implementation. Finally, we present several numerical examples to illustrate the usefulness and applicability of our algorithm.

**Keywords:** split pseudomonotone variational inequality problem, inertial technique, fixed point problem, non-Lipschitz operators, quasi-pseudocontractive mappings

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## 1 Introduction

Let  $C$  be a nonempty, closed, and convex subset of a real Hilbert space  $H$  with induced norm  $\|\cdot\|$  and inner product  $\langle \cdot, \cdot \rangle$ . The variational inequality problem (VIP) for  $f$  on  $C$  is defined as follows:

$$\text{find } \hat{x} \in C \text{ such that } \langle f\hat{x}, x - \hat{x} \rangle \geq 0, \quad \forall x \in C \quad (1)$$

If  $f$  is monotone, the problem (1) is known as a monotone VIP, while it is known as a pseudo-monotone VIP if  $f$  is a pseudo-monotone. We denote the solution set of VIP (1) by  $VI(C, f)$ . In the early 1960s, Stampacchia [1] and Fichera [2] independently introduced the theory of VIP. The VIP is a fundamental problem that has a wide range of applications in the applied field of mathematics, such as network equilibrium problems, complementarity problems, optimization theory, and systems of nonlinear equations (see [3,4]). As a result of its wide applications, several authors have proposed many iterative algorithms for approximating the

\* **Corresponding author: Oluwatosin Temitope Mewomo**, School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, e-mail: mewomoo@ukzn.ac.za

**Abd-Semii Oluwatosin-Enitan Owolabi:** School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, e-mail: 218086824@stu.ukzn.ac.za

**Timilehin Opeyemi Alakoya:** School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, e-mail: timimaths@gmail.com

solution of VIP and related optimization problems, (see [5–11]) and the references therein. The VIP is widely known to be equivalent to the following fixed point equation:

$$x^* = P_C(I - \lambda f)x^*, \quad (2)$$

for  $\lambda > 0$ , where  $P_C$  is the metric projection from  $H$  onto  $C$ .

A simple iterative formula that is an extension of (2) is the *projection gradient method* presented as follows:

$$x_{n+1} = P_C(I - \lambda f)x_n, \quad (3)$$

where  $\lambda \in \left(0, \frac{2\alpha}{L^2}\right)$  and  $f: H \rightarrow H$  is  $\alpha$ -strongly monotone and  $L$ -Lipschitz continuous. It is known that Algorithm 3 only converges weakly under some strict conditions that the operator  $f$  is either strongly monotone or inverse strongly monotone but fails to converge if  $f$  is monotone.

In order to overcome this barrier, a famous method called the *extragradient method* (EgM) was introduced by Korpelevich [12] for solving VIP in finite dimensional Euclidean spaces, which is defined as follows:

$$\begin{cases} y_n = P_C(x_n - \lambda f x_n), \\ x_{n+1} = P_C(x_n - \lambda f y_n), \end{cases} \quad n \geq 1, \quad (4)$$

where  $f$  is monotone and a Lipschitz continuous,  $\lambda \in \left(0, \frac{1}{L}\right)$ , and  $C \subseteq \mathbb{R}^n$  is a closed convex set. If the solution set  $VI(C, f)$  is nonempty, then the sequence  $\{x_n\}$  generated by the EgM converges to an element in  $VI(C, f)$ .

In recent years, the EgM has received great attention from numerous authors who have improved it in various ways (see, for instance, [13–15]). It is observed that the EgM requires the computation of two projections onto the closed convex set  $C$  per iteration. However, projection onto an arbitrary closed convex set  $C$  is often very difficult to compute. In order to overcome this barrier, authors have developed more efficient iterative algorithms; some of these algorithms are discussed below.

In 2000, Tseng [16] proposed the following iterative scheme known as the *Tseng's extragradient method* (TEgM):

**Algorithm 1.1.**

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda f x_n), \\ x_{n+1} = y_n - \lambda(f y_n - f x_n), \end{cases} \quad (5)$$

where  $A$  is a monotone and a Lipschitz continuous operator and  $\lambda \in \left(0, \frac{1}{L}\right)$ . Clearly, the TEgM requires one projection to be computed per iteration and hence has an advantage in computing projection over the EgM.

Furthermore, Censor *et al.* [8] introduced a new method that involves the modification of one of the projections in the EgM by replacing it with a projection onto an half space. This method is called the *subgradient extragradient method* (SEgM) and is defined as follows:

**Algorithm 1.2.** (SEgM)

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda f x_n), \\ T_n = \{z \in H : \langle x_n - \lambda f x_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda f y_n). \end{cases} \quad (6)$$

Censor *et al.* [8,9] proved that provided the solution set  $VI(C, f)$  is nonempty, the sequence  $\{x_n\}$  generated by the SEgM converges weakly to an element  $p \in VI(C, f)$ , where  $p = \lim_{n \rightarrow \infty} P_{VI(C, A)} x_n$ .

Also, Maingé and Gobinddass [17] obtained a result that relates to a weak convergence algorithm by using only a single projection by means of a projected reflected gradient-type method [14] and an inertial term for finding the solution of VIP in real Hilbert spaces.

Another related problem is the *fixed point problem* (FPP). Let  $S : C \rightarrow C$  be a nonlinear mapping. A point  $p \in C$  is called a fixed point of  $S$  if  $Sp = p$ . We denote by  $F(S)$ , the set of fixed points of  $S$ , that is,

$$F(S) = \{p \in C : Sp = p\}. \quad (7)$$

Many of the problems in sciences and engineering can be formulated as finding the solution of FPP of a nonlinear operator.

Recently, Thong and Hieu [18] introduced the following viscosity-type subgradient extragradient algorithm for approximating a common solution of VIP and FPP in Hilbert spaces:

### Algorithm 1.3

Let  $x_1 \in H$ ,  $\lambda_1 > 0$  and  $\mu \in (0, 1)$ . Compute  $x_{n+1}$  as follows:

**Step 1.** Calculate

$$y_n = P_C(x_n - \lambda_n f x_n).$$

**Step 2.** Compute

$$z_n = P_{T_n}(x_n - \lambda_n f y_n),$$

where  $T_n = \{x \in H : \langle x_n - \lambda_n f x_n - y_n, x - y_n \rangle \leq 0\}$ .

**Step 3.** Compute

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)[(1 - \beta_n)z_n + \beta_n S z_n]$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|x_n - y_n\|}{\|f x_n - f y_n\|} \right\}, & \text{if } f x_n - f y_n \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set  $n := n + 1$  and go to **Step 1**.

$S : H \rightarrow H$  is a demicontractive mapping such that  $I - S$  is demiclosed at zero,  $A : H \rightarrow H$  is monotone and Lipschitz continuous, and  $f : H \rightarrow H$  is a contraction.

Censor et al. in [7] introduced another problem called the *split variational inequality problem* (SVIP). The SVIP, which is a more general problem than the VIP, is formulated as follows: Find  $x \in C$  such that

$$\langle f x, y - x \rangle \geq 0, \quad \forall y \in C \quad (8)$$

and

$$\langle g(Ax), z - Ax \rangle \geq 0, \quad \forall z \in Q, \quad (9)$$

where  $C$  and  $Q$  are nonempty, closed, and convex subsets of real Hilbert spaces  $H_1$  and  $H_2$ , respectively;  $f$  and  $g$  are nonlinear mappings on  $C$  and  $Q$ , respectively; and  $A : H_1 \rightarrow H_2$  is a bounded linear operator. Observe that the SVIP can be viewed as a pair of VIPs in which the image of the solution of one VIP in a space  $H_1$  under a given bounded linear operator  $T$  is a solution of another VIP in another space  $H_2$ .

The following algorithm was introduced by Censor et al. [7] for solving SVIP (equations (8) and (9)):

$$x_{n+1} = P_C(I - \lambda f)(x_n + \tau A^*(P_Q(I - \lambda g) - I)Ax_n), \quad \forall n \in \mathbb{N}, \quad (10)$$

and the authors proved the following convergence theorem:

**Theorem 1.4.** Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator and  $f : H_1 \rightarrow H_1$  and  $g : H_2 \rightarrow H_2$ , be, respectively,  $\alpha_1$ - and  $\alpha_2$ -inverse strongly monotone operators with  $\alpha := \min\{\alpha_1, \alpha_2\}$ . Assume that SVIP (equations (8) and (9)) is

consistent,  $\tau \in \left(0, \frac{1}{L}\right)$  with  $L$  being the spectral radius of the operator  $A^*A$ ,  $\lambda \in (0, 2\alpha)$ , and suppose that for all  $x^*$  solving SVIP (equations (8) and (9)),

$$\langle f(x), P_C(I - \lambda f)(x) - x^* \rangle \geq 0, \quad \forall x \in H_1. \quad (11)$$

Then the sequence  $\{x_n\}$  generated by (10) converges weakly to a solution of SVIP (equations (8) and (9)).

It is clear that Algorithm 10 fully exploits the splitting structure of the SVIP (equations (8) and (9)). However, the weak convergence of this method was proved under some strong assumptions, such as assumption (11) and the fact that both mappings are required to be co-coercive (inverse strongly monotone). It is worth mentioning that assumption (11), which depends on the averaged operator technique, has been dispensed with by other authors for solving the SVIP and related problems (see, e.g., [19–22]), but their methods also relied on the co-coercivity of the cost operators.

In order to overcome some of these weaknesses, He et al. [23] proposed an easily implementable relaxed projection method, which fully exploits the splitting structure of the SVIP, for solving the SVIP (equations (8) and (9)) when the underlying operators are monotone and Lipschitz continuous in finite dimensional spaces. However, this method still requires the reformulation of the original problem into a VIP in a product space (for more details, see [23]).

Tian and Jiang in [24] studied a more general class of SVIP. Precisely, the authors investigated the following class of SVIP: find  $x \in C$  such that

$$\langle fx, y - x \rangle \geq 0 \quad \forall y \in C \quad \text{and} \quad Ax \in F(S), \quad (12)$$

where  $f : C \rightarrow H_1$  is monotone and Lipschitz continuous,  $S : H_2 \rightarrow H_2$  is a nonexpansive mapping, and  $A : H_1 \rightarrow H_2$  is a bounded linear operator.

Moreover, the authors proposed the following algorithm for approximating the solution of problem (12):

$$\begin{cases} y_n = P_C(x_n - \tau_n A^*(I - S)Ax_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(t_n)), \quad n \geq 1. \end{cases} \quad (13)$$

In addition, they proved the following convergence theorem:

**Theorem 1.5.** *Let  $H_1$  and  $H_2$  be real Hilbert spaces, and let  $C$  be a nonempty closed convex subset of  $H_1$ . Let  $A : H_1 \rightarrow H_2$  be a bounded linear operator such that  $A \neq 0$ , and  $S : H_2 \rightarrow H_2$  be a nonexpansive mapping. Let  $f : C \rightarrow H_1$  be a monotone and  $L$ -Lipschitz continuous mapping. Suppose that  $\Gamma = \{z \in \text{VI}(C, f) : Az \in F(S)\} \neq \emptyset$  and the sequences  $\{x_n\}$  is defined for arbitrary  $x_1 \in C$  by (13), where  $\{\tau_n\} \subset [a, b]$  for some  $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$  and  $\{\lambda_n\} \subset [c, d]$  for some  $c, d \in \left(0, \frac{1}{L}\right)$ . Then  $\{x_n\}$  converges weakly.*

It is clear that the class of SVIP (12) considered by Tian and Jiang [24] generalizes the class of SVIP (equations (8) and (9)) considered by Censor [7]. However, we observe that the result of Tian and Jiang [24] is only applicable when the associated cost operator  $f$  is monotone and Lipschitz continuous and  $S$  is non-expansive. Moreover, the implementation of the proposed Algorithm (13) by the authors requires knowledge of the Lipschitz constant of the cost operator  $f$  and prior knowledge of the operator norm  $\|A\|$ . In several instances, these parameters are unknown or difficult to estimate, which can hinder the implementation of their proposed algorithm. In spite of all these stringent conditions, the authors were only able to obtain a weak convergence result for their proposed algorithm. It is known that in solving optimization problems, strong convergence results are more applicable and therefore more desirable than weak convergence results.

In order to remedy some of the above limitations, Ogwo et al. [25] proposed and analyzed the convergence of the following algorithm for solving SVIP (12) when the underlying operator  $f$  is pseudomonotone (a weaker assumption than the monotone assumption) and Lipschitz continuous, and  $T$  is strictly pseudocontractive mapping:

**Algorithm 1.6**

**Initialization:** Let  $\gamma > 0$ ,  $l, \mu \in (0, 1)$  and  $x_1 \in H_1$  be given arbitrary.

**Iterative Steps:** Calculate  $x_{n+1}$  as follows:

**Step 1.** Compute

$$w_n = P_C(x_n - \tau_n A^*(I - T_\beta)Ax_n), \quad \text{and} \quad y_n = P_C(w_n - \lambda_n f w_n),$$

where  $0 < a \leq \tau_n \leq b < \frac{1}{\|A\|^2}$ ,  $T_\beta = \beta I + (1 - \beta)S$ , and  $\lambda_n$  is chosen to be the largest  $\lambda \in \{\gamma, \gamma^2, \gamma^3, \dots\}$  satisfying

$$\lambda_n \|f w_n - f y_n\| \leq \mu \|w_n - y_n\|.$$

**Step 2.** Compute

$$x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)z_n,$$

$$z_n = y_n - \lambda_n (f y_n - f w_n).$$

Set  $n := n + 1$  and go back to **Step 1**.

$f : H_1 \rightarrow H_1$  is a pseudo-monotone, Lipschitz continuous, and sequentially weakly continuous operator on bounded subsets of  $H_1$ ,  $g : H_1 \rightarrow H_1$  is a contraction mapping with constant  $\rho \in (0, 1)$ ,  $A : H_1 \rightarrow H_2$  is a bounded linear operator, and  $S : H_2 \rightarrow H_2$  is a  $\kappa$ -strictly pseudocontractive mapping with  $\kappa \in [0, 1)$ . Moreover, the authors proved the following strong convergence theorem:

**Theorem 1.7.** *Let  $\{x_n\}$  be a sequence generated by Algorithm 1.6. Assume that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = +\infty$ . Then  $\{x_n\}$  converges strongly to  $z = P_\Gamma g(z)$ , where  $\Gamma = \{z \in \text{VI}(C, f) : Az \in F(S)\} \neq \emptyset$ .*

We observe that while the result of Ogwo et al. [25] is an improvement over the result of Tian and Jiang [24], the following drawbacks are identified in their result: (1) their result is not applicable when the cost operator  $f$  is non-Lipschitz and/or not sequentially weakly continuous, and the operator  $T$  is more general than strict pseudocontractions; (2) the proposed algorithm involves linesearch technique, which could be computationally expensive to implement due to its loop nature; (3) implementation of their proposed algorithm requires knowledge of the operator norm.

One of our goals in this article is to remedy the above drawbacks. More precisely, we propose a new iterative method for approximating the solution SVIP (12) with the following features:

- (1) Unlike the result of Ogwo et al. [25] and Tian and Jiang [24], our proposed algorithm is applicable when the cost operator  $f$  is a non-Lipschitz pseudomonotone operator and  $T$  is a quasi-pseudocontraction. Moreover, our method does not require the weakly sequentially continuity condition assumed in [25] and in several other existing results in the literature on VIP with a pseudomonotone operator.
- (2) Our proposed algorithm does not involve any linesearch technique. It uses a simple but efficient self-adaptive step size technique that generates a nonmonotonic sequence of step sizes.
- (3) The implementation of our proposed algorithm does not require knowledge of the operator norm.
- (4) Unlike the result of Tian and Jiang [24] and several other results in the literature on SVIP, our method requires evaluating a minimal number of projections per iteration.
- (5) Our method employs the inertial technique to accelerate the rate of convergence.
- (6) In addition, the sequence generated by our proposed algorithm converges strongly to the solution of the SVIP (12).

Finally, we present some applications and numerical examples to illustrate the usefulness and efficiency of our proposed method in comparison with some related methods in the literature.

Subsequent sections of this article are organized as follows: in Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main result; in Section 3, we present our proposed method, while in Section 4, we first establish some lemmas that are useful in establishing the strong convergence result of our proposed algorithm and then prove the strong convergence theorem for the algorithm; in Section 5, we

present some numerical examples to illustrate the performance of our method and compare it with some related methods in the literature, and finally, Section 6, we give a concluding remark.

## 2 Preliminaries

In this section, we recall some basic lemmas and definitions required to establish our results. We denote by  $x_n \rightharpoonup x$  and  $x_n \rightarrow x$  the weak and strong convergence, respectively, of a sequence  $\{x_n\}$  in  $H$  to a point  $x \in H$ . Let  $C$  be a nonempty, closed, and convex subset of a real Hilbert space  $H$ . The metric projection [26]  $P_C : H \rightarrow C$  is defined for each  $x \in H$ , as the unique element  $P_C x \in C$  such that

$$\|x - P_C x\| = \inf\{\|x - z\| : z \in C\}.$$

The operator  $P_C$  is nonexpansive and has the following properties [27,28]:

(i) for all  $x, y \in C$ , we have

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle;$$

(ii) for any  $x \in H$ ,  $z = P_C x$  if and only if

$$\langle x - z, z - y \rangle \geq 0 \quad \forall y, z \in C;$$

(iii) for any  $x \in H$  and  $y \in C$ , we have

$$\|P_C x - y\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2.$$

**Definition 2.1.** Let  $T : C \rightarrow C$  be a mapping. Then,  $T$  is called

(i)  $L$ -Lipschitz continuous, if there exists a constant  $L > 0$  such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \forall x, y \in C;$$

If  $L \in [0, 1)$ , then  $T$  is a contraction mapping and it is nonexpansive if  $L = 1$ ;

(ii) quasi-nonexpansive, if  $F(T) \neq \emptyset$  and

$$\|Tx - p\| \leq \|x - p\| \quad \forall x \in C \quad \text{and} \quad p \in F(T);$$

(iii)  $k$ -strictly pseudocontractive mapping, if there exists  $\kappa \in [0, 1)$  such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C;$$

if  $\kappa = 1$ , then  $T$  is pseudocontractive;

(iv) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0 \quad \forall x, y \in C;$$

(v)  $\alpha$ -inverse strongly monotone ( $\alpha$ -ism) (or  $\alpha$ -co-coercive), if there exists  $\alpha > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha\|Tx - Ty\|^2 \quad \forall x, y \in C;$$

if  $\alpha = 1$ ,  $T$  is firmly nonexpansive;

(vi)  $\beta$ -strongly monotone, if there exists  $\beta > 0$  such that

$$\langle Tx - Ty, x - y \rangle \geq \beta\|x - y\|^2 \quad \forall x, y \in C;$$

(vii) pseudomonotone, if

$$\langle Tx, y - x \rangle \geq 0 \Rightarrow \langle Ty, y - x \rangle \geq 0 \quad \forall x, y \in C;$$

(viii)  $\alpha$ -averaged, if  $T = (1 - \alpha)I + \alpha S$ , where  $\alpha \in (0, 1)$  and  $S : C \rightarrow C$  is nonexpansive, see [29];

(ix) uniformly continuous, if for every  $\varepsilon > 0$ , there exists  $\delta = \delta(\varepsilon) > 0$  such that

$$\|Tx - Ty\| < \varepsilon \quad \text{whenever} \quad \|x - y\| < \delta \quad \forall x, y \in C.$$

In this connection, see Proposition 11.2 on page 42 of [30] and the early article by Bruck and Reich [31]. It is known that firmly nonexpansive mappings are  $\frac{1}{2}$ -averaged while averaged mappings are nonexpansive. Also, every  $\alpha$ -inverse strongly monotone mapping is  $\frac{1}{\alpha}$ -Lipschitz continuous. Moreover, if  $f$  is  $\alpha$ -strongly monotone and  $L$ -Lipschitz continuous, then  $f$  is  $\frac{\alpha}{L^2}$ -ism. Furthermore, both  $\alpha$ -strongly monotone and  $\alpha$ -inverse strongly monotone mappings are monotone, while monotone mappings are pseudomonotone. However, the converse is not true. For instance, the mapping  $f : (0, \infty) \rightarrow (0, \infty)$  defined by  $fx = \frac{1}{1+x}$  is pseudomonotone but not monotone. In addition, we note that uniform continuity is a weaker notion than Lipschitz continuity. For more examples on pseudomonotone operators that are not monotone, check [32,33].

**Definition 2.2.** An operator  $T : C \rightarrow C$  is said to be quasi-pseudocontractive, if  $F(T) \neq \emptyset$  and

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|Tx - x\|^2, \quad \forall x \in C, p \in F(T).$$

Clearly, the class of quasi-pseudocontractive mappings includes the class of pseudo-contractive mappings with nonempty fixed points set, and it contains several other classes of mappings.

It is well known that if  $D$  is a convex subset of  $H$ , then  $T : D \rightarrow H$  is uniformly continuous if and only if, for every  $\varepsilon > 0$ , there exists a constant  $K < +\infty$  such that

$$\|Tx - Ty\| \leq K\|x - y\| + \varepsilon \quad \forall x, y \in D. \quad (14)$$

**Lemma 2.3.** [30] *Let  $C$  be a nonempty, closed, and convex subset of a real Hilbert space  $H$ . Given  $x \in H$  and  $z \in C$ . Then,  $z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0 \quad \forall y \in C$ .*

**Lemma 2.4.** [25] *Let  $\{\alpha_n\}$  be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\lambda_n + \delta_n, \quad n \geq 0,$$

where  $\{\alpha_n\}$ ,  $\{\lambda_n\}$ , and  $\{\delta_n\}$  satisfy the following conditions:

- (i)  $\{\alpha_n\} \subset [0, 1]$ ,  $\sum_{n=0}^{\infty} \alpha_n = \infty$ ;
- (ii)  $\limsup_{n \rightarrow \infty} \lambda_n \leq 0$ ;
- (iii)  $\delta_n \geq 0 (n \geq 0)$ ,  $\sum_{n=0}^{\infty} \delta_n < \infty$ .

Then,  $\lim_{n \rightarrow \infty} a_n = 0$ .

**Lemma 2.5.** [34] *Let  $H$  be a real Hilbert space and  $S : H \rightarrow H$  be a mapping with  $L \geq 1$ . Denote*

$$T := (1 - \Phi)I + \Phi S((I - \eta)I + \eta S).$$

If  $0 < \Phi < \eta < \frac{1}{1 + \sqrt{1 + L^2}}$ , then the following hold:

- (i)  $F(S) = F(S((I - \eta)I + \eta T)) = F(T)$ ;
- (ii) If  $I - S$  is demiclosed at zero, then  $I - T$  is also demiclosed at zero;
- (iii) In addition, if  $S : H \rightarrow H$  is quasi-pseudocontractive mapping, then the mapping  $T$  is quasi-nonexpansive.

**Lemma 2.6.** [25,35] *Let  $H$  be a real Hilbert space, then, for all  $x, y \in H$  and  $\beta \in \mathbb{R}$ , the following hold:*

- (i)  $\|\beta x + (1 - \beta)y\|^2 = \beta\|x\|^2 + (1 - \beta)\|y\|^2 - \beta(1 - \beta)\|x - y\|^2$ ;
- (ii)  $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x - y \rangle$ ;
- (iii)  $\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$ .

**Lemma 2.7.** [36] *Let  $\{\Gamma_n\}$  be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence  $\{\Gamma_{n_j}\}_{j \geq 0}$  of  $\{\Gamma_n\}$  such that*

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \forall j \geq 0. \quad (15)$$

Furthermore, consider the sequence of integers  $\{\tau(n)\}_{n \geq n_0}$  defined by

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}. \quad (16)$$

Then,  $\{\Gamma_n\}_{n \geq n_0}$  is a nondecreasing sequence such that  $\tau(n) \rightarrow \infty$  as  $n \rightarrow \infty$ , and for all  $n \geq n_0$ , the following hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \Gamma_n \leq \Gamma_{\tau(n)+1}. \quad (17)$$

**Lemma 2.8.** [37] Suppose  $\{\lambda_n\}$  and  $\{\theta_n\}$  are two nonnegative real sequences such that

$$\lambda_{n+1} \leq \lambda_n + \theta_n \quad \forall n \geq 1.$$

If  $\sum_{n=1}^{\infty} \theta_n < \infty$ , then  $\lim_{n \rightarrow \infty} \lambda_n$  exists.

**Lemma 2.9.** [38] Let  $H$  be a real Hilbert space and  $T : H \rightarrow H$  be a nonexpansive mapping with  $F(T) \neq \emptyset$ . If  $\{x_n\}$  is a sequence in  $H$  converging weakly to  $x^*$  and if  $\{(I - T)x_n\}$  converges strongly to  $y$ , then  $(I - T)x^* = y$ .

**Lemma 2.10.** [39] Let  $C$  be a nonempty, closed, and convex of real Hilbert space  $H$ . Let operator  $f : C \rightarrow H$  be continuous and pseudomonotone. Then,  $x^*$  is a solution of  $VI(C, f)$  if and only if  $\langle fx, x - x^* \rangle \geq 0 \quad \forall x \in C$ .

### 3 Proposed algorithm

In this section, we present our proposed algorithm. Let  $H_1$  and  $H_2$  be real Hilbert spaces, and let  $C$  be a nonempty, closed and convex subset of  $H_1$ . Suppose  $g : H_1 \rightarrow H_1$  is a contraction mapping with constant  $\rho \in (0, 1)$ ,  $A : H_1 \rightarrow H_2$  is a bounded linear operator, and  $S : H_2 \rightarrow H_2$  is a quasi-pseudocontractive mapping such that  $I - S$  is demiclosed at zero. We assume that the solution set  $\Gamma = \{z \in VI(C, f) : Az \in F(S)\} \neq \emptyset$ .

We establish the strong convergence of our proposed algorithm under the following conditions:

- (C1)  $\{\alpha_n\} \subset (0, 1)$ ,  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\sum_{n=0}^{\infty} \alpha_n = +\infty$ ;
- (C2) The operator  $f : H_1 \rightarrow H_1$  is pseudomonotone and uniformly continuous on  $H_1$  and satisfies the following condition:
  - (a) Whenever  $\{x_n\} \subset C$  and  $x_n \rightharpoonup z$ , one has  $\|fz\| \leq \liminf_{n \rightarrow \infty} \|fx_n\|$ ;
- (C3)  $\{\varepsilon_n\}$  is a positive sequence such that  $\lim_{n \rightarrow \infty} \frac{\varepsilon_n}{\alpha_n} = 0$ ;
- (C4) Let  $\{\psi_n\}$  be a nonnegative sequence such that  $\sum_{n=1}^{\infty} \psi_n < +\infty$ .

Now we present our proposed algorithm as follows:

---

#### Algorithm 3.1

**Step 0:** Select  $x_0, x_1 \in H$ ,  $\lambda_1 > 0$ ,  $\mu \in (0, 1)$ , and  $0 < \phi_1 \leq \phi_n \leq \phi_2 < 1$  and set  $n = 1$ .

**Step 1:** Given the  $(n - 1)$ th and  $n$ th iterates, choose  $\theta_n$  such that  $0 \leq \theta_n \leq \tilde{\theta}_n$  with  $\tilde{\theta}_n$  defined as follows:

$$\tilde{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \quad (18)$$

**Step 2:** Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

**Step 3:** Compute

$$z_n = P_C(w_n - \tau_n A^*(I - T)Aw_n), \quad (19)$$

where

$$T = (1 - \eta)I + \eta S((1 - \xi)I + \xi S).$$

**Step 4:** Compute

$$y_n = P_C(z_n - \lambda_n f z_n). \quad (20)$$

**Step 5:** Compute

$$\begin{aligned} u_n &= y_n - \lambda_n(fy_n - fz_n), \\ x_{n+1} &= \alpha_n g(w_n) + (1 - \alpha_n)u_n, \end{aligned} \quad (21)$$

where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|z_n - y_n\|}{\|fz_n - fy_n\|}, \lambda_n + \psi_n \right\}, & fz_n - fy_n \neq 0, \\ \lambda_n + \psi_n, & \text{otherwise} \end{cases} \quad (22)$$

and

$$\tau_n := \begin{cases} \phi_n \frac{\|(I - T)Aw_n\|^2}{\|A^*(I - T)Aw_n\|^2}, & \text{if } Aw_n \neq TAw_n, \\ \tau, & \text{otherwise } (\tau \text{ being any nonnegative real number}). \end{cases}$$

Set  $n := n + 1$  and go to **Step 1**.

---

**Remark 3.2.**

(1) By conditions (C1) and (C3), it follows from (63) that

$$\lim_{n \rightarrow \infty} \theta_n \|x_n - x_{n-1}\| = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0. \quad (23)$$

(2) Observe that by Lemma 2.5, the mapping  $T$  is quasi-nonexpansive and  $I - T$  is demiclosed at zero.

**Remark 3.3.**

- (i) We point out that condition (C2)(a) is a much weaker assumption than the sequential weakly continuity assumption used in several of the existing results in the literature.
- (ii) Observe that while the pseudomonotone operator  $f$  is not necessarily Lipschitz, our proposed method does not require any linesearch technique but uses a simple step size rule in (67), which generates a nonmonotonic sequence of step sizes. The step size is constructed such that it reduces the dependence of the algorithm on the initial step size  $\lambda_1$ .

## 4 Convergence analysis

In this section, we analyze the convergence of our proposed algorithm. First, we establish some lemmas required to prove our strong convergence theorem.

**Lemma 4.1.** *Let  $\{\lambda_n\}$  be the sequence of step sizes generated by Algorithm 3.1. Then,  $\{\lambda_n\}$  is well defined and  $\lim_{n \rightarrow \infty} \lambda_n = \lambda \in \left[ \min \left\{ \frac{\mu}{N}, \lambda_1 \right\}, \lambda_1 + \Psi \right]$ , where  $\Psi = \sum_{n=1}^{\infty} \psi_n$  and for some  $N > 0$ .*

**Proof.** Since  $f$  is uniformly continuous, then by (14) it follows that for any given  $\varepsilon > 0$ , there exists  $K < +\infty$  such that  $\|fz_n - fy_n\| \leq K\|z_n - y_n\| + \varepsilon$ . Thus, for the case  $fz_n - fy_n \neq 0$  for all  $n \geq 1$ , we have

$$\frac{\mu \|z_n - y_n\|}{\|fz_n - fy_n\|} \geq \frac{\mu \|z_n - y_n\|}{K\|z_n - y_n\| + \varepsilon} = \frac{\mu \|z_n - y_n\|}{(K + \varepsilon_1)\|z_n - y_n\|} = \frac{\mu}{N},$$

where  $\varepsilon = \varepsilon_1 \|z_n - y_n\|$  for some  $\varepsilon_1 \in (0, 1)$  and  $N = K + \varepsilon_1$ . Therefore, by the definition of  $\lambda_{n+1}$ , the sequence  $\{\lambda_n\}$  has lower bound  $\min \left\{ \frac{\mu}{N}, \lambda_1 \right\}$  and has upper bound  $\lambda_1 + \Psi$ . By Lemma 2.8, the limit  $\lim_{n \rightarrow \infty} \lambda_n$  exists and denoted by  $\lambda = \lim_{n \rightarrow \infty} \lambda_n$ . Clearly,  $\lambda \in \left[ \min \left\{ \frac{\mu}{N}, \lambda_1 \right\}, \lambda_1 + \Psi \right]$ .  $\square$

**Lemma 4.2.** *Let  $\{x_n\}$  be a sequence generated by Algorithm 3.1. Then,  $\{x_n\}$  is bounded. Furthermore, if  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , then  $\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0$ .*

**Proof.** By (67), we have

$$\lambda_{n+1} = \min \left\{ \frac{\mu \|z_n - y_n\|}{\|fz_n - fy_n\|}, \lambda_n + \psi_n \right\} \leq \frac{\mu \|z_n - y_n\|}{\|fz_n - fy_n\|},$$

which implies that

$$\|fz_n - fy_n\| \leq \frac{\mu}{\lambda_{n+1}} \|z_n - y_n\| \quad \forall n \geq 1. \quad (24)$$

Let  $q \in \Gamma$ . By applying the triangle inequality, we obtain

$$\|w_n - q\| = \|x_n + \theta_n(x_n - x_{n-1}) - q\| \leq \|x_n - q\| + \theta_n \|x_n - x_{n-1}\| = \|x_n - q\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|. \quad (25)$$

Since, by (23),  $\lim_{n \rightarrow \infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$ , then there exists a constant  $M_1 > 0$  such that  $\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq M_1 \forall n \geq 1$ . Hence, it follows from (25) that

$$\|w_n - q\| \leq \|x_n - q\| + \alpha_n M_1. \quad (26)$$

By applying Lemma 2.6(iii) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \|w_n - p\|^2 &= \|x_n - \theta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle \\ &\leq \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \|x_n - p\| \|x_n - x_{n-1}\| \\ &= \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| (2\|x_n - p\| + \theta_n \|x_n - x_{n-1}\|) \\ &\leq \|x_n - p\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|, \end{aligned} \quad (27)$$

where  $M := \sup_{n \in \mathbb{N}} \{\|x_n - p\|, \theta_n \|x_n - x_{n-1}\|\}$ .

Next, by the definition of  $y_n$  with the firmly nonexpansivity of  $P_C$ , we obtain

$$\begin{aligned} \|y_n - q\|^2 &\leq \|P_C(z_n - \lambda_n fz_n) - q\|^2 \\ &\leq \langle y_n - q, (z_n - \lambda_n fz_n) - q \rangle \\ &= \frac{1}{2} (\|y_n - q\|^2 + \|z_n - \lambda_n fz_n - q\|^2 - \|y_n - z_n + \lambda_n fz_n\|^2) \\ &= \frac{1}{2} (\|y_n - q\|^2 + \|z_n - q\|^2 + \lambda_n^2 \|fz_n\|^2 - 2\lambda_n \langle z_n - q, fz_n \rangle) - \frac{1}{2} (\|y_n - z_n\|^2 + \lambda_n^2 \|fz_n\|^2 + 2\lambda_n \langle y_n - z_n, fz_n \rangle) \\ &= \frac{1}{2} (\|y_n - q\|^2 + \|z_n - q\|^2 - \|y_n - z_n\|^2 - 2\lambda_n \langle y_n - q, fz_n \rangle), \end{aligned}$$

which implies that

$$\|y_n - q\|^2 \leq \|z_n - q\|^2 - \|y_n - z_n\|^2 - 2\lambda_n \langle y_n - q, fz_n \rangle. \quad (28)$$

In addition, by (24) and (28), we have

$$\begin{aligned} \|u_n - q\|^2 &= \|y_n - \lambda_n(fy_n - fz_n) - q\|^2 \\ &= \|y_n - q\|^2 - 2\lambda_n \langle y_n - q, fy_n - fz_n \rangle + \lambda_n^2 \|fy_n - fz_n\|^2 \\ &\leq \|z_n - q\|^2 - \|y_n - z_n\|^2 - 2\lambda_n \langle y_n - q, fz_n \rangle - 2\lambda_n \langle y_n - q, fy_n - fz_n \rangle + \lambda_n^2 \|fy_n - fz_n\|^2 \\ &\leq \|z_n - q\|^2 - \|y_n - z_n\|^2 + \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \|z_n - y_n\|^2 - 2\lambda_n \langle y_n - q, fy_n \rangle \end{aligned} \quad (29)$$

$$= \|z_n - q\|^2 - \left( 1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \right) \|y_n - z_n\|^2 - 2\lambda_n \langle y_n - q, fy_n \rangle. \quad (30)$$

Since  $y_n \in C$  and  $q \in \Gamma$ , we obtain  $\langle y_n - q, fq \rangle \geq 0$ . Thus, by the pseudomonotonicity of  $f$ , we have  $\langle y_n - q, fy_n \rangle \geq 0$ . Hence, it follows from (30) that

$$\|u_n - q\|^2 \leq \|z_n - q\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - z_n\|^2. \quad (31)$$

Next, consider the limit

$$\lim_{n \rightarrow \infty} \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = (1 - \mu^2) > 0, \quad (0 < \mu < 1). \quad (32)$$

Thus, there exists  $N_0 > 0$  such that, for all  $n > N_0$ , we have  $\left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) > 0$ . Consequently, for all  $n > N_0$ , from (31), we have

$$\|u_n - q\|^2 \leq \|z_n - q\|^2. \quad (33)$$

By Lemma 2.6(iii) and the nonexpansivity of  $P_C$ , we have

$$\begin{aligned} \|z_n - q\|^2 &= \|P_C(w_n - \tau_n A^*(I - T)Aw_n) - q\|^2 \\ &\leq \|w_n - \tau_n A^*(I - T)Aw_n - q\|^2 \\ &= \|w_n - q\|^2 + \tau_n^2 \|A^*(I - T)Aw_n\|^2 - 2\tau_n \langle w_n - q, A^*(I - T)Aw_n \rangle. \end{aligned} \quad (34)$$

By Lemma 2.6(iii) and the quasi-nonexpansivity of  $T$ , we have

$$\begin{aligned} \langle w_n - q, A^*(I - T)Aw_n \rangle &= \langle Aw_n - Aq, (I - T)Aw_n \rangle \\ &= \langle TAw_n - Aq + (I - T)Aw_n, (I - T)Aw_n \rangle \\ &= \langle TAw_n - Aq, (I - T)Aw_n \rangle + \langle (I - T)Aw_n, (I - T)Aw_n \rangle \\ &= \langle TAw_n - Aq, (I - T)Aw_n \rangle + \|(I - T)Aw_n\|^2 \\ &= \frac{1}{2} [\|TAw_n - Aq + (I - T)Aw_n\|^2 - \|TAw_n - Aq\|^2 - \|(I - T)Aw_n\|^2] \\ &\quad + \|(I - T)Aw_n\|^2 \\ &= \frac{1}{2} [\|Aw_n - Aq\|^2 - \|TAw_n - Aq\|^2 + \|(I - T)Aw_n\|^2] \\ &\geq \frac{1}{2} [\|Aw_n - Aq\|^2 - \|Aw_n - Aq\|^2 + \|(I - T)Aw_n\|^2] \\ &= \frac{1}{2} \|(I - T)Aw_n\|^2. \end{aligned} \quad (35)$$

Substituting (35) into (34) and applying the definition of  $\tau_n$  and the condition on  $\phi_n$ , we obtain

$$\begin{aligned} \|z_n - q\|^2 &\leq \|w_n - q\|^2 + \tau_n^2 \|A^*(I - T)Aw_n\|^2 - \tau_n \|(I - T)Aw_n\|^2 \\ &= \|w_n - q\|^2 - \tau_n (\|(I - T)Aw_n\|^2 - \tau_n \|A^*(I - T)Aw_n\|^2) \\ &= \|w_n - q\|^2 - \tau_n (1 - \phi_n) \|(I - T)Aw_n\|^2 \end{aligned} \quad (36)$$

$$\leq \|w_n - q\|^2. \quad (37)$$

By applying (27), (31), and (36), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|\alpha_n g(w_n) + (1 - \alpha_n)u_n - q\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \|u_n - q\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \|z_n - q\|^2 - (1 - \alpha_n) \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - z_n\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) [\|w_n - q\|^2 - \tau_n (1 - \phi_n) \|(I - T)Aw_n\|^2] \\ &\quad - (1 - \alpha_n) \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - z_n\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \left( \|x_n - q\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right) \\ &\quad - \tau_n (1 - \phi_n) (1 - \alpha_n) \|(I - T)Aw_n\|^2 - (1 - \alpha_n) \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|y_n - z_n\|^2. \end{aligned} \quad (38)$$

Furthermore, by (26), (33), and (37), we obtain

$$\begin{aligned}
\|x_{n+1} - q\| &= \|\alpha_n g(w_n) + (1 - \alpha_n)u_n - q\| \\
&\leq \alpha_n \|g(w_n) - q\| + (1 - \alpha_n) \|u_n - q\| \\
&\leq \alpha_n \|g(w_n) - g(q)\| + \alpha_n \|g(q) - q\| + (1 - \alpha_n) \|u_n - q\| \\
&\leq \alpha_n \rho \|w_n - q\| + \alpha_n \|g(q) - q\| + (1 - \alpha_n) \|z_n - q\| \\
&\leq \alpha_n \rho \|w_n - q\| + \alpha_n \|g(q) - q\| + (1 - \alpha_n) \|w_n - q\| \\
&\leq \alpha_n \rho [\|x_n - q\| + \alpha_n M_1] + \alpha_n \|g(q) - q\| + (1 - \alpha_n) [\|x_n - q\| + \alpha_n M_1] \\
&= [1 - \alpha_n(1 - \rho)] \|x_n - q\| + \alpha_n \|g(q) - q\| + \alpha_n [1 - \alpha_n(1 - \rho)] M_1 \\
&\leq [1 - \alpha_n(1 - \rho)] \|x_n - q\| + \alpha_n(1 - \rho) \left[ \frac{\|g(q) - q\|}{1 - \rho} + \frac{M_1}{1 - \rho} \right] \\
&\leq \max \left\{ \|x_n - q\|, \left[ \frac{\|g(q) - q\|}{1 - \rho} + \frac{M_1}{1 - \rho} \right] \right\} \\
&\vdots \\
&\leq \max \left\{ \|x_{N_0} - q\|, \left[ \frac{\|g(q) - q\|}{1 - \rho} + \frac{M_1}{1 - \rho} \right] \right\},
\end{aligned}$$

which implies that  $\{x_n\}$  is bounded. Consequently,  $\{w_n\}$ ,  $\{z_n\}$ ,  $\{y_n\}$ , and  $\{u_n\}$  are bounded.

Moreover, by (66) and the fact that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , we have

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = \lim_{n \rightarrow \infty} \alpha_n \|g(w_n) - u_n\| = 0. \quad \square$$

**Lemma 4.3.** *Let  $\{z_n\}$ ,  $\{w_n\}$ , and  $\{y_n\}$  be sequences generated by Algorithm 3.1 such that  $\lim_{n \rightarrow \infty} \|z_n - w_n\| = 0 = \lim_{n \rightarrow \infty} \|z_n - y_n\|$ . If there exists a subsequence  $\{y_{n_k}\}$  of  $\{y_n\}$  that converges weakly to  $p \in H_1$ , then  $p \in \Gamma$ .*

**Proof.** Suppose  $\{y_{n_k}\}$  is a subsequence of  $\{y_n\}$  such that  $y_{n_k} \rightharpoonup p$ . Then, by the hypothesis of the lemma, we have  $z_{n_k} \rightharpoonup p$  and  $w_{n_k} \rightharpoonup p$ . Since  $A$  is a bounded linear operator, we have  $Aw_{n_k} \rightharpoonup Ap$ . From (36) and by the hypothesis of the lemma, we have

$$\tau_{n_k} (1 - \phi_{n_k}) \|(I - T)Aw_{n_k}\|^2 \leq \|w_{n_k} - q\|^2 - \|z_{n_k} - q\|^2 \leq \|w_{n_k} - z_{n_k}\| (\|w_{n_k} - q\| + \|z_{n_k} - q\|) \rightarrow 0, \quad k \rightarrow \infty.$$

By the definition of  $\tau_n$ , we obtain

$$\phi_{n_k} (1 - \phi_{n_k}) \frac{\|(I - T)Aw_{n_k}\|^4}{\|A^*(I - T)Aw_{n_k}\|^2} \rightarrow 0, \quad k \rightarrow \infty,$$

which implies that

$$\frac{\|(I - T)Aw_{n_k}\|^2}{\|A^*(I - T)Aw_{n_k}\|} \rightarrow 0, \quad k \rightarrow \infty.$$

Since  $\|A^*(I - T)Aw_{n_k}\|$  is bounded, then it follows that

$$\|(I - T)Aw_{n_k}\| \rightarrow 0, \quad k \rightarrow \infty.$$

Since  $I - T$  is demiclosed at zero and  $Aw_{n_k} \rightharpoonup Ap$ , then by Lemma 2.5(i), we have

$$Ap \in F(T) = F(S). \quad (39)$$

Since  $z_{n_k} \rightharpoonup p$ ,  $\lim_{k \rightarrow \infty} \|z_{n_k} - y_{n_k}\| = 0$ , and  $\{y_n\} \subset C$ , we have  $p \in C$ . From  $y_{n_k} = P_C(z_{n_k} - \lambda_{n_k} f z_{n_k})$ , we have  $\langle z_{n_k} - \lambda_{n_k} f z_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq 0$ ,  $\forall x \in C$ , which implies that

$$\frac{1}{\lambda_{n_k}} \langle z_{n_k} - y_{n_k}, x - y_{n_k} \rangle \leq \langle f z_{n_k}, x - y_{n_k} \rangle, \quad \forall x \in C,$$

which is equivalent to

$$\frac{1}{\lambda_{n_k}} \langle z_{n_k} - y_{n_k}, x - y_{n_k} \rangle + \langle fz_{n_k}, y_{n_k} - z_{n_k} \rangle \leq \langle fz_{n_k}, x - z_{n_k} \rangle, \quad \forall x \in C. \quad (40)$$

Since the subsequence  $\{z_{n_k}\}$  is weakly convergent to  $z \in H_1$ , then  $\{z_{n_k}\}$  is a bounded subsequence. Moreover, since  $f$  is uniformly continuous and  $\|z_{n_k} - y_{n_k}\| \rightarrow 0$ , we have that  $\{fz_{n_k}\}$  and  $\{y_{n_k}\}$  are bounded as well. Since  $\lim_{k \rightarrow \infty} \lambda_{n_k} = \lambda > 0$ , then from (40), we have

$$\liminf_{k \rightarrow \infty} \langle fz_{n_k}, x - z_{n_k} \rangle \geq 0 \quad \forall x \in C. \quad (41)$$

Furthermore, we have

$$\langle fy_{n_k}, x - y_{n_k} \rangle = \langle fy_{n_k} - fz_{n_k}, x - z_{n_k} \rangle + \langle fz_{n_k}, x - z_{n_k} \rangle + \langle fy_{n_k}, z_{n_k} - y_{n_k} \rangle. \quad (42)$$

Since  $\|z_{n_k} - y_{n_k}\| \rightarrow 0$ , then by the uniform continuity of  $f$ , we obtain  $\lim_{k \rightarrow \infty} \|fz_{n_k} - fy_{n_k}\| = 0$ . This combined with (41) and (42) provides

$$\liminf_{n \rightarrow \infty} \langle fy_{n_k}, x - y_{n_k} \rangle \geq 0.$$

Next, let  $\{\Phi_k\}$  be a decreasing sequence of positive numbers such that  $\Phi_k \rightarrow 0$  as  $k \rightarrow \infty$ . Let  $N_k$  represent the smallest positive integer for any  $k$  such that

$$\langle fy_{n_j}, x - y_{n_j} \rangle + \Phi_k \geq 0 \quad \forall j \geq N_k. \quad (43)$$

It is clear that the sequence  $\{N_k\}$  is increasing since  $\{\Phi_k\}$  is decreasing. From  $\{y_{N_k}\} \subset C$ , for any  $k$ , suppose that  $fy_{N_k} \neq 0$  (otherwise  $y_{N_k}$  is a solution) and let

$$u_{N_k} = \frac{fy_{N_k}}{\|fy_{N_k}\|^2}.$$

Then,  $\langle fy_{N_k}, u_{N_k} \rangle = 1$  for each  $k$ . From (43), we have

$$\langle fy_{N_k}, x + \Phi_k u_{N_k} - y_{N_k} \rangle \geq 0 \quad \forall k.$$

By the pseudomonotonicity of  $f$ , we obtain

$$\langle f(x + \Phi_k u_{N_k}), x + \Phi_k u_{N_k} - y_{N_k} \rangle \geq 0,$$

which is equivalent to

$$\langle fx, x - y_{N_k} \rangle \geq \langle fx - f(x + \Phi_k u_{N_k}), x + \Phi_k u_{N_k} - y_{N_k} \rangle - \Phi_k \langle fx, u_{N_k} \rangle. \quad (44)$$

Since  $z_{n_k} \rightarrow p$  and  $\lim_{k \rightarrow \infty} \|z_{n_k} - y_{n_k}\| = 0$ , we have  $y_{N_k} \rightarrow p \in C$ . We assume that  $fp \neq 0$  (otherwise  $p$  is a solution). Since  $f$  satisfies condition (C2)(a), we obtain

$$0 < \|fp\| \leq \liminf_{k \rightarrow \infty} \|fy_{n_k}\|.$$

Using  $\{y_{N_k}\} \subset \{y_{n_k}\}$  and  $\Phi_k \rightarrow 0$  as  $k \rightarrow \infty$ , we have

$$0 \leq \limsup_{k \rightarrow \infty} \|\Phi_k u_{N_k}\| = \limsup_{k \rightarrow \infty} \left( \frac{\Phi_k}{\|fy_{N_k}\|} \right) \leq \frac{\limsup_{k \rightarrow \infty} \Phi_k}{\liminf_{k \rightarrow \infty} \|fy_{n_k}\|} = 0,$$

which implies that  $\lim_{k \rightarrow \infty} \Phi_k u_{N_k} = 0$ . From the facts that  $f$  is uniformly continuous,  $\{y_{N_k}\}$  and  $\{u_{N_k}\}$  are bounded and  $\lim_{k \rightarrow \infty} \Phi_k u_{N_k} = 0$ , it follows from (44) that

$$\liminf_{k \rightarrow \infty} \langle fx, x - y_{N_k} \rangle \geq 0.$$

Thus, we have

$$\langle fx, x - p \rangle = \lim_{k \rightarrow \infty} \langle fx, x - y_{N_k} \rangle = \liminf_{k \rightarrow \infty} \langle fx, x - y_{N_k} \rangle \geq 0 \quad \forall x \in C.$$

Thus, by Lemma 2.10, we obtain  $p \in \text{VI}(C, f)$ . This together with (39) implies that  $p \in \Gamma$  as required.  $\square$

**Theorem 4.4.** *Let  $\{x_n\}$  be a sequence generated by Algorithm 3.1 under conditions (C1)–(C4). Then,  $\{x_n\}$  converges strongly to  $z = P_{\Gamma}g(z)$ .*

**Proof.** Let  $z = P_{\Gamma}g(z)$ , we consider two cases to prove the theorem.

**Case 1:** Suppose  $\{\|x_n - z\|\}$  is monotone decreasing. Then, by Lemma 4.2  $\{\|x_n - z\|\}$  is convergent. Hence,

$$\lim_{n \rightarrow \infty} \|x_n - z\| = \lim_{n \rightarrow \infty} \|x_{n+1} - z\|. \quad (45)$$

By (32) and (45), and  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , from (38), we obtain

$$\lim_{n \rightarrow \infty} \|y_n - z_n\| = 0. \quad (46)$$

Next, by the definition of  $u_n$  and applying the uniform continuity of  $f$ , we have

$$\|u_n - y_n\| = \lambda_n \|f y_n - f z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (47)$$

From the definition of  $w_n$  and by Remark 3.2 (1), we have

$$\|w_n - x_n\| = \theta_n \|x_n - x_{n-1}\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (48)$$

From (46) and (47), we have

$$\|u_n - z_n\| \leq \|u_n - y_n\| + \|y_n - z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (49)$$

From (38), we have

$$\tau_n(1 - \phi_n)(1 - \alpha_n)\|(I - T)Aw_n\|^2 \leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \left[ \|x_n - q\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] - \|x_{n+1} - q\|^2.$$

By applying (45) together with the fact that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , we obtain

$$\tau_n(1 - \phi_n)\|(I - T)Aw_n\|^2 \rightarrow 0, \quad n \rightarrow \infty.$$

By the definition of  $\tau_n$  and the condition on  $\phi_n$ , we obtain

$$\tau_n(1 - \phi_n) \frac{\|(I - T)Aw_n\|^4}{\|A^*(I - T)Aw_n\|^2} \rightarrow 0, \quad n \rightarrow \infty.$$

Consequently, we have

$$\frac{\|(I - T)Aw_n\|^2}{\|A^*(I - T)Aw_n\|} \rightarrow 0, \quad n \rightarrow \infty.$$

Since  $\|A^*(I - T)Aw_n\|$  is bounded, then it follows that

$$\|(I - T)Aw_n\| \rightarrow 0, \quad n \rightarrow \infty. \quad (50)$$

From this, we obtain

$$\|A^*(I - T)Aw_n\| \leq \|A^*\| \|(I - T)Aw_n\| = \|A\| \|(I - T)Aw_n\| \rightarrow 0, \quad n \rightarrow \infty. \quad (51)$$

From (34) and (37), we observe that

$$\|w_n - \tau_n A^*(I - T)Aw_n - q\|^2 \leq \|w_n - q\|^2. \quad (52)$$

By Lemma 2.6(iii), (52) together with the firmly nonexpansivity of  $P_C$ , we have

$$\begin{aligned} \|z_n - q\|^2 &= \|P_C(w_n - \tau_n A^*(I - T)Aw_n) - q\|^2 \\ &\leq \langle z_n - q, w_n - \tau_n A^*(I - T)Aw_n - q \rangle \\ &= \frac{1}{2} (\|z_n - q\|^2 + \|w_n - \tau_n A^*(I - T)Aw_n - q\|^2 - \|z_n - w_n + \tau_n A^*(I - T)Aw_n\|^2) \\ &\leq \frac{1}{2} [\|z_n - q\|^2 + \|w_n - q\|^2 - (\|z_n - w_n\|^2 + \tau_n^2 \|A^*(I - T)Aw_n\|^2 + 2\tau_n \langle z_n - w_n, A^*(I - T)Aw_n \rangle)]. \end{aligned}$$

From which we obtain

$$\begin{aligned} \|z_n - q\|^2 &\leq \|w_n - q\|^2 - \|z_n - w_n\|^2 - 2\tau_n \langle z_n - w_n, A^*(I - T)Aw_n \rangle \\ &\leq \|w_n - q\|^2 - \|z_n - w_n\|^2 + 2\tau_n \|w_n - z_n\| \|A^*(I - T)Aw_n\| \\ &\leq \|w_n - q\|^2 - \|z_n - w_n\|^2 + 2M_2 \|A^*(I - T)Aw_n\|, \end{aligned} \quad (53)$$

where  $M_2 = \sup\{\tau_n \|w_n - z_n\| : n \geq 1\}$ . Now, by applying Lemma 2.6 and equations (27), (33), and (53), we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \|u_n - q\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \|z_n - q\|^2 \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) [\|w_n - q\|^2 - \|z_n - w_n\|^2 + 2M_2 \|A^*(I - T)Aw_n\|] \\ &\leq \alpha_n \|g(w_n) - q\|^2 + (1 - \alpha_n) \left[ \|x_n - q\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| - \|z_n - w_n\|^2 + 2M_2 \|A^*(I - T)Aw_n\| \right] \\ &= (1 - \alpha_n) \|x_n - q\|^2 + \alpha_n \left[ \|g(w_n) - q\|^2 + 3M(1 - \alpha_n) \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] \\ &\quad + 2M_2(1 - \alpha_n) \|A^*(I - T)Aw_n\| - (1 - \alpha_n) \|z_n - w_n\|^2, \end{aligned}$$

which implies that

$$\begin{aligned} (1 - \alpha_n) \|z_n - w_n\|^2 &\leq (1 - \alpha_n) \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n [\|g(w_n) - q\|^2 + 3M(1 - \alpha_n) \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|] \\ &\quad + 2M_2(1 - \alpha_n) \|A^*(I - T)Aw_n\|. \end{aligned}$$

By applying (45), (51) and the fact that  $\lim_{n \rightarrow \infty} \alpha_n = 0$ , we obtain

$$\lim_{n \rightarrow \infty} \|z_n - w_n\| = 0. \quad (54)$$

Moreover, from (48) and (54), we obtain

$$\|x_n - z_n\| \leq \|x_n - w_n\| + \|w_n - z_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (55)$$

From the definition of  $x_{n+1}$ , we have

$$\|x_{n+1} - z_n\| = \|\alpha_n(g(w_n) - z_n) + (1 - \alpha_n)(u_n - z_n)\| \leq \alpha_n \|g(w_n) - z_n\| + (1 - \alpha_n) \|u_n - z_n\|.$$

By (49) and the condition on  $\alpha_n$ , we obtain

$$\lim_{n \rightarrow \infty} \|x_{n+1} - z_n\| = 0. \quad (56)$$

Similarly, from (55) and (56), we have

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\| \rightarrow 0. \quad (57)$$

Since  $\{x_n\}$  is bounded, there exists a subsequence  $\{x_{n_k}\}$  of  $\{x_n\}$  such that  $\{x_{n_k}\}$  converges weakly to some  $p \in H_1$  and

$$\limsup_{n \rightarrow \infty} \langle g(z) - z, x_n - z \rangle = \lim_{k \rightarrow \infty} \langle g(z) - z, x_{n_k} - z \rangle = \langle g(z) - z, p - z \rangle. \quad (58)$$

In addition, by equations (46) and (54) and Lemma 4.3, we obtain that  $p \in \Gamma$ . Since  $z = P_\Gamma g(z)$  and by (58), we obtain

$$\limsup_{n \rightarrow \infty} \langle g(z) - z, x_n - z \rangle \leq 0,$$

which follows from (57) that

$$\limsup_{n \rightarrow \infty} \langle g(z) - z, x_{n+1} - z \rangle = \limsup_{n \rightarrow \infty} (\langle g(z) - z, x_{n+1} - x_n \rangle + \langle g(z) - z, x_n - z \rangle) \leq 0. \quad (59)$$

Next, by applying equations (27), (33), and (37) and Lemma 2.6(ii), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle g(w_n) - z, x_{n+1} - z \rangle \\
&= (1 - \alpha_n)^2 \|u_n - z\|^2 + 2\alpha_n \langle g(w_n) - g(z), x_{n+1} - z \rangle + 2\alpha_n \langle g(z) - z, x_{n+1} - z \rangle \\
&\leq (1 - \alpha_n)^2 \left[ \|x_n - z\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] + 2\alpha_n \rho \|w_n - z\| \|x_{n+1} - z\| + 2\alpha_n \langle g(z) - z, x_{n+1} - z \rangle \\
&\leq (1 - \alpha_n)^2 \left[ \|x_n - z\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] + \alpha_n \rho \|w_n - z\|^2 + \alpha_n \rho \|x_{n+1} - z\|^2 + 2\alpha_n \langle g(z) - z, x_{n+1} - z \rangle \\
&\leq (1 - \alpha_n)^2 \left[ \|x_n - z\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] + \alpha_n \rho \left[ \|x_n - z\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] + \alpha_n \rho \|x_{n+1} - z\|^2 \\
&\quad + 2\alpha_n \langle g(z) - z, x_{n+1} - z \rangle \\
&= ((1 - \alpha_n)^2 + \alpha_n \rho) \left[ \|x_n - z\|^2 + 3M\alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right] + \alpha_n \rho \|x_{n+1} - z\|^2 + 2\alpha_n \langle g(z) - z, x_{n+1} - z \rangle.
\end{aligned}$$

From which we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\leq \frac{(1 - 2\alpha_n + \alpha_n^2 + \alpha_n \rho)}{1 - \alpha_n \rho} \|x_n - z\|^2 + 3M \frac{((1 - \alpha_n)^2 + \alpha_n \rho)}{(1 - \alpha_n \rho)} \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \\
&\quad + \frac{2\alpha_n}{(1 - \alpha_n \rho)} \langle g(z) - z, x_{n+1} - z \rangle \\
&= \frac{(1 - 2\alpha_n + \alpha_n \rho)}{1 - \alpha_n \rho} \|x_n - z\|^2 + \frac{\alpha_n^2}{(1 - \alpha_n \rho)} \|x_n - z\|^2 + 3M \frac{((1 - \alpha_n)^2 + \alpha_n \rho)}{(1 - \alpha_n \rho)} \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \\
&\quad + \frac{2\alpha_n}{(1 - \alpha_n \rho)} \langle g(z) - z, x_{n+1} - z \rangle \tag{60} \\
&\leq \left( 1 - \frac{2\alpha_n(1 - \rho)}{(1 - \alpha_n \rho)} \right) \|x_n - z\|^2 + \frac{2\alpha_n(1 - \rho)}{(1 - \alpha_n \rho)} \left\{ \frac{\alpha_n}{2(1 - \rho)} M_3 + \frac{3M((1 - \alpha_n)^2 + \alpha_n \rho)}{2(1 - \rho)} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \right. \\
&\quad \left. + \frac{1}{(1 - \rho)} \langle g(z) - z, x_{n+1} - z \rangle \right\},
\end{aligned}$$

where  $M_3 = \sup\{\|x_n - z\|^2 : n \in \mathbb{N}\}$ . By Remark 3.2(1), (59), and Lemma 2.4, we obtain that  $\lim_{n \rightarrow \infty} \|x_n - z\| = 0$ . Thus,  $\{x_n\}$  converges strongly to  $z = P_T g(z)$ .

**Case 2:** Suppose that  $\{\|x_n - z\|\}$  is not monotone decreasing. Then, there exists a subsequence  $\{\|x_{n_j} - z\|\}$  of  $\{\|x_n - z\|\}$  such that

$$\|x_{n_j} - z\|^2 < \|x_{n_{j+1}} - z\|^2 \quad \forall j \in \mathbb{N}.$$

Then, by Lemma 2.7, there exists a nondecreasing sequence  $\{m_k\}$  of  $\mathbb{N}$  such that as  $\lim_{k \rightarrow \infty} m_k = +\infty$  and the following inequalities hold:

$$\|x_{m_k} - z\|^2 \leq \|x_{m_{k+1}} - z\|^2 \quad \text{and} \quad \|x_k - z\|^2 \leq \|x_{m_{k+1}} - z\|^2. \tag{61}$$

By following similar arguments as in **Case 1**, we obtain

$$\lim_{k \rightarrow \infty} \|y_{m_k} - z_{m_k}\| = 0,$$

$$\lim_{k \rightarrow \infty} \|x_{m_k} - w_{m_k}\| = 0,$$

$$\lim_{k \rightarrow \infty} \|x_{m_k} - x_{m_{k+1}}\| = 0,$$

and

$$\limsup_{k \rightarrow \infty} \langle g(z) - z, x_{m_{k+1}} - z \rangle \leq 0. \tag{62}$$

From (60), we obtain

$$\begin{aligned} \|x_{m_{k+1}} - z\|^2 \leq & \left(1 - \frac{2\alpha_{m_k}(1-\rho)}{1-\alpha_{m_k}\rho}\right) \|x_{m_k} - z\|^2 + \frac{2\alpha_{m_k}(1-\rho)}{1-\alpha_{m_k}\rho} \left[ \frac{\alpha_{m_k}}{2(1-\rho)} M_3 \right. \\ & \left. + \frac{3M((1-\alpha_{m_k})^2 + \alpha_{m_k}\rho)}{2(1-\rho)} \frac{\theta_{m_k}}{\alpha_{m_k}} \|x_{m_k} - x_{m_{k-1}}\| + \frac{1}{(1-\rho)} \langle g(z) - z, x_{m_{k+1}} - z \rangle \right], \end{aligned}$$

which implies that

$$\|x_{m_{k+1}} - z\|^2 \leq \frac{\alpha_{m_k}}{2(1-\rho)} M_3 + \frac{3M((1-\alpha_{m_k})^2 + \alpha_{m_k}\rho)}{2(1-\rho)} \frac{\theta_{m_k}}{\alpha_{m_k}} \|x_{m_k} - x_{m_{k-1}}\| + \frac{1}{1-\rho} \langle g(z) - z, x_{m_{k+1}} - z \rangle.$$

From (61), we have that

$$\begin{aligned} \|x_k - z\|^2 & \leq \|x_{m_{k+1}} - z\|^2 \\ & \leq \frac{\alpha_{m_k}}{2(1-\rho)} M_3 + \frac{3M((1-\alpha_{m_k})^2 + \alpha_{m_k}\rho)}{2(1-\rho)} \frac{\theta_{m_k}}{\alpha_{m_k}} \|x_{m_k} - x_{m_{k-1}}\| + \frac{1}{1-\rho} \langle g(z) - z, x_{m_{k+1}} - z \rangle \end{aligned}$$

By Remark 3.2, (62), and the fact that  $\lim_{k \rightarrow \infty} \alpha_{m_k} = 0$ , we have  $\limsup_{k \rightarrow \infty} \|x_k - z\| \leq 0$ . Thus,  $\{x_k\}$  converges strongly to  $z = P_{\Gamma}g(z)$ .  $\square$

If we set  $g(x) = v$  for arbitrary but fixed  $v \in H_1$  and for all  $x \in H_1$  in Algorithm 3.1, we obtain the following result as a corollary to Theorem 4.4.

**Corollary 4.5.** *Let  $v \in H_1$  be a fixed element and let  $\{x_n\}$  be a sequence generated by the following algorithm such that conditions (C1)–(C4) of Theorem 4.4 hold. Then,  $x_n$  converges strongly to  $z = P_{\Gamma}(v)$ .*

---

#### Algorithm 4.6

**Step 0:** Select  $x_0, x_1 \in H, \lambda_1 > 0, \mu \in (0, 1)$ , and  $0 < \phi_1 \leq \phi_n \leq \phi_2 < 1$  and set  $n = 1$ .

**Step 1:** Given the  $(n - 1)$ th and  $n$ th iterates, choose  $\theta_n$  such that  $0 \leq \theta_n \leq \tilde{\theta}_n$  with  $\tilde{\theta}_n$  defined as follows:

$$\tilde{\theta}_n = \begin{cases} \min \left\{ \theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|} \right\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases} \quad (63)$$

**Step 2:** Compute

$$w_n = x_n + \theta_n(x_n - x_{n-1}).$$

**Step 3:** Compute

$$z_n = P_C(w_n - \tau_n A^*(I - T)Aw_n), \quad (64)$$

where

$$T = (1 - \eta)I + \eta S((1 - \mu)I + \mu S).$$

**Step 4:** Compute

$$y_n = P_C(z_n - \lambda_n f z_n). \quad (65)$$

**Step 5:** Compute

$$\begin{aligned} u_n &= y_n - \lambda_n(fy_n - fz_n), \\ x_{n+1} &= \alpha_n v + (1 - \alpha_n)u_n, \end{aligned} \quad (66)$$

where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|z_n - y_n\|}{\|fz_n - fy_n\|}, \lambda_n + \psi_n \right\}, & fz_n - fy_n \neq 0, \\ \lambda_n + \psi_n, & \text{otherwise} \end{cases} \quad (67)$$

and

$$\tau_n := \begin{cases} \phi_n \frac{\|(I-T)Aw_n\|^2}{\|A^*(I-T)Aw_n\|^2}, & \text{if } Aw_n \neq TAw_n, \\ \tau, & \text{otherwise } (\tau \text{ being any nonnegative real number}). \end{cases}$$

Set  $n := n + 1$  and go to **Step 1**.

## 5 Numerical examples

In this section, we present some numerical examples and compare our proposed method with some of the existing methods in the literature. We compare our proposed Algorithm 3.1 (Proposed Alg.) with Appendix A.1 (Tian and Jiang Alg.), Appendix A.2 (Pham *et al.* Alg.), Appendix A.3 (He *et al.*), and Algorithm 1.6. In our computations, we choose  $\alpha_n = \frac{1}{2n+1}$ ,  $\varepsilon_n = \frac{1}{(2n+1)^2}$ ,  $g(x) = \frac{x}{2}$ ,  $\mu = 0.95$ ,  $\lambda = 1.5$ ,  $\phi_n = \frac{2n}{3n+1}$ ,  $\psi_n = \frac{100}{(2n+1)^3}$ ,  $\theta = 0.95$ ,  $\eta = 0.22$ ,  $\xi = 0.27$ ,  $\tau_n = \tau = 0.01$ ,  $\lambda_n = 0.5$ , and  $Sx = \frac{x}{2}$ .

As for Algorithm of He *et al.* [23], we set  $\mu_1 = 5(\|T'H T\| + L_1)/\nu$ ,  $\mu_2 = 10(\|T'H T\| + L_2)/\nu$ ,  $\nu = 0.8$ ,  $\rho = 2.5$ ,  $\lambda_n H = \frac{2}{\|T'H T\|} I_N$ , and  $\gamma = 1.5$ .

**Example 5.1.** In this example, we consider Example 5.2 of He *et al.* [23]. Let

$$\min\{G_1(x) + G_2(y) \mid Tx = y, x \in \kappa, y \in \Gamma\} \quad (68)$$

be a separable, convex, and quadratic programming problem, where

$$G_1(x) = \frac{1}{2}x'M_1x + q_1'x \quad (x' \text{ means the transpose of } x)$$

and

$$G_2(y) = \frac{1}{2}y'M_2y + q_2'y$$

The problem (68) can be rewritten as SVIP (equations (8) and (9)), where

$$f(x) = M_x + q_1 \quad \text{and} \quad g(y) = M_2(y) + q_2. \quad (69)$$

The matrices  $M_1$  and  $M_2$  are defined as  $M_i = V_i \Sigma_i V_i'$ , where  $V_i = I - \frac{2v_i v_i'}{\|v_i\|^2}$  and  $\Sigma_i = \text{diag}(\sigma_{i1}, \sigma_{i2}, \dots, \sigma_{iN_i})$  are the householder and the diagonal matrix, respectively, with  $N_i = N$  and  $N_2 = m$  being the dimensional of  $x$  and  $y$ , respectively. Moreover, let  $\sigma_{i,j}$  be defined as follows:

$$\sigma_{i,j} = \cos \frac{j\pi}{N_i + 1} + 1 + \frac{\cos \frac{\pi}{N_i + 1} + 1 - \hat{C}_i \left( \cos \frac{N_i \pi}{N_i + 1} + 1 \right)}{\hat{C}_i - 1}, \quad j = 1, 2, \dots, N_i,$$

where  $\hat{C}_i$  is the present condition of  $M_i$ . We set  $\hat{C}_i = 10^4$  and  $q_i = 0$ ,  $i = 1, 2$ , and uniformly take the vector  $v_i \in \mathbb{R}^{N_i}$  ( $i = 1, 2$ ) in  $(-1, 1)$ . Hence,  $f$  and  $g$  are monotone and Lipschitz continuous operators with  $L_i = \|M_i\|$ ,  $i = 1, 2$ . Moreover, the bounded linear operator  $T \in \mathbb{R}^{M \times N}$  is generated with independent Gaussian components distributed in the interval  $(0, 1)$ , and then each column of  $T$  is normalized with the unit norm. Let  $\kappa = \{x \in \mathbb{R}^N : \|x\| \leq 1\}$  and  $\Gamma = \{y \in \mathbb{R}^m : \mathfrak{l} \leq y \leq u\}$ , where the smallest and largest components of  $\tilde{y} = T\tilde{x}$ , where  $\tilde{x}$  is the sparse vector whose components are evenly distributed in  $(0, 1)$ , are the entries of  $\mathfrak{l} \in \mathbb{R}^m$  and  $u \in \mathbb{R}^m$ , respectively. More so, we consider different scenarios of the problem's dimensionality with  $n = 100, 300, 500, 700$  and  $m = \frac{N}{2}$ . Let the starting point for Algorithm 3.1 be  $x_1 = (1, 1, \dots, 1)$ , while the entries of  $x_0$  are randomly generated in  $[0, 1]$ . For Algorithm of He *et al.* [23],

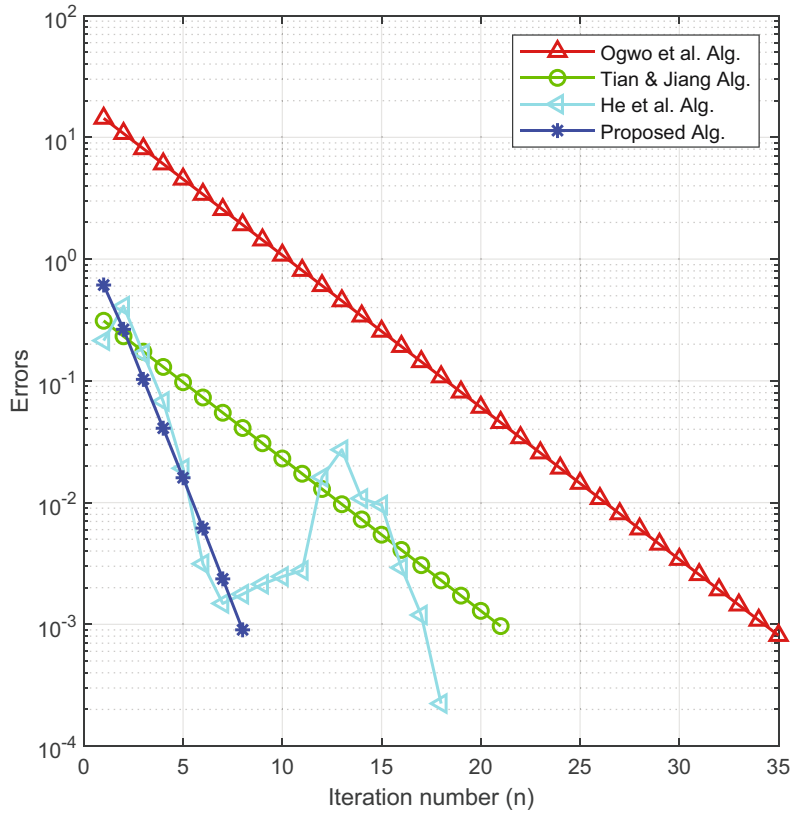


Figure 1: Example 5.1: ( $N = 5, m = 5$ ).

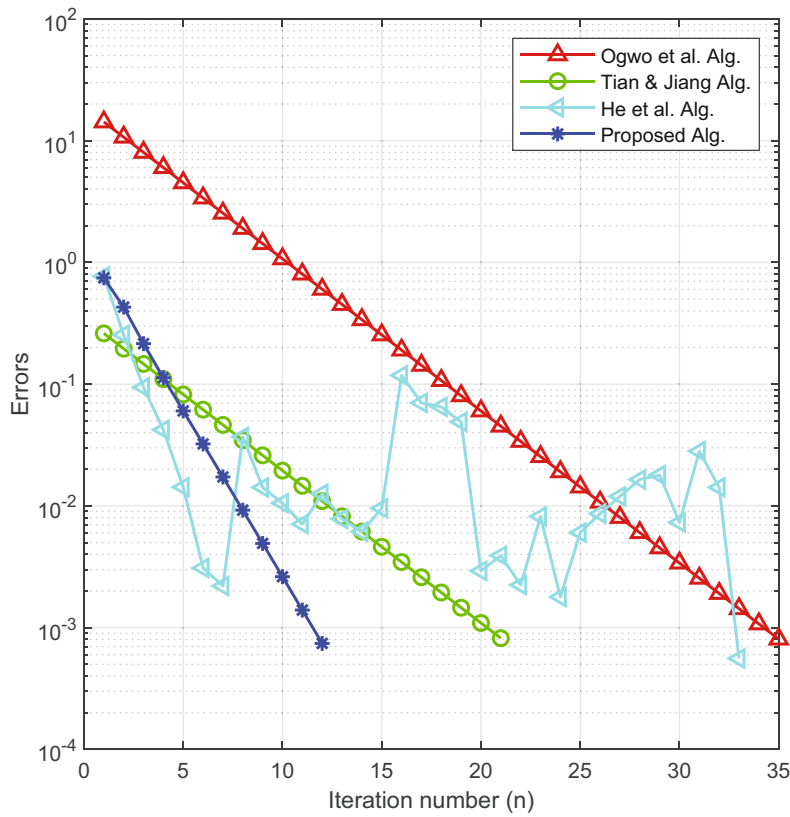


Figure 2: Example 5.1: ( $N = 10, m = 5$ ).

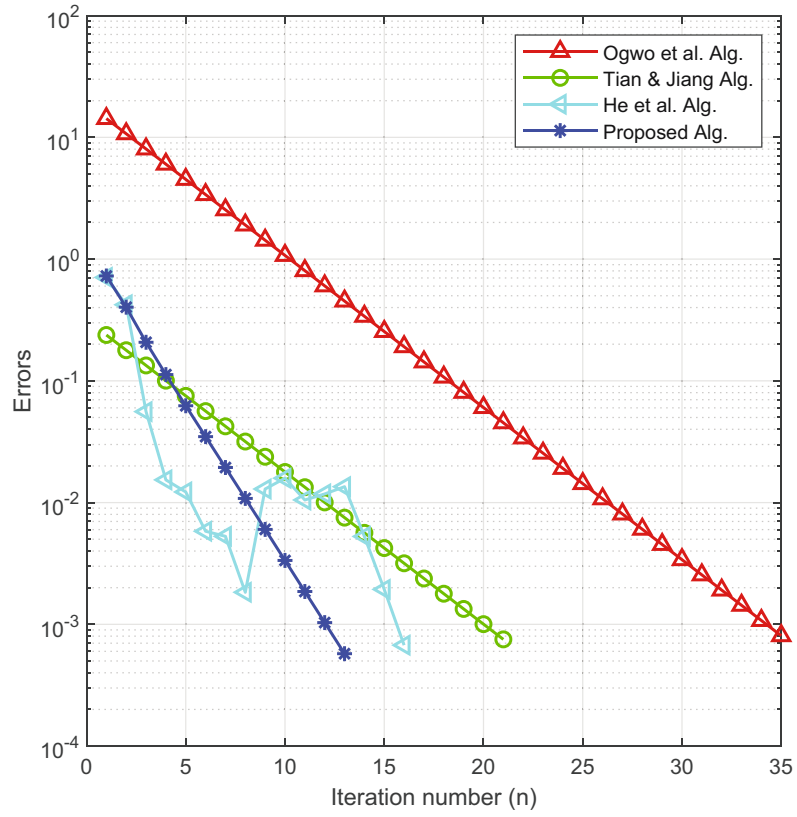


Figure 3: Example 5.1: ( $N = 10, m = 10$ ).

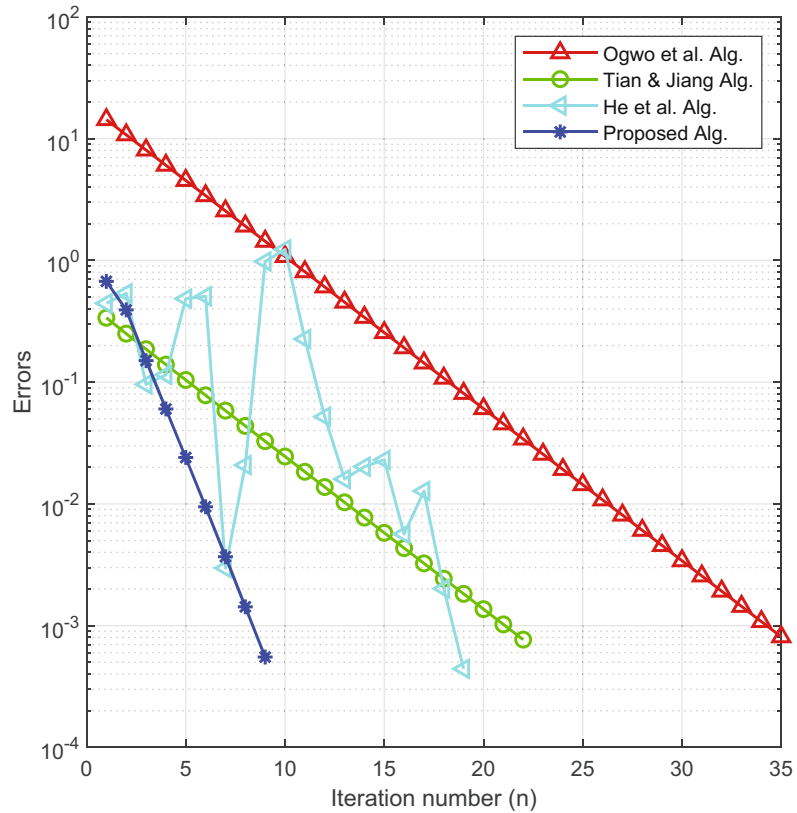


Figure 4: Example 5.1: ( $N = 5, m = 10$ ).

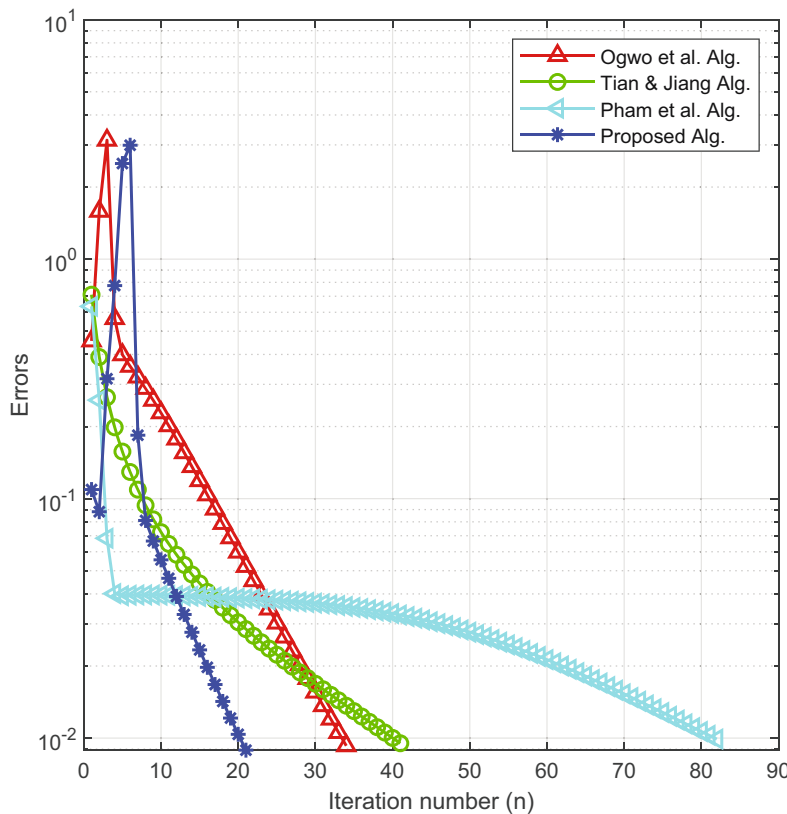
**Table 1:** Numerical results for Example 5.1

	$(N = 5, m = 5)$		$(N = 10, m = 5)$		$(N = 10, m = 10)$		$(N = 10, m = 10)$	
	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time
Ogwo et al. Algorithm 1.6	35	0.0053	35	0.0061	35	0.0050	35	0.0197
Tian and Jiang App. A.1	21	0.0042	21	0.0043	21	0.0040	22	0.0148
He et al. App. A.3	18	0.0146	33	0.0157	16	0.0145	19	0.0258
Proposed Algorithm 3.1	8	0.0143	12	0.0153	13	0.0233	9	0.0236

Iter. – number of iterations; CPU – central processing unit.

we set  $\mu_1 = 5(\|T^*HT\| + L_1)/\nu$ ,  $\mu_2 = 10(\|T^*HT\| + L_2)/\nu$ ,  $\nu = 0.8$ ,  $\rho = 2.5$ ,  $\lambda_n H = \frac{2}{\|T^*T\|} I_N$ , and  $\gamma = -1.5$ , with starting points  $x_1 = (1, 1, 1, \dots, 1)'$ ,  $y_1 = (0, 0, \dots, 0)'$  and  $\lambda_1 = (0, 0, \dots, 0)$ . The stopping criterion used for our computation is  $\|x_{n+1} - x_n\| < 10^{-3}$ . We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figures 1–4 and Table 1.

**Example 5.2.** Let  $H_1 = (\ell_2(\mathbb{R}), \|\cdot\|_{\ell_2}) = H_2$ , where  $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, \dots, x_i, \dots), x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ , with inner product  $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$  and norm  $\|x\| := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$ ,  $\forall x, y \in \ell_2(\mathbb{R})$ . Now, let the operator  $f_1, f, h : \ell_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{R})$  be defined by  $f_1 x = f x = h x = (\frac{1}{\|x\| + s} + \|x\|)x$ , ( $s > 0$ ),  $x \in \ell_2(\mathbb{R})$ . Then,  $f_1, f$ , and  $h$  are uniformly continuous and pseudomonotone. Let  $A : \ell_2(\mathbb{R}) \rightarrow \ell_2(\mathbb{R})$  be defined by  $Ax = (0, \frac{x_1}{2}, \frac{x_2}{3}, \frac{x_3}{3}, \dots)$  for all  $x \in \ell_2$ . Then,  $A$  is a bounded linear operator on  $\ell_2$  with adjoint  $A^*y = (y_2, \frac{y_3}{2}, \frac{y_4}{3}, \dots)$  for all  $y \in \ell_2(\mathbb{R})$ . Let  $C = \{x \in \ell_2 : \|x - y\|_{\ell_2} \leq a\}$ , where  $y = (1, \frac{1}{2}, \frac{1}{3}, \dots)$  and  $a = 3$ . So,  $C$  is a nonempty, closed, convex subset of  $\ell_2$ . Hence,



**Figure 5:** Example 5.2: Case I.

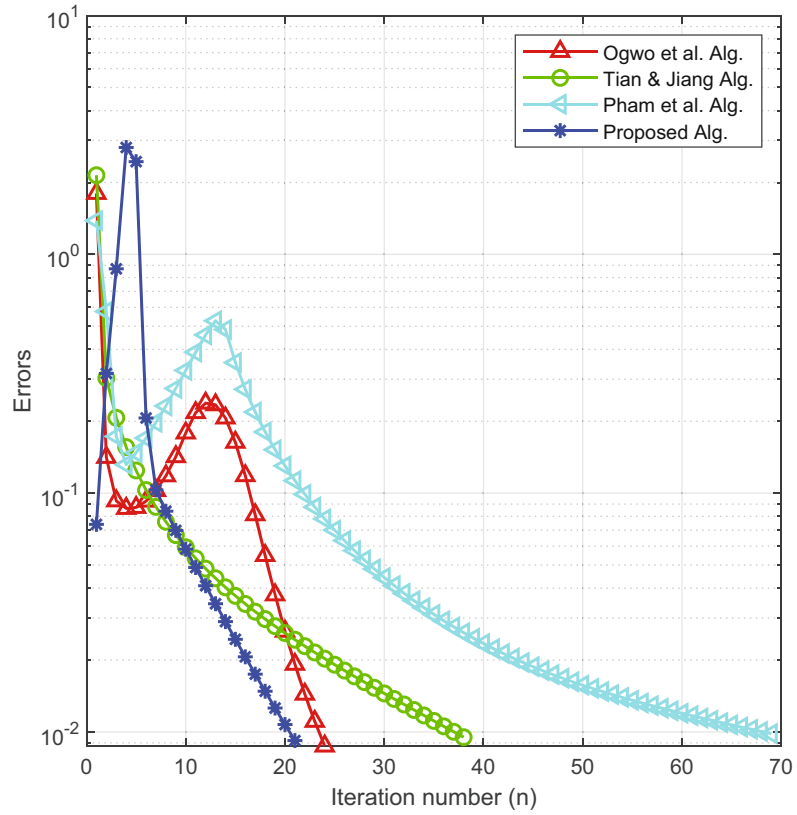


Figure 6: Example 5.2: Case II.

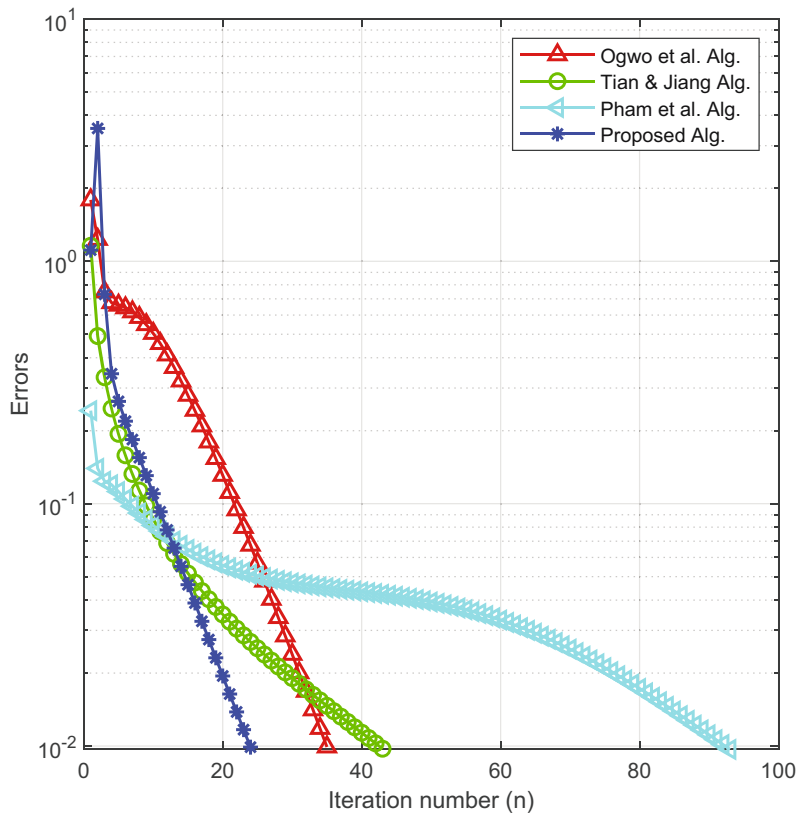


Figure 7: Example 5.2: Case III.

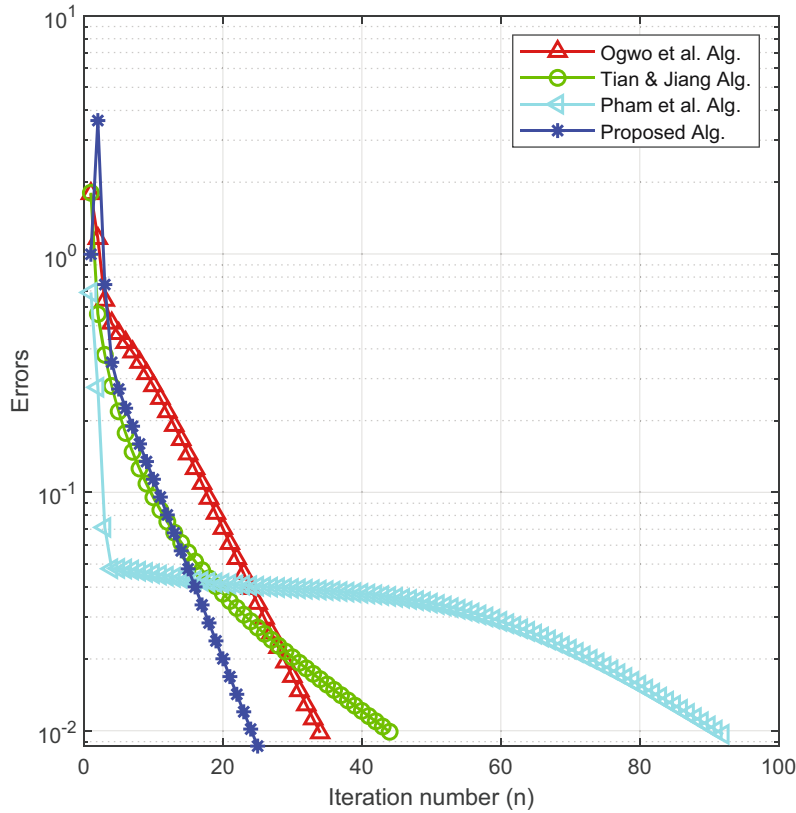


Figure 8: Example 5.2: Case IV.

Table 2: Numerical results for Example 5.2

	Case I		Case II		Case III		Case IV	
	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time	Iter.	CPU time
Ogwo et al. Algorithm 1.6	34	0.0094	24	0.0073	35	0.0080	34	0.0078
Tian and Jiang App. A.1	41	0.0144	38	0.0149	43	0.0179	44	0.0137
Pham et al. App. A.2	82	0.0148	69	0.0125	93	0.0151	92	0.0144
Proposed Algorithm 3.1	21	0.0138	21	0.0106	24	0.0116	25	0.0121

Iter. – number of iterations; CPU – central processing unit.

$$P_C(x) = \begin{cases} x, & \text{if } \|x - y\|_2 \leq a, \\ \frac{x - y}{\|x - y\|_2} a + y, & \text{otherwise.} \end{cases}$$

Now, we consider the following cases for the starting points:

Case I:  $x_0 = (3, 1, \frac{1}{3}, \dots)$  and  $x_1 = (2, 1, \frac{1}{2}, \dots)$ ,

Case II:  $x_0 = (-4, 1, -\frac{1}{4}, \dots)$  and  $x_1 = (2, 1, -\frac{1}{2}, \dots)$ ,

Case III:  $x_0 = (4, -1, \frac{1}{4}, \dots)$  and  $x_1 = (3, 1, \frac{1}{3}, \dots)$ ,

Case IV:  $x_0 = (5, 1, \frac{1}{5}, \dots)$  and  $x_1 = (3, 1, \frac{1}{3}, \dots)$ . The stopping criterion used for our computation is  $\|x_{n+1} - x_n\| < 10^{-2}$ . We plot the graphs of errors against the number of iterations in each case. The numerical results are reported in Figures 5–8 and Table 2.

## 6 Conclusion

In this article, we introduced and studied an inertial iterative method for approximating the solution of a class of pseudomonotone SVIP in the framework of Hilbert spaces. We established that the sequence generated by our method converges strongly to a solution of the SVIP when the cost operator is uniformly continuous and without prior knowledge of the operator norm. We gave some numerical examples to illustrate the efficacy and advantage of our method and compare it with related methods in the literature.

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## Appendix

### A.1 Appendix

The Algorithm in Tian and Jiang [40]. Let  $x_1 \in C$  and let  $\{x_n\}$ ,  $y_n$ , and  $\{v_n\}$  be sequences defined as follows:

$$\begin{cases} y_n = P_C(x_n - \tau_n A^*(I - S)Ax_n), \\ v_n = P_C(y_n - \lambda_n f(y_n)), \\ u_n = P_C(y_n - \lambda_n f(v_n)), \\ x_{n+1} = \alpha_n g(x_n) + (1 - \alpha_n)u_n, \quad n \geq 1, \end{cases} \quad (\text{A1})$$

where  $\lim_{n \rightarrow \infty} \alpha_n = 0$ ,  $\{\alpha_n\} \subset (0, 1)$ , and  $\sum_{n=1}^{\infty} \alpha_n = \infty$ ,  $\{\tau_n\} \subset [a, b]$ , for some  $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$ ,  $\{\lambda_n\} \subset [c, d]$  for some  $c, d \in \left(0, \frac{1}{L}\right)$ ,  $S : H_2 \rightarrow H_2$  is a nonexpansive mapping,  $f : C \rightarrow H_1$  is a monotone and  $L$ -Lipschitz continuous operator, and  $g$  is a contraction on  $C$ .

### A.2 Appendix

Algorithm of Pham et al. [41]

**Step 0.** Choose  $\mu_0, \lambda_0 > 0$ ,  $\mu, \lambda \in (0, 1)$ ,  $\{\tau_n\} \subset [\underline{\tau}, \bar{\tau}] \subset \left(0, \frac{1}{\|A\|^2 + 1}\right)$ ,  $\{\alpha_n\} \subset (0, 1)$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 0$  and  $\sum_{n=1}^{\infty} \alpha_n = +\infty$ .

**Step 1.** Let  $x_0 \in H_1$ . Set  $n := 0$ .

**Step 2.** Compute

$$\begin{aligned} u_n &= Ax_n, \\ v_n &= P_Q(u_n - \mu_n h u_n), \\ w_n &= P_{Q_n}(u_n - \mu_n h v_n), \end{aligned}$$

where

$$Q_n = \{y \in H_2 : \langle u_n - \mu_n h u_n - v_n, y - v_n \rangle \leq 0\}$$

and

$$\mu_{n+1} = \begin{cases} \min \left\{ \frac{\mu \|u_n - v_n\|}{\|h u_n - h v_n\|}, \mu_n \right\}, & \text{if } h u_n \neq h v_n, \\ \mu_n, & \text{otherwise.} \end{cases}$$

**Step 3.** Compute

$$\begin{aligned} y_n &= x_n + \tau_n A^*(w_n - u_n), \\ z_n &= P_C(y_n - \lambda_n f y_n), \\ t_n &= P_{C_n}(y_n - \lambda_n f z_n), \end{aligned}$$

where

$$C_n = \{x \in H_1 : \langle y_n - \lambda_n f y_n - z_n, x - z_n \rangle \leq 0\}$$

and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \frac{\lambda \|y_n - z_n\|}{\|f y_n - f z_n\|}, \lambda_n \right\}, & \text{if } f y_n \neq f z_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

**Step 4.** Compute

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) t_n$$

Set  $n := n + 1$  and go back to **Step 2**.

$C$  and  $Q$  are nonempty, closed and convex subsets of  $H_1$  and  $H_2$ , respectively, and  $f : H_1 \rightarrow H_1$  and  $h : H_2 \rightarrow H_2$  are pseudomonotone and Lipschitz continuous operators.

### A.3 Appendix

Algorithm of He *et al.* [23]

**Step 0.** Given a symmetric positive definite matrix  $\mathbb{H} \in \mathbb{R}^{m \times m}$ ,  $\gamma \in (0, 2)$ , and  $\rho \in (\rho_{\min}, 3)$ , where  $\rho_{\min} := \max \left\{ -3, \frac{2(v-1)\mu_1}{4\zeta_{\max}(T'HT)} \right\}$ , and  $T : \mathbb{R}^N \rightarrow \mathbb{R}^m$  is a linear operator, where  $T'$  means the transpose of  $T$ . Set an initial point  $u_1 := (x_1, y_1, \lambda_1) \in \Omega : \kappa \times \Gamma \times \mathbb{R}^m$ , where  $\kappa$  and  $\Gamma$  are nonempty, closed, and convex subsets of  $\mathbb{R}^N$  and  $\mathbb{R}^m$ , respectively.

**Step 1.** Generate a predictor  $\tilde{u}_n := (\tilde{x}_n, \tilde{y}_n, \tilde{\lambda}_n)$  with appropriate parameters  $u_1$  and  $u_2$ :

$$\begin{cases} \tilde{\lambda}_n = \lambda_n - \mathbb{H}(Tx_n - y_n), \\ \tilde{x}_n = P_\kappa \left[ x_n - \frac{1}{\mu_1} (A(x_n) - T'\tilde{\lambda}_n) \right], \\ \hat{\lambda}_n = \lambda - \mathbb{H}(T\hat{x}_n - y_n), \quad \text{where } \hat{x} := \rho x_n + (1 - \rho)\tilde{x}_n, \\ \tilde{y}_n = P_\Gamma \left[ y_n - \frac{1}{\mu_2} (F(y_n) + \tilde{\lambda}_n) \right], \\ \tilde{\lambda}_n = \lambda - \mathbb{H}(T\tilde{x}_n - \tilde{y}_n). \end{cases} \quad (\text{A2})$$

**Step 2.** Update the next iterative  $u_{n+1} := (x_{n+1}, y_{n+1}, \lambda_{n+1})$  via  $u_{n+1} = u_n - \gamma \alpha_k d(u_n, \tilde{u}_n)$ , where

$$\begin{cases} \alpha_k := \frac{\psi(u_n, \tilde{u}_n)}{\|d(u_n, \tilde{u}_n)\|^2}, \\ d(u_n, \tilde{u}_n) := G(u_n - \tilde{u}_n) - \xi_n, \\ \psi(u_n, \tilde{u}_n) := \langle \lambda_n - \tilde{\lambda}_n, \rho T(x_n - \tilde{x}_n) - (y_n - \tilde{y}_n) \rangle + \langle u_n - \tilde{u}_n, d(u_n, \tilde{u}_n) \rangle, \end{cases}$$

$$\xi_n = \begin{pmatrix} \xi_{n_x} \\ \xi_{n_y} \\ 0 \end{pmatrix} = \begin{pmatrix} A(x_n) - A(\tilde{x}_n) + T'\mathbb{H}T(x_n - \tilde{x}_n) \\ F(y_n) - F(\tilde{y}_n) + \mathbb{H}(y_n - \tilde{y}_n) \\ 0 \end{pmatrix},$$

where  $A$  and  $F$  are monotone and Lipschitz continuous with constants  $L_1$  and  $L_2$ , respectively, and

$$G := \begin{pmatrix} \mu_n I_N + \rho T'\mathbb{H}T & 0 & 0 \\ 0 & \mu_2 I_m + \mathbb{H} & 0 \\ 0 & 0 & \mathbb{H}^{-1} \end{pmatrix} \quad (\text{A3})$$

is the block diagonal matrix, with identity matrices  $I_N$  and  $I_m$  of size  $N$  and  $m$ , respectively. The parameters  $\mu_1$  and  $\mu_2$  are chosen such that

$$\|\xi_{n_x}\| \leq v\mu_1 \|x_n - \tilde{x}_n\| \quad \text{and} \quad \|\xi_{n_y}\| \leq v\mu_1 \|y_n - \tilde{y}_n\|, \quad \text{where } v \in (0, 1).$$