

Research Article

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Homotopy cartesian squares in extriangulated categories

<https://doi.org/10.1515/math-2022-0570>

received July 10, 2022; accepted March 1, 2023

Abstract: Let $(C, \mathbb{E}, s)$ be an extriangulated category. Given a composition of two commutative squares in C , if two commutative squares are homotopy cartesian, then their composition is also a homotopy cartesian square. This covers the result by Mac Lane [*Categories for the Working Mathematician*, Second edition, Graduate Texts in Mathematics, Vol. 5, Springer-Verlag, New York, 1998] for abelian categories and by Christensen and Frankland [*On good morphisms of exact triangles*, J. Pure Appl. Algebra **226** (2022), no. 3, 106846] for triangulated categories.

Keywords: extriangulated category, homotopy cartesian square, triangulated category, abelian category, push-out and pull-back

MSC 2020: 18G80, 18E10

1 Introduction

Most of homological algebra can be carried out in the setting of abelian categories, which includes all categories of modules. We recall the notations and properties of pull-back and push-out squares in an abelian category. Let $\mathcal{A}$ be an abelian category. Given two morphisms $x' : A' \rightarrow B'$ and $b : B \rightarrow B'$ in $\mathcal{A}$, then a commutative diagram is as follows:

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

is a *pull-back* if and only if the sequence $0 \rightarrow A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B'$ is left exact. Given two morphisms $x : A \rightarrow B$ and $a : A \rightarrow A'$, then a commutative diagram is as follows:

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

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is a *push-out* if and only if the sequence $A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \rightarrow 0$ is right exact. By the above properties, we obtain that a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

is both *pull-back* and *push-out* if and only if the sequence

$$0 \rightarrow A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \rightarrow 0$$

is exact.

The notion of a triangulated category was introduced by Grothendieck [1] and Verdier [2]. Having their origins in algebraic geometry and algebraic topology, triangulated categories have by now become indispensable in many different areas of mathematics. The notion of homotopy cartesian square first appeared in the study by Parshall and Scott [3] (also see [4, Definition 1.4.1]). We recall the notion here. Let C be a triangulated category with shift functor Σ . A commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

is called *homotopy cartesian* if there exists a triangle

$$A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \xrightarrow{\partial} \Sigma A,$$

where the morphism ∂ is called a *differential* of the homotopy cartesian square. We view a homotopy cartesian square as the triangulated analog of a pull-back and push-out square in an abelian category.

Recently, extriangulated categories were introduced by Nakaoka and Palu [5] by extracting those properties of $\text{Ext}^1(-, -)$ on exact categories and on triangulated categories that seem relevant from the point of view of cotorsion pairs. Exact categories (abelian categories are also exact categories) and triangulated categories are extriangulated categories, while there are some other examples of extriangulated categories, which are neither exact nor triangulated, see [5–9]. In particular, Nakaoka and Palu [5, Remark 2.18] proved that extension-closed subcategories of extriangulated categories are extriangulated categories. For example, let A be an artin algebra and $K^{[-1,0]}(\text{proj} A)$ the category of complexes of finitely generated projective A -modules concentrated in degrees -1 and 0 , with morphisms considered up to homotopy. Then $K^{[-1,0]}(\text{proj} A)$ is an extension-closed subcategory of the bounded homotopy category $K^b(\text{proj} A)$, which is neither exact nor triangulated, see [10, Example 6.2]. This construction gives extriangulated categories that are neither exact nor triangulated.

In this article, we study homotopy cartesian squares in extriangulated categories. This unifies homotopy cartesian squares in abelian categories and triangulated categories independently. Let us recall the definition of homotopy cartesian squares in an extriangulated category.

Definition 1.1. [11, Definition 3.1] Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Then a commutative square

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ a \downarrow & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

is called *homotopy cartesian* if there exists an $\mathbb{E}$ -triangle

$$A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \xrightarrow{\theta} \Sigma A,$$

The $\mathbb{E}$ -extension $\theta \in \mathbb{E}(B', A)$ is called a *differential* of the homotopy cartesian square.

This gives a simultaneous generalization of the notions of push-out and pull-back square in abelian categories and homotopy cartesian square in triangulated categories.

The main result of this article is the following, which is a simultaneous generalization of the result of Mac Lane [12] for abelian categories and that of Christensen and Frankland [13] for triangulated categories.

Theorem 1.2. (See Theorem 3.2 for more details) *Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Consider the following commutative diagram in C :*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ a \downarrow & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \quad \begin{array}{c} \text{(I)} \\ \text{(II)} \end{array}$$

If the two squares (I) and (II) are homotopy cartesian in C , then the outside rectangle is a homotopy cartesian square in C .

Moreover, given differentials $\theta_L \in \mathbb{E}(B', A)$ and $\theta_R \in \mathbb{E}(C', B)$ of the left square and right square, respectively, there exists a differential θ_P for the outside rectangle satisfying

$$\theta_R = x_*\theta_P \quad \text{and} \quad \theta_L = (y')^*\theta_P.$$

This article is organized as follows. In Section 2, we recall the definition of an extriangulated category and related notions. In Section 3, we show that, in an extriangulated category C , given a composition of two commutative squares, if both squares are homotopy cartesian, then so is the composition.

2 Preliminaries

Let us briefly recall some definitions and basic properties of extriangulated categories from [5]. We omit some details here, but the reader can find them in [5].

Let C be an additive category equipped with an additive bifunctor $\mathbb{E} : C^{\text{op}} \times C \rightarrow \text{Ab}$, where Ab is the category of abelian groups. For any objects $A, C \in C$, an element $\delta \in \mathbb{E}(C, A)$ is called an $\mathbb{E}$ -extension. For any $a \in C(A, A')$ and $c \in C(C', C)$, $\mathbb{E}(C, a)(\delta) \in \mathbb{E}(C, A')$ and $\mathbb{E}(c, A)(\delta) \in \mathbb{E}(C', A)$ are simply denoted by $a_*\delta$ and $c^*\delta$, respectively.

Definition 2.1. [5, Definition 2.3] Let $(A, \delta, C), (A', \delta', C')$ be any pair of $\mathbb{E}$ -extensions. A *morphism*

$$(a, c) : (A, \delta, C) \rightarrow (A', \delta', C')$$

of $\mathbb{E}$ -extensions is a pair of morphisms $a \in C(A, A')$ and $c \in C(C, C')$ in C , satisfying the equality $a_*\delta = c^*\delta'$. Simply we denote it as $(a, c) : \delta \rightarrow \delta'$.

Definition 2.2. [5, Definition 2.6] Let $\delta = (A, \delta, C), \delta' = (A', \delta', C')$ be any pair of $\mathbb{E}$ -extensions. Let

$$C \xrightarrow{t_C} C \oplus C' \xleftarrow{t_{C'}} C'$$

and

$$A \xleftarrow{p_A} A \oplus A' \xrightarrow{p_{A'}} A'$$

be coproduct and product in $\mathcal{B}$, respectively. Since $\mathbb{E}$ is an additive bifunctor, we have a natural isomorphism

$$\mathbb{E}(C \oplus C', A \oplus A') \cong \mathbb{E}(C, A) \oplus \mathbb{E}(C, A') \oplus \mathbb{E}(C', A) \oplus \mathbb{E}(C', A').$$

Let $\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$ be the element corresponding to $(\delta, 0, 0, \delta')$ through the above isomorphism. This is the unique element that satisfies

$$\mathbb{E}(\iota_C, p_A)(\delta \oplus \delta') = \delta, \mathbb{E}(\iota_C, p_{A'}) (\delta \oplus \delta') = 0, \mathbb{E}(\iota_{C'}, p_A)(\delta \oplus \delta') = 0, \mathbb{E}(\iota_{C'}, p_{A'}) (\delta \oplus \delta') = \delta'.$$

Let C and $\mathbb{E}$ be as before.

Definition 2.3. [5, Definitions 2.7 and 2.8] Let $A, C \in \mathcal{C}$ be any pair of objects. Sequences of morphisms in $\mathcal{C}$

$$A \xrightarrow{x} B \xrightarrow{y} C \quad \text{and} \quad A \xrightarrow{x'} B' \xrightarrow{y'} C$$

are said to be *equivalent* if there exists an isomorphism $b \in \mathcal{C}(B, B')$, which makes the following diagram commutative:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \parallel & & \downarrow \cong b & & \parallel \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A, C \in \mathcal{C}$, we denote $0 = [A \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} A \oplus C \xrightarrow{(0,1)} C]$.

For any two equivalence classes, we denote

$$[A \xrightarrow{x} B \xrightarrow{y} C] \oplus [A' \xrightarrow{x'} B' \xrightarrow{y'} C'] = [A \oplus A' \xrightarrow{x \oplus x'} B \oplus B' \xrightarrow{y \oplus y'} C \oplus C'].$$

Definition 2.4. [5, Definition 2.9] Let $\mathfrak{s}$ be a correspondence that associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ with any $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$. We say that $\mathfrak{s}$ is called a *realization* of $\mathbb{E}$ if it satisfies the following condition.

- Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$ -extensions, with

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

Then, for any morphism $(a, c) : \delta \rightarrow \delta'$, there exists $b \in \mathcal{C}(B, B')$ such that the following diagram commutes:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

In this situation, we say that the triple of morphisms (a, b, c) *realizes* (a, c) .

Definition 2.5. [5, Definition 2.10] A realization $\mathfrak{s}$ of $\mathbb{E}$ is called *additive* if it satisfies the following conditions.

- (1) For any $A, C \in \mathcal{C}$, the split $\mathbb{E}$ -extension $0 \in \mathbb{E}(C, A)$ satisfies $\mathfrak{s}(0) = 0$.
- (2) For any pair of $\mathbb{E}$ -extensions $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$, $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$ holds.

Definition 2.6. [5, Definition 2.12] Let $\mathcal{C}$ be an additive category. An extriangulated category is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$ satisfying the following axioms:

- (ET1) $\mathbb{E} : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Ab}$ is an additive bifunctor.
- (ET2) $\mathfrak{s}$ is an additive realization of $\mathbb{E}$.
- (ET3) Let $\delta \in \mathbb{E}(C, A)$ and $\delta' \in \mathbb{E}(C', A')$ be any pair of $\mathbb{E}$ -extensions that are realized as follows:

$$\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C], \quad \mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C'].$$

For any commutative square

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

in C , there exists a morphism $(a, c) : \delta \rightarrow \delta'$ satisfying $cy = y'b$.

(ET3)^{op} Dual of (ET3).

(ET4) Let (A, δ, D) and (B, δ', F) be $\mathbb{E}$ -extensions realized by

$$A \xrightarrow{f} B \xrightarrow{f'} \quad \text{and} \quad B \xrightarrow{g} C \xrightarrow{g'} F,$$

respectively. Then there exists an object $E \in C$, a commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \parallel & & \downarrow g & & \downarrow d \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow g' & & \downarrow e \\ & & F & = & F \end{array}$$

in C , and an $\mathbb{E}$ -extension $\delta'' \in \mathbb{E}(E, A)$ realized by $A \xrightarrow{h} C \xrightarrow{h'} E$, which satisfy the following compatibilities:

- (i) $D \xrightarrow{d} E \xrightarrow{e} F$ realizes $f'_*\delta'$,
- (ii) $d^*\delta'' = \delta$,
- (iii) $f_*\delta'' = e^*\delta'$.

(ET4)^{op} Dual of (ET4).

We collect the following terminology from [5].

Definition 2.7. [5, Definitions 2.15 and 2.19] Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category.

- (1) A sequence $A \xrightarrow{x} B \xrightarrow{y} C$ is called a *conflation* if it realizes some $\mathbb{E}$ -extension $\delta \in \mathbb{E}(C, A)$. In this case, x is called an *inflation* and y is called a *deflation*.
- (2) If a conflation $A \xrightarrow{x} B \xrightarrow{y} C$ realizes $\delta \in \mathbb{E}(C, A)$, we call the pair $(A \xrightarrow{x} B \xrightarrow{y} C, \delta)$ an $\mathbb{E}$ -*triangle*, and write it as follows:

$$A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta}$$

We usually do not write this “ δ ” if it is not used in the argument.

- (3) Let $A \xrightarrow{x} B \xrightarrow{y} C \dashrightarrow^{\delta}$ and $A' \xrightarrow{x'} B' \xrightarrow{y'} C' \dashrightarrow^{\delta'}$ be any pair of $\mathbb{E}$ -triangles. If a triplet (a, b, c) realizes $(a, c) : \delta \rightarrow \delta'$, then we write it as

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \dashrightarrow^{\delta} > \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \dashrightarrow^{\delta'} > \end{array}$$

and call (a, b, c) a *morphism of $\mathbb{E}$ -triangles*.

3 Proof of the main result

First, we prove the following crucial lemma.

Lemma 3.1. *Let $(C, \mathbb{E}, s)$ be an extriangulated category.*

(1) *Consider a homotopy cartesian square*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & & \\ \downarrow a & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \xrightarrow{\delta'} \gg \end{array}$$

with a differential $\theta \in \mathbb{E}(B', A)$ such that the bottom row is an $\mathbb{E}$ -triangle in C . Then, the square extends to a morphism of $\mathbb{E}$ -triangles of the form $(a, b, 1)$, as illustrated in the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C' & \xrightarrow{\delta} & \gg \\ \downarrow a & & \downarrow b & & \parallel & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \gg \end{array}$$

satisfying $\theta = (y')^(-\delta)$.*

(2) *Consider a homotopy cartesian square*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \xrightarrow{\delta} \gg \\ \downarrow a & & \downarrow b & & \\ A' & \xrightarrow{x'} & B' & & \end{array}$$

with a differential $\theta \in \mathbb{E}(B', A)$ such that the top row is an $\mathbb{E}$ -triangle in C . Then, the squares extend to a morphism of $\mathbb{E}$ -triangles of the form $(a, b, 1)$, as illustrated in the following diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C & \xrightarrow{\delta} & \gg \\ \downarrow a & & \downarrow b & & \parallel & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C & \xrightarrow{\delta'} & \gg \end{array}$$

satisfying $\theta = (y')^(-\delta)$.*

Proof. (1) Since the square

$$\begin{array}{ccc} A & \xrightarrow{x} & B \\ \downarrow a & & \downarrow b \\ A' & \xrightarrow{x'} & B' \end{array}$$

is homotopy cartesian, there exists an $\mathbb{E}$ -triangle: $A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \xrightarrow{\theta} \gg$. By the dual of [5, Proposition 3.17], we obtain the following commutative diagram made of $\mathbb{E}$ -triangles:

$$\begin{array}{ccccccc} & & A & \xlongequal{\quad} & A & & \\ & & \downarrow \begin{pmatrix} a \\ x \end{pmatrix} & & \downarrow x & & \\ A' & \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & A' \oplus B & \xrightarrow{(0, 1)} & B & \xrightarrow{0} & \gg \\ \parallel & & \downarrow (x', -b) & & \downarrow -y & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \gg \\ & & \downarrow \theta & & \downarrow -\delta & & \\ & & \downarrow & & \downarrow & & \end{array}$$

which satisfies $\theta = (y')^*(-\delta)$ and $\begin{pmatrix} a \\ x \end{pmatrix}_*(-\delta) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_*\delta' = 0$. From the diagram, we have $y = y'b$. By equality $\begin{pmatrix} a \\ x \end{pmatrix}_*(-\delta) + \begin{pmatrix} 1 \\ 0 \end{pmatrix}_*\delta' = 0$, we have $a_*\delta = \delta'$. Thus, we obtain the following morphism of $\mathbb{E}$ -triangles:

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C' & \xrightarrow{\delta} & \triangleright \\ \downarrow a & & \downarrow b & & \parallel & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' & \xrightarrow{\delta'} & \triangleright \end{array}$$

This completes the proof.

(2) The proof is similar to (1). □

We now prove our main result.

Theorem 3.2. *Let $(C, \mathbb{E}, \mathfrak{s})$ be an extriangulated category. Consider the following commutative diagram:*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow a & & \downarrow b & & \downarrow c \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array} \quad \begin{array}{c} \text{(I)} \\ \text{(II)} \end{array} \quad (3.1)$$

If the two squares (I) and (II) are homotopy cartesian in C , then the outside rectangle is also homotopy cartesian in C .

Moreover, given the differentials $\theta_L \in \mathbb{E}(B', A)$ and $\theta_R \in \mathbb{E}(C', B)$ for the left-hand and right-hand squares, respectively, there exists a differential θ_P for the outside rectangle satisfying

$$\theta_R = x_*\theta_P \quad \text{and} \quad \theta_L = (y')^*\theta_P.$$

Proof. Suppose we are given differentials $\theta_L \in \mathbb{E}(B', A)$ and $\theta_R \in \mathbb{E}(C', B)$ for the left-hand and right-hand squares, respectively. Since the square (I) is homotopy cartesian, there exists an $\mathbb{E}$ -triangle

$$A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} A' \oplus B \xrightarrow{(x', -b)} B' \xrightarrow{\theta_L} \triangleright$$

with the differential $\theta_L \in \mathbb{E}(B', A)$. We have the following commutative diagram, all of whose vertical morphisms are isomorphisms, with the first and the third column being identities:

$$\begin{array}{ccccc} A & \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} & A' \oplus B & \xrightarrow{(x', -b)} & B' & \xrightarrow{\theta_L} & \triangleright \\ \parallel & & \downarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & & \parallel & & \\ A & \xrightarrow{\begin{pmatrix} x \\ a \end{pmatrix}} & B \oplus A' & \xrightarrow{(b, -x')} & B' & \xrightarrow{\theta_L} & \triangleright \end{array}$$

We have the $\mathbb{E}$ -triangle $A \xrightarrow{\begin{pmatrix} x \\ a \end{pmatrix}} B \oplus A' \xrightarrow{(b, -x')} B' \xrightarrow{\theta_L} \triangleright$. It follows that

$$A \xrightarrow{\begin{pmatrix} x \\ a \end{pmatrix}} B \oplus A' \xrightarrow{(b, -x')} B' \xrightarrow{-\theta_L} \triangleright$$

is an $\mathbb{E}$ -triangle. Note that the sequence $0 \rightarrow C \xrightarrow{-1} C \xrightarrow{0} 0$ is the $\mathbb{E}$ -triangle, we obtain that

$$A \xrightarrow{\begin{pmatrix} -x \\ -a \\ 0 \end{pmatrix}} B \oplus A' \oplus C \xrightarrow{\begin{pmatrix} b & -x' & 0 \\ 0 & 0 & -1 \end{pmatrix}} B' \oplus C \xrightarrow{(-\theta_L) \oplus 0} \triangleright$$

is also an $\mathbb{E}$ -triangle. We have the following commutative diagram, all of whose vertical morphisms are isomorphisms, with the first and the third column being identities:

$$\begin{array}{ccccccc}
A & \xrightarrow{\begin{pmatrix} x \\ a \\ 0 \end{pmatrix}} & B \oplus A' \oplus C & \xrightarrow{\begin{pmatrix} b & -x' & 0 \\ 0 & 0 & -1 \end{pmatrix}} & B' \oplus C & \xrightarrow{(-\theta_L) \oplus 0} & \gg \\
\parallel & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ y & 0 & 1 \end{pmatrix} & & \parallel & & \\
A & \xrightarrow{\begin{pmatrix} x \\ a \\ yx \end{pmatrix}} & B \oplus A' \oplus C & \xrightarrow{\begin{pmatrix} b & -x' & 0 \\ y & 0 & -1 \end{pmatrix}} & B' \oplus C & \xrightarrow{(-\theta_L) \oplus 0} & \gg
\end{array}$$

We have the $\mathbb{E}$ -triangle

$$A \xrightarrow{\begin{pmatrix} x \\ a \\ yx \end{pmatrix}} B \oplus A' \oplus C \xrightarrow{\begin{pmatrix} b & -x' & 0 \\ y & 0 & -1 \end{pmatrix}} B' \oplus C \xrightarrow{(-\theta_L) \oplus 0}$$

This shows that the square

$$\begin{array}{ccc}
A & \xrightarrow{\begin{pmatrix} a \\ yx \end{pmatrix}} & A' \oplus C \\
x \downarrow & & \downarrow \begin{pmatrix} x' & 0 \\ 0 & 1 \end{pmatrix} \\
B & \xrightarrow{\begin{pmatrix} b \\ y \end{pmatrix}} & B' \oplus C
\end{array}$$

is homotopy cartesian.

Recall that we have the following $\mathbb{E}$ -triangle:

$$B \xrightarrow{\begin{pmatrix} b \\ y \end{pmatrix}} B' \oplus C \xrightarrow{(y', -c)} C' \xrightarrow{\theta_R}$$

with the differential $\theta_R \in \mathbb{E}(C', B)$. By applying Lemma 3.1 to the following diagram

$$\begin{array}{ccccccc}
A & \xrightarrow{\begin{pmatrix} a \\ yx \end{pmatrix}} & A' \oplus C & & & & \\
\downarrow x & & \downarrow \begin{pmatrix} x' & 0 \\ 0 & 1 \end{pmatrix} & & & & \\
B & \xrightarrow{\begin{pmatrix} b \\ y \end{pmatrix}} & B' \oplus C & \xrightarrow{(y', -c)} & C' & \xrightarrow{\theta_R} & \gg
\end{array}$$

we obtain the following morphism of $\mathbb{E}$ -triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{\begin{pmatrix} a \\ yx \end{pmatrix}} & A' \oplus C & \xrightarrow{(y'x', -c)} & C' & \xrightarrow{\theta_P} & \gg \\
\downarrow x & & \downarrow \begin{pmatrix} x' & 0 \\ 0 & 1 \end{pmatrix} & & \parallel & & \\
B & \xrightarrow{\begin{pmatrix} b \\ y \end{pmatrix}} & B' \oplus C & \xrightarrow{(y', -c)} & C' & \xrightarrow{\theta_R} & \gg
\end{array} \tag{3.2}$$

satisfying $(-\theta_L) \oplus 0 = (y', -c)^*(-\theta_P)$.

The top row in (3.2) shows that the outside rectangle in (3.1) is homotopy cartesian.

Since (3.2) is a morphism of $\mathbb{E}$ -triangles, we have $\theta_R = x_*\theta_P$.

By the equality $(-\theta_L) \oplus 0 = (y', -c)^*(-\theta_P)$, we have $\theta_L = (y')^*\theta_P$.

This completes the proof. $\square$

Theorem 3.3. Let $(C, \mathbb{E}, s)$ be an extriangulated category. Consider the following commutative diagram:

$$\begin{array}{ccccc}
& & u & & \\
& \searrow & & \searrow & \\
A & & B & \xrightarrow{y} & C \\
a \downarrow & & \downarrow b & & \downarrow c \\
A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C'
\end{array}$$

If the total rectangle and the right-hand square are homotopy cartesian squares in C with given differentials $\theta_P \in \mathbb{E}(C', A)$ and $\theta_R \in \mathbb{E}(C', B)$, respectively, then there exists a morphism $x : A \rightarrow B$ such that the left-hand square is also homotopy cartesian with a differential $\theta_L = (y')^* \theta_P$ and that we have $\theta_R = x_* \theta_P$.

Proof. Since the total rectangle and the right-hand square are homotopy cartesian squares in C with given differentials $\theta_P \in \mathbb{E}(C', A)$ and $\theta_R \in \mathbb{E}(C', B)$, respectively, we have two E-triangles

$$A \xrightarrow{\begin{pmatrix} a \\ u \end{pmatrix}} A' \oplus C \xrightarrow{(y'x', -c)} C' \xrightarrow{\theta_P} 0$$

and

$$B \xrightarrow{\begin{pmatrix} b \\ y \end{pmatrix}} B' \oplus C \xrightarrow{(y', -c)} C' \xrightarrow{\theta_R} 0$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\begin{pmatrix} a \\ u \end{pmatrix}} & A' \oplus C & \xrightarrow{(y'x', -c)} & C' & \xrightarrow{\theta_P} & 0 \\ & & \downarrow \begin{pmatrix} x' & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} & & \parallel & & \\ B \oplus A' & \xrightarrow{\begin{pmatrix} b & 0 \\ y & 0 \\ 0 & 1 \end{pmatrix}} & B \oplus C \oplus A' & \xrightarrow{(y', -c, 0)} & B' & \xrightarrow{\theta_R \oplus 0} & 0 \\ & & \downarrow (1, 0, -x') & & & & \\ B' & \xrightarrow{\quad} & B' & & & & \\ & & \downarrow 0 \oplus 0 & & & & \\ & & 0 & & & & \end{array}$$

By [5, Proposition 3.17], there exists an E-triangle

$$A \xrightarrow{t} B \oplus A' \xrightarrow{s} B' \xrightarrow{\omega} 0$$

which makes the above diagram commutative in C and satisfies the following properties:

- (1) $t_*(\theta_P) = \theta_R \oplus 0$, ($\spadesuit$)
- (2) $\begin{pmatrix} a \\ u \end{pmatrix}^*(\omega) = 0 \oplus 0$,
- (3) $(1, 0, -x')^*(\omega) + (y', -c, 0)^*(\theta_P) = 0$. ($\clubsuit$)

By the square (III) is commutative, we obtain that s is equal to $(b, -x')$. By the square (IV) is commutative, we have that t is of the form $\begin{pmatrix} x \\ a \end{pmatrix}$ and $bx = x'a$. It follows that

$$A \xrightarrow{\begin{pmatrix} a \\ x \end{pmatrix}} B \oplus A' \xrightarrow{(b, -x')} B' \xrightarrow{\omega} 0$$

is also an E-triangle in C . Put $\theta_L = -\omega$, which shows that the left-hand square is homotopy cartesian with the differential θ_L .

By the equality ($\spadesuit$), we have $\theta_R = x_* \theta_P$. By the equality ($\clubsuit$), we have $\theta_L = (y')^* \theta_P$.

This completes the proof. $\square$

Acknowledgments: The authors would like to thank the referee for reading the article carefully and for many suggestions on mathematics and English expressions. In particular, we are also very grateful to the reviewer for pointing out that Theorem 3.3 is correct and giving a sketch of a proof.

Funding information: This work was supported by the National Natural Science Foundation of China (Grant No. 11901190).

Author contributions: All authors contributed equally to each part of this work and read and approved the final manuscript.

Conflict of interest: The authors declare that they have no competing interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during this study.

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