Research Article

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A derivative-Hilbert operator acting on Dirichlet spaces

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Abstract: Let μ be a positive Borel measure on the interval $[0,1)$. The Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1)} t^n \mathrm{d}\mu(t)$, induces formally the operator as follows:

$$\mathcal{DH}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)(n+1)z^n, \quad z \in \mathbb{D},$$

where $f(z) = \sum_{n=0}^{\infty}a_n z^n$ is an analytic function in $\mathbb{D}$. In this article, we characterize those positive Borel measures on $[0,1)$ for which $\mathcal{DH}_\mu$ is bounded (resp. compact) from Dirichlet spaces $\mathcal{D}_\alpha$ $(0 < \alpha \leq 2)$ into $\mathcal{D}_\beta$ $(2 \leq \beta < 4)$.

Keywords: Hilbert operator, Dirichlet space, Carleson measure

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ denote the class of all analytic functions in $\mathbb{D}$.

For $0 < p < \infty$ and $f \in H(\mathbb{D})$, the integral means $M_p(r,f)$ are defined by

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\mathrm{d}\theta\right)^{\frac{1}{p}}, \quad 0 < r < 1.$$

The Hardy space $H^p(0 < p < \infty)$ consists of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty,$$

and H^∞ is the space of all bounded functions f in $H(\mathbb{D})$. We refer to [1] for the theory of Hardy spaces.

For $0 < p < \infty$, the Bergman space A^p consists of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{A^p}^p = \int_{\mathbb{D}}|f(z)|^p\mathrm{d}A(z) < \infty,$$

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where $\mathrm{d}A$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. We refer to [2] for more information about Bergman spaces.

For $\alpha \in \mathbb{R}$, the Dirichlet space $\mathcal{D}_\alpha$ consists of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ for which

$$\|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^{1-\alpha} |a_n|^2 < \infty.$$

We obtain the classical Dirichlet space $\mathcal{D} = \mathcal{D}_0$ if $\alpha = 0$ (see [3]), we obtain the Hardy space $H^2 = \mathcal{D}_1$ if $\alpha = 1$ (see [1,4]), and we obtain the Bergman space $A^2 = \mathcal{D}_2$ if $\alpha = 2$. We mention [3] for complete information on Dirichlet spaces.

Suppose that μ is a positive Borel measure on [0,1), we furtherly define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1)} t^n \mathrm{d}\mu(t)$. The matrix $\mathcal{H}_\mu$ can be seen as an operator on $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(\mathbb{D})$ by its action on the Taylor coefficients: $\{a_n\}_{n\geq 0} \to \{\sum_{k=0}^{\infty} \mu_{n,k} a_k\}_{n\geq 0}$. Furthermore, we can formally induce the Hankel operator $\mathcal{H}_\mu$ as follows:

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad z \in \mathbb{D},$$

whenever the right-hand side of this equation can be defined as an analytic function in $\mathbb{D}$. If μ is the Lebesgue measure, $\mathcal{H}_\mu$ is the classical Hilbert operator $\mathcal{H}$. This is why $\mathcal{H}_\mu$ is called a generalized Hilbert operator.

The operator $\mathcal{H}_\mu$ has been extensively studied in [5–13]. Galanopoulos and Peláez [13] characterized those measures μ supported on $[0,1)$ such that the generalized Hilbert operator $\mathcal{H}_\mu$ is well defined and is bounded on H^1. Chatzifountas et al. [5] described those measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into H^q, where $0 < p, q < \infty$. Diamantopoulos [9] gave many results about the operator induced by Hankel matrices on Dirichlet spaces. Recently, Girela and Merchán [6] have studied the operators $\mathcal{H}_\mu$ acting on certain conformally invariant spaces.

In Ye and Zhou's works [14,15], they defined the derivative-Hilbert operator $\mathcal{DH}_\mu$ as follows:

$$\mathcal{DH}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n.$$

It is closely related to the generalized Hilbert operator, that is,

$$\mathcal{DH}_\mu(f)(z) = (z\mathcal{H}_\mu(f)(z))'.$$

Therefore, we called $\mathcal{DH}_\mu$ to be the derivative-Hilbert operator. In that work, the second and the third authors characterized the measures μ for which $\mathcal{DH}_\mu$ is a bounded (resp. compact) operator from A^p into A^q for some p and q, and they also characterized the measures μ for which $\mathcal{DH}_\mu$ is a bounded (resp. compact) operator on the Bloch space.

Let us recall the definition of Carleson-type measures that play a very important role in the theory of Banach spaces of analytic functions. We refer to [16,17] for some results about Carleson measures.

If $I \subset \partial\mathbb{D}$ in an arc, $|I|$ denotes the length of I, and the Carleson square $S(I)$ is defined as follows:

$$S(I) = \left\{ z = re^{it} : e^{it} \in I, 1 - \frac{|I|}{2\pi} \leq r < 1 \right\}.$$

Suppose $0 < p < \infty$ and μ is a positive Borel measure on $\mathbb{D}$, then μ is said to be an s-Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

Here, μ is said to be a vanishing s-Carleson measure if

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^s} = 0.$$

If μ is a Borel measure on $[0, 1)$, it can been seen as a Borel measure on $\mathbb{D}$ by identifying it as $\hat{\mu}(A) = \mu(A \cap [0, 1))$, for every Borel set $A \subset \mathbb{D}$. In this way, for $0 < s < \infty$, we called μ to be an s-Carleson measure if there exists a positive constant C such that

$$\mu([t, 1)) \leq C(1 - t)^s, \quad t \in [0, 1).$$

Also, μ is a vanishing s-Carleson measure on $[0, 1]$ if μ satisfies

$$\lim_{t \to 1^-} \frac{\mu([t, 1))}{(1 - t)^s} = 0.$$

In this article, we mainly characterize the positive Borel measures μ on $[0, 1)$ for which the derivative-Hilbert operator $\mathcal{DH}_\mu$ is bounded (resp. compact) from Dirichlet spaces $\mathcal{D}_\alpha (0 < \alpha \leq 2)$ into $\mathcal{D}_\beta (2 \leq \beta < 4)$.

In this work, C denotes a positive constant that only depends on the displayed parameters but not necessarily the same from one occurrence to the next. In addition, we say that $A \gtrsim B$ if there exist a constant C (independent of A and B) such that $A \geq CB$, and $A \lesssim B$ is the same as $A \gtrsim B$.

2 Main results

We shall first give a sufficient condition such that the operator $\mathcal{DH}_\mu$ is well defined on the Dirichlet space $\mathcal{D}_\alpha$, for $\alpha \geq -1$. And we characterize the measure μ such that $\mathcal{DH}_\mu$ is bounded from Dirichlet spaces $\mathcal{D}_\alpha (0 < \alpha \leq 2)$ into $\mathcal{D}_\beta (2 \leq \beta < 4)$.

Theorem 2.1. *Suppose that $\alpha \geq -1$, and let μ be a positive Borel measure on $[0, 1)$. If the moments of μ satisfy that $\mu_n = O\left(n^{-(\frac{\alpha}{2} + \varepsilon)}\right)$ for some $\varepsilon > 0$, then $\mathcal{DH}_\mu$ is well defined on $\mathcal{D}_\alpha$.*

Proof. Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}_\alpha$. By Cauchy-Schwarz inequality, we obtain that

$$\begin{aligned}
\left| \sum_{k=0}^{\infty} \mu_{n,k} a_k \right| &\leq \sum_{k=0}^{\infty} \mu_{n,k} |a_k| \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{(n + k + 1)^{\frac{\alpha}{2} + \varepsilon}} \\
&= \sum_{k=0}^{\infty} (k + 1)^{\frac{\alpha - 1}{2}} \frac{1}{(n + k + 1)^{\frac{\alpha}{2} + \varepsilon}} (k + 1)^{\frac{1 - \alpha}{2}} |a_k| \\
&\leq \left(\sum_{k=0}^{\infty} \frac{(k + 1)^{\alpha - 1}}{(n + k + 1)^{\alpha + 2\varepsilon}} \right)^{\frac{1}{2}} \left(\sum_{k=0}^{\infty} (k + 1)^{1 - \alpha} |a_k|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{(k + 1)^{1 + 2\varepsilon}} \right)^{\frac{1}{2}} \|f\|_{\mathcal{D}_\alpha} < \infty.
\end{aligned}$$

This shows that the operator $\mathcal{DH}_\mu$ is well defined on $\mathcal{D}_\alpha$. □

Next, we import an auxiliary lemma, which is needed for the main theorem in this article.

Lemma 2.1. [18, Theorem 318] *Let $K(x, y)$ be a real function of two variables and has the following properties*:

(*i*) *$K(x, y)$ is non-negative and homogeneous of degree -1;*

(*ii*)

$$\int_0^\infty K(x, 1) x^{-\frac{1}{2}} \mathrm{d}x = \int_0^\infty K(1, y) y^{-\frac{1}{2}} \mathrm{d}y = C;$$

(*iii*) *$K(x, 1)x^{-\frac{1}{2}}$ is a strictly decreasing function of x, and $K(1, y)y^{-\frac{1}{2}}$ of y; or, more generally;*

iii’ *$K(x,1)x^{-\frac{1}{2}}$ decreases from $x = 1$ onwards, while the interval $(0,1)$ can be divided into two parts, $(0,\xi)$ and $(\xi,1)$, of which one may be null, in the first of which it decreases and in the second of which it increases; and $K(1,y)y^{-\frac{1}{2}}$ has similar properties; and $K(x,x) = 0$.*

Then for every sequence $\{a_n\}_{n\geq 0}$ such that $\sum_{n=0}^{\infty}|a_n|^2 < \infty$, we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}K(n,k)|a_k|\right)^2 \leq C^2\sum_{n=1}^{\infty}|a_n|^2.$$

In short, if $f(z) = \sum_{n=0}^{\infty}a_nz^n \in H^2$, we have

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}K(n,k)|a_k|\right)^2 \leq C^2\|f\|_{H^2}^2.$$

Theorem 2.2. *Suppose that $0 < \alpha \leq 2$, $2 \leq \beta < 4$, and let μ be a positive Borel measure on $[0,1)$, which satisfies the condition in Theorem 2.1. Then the following conditions are equivalent:*

(i) *μ is a $\left(2 - \frac{\beta-\alpha}{2}\right)$-Carleson measure.*

(ii) $\mu_n = O\left(\frac{1}{n^{2-\frac{\beta-\alpha}{2}}}\right)$.

(iii) *$\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.*

Before giving the proof, let us recall some classical conclusions about the Beta function. The Beta function $B(s,t)$ can be defined as follows:

$$B(s,t) = \int_0^{\infty}\frac{x^{s-1}}{(1+x)^{s+t}}\mathrm{d}x,$$

for each s, t with $\mathrm{Re}(s) > 0$, $\mathrm{Re}(t) > 0$. The value $B(s,t)$ can be expressed in terms of the Gamma function as follows:

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Now we continue to complete the proof of the Theorem 2.2.

Proof. (i) $\Leftrightarrow$ (ii). The result can be found in [5,10].

(ii) $\Rightarrow$ (iii). First, we define two operators. For $f(z) = \sum_{n=0}^{\infty}a_nz^n \in \mathcal{D}_\alpha$, we define $V_\alpha(f)$ by the formula

$$V_\alpha(f)(z) = \sum_{n=0}^{\infty}(n+1)^{\frac{1-\alpha}{2}}a_nz^n,$$

and for $g(z) = \sum_{n=0}^{\infty}b_nz^n \in H^2$, we define $T_\beta(g)$ by the formula:

$$T_\beta(g)(z) = \sum_{n=0}^{\infty}(n+1)^{\frac{\beta-1}{2}}b_nz^n.$$

It is easy to check that V_α is a bounded operator from $\mathcal{D}_\alpha$ into H^2, and T_β is a bounded operator from H^2 into $\mathcal{D}_\beta$.

Now suppose that $0 < \alpha \leq 2$ and $2 \leq \beta < 4$. We consider a new operator S_μ defined as follows: If $h(z) = \sum_{n=0}^{\infty}c_nz^n \in H^2$, we define $S_\mu(h)$ by

$$S_\mu(h)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}c_k\right)z^n.$$

A direct calculation shows that

$$
\begin{aligned}
\|S_\mu(h)\|_{H^2}^2 &= \sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}c_k\right|^2\\
&\le \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}|c_k|\right)^2\\
&\lesssim \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\frac{|c_k|}{(n+k+2)^{2-\frac{\beta-\alpha}{2}}}\right)^2\\
&= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}n^{\frac{3-\beta}{2}}k^{\frac{\alpha-1}{2}}\frac{|c_{k-1}|}{(n+k)^{2-\frac{\beta-\alpha}{2}}}\right)^2.
\end{aligned}
$$

Let

$$
K(x,y) = x^{\frac{3-\beta}{2}}y^{\frac{\alpha-1}{2}}\frac{1}{(x+y)^{2-\frac{\beta-\alpha}{2}}},\quad x>0,\quad y>0.
$$

Then we obtain that

$$
\begin{aligned}
\int_0^\infty K(x,1)x^{-\frac{1}{2}}\mathrm{d}x &= \int_0^\infty \frac{x^{1-\frac{\beta}{2}}}{(x+1)^{2-\frac{\beta-\alpha}{2}}}\mathrm{d}x = B\left(2-\frac{\beta}{2},\frac{\alpha}{2}\right),\\
\int_0^\infty K(1,y)y^{-\frac{1}{2}}\mathrm{d}y &= \int_0^\infty \frac{y^{\frac{\alpha}{2}-1}}{(y+1)^{2-\frac{\beta-\alpha}{2}}}\mathrm{d}y = B\left(\frac{\alpha}{2},2-\frac{\beta}{2}\right).
\end{aligned}
$$

And it is clear that the functions $K(x,1)x^{-\frac{1}{2}}$ and $K(1,y)y^{-\frac{1}{2}}$ are strictly decreasing. By applying Lemma 2.1, we have

$$
\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty}n^{\frac{3-\beta}{2}}k^{\frac{\alpha-1}{2}}\frac{|c_{k-1}|}{(n+k)^{2-\frac{\beta-\alpha}{2}}}\right)^2 \lesssim \left(B\left(2-\frac{\beta}{2},\frac{\alpha}{2}\right)\right)^2\|h\|_{H^2}^2.
$$

This implies that the operator S_μ is bounded on H^2.

For each $f\in\mathcal{D}_\alpha$, it is easy to check that

$$
\begin{aligned}
T_\beta\circ S_\mu\circ V_\alpha(f)(z) &= \sum_{n=0}^{\infty}\left((n+1)^{\frac{\beta-1}{2}}\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}(k+1)^{\frac{1-\alpha}{2}}\mu_{n,k}a_k\right)z^n\\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)(n+1)z^n = \mathcal{DH}_\mu(f)(z).
\end{aligned}
$$

Hence, $\mathcal{DH}_\mu$ is bounded from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.

(iii) $\Rightarrow$ (i). For $0<t<1$, let $f_t(z)=(1-t^2)^{1-\frac{\alpha}{2}}\sum_{n=0}^{\infty}t^nz^n$. We have that

$$
\|f_t\|_{\mathcal{D}_\alpha}^2 = (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{1-\alpha}t^{2n}\approx 1.
$$

Therefore,

$$
\begin{aligned}
\|\mathcal{DH}_\mu(f_t)\|_{\mathcal{D}_\beta}^2 &= \sum_{n=0}^{\infty}(n+1)^{1-\beta}\left(\sum_{k=0}^{\infty}(n+1)\mu_{n,k}(1-t^2)^{1-\frac{\alpha}{2}}t^k\right)^2\\
&\gtrsim (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{3-\beta}\left(\sum_{k=0}^{\infty}t^k\int_t^1\varsigma^{n+k}\mathrm{d}\mu(\varsigma)\right)^2\\
&\gtrsim (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{3-\beta}\left(\sum_{k=0}^{n}t^{n+2k}\mu([t,1))\right)^2.
\end{aligned}
$$

Since $\mathcal{DH}_\mu$ is bounded from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$, we obtain that

$$\begin{aligned}\|\mathcal{DH}_\mu\|^2_{\mathcal{D}_\beta}\|f_t\|^2_{\mathcal{D}_\beta} &\geq \|\mathcal{DH}_\mu(f_t)\|^2_{\mathcal{D}_\beta}\\ &\gtrsim (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{3-\beta}\left(\sum_{k=0}^{n}t^{n+2k}\mu([t,1))\right)^2\\ &\gtrsim (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{5-\beta}t^{6n}\mu([t,1))^2\\ &\approx \frac{\mu([t,1))^2}{(1-t^2)^{4+\alpha-\beta}}.\end{aligned}$$

This implies that

$$\mu([t,1)) \lesssim (1-t^2)^{2-\frac{\beta-\alpha}{2}},$$

which is equivalent to saying that μ is a $(2-\frac{\beta-\alpha}{2})$-Carleson measure. □

In particular, if we take $\alpha=\beta=2$ in Theorem 2.2, we can obtain the following corollary, which the second and the third authors have proved in [14].

Corollary 2.1. *The operator $\mathcal{DH}_\mu$ is bounded on A^2 if and only if the measure μ is a 2-Carleson measure.*

Lemma 2.2. *Let $0<\alpha\leq 2$, $2\leq\beta<4$, and $\mathcal{DH}_\mu$ is a bounded operator from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$. Then $\mathcal{DH}_\mu$ is a compact operator if and only if $\mathcal{DH}_\mu(f_n)\to 0$ in H^2, for any bounded sequence $\{f_n\}$ in $\mathcal{D}_\alpha$, which converges to 0 uniformly on every compact subset of $\mathbb{D}$.*

Proof. The proof is similar to that of in [19, Proposition 3.11], and we omit the details. □

Theorem 2.3. *Suppose that $0<\alpha\leq 2$, $2\leq\beta<4$, and let μ be a positive Borel measure on $[0,1)$, which satisfies the condition in Theorem 2.1. Then the following conditions are equivalent:*

(*i*) *μ is a vanishing $(2-\frac{\beta-\alpha}{2})$-Carleson measure.*

(*ii*) $\mu_n = o\left(\frac{1}{n^{2-\frac{\beta-\alpha}{2}}}\right)$.

(*iii*) *$\mathcal{DH}_\mu$ is a compact operator from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.*

Proof. (i) ⇔ (ii). The result can be found in [5,10].

(ii) ⇒ (iii). Take $f(z)=\sum_{n=0}^{\infty}a_nz^n\in\mathcal{D}_\alpha$ and $h(z)=\sum_{n=0}^{\infty}c_nz^n\in H^2$. Let

$$S_{\mu,m}(h)(z)=\sum_{n=0}^{m}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}c_k\right)z^n.$$

Notice that $S_{\mu,m}$ is a finite rank operator, then $S_{\mu,m}$ is compact on H^2. Since μ_n satisfies $\mu_n=o(n^{-(2-\frac{\beta-\alpha}{2})})$, we obtain that for any $\varepsilon>0$, and there exists an $N>0$ such that $|\mu_m|<\varepsilon n^{-(2-\frac{\beta-\alpha}{2})}$ when $m>N$. Then we note

$$(S_\mu-S_{\mu,m})(h)(z)=\sum_{n=m+1}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}c_k\right)z^n,$$

$$\begin{aligned}(T_\beta\circ S_\mu\circ V_a-T_\beta\circ S_{\mu,m}\circ V_a)(f)(z) &= \sum_{n=m+1}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)\mu_{n,k}a_k\right)z^n\\ &= T_\beta\circ(S_\mu-S_{\mu,m})\circ V_a(f)(z)\\ &= (\mathcal{DH}_\mu-\mathcal{DH}_{\mu,m})(f)(z).\end{aligned}$$

Therefore,

$$\|(S_\mu - S_{\mu,m})(h)\|_{H^2}^2 = \sum_{n=m+1}^{\infty}\left|\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\mu_{n,k}c_k\right|^2.$$

Then for $m > N$, we have

$$\|(S_\mu - S_{\mu,m})(h)\|_{H^2}^2 \lesssim \varepsilon^2\sum_{n=m+1}^{\infty}\left(\sum_{k=0}^{\infty}(n+1)^{\frac{3-\beta}{2}}(k+1)^{\frac{\alpha-1}{2}}\frac{|c_k|}{(n+k+2)^{2-\frac{\beta-\alpha}{2}}}\right)^2.$$

By Lemma 2.1 and the proof of Theorem 2.2, we obtain

$$\|(S_\mu - S_{\mu,m})(h)\|_{H^2}^2 \lesssim \varepsilon^2\|h\|_{H^2}^2.$$

Thus,

$$\|S_\mu - S_{\mu,m}\|_{H^2\to H^2} \lesssim \varepsilon.$$

It is clear that

$$\|\mathcal{DH}_\mu - \mathcal{DH}_{\mu,m}\|_{\mathcal{D}_\alpha\to\mathcal{D}_\beta} \lesssim \varepsilon.$$

Hence, $\mathcal{DH}_\mu$ is compact from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$.

(iii) ⇒ (i). For $0 < t < 1$, let $f_t(z) = (1-t^2)^{1-\frac{\alpha}{2}}\sum_{n=0}^{\infty}t^nz^n$, we have

$$\|f_t\|_{\mathcal{D}_\alpha}^2 = (1-t^2)^{2-\alpha}\sum_{n=0}^{\infty}(n+1)^{1-\alpha}t^{2n} \approx 1,$$

and $\lim_{t\to1}f_t(z) = 0$ for any $z \in \mathbb{D}$. Since all Hilbert spaces are reflexive, we obtain that f_t is convergent weakly to 0 in $\mathcal{D}_\alpha$ as $t \to 1$. By the assumption that $\mathcal{DH}_\mu$ is compact from $\mathcal{D}_\alpha$ into $\mathcal{D}_\beta$, we have

$$\lim_{t\to1}\|\mathcal{DH}_\mu(f_t)\|_{\mathcal{D}_\beta} = 0.$$

Similar to the proof of Theorem 2.2, we obtain that

$$\mu([t,1)) \lesssim (1-t)^{2-\frac{\beta-\alpha}{2}}\|\mathcal{DH}_\mu(f_t)\|_{\mathcal{D}_\beta}.$$

Therefore,

$$\lim_{t\to1}\frac{\mu([t,1))}{(1-t)^{2-\frac{\beta-\alpha}{2}}} = 0.$$

Thus, μ is a vanishing $(2-\frac{\beta-\alpha}{2})$-Carleson measure. □

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