Research Article

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Solvability for a nonlocal dispersal model governed by time and space integrals

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Abstract: This work is to analyze a nonlocal dispersal model governed by a Volterra type integral and two space integrals. A weighted integral is included, and an existence result of solutions for this model is proved. We assume a suitably Hartman-type sign condition and use a sufficiently regular measure of noncompactness. Finally, the degree theory referring to condensing operators is applied.

Keywords: nonlocal dispersal model, Volterra type integral, space integral, regular measure of noncompactness, Hartman-type sign condition

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1 Introduction and the main result

In this article, we are concerned with the following periodic problem of a nonlocal dispersal model governed by a Volterra type integral and two space integrals:

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = f\left(t, u(t,\xi), \displaystyle\int_0^t k(t,s)u(s,\xi)\mathrm{d}s\right) + a(t)\displaystyle\int_{\mathbb{R}} K(\xi,\eta)u(t,\eta)\mathrm{d}\eta \\ \qquad\qquad + W\left(\xi, \displaystyle\int_{\mathbb{R}} \varphi(\eta)u(t,\eta)\mathrm{d}\eta\right) - b(t,\xi)u(t,\xi), \quad t \in [0,\omega], \xi \in \mathbb{R}, \\ u(\omega,\xi) = u(0,\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $\omega \geq 0$, f, k, a, K, W, φ, and b are given functions. The main assumptions imposed on the aforementioned symbols will be introduced later in this article. Here, the nonlocal dispersal model (1.1) is a model with a feedback control and two space integral kernel for describing biological invasion and disease spread. The feedback control W in the model (1.1) depends on the weighted values of $u(t,\cdot)$ over the whole habitat $\int_{\mathbb{R}} \varphi(\eta)u(t,\eta)\mathrm{d}\eta$; (for more introduction, see Malaguti and Rubbioni [1]). The time-continuous kernel k in the model (1.1) is involved in the Volterra type integral:

$$\int_0^t k(t,s)u(s,\xi)\mathrm{d}s. \tag{1.2}$$

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In particular, when $k(t, s) = \frac{e^{-(t-s)/T}}{T}$, it accounts for a rapidly decreasing memory effect. The integral kernel K in the model (1.1) being involved in the space integral

$$a(t)\int_{\mathbb{R}} K(\xi, \eta)u(t, \eta)\mathrm{d}\eta \tag{1.3}$$

accounts for long-distance interactions between individuals. We study the model that governed the time integral and the space integral at the same time. The model in this article is important in applications that introduce a time memory effect and a long-distance interaction concurrently. As far as we know, there are few papers discussing a model with Volterra-type integral and space integral simultaneously.

If the nonlinearity f in (1.1) is replaced by $f(t, u(t, \xi))$, then (1.1) becomes a model only with one kernel K and the feedback control W (see [1, (4)]). In this case, it is also for many phenomena. We refer the reader to [1] and the references therein. In particular, for the case that without the feedback control W, several studies involving the space integral (1.3) and its analogous have recently been published. Alves et al. [2] obtained the existence of positive solution for a class of nonlocal problems. Benedetti et al. [3] studied a second-order partial differential equation with a space integral. They used an approximation solvability method, which combines a Schauder degree argument with an Hartman-type inequality, to prove existence of periodic solutions. Eigentler and Sherratt [4] considered the "nonlocal Klausmeier model," i.e., a coupled reaction-advection-diffusion system. Hutson et al. [5], Jin and Zhao [6] and Wang et al. [7] all studied first-order differential equation with an integral diffusion term.

Equation (1.1) becomes a one without the feedback control W and the space integral (1.3) (see [8]) and is also a model in many applications, such as population dynamics, biology, and epidemiology. In recent years, the model involving the time integral (1.2) and its analogous has been studied in many papers. In [8], Bungardi et al. obtained an existence of mild solutions to a nonlocal semilinear integro-differential problem with a Volterra integral. They developed a vector valued measure of noncompactness involving a Volterra integral operator. It is worth mentioning that the nonlocal function there assumed to be a compact operator. Zhuang et al. [9] studied the periodic boundary value problems for integro-differential equations of Volterra type, concerning with a monotone method. Yu et al. [10] obtained the global solvability for a nonlinear Volterra delay evolution inclusion subjected to a nonlocal implicit initial condition.

Our article is inspired from the main results in [1,8]. Our aim is to prove an existence result for the periodic problem (1.1). The main methods we use in this article is that the one given in [1], i.e., the degree theory referring to condensing operators in Hilbert spaces. We combine it with a suitably assumed Hartman-type sign condition. We use a sufficiently regular measure of noncompactness (MNC) similarly defined as the one given in [8]. This case is important in applications and does not follow by a direct modification of the arguments in both [1,8]. Here, let us mention that the periodic condition is not a compactness assumption.

A function $u : [0, \omega] \times \mathbb{R} \to \mathbb{R}$ is said to be a solution of equation (1.1) if it satisfies the following conditions:

- $u(t, \cdot) \in L^2(\mathbb{R})$;
- $x : [0, \omega] \to L^2(\mathbb{R})$ defined by $x(t) = u(t, \cdot)$ for all $t \in [0, \omega]$;
- $x \in AC([0, \omega]; L^2(\mathbb{R}))$;
- x satisfies the equation in (1.1) for a.e. $t \in [0, \omega]$.

To begin with, let us consider the following conditions:

(H1) $f : [0, \omega] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

(1) $f(\cdot, \tilde{p}, \tilde{q})$ is measurable for every $\tilde{p}, \tilde{q} \in \mathbb{R}$;

(2) $f(t, 0, 0) = 0$ for a.e. $t \in [0, \omega]$;

(3) $f(t, \cdot, \cdot)$ is $\alpha(t)$-Lipschitz continuous for a.e. $t \in [0, \omega]$, i.e., there exists $\alpha \in L^1_+(0, \omega)$ such that for each $(\tilde{p}_1, \tilde{q}_1), (\tilde{p}_2, \tilde{q}_2) \in \mathbb{R} \times \mathbb{R}$,

$$|f(t, \tilde{p}_1, \tilde{q}_1) - f(t, \tilde{p}_2, \tilde{q}_2)| \le \alpha(t)(|\tilde{p}_1 - \tilde{p}_2| + |\tilde{q}_1 - \tilde{q}_2|) \quad \text{a.e. } t \in [0, \omega].$$

(H2) $k : \Delta \to \mathbb{R}$ is a continuous function, where $\Delta := \{(t, s) \in [0, \omega] \times [0, \omega];\ 0 \le s \le t \le \omega\}$ and $\varphi \in L^2(\mathbb{R})$, $\|\varphi\|_{L^2(\mathbb{R})} = 1$;

(H3) $a \in L^1_+(0, \omega)$ and $K : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies

$$K \in L^2(\mathbb{R} \times \mathbb{R}), \quad \|K\|_{L^2(\mathbb{R} \times \mathbb{R})} = 1.$$

(H4) $W : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $f_1(\xi, r) \le W(\xi, r) \le f_2(\xi, r)$ for every $(\xi, r) \in \mathbb{R} \times \mathbb{R}$ with

(1) $W(\cdot, r) \in L^2(\mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R})$ for every $r \in \mathbb{R}$ and satisfying

$$\left| \frac{\partial W(\xi, r)}{\partial \xi} \right| \le l(\xi) \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and every } r \in \mathbb{R},$$

where $l \in L^1(\mathbb{R})$;

(2) $f_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are functions such that for $i = 1, 2$,

$$f_i(\cdot, r) \in L^2(\mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R}) \quad \text{for every } r \in \mathbb{R}$$

and

$$\left| \frac{\partial f_i(\xi, r)}{\partial \xi} \right| \le l(\xi) \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and every } r \in \mathbb{R};$$

(3) there exists $\psi : \mathbb{R} \times \mathbb{R}^0_+ \to \mathbb{R}$ such that $\psi(\cdot, M) \in L^2(\mathbb{R})$ for every $M > 0$ and

$$|f_i(\xi, r)| \le \psi(\xi, M) \quad \text{for every } (\xi, r) \in \mathbb{R} \times [-M, M];$$

(4) $f_1(\xi, r) \le f_2(\xi, r)$ for every $(\xi, r) \in \mathbb{R} \times \mathbb{R}$ and

$$f_1(\xi, r_0) \ge \limsup_{r \to r_0} f_1(\xi, r), \quad f_2(\xi, r_0) \le \liminf_{r \to r_0} f_2(\xi, r)$$

for every $\xi \in \mathbb{R}$ and $r_0 \in \mathbb{R}$.

(H5) $b : [0, \omega] \times \mathbb{R} \to \mathbb{R}$ is a measurable function and satisfies that there exist $s_1, s_2 \in L^1_+(0, \omega)$ such that

$$0 < s_1(t) \le b(t, \xi) \le s_2(t) \quad \text{for every } (t, \xi) \in [0, \omega] \times \mathbb{R}.$$

Remarks 1.1.

(1) An example of functions that satisfies condition (H_1) is

$$f(t, \tilde{p}, \tilde{q}) = \frac{\alpha(t)(\tilde{p}^2 + \tilde{q}^2)}{1 + \tilde{p}^2 + \tilde{q}^2} \quad \text{for } (t, \tilde{p}, \tilde{q}) \in [0, \omega] \times \mathbb{R} \times \mathbb{R}.$$

In this case, $f(t, \cdot, \cdot)$ is $\alpha(t)$-Lipschitz continuous for a.e. $t \in [0, \omega]$.

(2) Conditions (H_3), (H_4), and (H_5) are satisfied by some functions K, W, and b of the examples in [1, Example 2.1 (2),(3),(4)], respectively.

We now proceed to the statement of the main result in this article.

Theorem 1.1. *Assume conditions* (H_1)–(H_5). *Further suppose that the following inequalities hold for fixed* $R > 0$ *and* $\delta > 0$:

$$\tilde{M}(1 + M_1\omega) < 1; \tag{1.4}$$

$$s_1(t) \ge \sqrt{2}\alpha(t) + a(t) + \frac{\|\psi(\cdot, R)\|_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)M_1R\omega}{R} + \delta, \quad \textbf{a.e. } t \in [0, \omega], \tag{1.5}$$

where

$$\tilde{M} := \left(\frac{e^{\|s_2\|_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}} + 1\right) e^{\|s_2\|_{L^1(0,\omega)}} \|\sqrt{2}\alpha + a\|_{L^1(0,\omega)}, \quad M_1 := \max_{(t,s)\in\Delta}\{|k(t,s)|\}.$$

Then problem (1.1) *admits at least one solution u with $\|u(t,\cdot)\| \leq R$, for all $t \in [0,\omega]$.*

2 Reformulation of the model

Let X be a Banach space. Denote by $\|\cdot\|_X$ the norms of X, $\|\cdot\|_{\mathcal{L}(X)}$ the operator norm of linear and bounded operators on X. We reformulate equation (1.1) as an ordinary differential equation in the Hilbert space $L^2(\mathbb{R})$. To this aim, we give the following definitions, and for these well-defined properties, see Section 3.

- The Volterra integral operator $\hat{k} : C([0,\omega]; L^2(\mathbb{R})) \to C([0,\omega]; L^2(\mathbb{R}))$, which is defined by

$$\hat{k}x(t) = \int_0^t k(t,s)x(s)\mathrm{d}s \quad \text{for all } t \in [0,\omega] \quad \text{and} \quad x \in C([0,\omega]; L^2(\mathbb{R})).$$

- The map $\hat{f} : [0,\omega] \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$, which is defined, for every $(t,\tilde{x},\tilde{y}) \in [0,\omega] \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$, by

$$\hat{f}(t,\tilde{x},\tilde{y})(\xi) = f(t,\tilde{x}(\xi),\tilde{y}(\xi)) \quad \text{for } \xi \in \mathbb{R}.$$

  Particularly, we see that $\hat{f}(t,\tilde{x},\tilde{y}) = f(t,\tilde{x}(\cdot),\tilde{y}(\cdot))$ for every $(t,\tilde{x},\tilde{y}) \in [0,\omega] \times L^2(\mathbb{R}) \times L^2(\mathbb{R})$.
- Define $\hat{K} : [0,\omega] \times L^2(\mathbb{R}) \to L^2(\mathbb{R})$ as follows:

$$\hat{K}(t,\tilde{x})(\xi) = a(t)\int_{\mathbb{R}} K(\xi,\eta)\tilde{x}(\eta)\mathrm{d}\eta \quad \text{for } \xi \in \mathbb{R}$$

  for every $(t,\tilde{x}) \in [0,\omega] \times L^2(\mathbb{R})$.
- Define the multimap $\hat{W} : L^2(\mathbb{R}) \rightsquigarrow L^2(\mathbb{R})$ as

$$\hat{W}(\tilde{x}) = \left\{\begin{matrix} \tilde{y} \in L^2(\mathbb{R}) \cap AC_{\mathrm{loc}}(\mathbb{R}) : |\tilde{y}'(\xi)| \leq l(\xi) \text{ for a.e. } \xi \in \mathbb{R} \text{ and} \\ f_1(\xi, \int_{\mathbb{R}} \varphi(\eta)\tilde{x}(\eta)\mathrm{d}\eta) \leq \tilde{y}(\xi) \leq f_2(\xi, \int_{\mathbb{R}} \varphi(\eta)\tilde{x}(\eta)\mathrm{d}\eta) \quad \text{for all } \xi \in \mathbb{R} \end{matrix}\right\}$$

  for every $\tilde{x} \in L^2(\mathbb{R})$.
- The linear operator $A(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, for every $t \in [0,\omega]$ is defined, for every $\tilde{x} \in L^2(\mathbb{R})$, by

$$A(t)\tilde{x}(\xi) = -b(t,\xi)\tilde{x}(\xi) \quad \text{for every } \xi \in \mathbb{R}.$$

Now, we can write the reformulation of equation (1.1) as the following semilinear evolution inclusion in the Hilbert space $L^2(\mathbb{R})$:

$$x'(t) \in A(t)x(t) + \hat{f}(t,x(t),\hat{k}x(t)) + \hat{K}(t,x(t)) + \hat{W}(x(t)), \quad t \in [0,\omega]. \tag{2.1}$$

Obviously, the solutions for (2.1) with $x(0) = x(\omega)$ give rise to the one for (1.1).

3 Preliminaries

For the Volterra integral operator $\hat{k}$ presented in Section 2, we investigate firstly its main properties.

Lemma 3.1. *The map $\hat{k}$ satisfies the following properties*:
(1) $\hat{k}$ *is well defined*;
(2) $\hat{k}$ *is a linear and continuous operator*;
(3) $\|\hat{k}x\|_{C([0,\omega];\, L^2(\mathbb{R}))} \le M_1\omega\|x\|_{C([0,\omega];\, L^2(\mathbb{R}))}$, *for each* $x \in C([0,\omega];\, L^2(\mathbb{R}))$.

Proof. For each $x \in C([0,\omega];\, L^2(\mathbb{R}))$, from condition (H_2), we conclude that

$$\|\hat{k}x(t)\|_{L^2(\mathbb{R})} \le \int_0^t |k(t,s)|\|x(s)\|_{L^2(\mathbb{R})}\mathrm{d}s \le \max_{s\in[0,\omega]}|k(t,s)|\|x\|_{C([0,\omega];\, L^2(\mathbb{R}))}\int_0^t \mathrm{d}s \tag{3.1}$$

for every $t \in [0,\omega]$, thanks to absolute integrability of the integral. Moreover, we can prove that $\hat{k}x \in C([0,\omega];\, L^2(\mathbb{R}))$. In fact, let the sequence $\{t_n\} \subset [0,\omega]$ be such that $t_n \to t$ in $\mathbb{R}$. It follows that,

$$\begin{aligned}\|\hat{k}x(t_n) - \hat{k}x(t)\|_{L^2(\mathbb{R})} &\le \left\| \int_0^{t_n} k(t_n,s)x(s)\mathrm{d}s - \int_0^t k(t,s)x(s)\mathrm{d}s \right\|_{L^2(\mathbb{R})}\\ &\le \left\| \int_0^{t_n} k(t_n,s)x(s)\mathrm{d}s - \int_0^t k(t_n,s)x(s)\mathrm{d}s + \int_0^t k(t_n,s)x(s)\mathrm{d}s - \int_0^t k(t,s)x(s)\mathrm{d}s \right\|_{L^2(\mathbb{R})}\\ &\le M_1\|x\|_{C([0,\omega];\, L^2(\mathbb{R}))}|t_n - t| + \|x\|_{C([0,\omega];\, L^2(\mathbb{R}))}\omega \max_{s\in[0,\omega]}|k(t_n,s) - k(t,s)|.\end{aligned}$$

From the continuity of k, we obtain the desired, which implies that conclusion (1) holds. By taking the upper bound about t at both ends of (3.1), we obtain

$$\sup_{t\in[0,\omega]}\|\hat{k}x(t)\|_{L^2(\mathbb{R})} \le M_1\omega\|x\|_{C([0,\omega];\, L^2(\mathbb{R}))},$$

where M_1 is given in Theorem 1.1. Then conclusion (3) holds.

Next, we can claim that $\hat{k}$ is linear and continuous. Indeed, for $a_1, a_2 \in \mathbb{R}$ and $x, y \in C([0,\omega];\, L^2(\mathbb{R}))$, we have that

$$\hat{k}(a_1x(t) + a_2y(t)) = \int_0^t k(t,s)(a_1x(s) + a_2y(s))\mathrm{d}s = a_1\hat{k}x(t) + a_2\hat{k}y(t)$$

for every $t \in [0,\omega]$. Meanwhile, from the continuity of the integral and kernel k, one sees that $\hat{k}$ is continuous, and further,

$$\|\hat{k}\|_{\mathcal{L}(C([0,\omega];\, L^2(\mathbb{R})))} \le M_1\omega.$$

Then the proof is complete. □

Then we can introduce a MNC υ in $C([0,\omega];\, L^2(\mathbb{R}))$ defined by, for every bounded $\Omega \subset C([0,\omega];\, L^2(\mathbb{R}))$,

$$\upsilon(\Omega) = \max_{\{u_n\}_{n\in\mathbb{N}}\subset\Omega}(\gamma(\{u_n\}_{n\in\mathbb{N}}), \eta(\{u_n\}_{n\in\mathbb{N}})),$$

where

$$\begin{aligned}\gamma(\{u_n\}_{n\in\mathbb{N}}) &= \sup_{t\in[0,\omega]}(\chi(\{u_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\hat{k}u_n(t)\}_{n\in\mathbb{N}})),\\ \eta(\{u_n\}_{n\in\mathbb{N}}) &= \mathrm{mod}_C(\{u_n\}_{n\in\mathbb{N}}) + \mathrm{mod}_C(\{\hat{k}u_n\}_{n\in\mathbb{N}}),\end{aligned}$$

where mod_C is the modulus of equicontinuity in $C([0,\omega];\, L^2(\mathbb{R}))$, which has the following form:

$$\mathrm{mod}_C(\Omega) = \lim_{\delta\to 0}\sup_{x\in\Omega}\max_{|t_1-t_2|\le\delta}\|x(t_1) - x_{t2}\|_{L^2(\mathbb{R})}$$

for each bounded $\Omega \subset C([0,\omega]; L^2(\mathbb{R}))$. Obviously, $\mathrm{mod}_C(\Omega) = 0$ implies that $\Omega(\cdot)$ is equicontinuous. The range for υ is the cone $\mathbb{R}^2_+$ and the maximum is taken in the sense of the partial ordering induced by this cone.

Remark 3.1. The MNC υ is well defined in $C([0,\omega]; L^2(\mathbb{R}))$ and nonsingular (see [8, Theorem 4.1, p. 2531]). In general, one says MNC υ is nonsingular if $\upsilon(\{a\} \cup \Omega) = \upsilon(\Omega)$ for every $a \in L^2(\mathbb{R})$ and bounded $\Omega \subset C([0,\omega]; L^2(\mathbb{R}))$.

Lemma 3.2. *The MNC υ is regular, i.e., $\upsilon(\Omega) = (0,0)$ is equivalent to that Ω is relative compactness in $C([0,\omega]; L^2(\mathbb{R}))$.*

Proof. If the set $\Omega \subset C([0,\omega]; L^2(\mathbb{R}))$ is relatively compact, then for any sequence $\{u_n\}_{n\in\mathbb{N}} \subset \Omega$, we have $\chi(\{u_n(t)\}_{n\in\mathbb{N}}) = 0$ for every $t \in [0,\omega]$ and $\mathrm{mod}_C(\{u_n\}_{n\in\mathbb{N}}) = 0$. Since $\hat{k}$ is continuous (see Lemma 3.1), one sees that

$$\chi(\{\hat{k}u_n(t)\}_{n\in\mathbb{N}}) = 0, \quad t \in [0,\omega],$$

and that

$$\mathrm{mod}_C(\{\hat{k}u_n\}_{n\in\mathbb{N}}) = 0.$$

Thus, $\upsilon(\Omega) = (0,0)$.

On the other hand, if $\upsilon(\Omega) = (0,0)$, then for every $\{u_n\}_{n\in\mathbb{N}} \subset \Omega$, we have that

$$\sup_{t\in[0,\omega]} (\chi(\{u_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\hat{k}u_n(t)\}_{n\in\mathbb{N}})) = 0,$$

and hence, $\chi(\{u_n(t)\}_{n\in\mathbb{N}}) = 0$ for all $t \in [0,\omega]$. Moreover,

$$\mathrm{mod}_C(\{u_n\}_{n\in\mathbb{N}}) + \mathrm{mod}_C(\{\hat{k}u_n\}_{n\in\mathbb{N}}) = 0,$$

and then $\mathrm{mod}_C(\{u_n\}_{n\in\mathbb{N}}) = 0$, which implies that $\{u_n\}_{n\in\mathbb{N}}$ is equicontinuous.

Therefore, the well-known Arzela-Ascoli theorem (see [11, Theorem 1.3.1, p. 7]) implies that Ω is relatively compact, and then, we obtain the desired. □

To prove our main result, we state the following version for Hausdorff measures of noncompactness in [12, Corollary 4.2.5, p. 113].

Proposition 3.1. *Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1(0,\omega; L^2(\mathbb{R}))$ satisfy the following conditions*:

(1) *$\{f_n\}_{n\in\mathbb{N}}$ is integrably bounded in $L^1(0,\omega; L^2(\mathbb{R}))$*;

(2) *there exists $q \in L^1_+(0,\omega)$ such that, for a.e. $t \in [0,\omega]$,*

$$\chi(\{f_n(t)\}_{n\in\mathbb{N}}) \le q(t).$$

Then, we have the estimate, for every $t \in [0,\omega]$,

$$\chi\left(\left\{\int_0^t f_n(s)\mathrm{d}s\right\}_{n\in\mathbb{N}}\right) \le \int_0^t q(s)\mathrm{d}s.$$

We next describe the main properties of the nonlinear term $\hat{f}$.

Lemma 3.3. *The map $\hat{f}$ satisfies the following properties*:

(1) *$\hat{f}$ is well defined*;

(2) *$\hat{f}(\cdot,\tilde{x},\tilde{y})$ is measurable for every $\tilde{x},\tilde{y} \in L^2(\mathbb{R})$*;

(3) *$\hat{f}(t,\cdot,\cdot): L^2(\mathbb{R})\times L^2(\mathbb{R})\to L^2(\mathbb{R})$ is $\alpha(t)$-Lipschitz continuous for a.e. $t\in[0,\omega]$, i.e., for each $(\tilde{x}_n,\tilde{y}_n)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$, $n=1,2$ and a.e. $t\in[0,\omega]$,*

$$\|f(t,\tilde{x}_1,\tilde{y}_1)-f(t,\tilde{x}_2,\tilde{y}_2)\|_{L^2(\mathbb{R})}\le\sqrt{2}\alpha(t)\Big(\|\tilde{x}_1-\tilde{x}_2\|_{L^2(\mathbb{R})}+\|\tilde{y}_1-\tilde{y}_2\|_{L^2(\mathbb{R})}\Big);$$

(4) *for every $(t,\tilde{x},\tilde{y})\in[0,\omega]\times L^2(\mathbb{R})\times L^2(\mathbb{R})$,*

$$||\hat{f}(t,\tilde{x},\tilde{y})||_{L^2(\mathbb{R})}\le\sqrt{2}\alpha(t)\Big(||\tilde{x}||_{L^2(\mathbb{R})}+||\tilde{y}||_{L^2(\mathbb{R})}\Big),$$

where α is given in condition (H_1).

Proof. For every $(t,\tilde{x},\tilde{y})\in[0,\omega]\times L^2(\mathbb{R})\times L^2(\mathbb{R})$, from condition (H_1)(2)(3), we have that

$$|\hat{f}(t,\tilde{x},\tilde{y})(\xi)|=|f(t,\tilde{x}(\xi),\tilde{y}(\xi))|=|f(t,\tilde{x}(\xi),\tilde{y}(\xi))-f(t,0,0)|\le\alpha(t)(|\tilde{x}(\xi)|+|\tilde{y}(\xi)|).\tag{3.2}$$

Then $\hat{f}$ is well defined.

From (3.2), one sees that

$$||\hat{f}(t,\tilde{x},\tilde{y})||^2_{L^2(\mathbb{R})}\le\int_{\mathbb{R}}(\alpha(t))^2(|\tilde{x}(\xi)|+|\tilde{y}(\xi)|)^2\mathrm{d}\xi\le2(\alpha(t))^2\Big(||\tilde{x}||_{L^2(\mathbb{R})}+||\tilde{y}||_{L^2(\mathbb{R})}\Big)^2.$$

Thus, conclusion (4) holds.

Let $(\tilde{x},\tilde{y})\in L^2(\mathbb{R})\times L^2(\mathbb{R})$ be fixed. Let $Q\in(L^2(\mathbb{R}))^*$, from the Riesz representation theorem, there exists $g\in L^2(\mathbb{R})$ such that

$$Q(h)=\int_{\mathbb{R}}h(\xi)g(\xi)\mathrm{d}\xi\quad\text{for every } h\in L^2(\mathbb{R}).$$

We consider the functional $Q\circ\hat{f}(\cdot,\tilde{x},\tilde{y}):[0,\omega]\to\mathbb{R}$ defined by

$$Q\circ\hat{f}(t,\tilde{x},\tilde{y})=\int_{\mathbb{R}}f(t,\tilde{x}(\xi),\tilde{y}(\xi))g(\xi)\mathrm{d}\xi,\qquad t\in[0,\omega].$$

The condition (H_1)(1)(3) implies that f is globally measurable, and hence,

$$\overline{f}(t,\xi):=f(t,\tilde{x}(\xi),\tilde{y}(\xi))g(\xi)$$

is measurable on $[0,\omega]\times\mathbb{R}$. In addition, we obtain

$$|f(t,\tilde{x}(\xi),\tilde{y}(\xi))g(\xi)|\le\alpha(t)(|\tilde{x}(\xi)|+|\tilde{y}(\xi)|)|g(\xi)|,$$

so, $\overline{f}(\cdot,\cdot)\in L^1([0,\omega]\times\mathbb{R})$. Then from the Fubini theorem, one can see $Q\circ\hat{f}(\cdot,\tilde{x},\tilde{y})\in L^1(0,\omega)$ and so is measurable. By the arbitrariness of Q and the separability of $L^2(\mathbb{R})$, we obtain the measurability of $\hat{f}(\cdot,\tilde{x},\tilde{y})$ from [13, Corollary 1.11, p. 279] (see also [14, Corollary 3.10.5, p. 365]).

To prove conclusion (3). Let $(\tilde{x}_n,\tilde{y}_n)\in L^2(\mathbb{R})\times L^2(\mathbb{R})$, $n=1,2$, from condition (H_1)(3), one has that for a.e. $t\in[0,\omega]$,

$$\begin{aligned}\|\hat{f}(t,\tilde{x}_1,\tilde{y}_1)-f(t,\tilde{x}_2,\tilde{y}_2)\|^2_{L^2(\mathbb{R})}&=\int_{\mathbb{R}}|f(t,\tilde{x}_1(\xi),\tilde{y}_1(\xi))-f(t,\tilde{x}_2(\xi),\tilde{y}_2(\xi))|^2\mathrm{d}\xi\\&\le\int_{\mathbb{R}}(\alpha(t))^2(|\tilde{x}_1(\xi)-\tilde{x}_2(\xi)|+|\tilde{y}_2(\xi)-\tilde{y}_2(\xi)|)^2\mathrm{d}\xi\\&\le2(\alpha(t))^2\int_{\mathbb{R}}(|\tilde{x}_1(\xi)-\tilde{x}_2(\xi)|^2+|\tilde{y}_2(\xi)-\tilde{y}_2(\xi)|^2)\mathrm{d}\xi\\&\le2(\alpha(t))^2(\|\tilde{x}_1-\tilde{x}_2\|_{L^2(\mathbb{R})}+\|\tilde{y}_2-\tilde{y}_2\|_{L^2(\mathbb{R})})^2.\end{aligned}$$

Then the proof is complete. □

The map $\hat{K}$, multimap $\hat{W}$ and linear operator $A(t)$ for every $t \in [0, \omega]$ are all well defined and have following properties (see [1, pp. 829–837]).

Lemma 3.4. [1, Proposition 3.1, p. 832] *The map $\hat{K}$ satisfies the following properties*:
(1) *$\hat{K}(\cdot, \tilde{x})$ is measurable for every $\tilde{x} \in L^2(\mathbb{R})$;*
(2) *$\hat{K}(t, \cdot) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is $a(t)$-Lipschitz continuous for every $t \in [0, \omega]$, i.e., for each $\tilde{x}, \tilde{y} \in L^2(\mathbb{R})$, one has*

$$\|\hat{K}(t, \tilde{x}) - \hat{K}(t, \tilde{y})\|_{L^2(\mathbb{R})} \le a(t)\|\tilde{x} - \tilde{y}\|_{L^2(\mathbb{R})} \quad \text{for every } t \in [0, \omega].$$

Lemma 3.5. [1, Proposition 3.2, p. 833, Propositions 3.3, 3.4, p. 835] *The multimap $\hat{W}$ satisfies the following properties*:
(1) *$\hat{W}(\tilde{x})$ is nonempty, closed, and convex in $L^2(\mathbb{R})$ for every $\tilde{x} \in L^2(\mathbb{R})$;*
(2) *$\hat{W}$ is locally compact, i.e., its restriction to a neighborhood of every point is compact*;
(3) *$\hat{W}$ is lower semicontinuous.*

Lemma 3.6. [1, p. 830] *The linear operator $A(t)$ for every $t \in [0, \omega]$ satisfy the following properties*:
(1) *$A(t)$ is bounded, $t \in [0, \omega]$;*
(2) *the family $\{A(t)\}_{t\in[0,\omega]}$ generates an evolution system $\{U(t, s)\}_{(t,s)\in\Delta}$, and each $U(t, s) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is defined, for every $\tilde{x} \in L^2(\mathbb{R})$, by*

$$U(t, s)\tilde{x}(\xi) = e^{\int_s^t -b(\sigma,\xi)\mathrm{d}\sigma}\tilde{x}(\xi) \quad \text{for every } \xi \in \mathbb{R}.$$

Remarks 3.2. [1, p. 831]
(1) $U(t, s)$ is bounded with

$$||U(t, s)||_{\mathcal{L}(L^2(\mathbb{R}))} \le e^{||s_2||_{L^1(0,\omega)}} \quad \text{for all } (t, s) \in \Delta; \tag{3.3}$$

(2) the unique solution of the linear-associated initial problem

$$\begin{cases} x'(t) = A(t)x(t) + f(t), t \in [0, \omega], & f \in L^1(0, \omega; L^2(\mathbb{R})), \\ x(0) = x_0, \quad x_0 \in L^2(\mathbb{R}) \end{cases}$$

can be written in the integral form

$$x(t) = U(t, 0)x_0 + \int_0^t U(t, s)f(s)\mathrm{d}s, \quad t \in [0, \omega];$$

(3) $I - U(\omega, 0)$ is an invertible operator with

$$||(I - U(\omega, 0))^{-1}||_{\mathcal{L}(L^2(\mathbb{R}))} \le \frac{1}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}}. \tag{3.4}$$

4 The proof of the main result

To prove that problem (1.1) admits at least one solution, we study the periodic problem in $L^2(\mathbb{R})$ given by (2.1) and

$$x(0) = x(\omega).$$

Since the multimap $\hat{W}$ in (2.1) is lower semicontinuous with closed and convex values (see Lemma 3.5(1),(3)), it admits a continuous selection $\hat{g} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ from [12, Theorem 1.2.18, p. 19]. The local compactness of $\hat{W}$ (see Lemma 3.5(2)) implies that $\hat{g}$ is compact. So, we study the semilinear evolution equation in $L^2(\mathbb{R})$ instead of (2.1)

$$x'(t) = A(t)x(t) + \hat{f}(t, x(t), \hat{k}x(t)) + \hat{K}(t, x(t)) + \hat{g}(x(t)), \quad t \in [0, \omega]. \tag{4.1}$$

For every $q \in C([0, \omega]; L^2(\mathbb{R}))$, consider the linear equation:

$$x'(t) = A(t)x(t) + \lambda(\hat{f}(t, q(t), \hat{k}q(t)) + \hat{K}(t, q(t)) + \hat{g}(q(t))), \quad t \in [0, \omega], \tag{4.2}$$

with the parameter $\lambda \in [0, 1]$. From Remarks 3.2(2), the unique solution of (4.2) with a given initial condition $x(0) = x_0 \in L^2(\mathbb{R})$ can be written as follows:

$$x(t) = U(t, 0)x_0 + \lambda \int_0^t U(t, s)(\hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s)))\mathrm{d}s, \quad t \in [0, \omega].$$

Since $I - U(\omega, 0)$ is invertible (see Remarks 3.2(3)), the unique solution x_q^λ of (4.2) with the periodic condition $x(0) = x(\omega)$ is that, for $t \in [0, \omega]$,

$$\begin{aligned} x_q^\lambda(t) = {} & \lambda U(t, 0)(I - U(\omega, 0))^{-1} \int_0^\omega U(\omega, s)(\hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s)))\mathrm{d}s \\ & + \lambda \int_0^t U(t, s)(\hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s)))\mathrm{d}s. \end{aligned} \tag{4.3}$$

Then, we introduce the operator $T : C([0, \omega]; L^2(\mathbb{R})) \times [0, 1] \to C([0, \omega]; L^2(\mathbb{R}))$ defined by

$$T(q, \lambda) = x_q^\lambda.$$

Obviously, the fixed points of $T(\cdot,1)$ are solutions of equation (4.1) with $x(0) = x(\omega)$, which give the solutions for equation (2.1).

Lemma 4.1. *Assume all conditions of* Theorem 1.1 *hold. Then T is a bounded and continuous operator.*

Proof. Let $\Lambda \subset C([0, \omega]; L^2(\mathbb{R}))$ be bounded. Since $\hat{g}$ is compact, there exists a constant $\hat{\gamma} > 0$ such that $||\hat{g}(q(t))||_{L^2(\mathbb{R})} \le \hat{\gamma}$ for $t \in [0, \omega]$ and every $q \in \Lambda$.

Let $(q, \lambda) \in \Lambda \times [0, 1]$ be fixed. From (3.3), (3.4), and Lemma 3.1(3), we have that

$$\begin{aligned} ||x_q^\lambda(t)||_{L^2(\mathbb{R})} \le {} & \frac{e^{||s_2||_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}} \int_0^\omega e^{||s_2||_{L^1(0,\omega)}} ||\hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s))||_{L^2(\mathbb{R})}\mathrm{d}s \\ & + \int_0^t e^{||s_2||_{L^1(0,\omega)}} ||\hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s))||_{L^2(\mathbb{R})}\mathrm{d}s \\ \le {} & \left(\frac{e^{||s_2||_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}} + 1 \right) e^{||s_2||_{L^1(0,\omega)}} \int_0^\omega (\sqrt{2}\alpha(s)\Big(||q(s)||_{L^2(\mathbb{R})} + ||\hat{k}q(s)||_{L^2(\mathbb{R})}\Big) + a(s)||q(s)||_{L^2(\mathbb{R})} + \hat{\gamma})\mathrm{d}s \\ \le {} & \left(\frac{e^{||s_2||_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}} + 1 \right) e^{||s_2||_{L^1(0,\omega)}} (M_\Lambda||\sqrt{2}\alpha + a||_{L^1(0,\omega)} + M_1 M_\Lambda \omega ||\sqrt{2}\alpha||_{L^1(0,\omega)} + \hat{\gamma}\omega), \end{aligned}$$

where $||q(s)||_{L^2(\mathbb{R})} \le M_\Lambda$ for every $s \in [0, \omega]$ and $q \in \Lambda$. Thus, T is bounded on bounded sets.

To prove T is a continuous operator. Let $q_n \to q$ in $C([0,\omega]; L^2(\mathbb{R}))$ and $\lambda_n \to \lambda$ in $\mathbb{R}$. From (4.3), we have

$$x_{q_n}^{\lambda_n} - x_q^{\lambda} = x_{q_n}^{\lambda_n} - x_{q_n}^{\lambda} + x_{q_n}^{\lambda} - x_q^{\lambda} = (\lambda_n - \lambda)x_{q_n}^1 + (x_{q_n}^{\lambda} - x_q^{\lambda}).$$

Since $q_n \to q$ in $C([0,\omega]; L^2(\mathbb{R}))$, $\{q_n\}_{n\in\mathbb{N}}$ is bounded in $C([0,\omega]; L^2(\mathbb{R}))$. From an argument similar as mentioned earlier, one obtains that $\{x_{q_n}^1\}_{n\in\mathbb{N}}$ is also bounded in $C([0,\omega]; L^2(\mathbb{R}))$ for $n \in \mathbb{N}$. The boundedness property in $C([0,\omega]; L^2(\mathbb{R}))$ of $\{x_{q_n}^1\}_{n\in\mathbb{N}}$ implies that

$$x_{q_n}^{\lambda_n} \to x_{q_n}^{\lambda} \quad \text{in } C([0,\omega]; L^2(\mathbb{R})).$$

Again from (4.3), we obtain that for $t \in [0,\omega]$,

$$x_{q_n}^{\lambda}(t) - x_q^{\lambda}(t) = \lambda U(t,0)(I - U(\omega,0))^{-1}\int_0^{\omega} U(\omega,s)(h_n(s) - h(s))\mathrm{d}s + \lambda\int_0^t U(t,s)(h_n(s) - h(s))\mathrm{d}s,$$

where

$$\begin{aligned} h_n(s) &:= \hat{f}(s, q_n(s), \hat{k}q_n(s)) + \hat{K}(s, q_n(s)) + \hat{g}(q_n(s)),\\ h(s) &:= \hat{f}(s, q(s), \hat{k}q(s)) + \hat{K}(s, q(s)) + \hat{g}(q(s)), \end{aligned} \qquad s \in [0,\omega].$$

From (3.3) and (3.4), we have

$$\begin{aligned} \|x_{q_n}^{\lambda}(t) - x_q^{\lambda}(t)\|_{L^2(\mathbb{R})} &\le \frac{e^{\|s_2\|_{L^1(0,\omega)}}}{1 - e^{\int_0^{\omega} -s_1(t)\mathrm{d}t}}\int_0^{\omega} e^{\|s_2\|_{L^1(0,\omega)}}\|h_n(s) - h(s)\|_{L^2(\mathbb{R})}\mathrm{d}s + \int_0^t e^{\|s_2\|_{L^1(0,\omega)}}\|h_n(s) - h(s)\|_{L^2(\mathbb{R})}\mathrm{d}s\\ &\le \left(\frac{e^{\|s_2\|_{L^1(0,\omega)}}}{1 - e^{\int_0^{\omega} -s_1(t)\mathrm{d}t}} + 1\right)e^{\|s_2\|_{L^1(0,\omega)}}\int_0^{\omega}\|h_n(s) - h(s)\|_{L^2(\mathbb{R})}\mathrm{d}s. \end{aligned}$$

From the continuity of the $\hat{f}$, $\hat{k}$, $\hat{K}$, and $\hat{g}$ (see Lemma 3.1–3.4), one sees that $h_n(s) \to h(s)$ for a.e. $s \in [0,\omega]$. Also from Lemmas 3.1–3.4 and a similar argument as mentioned earlier, we have that

$$\begin{aligned} \|h_n(s) - h(s)\|_{L^2(\mathbb{R})} &\le \sqrt{2}\alpha(s)(\|q_n(s) - q(s)\|_{L^2(\mathbb{R})} + \|\hat{k}(q_n(s) - q(s))\|_{L^2(\mathbb{R})}) + a(s)\|q_n(s) - q(s)\|_{L^2(\mathbb{R})} + 2\hat{\gamma}\\ &\le 2\rho(\sqrt{2}\alpha(s) + a(s)) + 2\sqrt{2}\omega\rho M_1\alpha(s) + 2\hat{\gamma}, \end{aligned}$$

where $\|q_n(s)\|_{L^2(\mathbb{R})} \le \rho$ for every $s \in [0,\omega]$ and $n \in \mathbb{N}$. Thanks to Lebesgue's dominated convergence theorem, we have that $x_{q_n}^{\lambda}(t) \to x_q^{\lambda}(t)$ in $L^2(\mathbb{R})$ for every $t \in [0,\omega]$, i.e.,

$$x_{q_n}^{\lambda} \to x_q^{\lambda} \quad \text{in } C([0,\omega]; L^2(\mathbb{R})).$$

Therefore, we obtain the desired. □

Lemma 4.2. *Assume all conditions of Theorem* 1.1 *hold. Then T is condensing with respect to the MNC υ.*

Proof. To prove that T is an υ-condensing operator, i.e., for bounded $\Omega \subset C([0,\omega]; L^2(\mathbb{R}))$, the relation $\upsilon(T(\Omega \times [0,1])) \ge \upsilon(\Omega)$ implies that Ω is relatively compact. Let $\Omega \subset C([0,\omega]; L^2(\mathbb{R}))$ be bounded. $T(\Omega \times [0,1])$ is bounded in $C([0,\omega]; L^2(\mathbb{R}))$ from Lemma 4.1. Consider a sequence $\{x_n\}_{n\in\mathbb{N}} \subset T(\Omega \times [0,1])$, which achieves that

$$\upsilon(T(\Omega \times [0,1])) = (\gamma(\{x_n\}_{n\in\mathbb{N}}), \eta(\{x_n\}_{n\in\mathbb{N}})) \in \mathbb{R}_+^2.$$

There exist $\{q_n\}_{n\in\mathbb{N}} \subset \Omega$ and $\{\lambda_n\}_{n\in\mathbb{N}} \subset [0,1]$ such that $x_{q_n}^{\lambda_n} = x_n$.

From Lemmas 3.3(3), 3.4(2), and the compactness of the map $\hat{g}$, we have that, for a.e. $t \in [0,\omega]$,

$$\begin{aligned} &\chi(\{\hat{f}(t, q_n(t), \hat{k}q_n(t))\}_{n\in\mathbb{N}}) \le \sqrt{2}\alpha(t)(\chi(\{q_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\hat{k}q_n(t)\}_{n\in\mathbb{N}})),\\ &\chi(\{\hat{K}(t, q_n(t))\}_{n\in\mathbb{N}}) \le a(t)\chi(\{q_n(t)\}_{n\in\mathbb{N}}),\\ &\chi(\{\hat{g}(q_n(t))\}_{n\in\mathbb{N}}) = 0. \end{aligned}$$

Since the Hausdorff MNC χ is algebraically semiadditive (see [12, (vii), p. 34]), we conclude that

$$\begin{aligned}\chi(\{h_n(s)\}_{n\in\mathbb{N}}) &\le \chi(\{\hat{f}(s, q_n(s), \hat{k}q_n(s))\}_{n\in\mathbb{N}}) + \chi(\{\hat{K}(s, q_n(s))\}_{n\in\mathbb{N}}) + \chi(\{\hat{g}(q_n(s))\}_{n\in\mathbb{N}})\\ &\le \sqrt{2}\alpha(s)(\chi(\{q_n(s)\}_{n\in\mathbb{N}}) + \chi(\{\hat{k}q_n(s)\}_{n\in\mathbb{N}})) + a(s)\chi(\{q_n(s)\}_{n\in\mathbb{N}})\\ &\le (\sqrt{2}\alpha(s) + a(s))\gamma(\{q_n\}_{n\in\mathbb{N}}) \quad \text{for a.e. } s \in [0, \omega],\end{aligned} \tag{4.4}$$

where the form of $h_n(s)$ is given in the proof of Lemma 4.1. By a inequality (see [12, (2.1.2), p. 35]), one sees that

$$\chi(\{U(t, s)h_n(s)\}_{n\in\mathbb{N}}) \le (\sqrt{2}\alpha(s) + a(s))\gamma(\{q_n\}_{n\in\mathbb{N}})e^{\|s_2\|_{L^1(0,\omega)}} \quad \text{for a.e. } (t, s) \in \Delta.$$

Clearly, $\{U(t, \cdot)h_n\}_{n\in\mathbb{N}}$ is integrably bounded in $L^1(0, t; L^2(\mathbb{R}))$ for all $t \in [0, \omega]$, so is $\{\lambda_n U(t, \cdot)h_n\}_{n\in\mathbb{N}}$. Then, we can exchange the MNC χ with the integral by Proposition 3.1 and obtain that

$$\begin{aligned}\chi\left(\left\{\lambda_n \int_0^t U(t, s)h_n(s)\mathrm{d}s\right\}_{n\in\mathbb{N}}\right) &\le \gamma(\{q_n\}_{n\in\mathbb{N}})e^{\|s_2\|_{L^1(0,\omega)}}\int_0^t (\sqrt{2}\alpha(s) + a(s))\mathrm{d}s\\ &\le \gamma(\{q_n\}_{n\in\mathbb{N}})e^{\|s_2\|_{L^1(0,\omega)}}\|\sqrt{2}\alpha + a\|_{L^1(0,\omega)}\end{aligned}$$

for all $t \in [0, \omega]$, and here, we used the semi-homogeneity of χ. From (3.3) and (3.4), we have that

$$\chi\left(\left\{\lambda_n U(t, 0)(I - U(\omega, 0))^{-1}\int_0^\omega U(\omega, s)h_n(s)\mathrm{d}s\right\}_{n\in\mathbb{N}}\right) \le \gamma(\{q_n\}_{n\in\mathbb{N}})\frac{e^{\|s_2\|_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}}e^{\|s_2\|_{L^1(0,\omega)}}\|\sqrt{2}\alpha + a\|_{L^1(0,\omega)}.$$

Then we deduce that

$$\chi(\{x_n(t)\}_{n\in\mathbb{N}}) \le \gamma(\{q_n\}_{n\in\mathbb{N}})\left(\frac{e^{\|s_2\|_{L^1(0,\omega)}}}{1 - e^{\int_0^\omega -s_1(t)\mathrm{d}t}} + 1\right)e^{\|s_2\|_{L^1(0,\omega)}}\|\sqrt{2}\alpha + a\|_{L^1(0,\omega)} = \gamma(\{q_n\}_{n\in\mathbb{N}})\tilde{M}$$

for all $t \in [0, \omega]$. And since $k(t, s)$ can be regarded as a real number, it follows from semi-homogeneity of χ that

$$\chi(\{k(t, s)x_n(t)\}_{n\in\mathbb{N}}) \le |k(t, s)|\gamma(\{q_n\}_{n\in\mathbb{N}})\tilde{M}$$

for all $(t, s) \in \Delta$.

Since $\{k(t, \cdot)x_n\}_{n\in\mathbb{N}}$ is integrably bounded in $L^1(0, t; L^2(\mathbb{R}))$ for all $t \in [0, \omega]$ (see the proof of Lemma 4.1), from Proposition 3.1, we have that

$$\begin{aligned}\sup_{t\in[0,\omega]}\chi(\{\hat{k}x_n(t)\}_{n\in\mathbb{N}}) &= \sup_{t\in[0,\omega]}\chi\left(\left\{\int_0^t k(t, s)x_n(s)\mathrm{d}s\right\}_{n\in\mathbb{N}}\right)\\ &\le \sup_{t\in[0,\omega]}\int_0^t \chi(\{k(t, s)x_n(s)\}_{n\in\mathbb{N}})\mathrm{d}s\\ &\le \int_0^\omega \max_{(t,s)\in\Delta}|k(t, s)|\chi(\{x_n(s)\}_{n\in\mathbb{N}})\mathrm{d}s\\ &\le \gamma(\{q_n\}_{n\in\mathbb{N}})\tilde{M}M_1\omega.\end{aligned}$$

Thus, we deduce that

$$\gamma(\{x_n\}_{n\in\mathbb{N}}) \le \sup_{t\in[0,\omega]}(\chi(\{x_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\hat{k}x_n(t)\}_{n\in\mathbb{N}})) \le \gamma(\{q_n\}_{n\in\mathbb{N}})\tilde{M}(1 + M_1\omega).$$

Since $\upsilon(T(\Omega \times [0, 1])) \ge \upsilon(\Omega)$, we obtain that

$$\gamma(\{q_n\}_{n\in\mathbb{N}}) \le \gamma(\{x_n\}_{n\in\mathbb{N}}) \le \gamma(\{q_n\}_{n\in\mathbb{N}})\tilde{M}(1 + M_1\omega).$$

By the condition (1.4), we obtain that $\gamma(\{q_n\}_{n\in\mathbb{N}}) = 0$. So, $\gamma(\{x_n\}_{n\in\mathbb{N}}) = 0$.

From (4.4), we have that $\chi(\{h_n(s)\}_{n\in\mathbb{N}}) = 0$ for a.e. $s \in [0, \omega]$. Recall $\{h_n\}_{n\in\mathbb{N}}$ is integrably bounded in $L^1(0, \omega; L^2(\mathbb{R}))$, so $\{x_n\}_{n\in\mathbb{N}}$ is relatively compact in $C([0, \omega]; L^2(\mathbb{R}))$. This implies that $\mathrm{mod}_C(\{x_n\}_{n\in\mathbb{N}}) = 0$ (see [15, Lemma 2, p. 626]). And the continuity of $\hat{k}$ (see Lemma 3.1(2)) guarantees that $\mathrm{mod}_C(\{\hat{k}x_n\}_{n\in\mathbb{N}}) = 0$, too. Then we deduce that $\eta(\{x_n\}_{n\in\mathbb{N}}) = 0$.

Therefore, we have that

$$(0, 0) = (\gamma(\{x_n\}_{n\in\mathbb{N}}), \eta(\{x_n\}_{n\in\mathbb{N}})) = \upsilon(T(\Omega \times [0, 1])) \geq \upsilon(\Omega),$$

and hence, $\upsilon(\Omega) = (0, 0)$. Since υ is regular MNC, we obtain that Ω is relatively compact. This implies that T is an υ-condensing operator. □

Proof of Theorem 1.1. We first prove that T has no fixed points on the bounded of Q for every $\lambda \in [0, 1)$.

Define

$$Q = \Big\{q \in C([0, \omega]; L^2(\mathbb{R})) : \|q(t)\|_{L^2(\mathbb{R})} \leq R, \quad \text{for every } t \in [0, \omega]\Big\},$$

where R is given in (1.5). It is obviously that Q is nonempty, closed, and convex set of $C([0, \omega]; L^2(\mathbb{R}))$.

For the case $\lambda = 0$, it is easy to see $x_q^0 = 0$ for all $q \in Q$; thus, $T(Q \times \{0\}) \cap \partial Q = \varnothing$.

For the case $\lambda \in (0, 1)$, we reason by contradiction and assume that there exists $(q, \lambda) \in \partial Q \cap (0, 1)$ such that $x_q^\lambda = q$. From the definition of Q, one sees that there exists $t_0 \in [0, \omega]$ such that $\|q(t_0)\|_{L^2(\mathbb{R})} = R$. Since $q(0) = q(\omega)$, we can assume that $t_0 \in (0, \omega]$ without loss of generality.

Let us denote by $(\cdot, \cdot)$ the scalar product in $L^2(\mathbb{R})$. According to (4.2), it follows that for a.e. $t \in [0, \omega]$,

$$\begin{aligned}(q(t), q'(t)) &= (q(t), A(t)q(t) + \lambda(\hat{f}(t, q(t), \hat{k}q(t)) + \hat{K}(t, q(t)) + \hat{g}(q(t))))\\ &= \int_{\mathbb{R}} q(t)(\xi)(A(t)q(t))(\xi)\mathrm{d}\xi + \int_{\mathbb{R}} \lambda q(t)(\xi)\hat{f}(t, q(t), \hat{k}q(t))(\xi)\mathrm{d}\xi\\ &\quad + \int_{\mathbb{R}} \lambda q(t)(\xi)\hat{K}(t, q(t))(\xi)\mathrm{d}\xi + \int_{\mathbb{R}} \lambda q(t)(\xi)\hat{g}(q(t))(\xi)\mathrm{d}\xi.\end{aligned}$$

For every $t \in [0, \omega]$, we estimate that as follows:

- from condition (H_5),

$$\int_{\mathbb{R}} q(t)(\xi)(A(t)q(t))(\xi)\mathrm{d}\xi = \int_{\mathbb{R}} -b(t, \xi)(q(t))^2(\xi)\mathrm{d}\xi \leq \int_{\mathbb{R}} -s_1(t)(q(t))^2(\xi)\mathrm{d}\xi = -s_1(t)\|q(t)\|^2_{L^2(\mathbb{R})};$$

- by using the Hölder inequality and Lemma 3.3,

$$\begin{aligned}\int_{\mathbb{R}} \lambda q(t)(\xi)\hat{f}(t, q(t), \hat{k}q(t))(\xi)\mathrm{d}\xi &\leq \lambda \int_{\mathbb{R}} |q(t)(\xi)\hat{f}(t, q(t), \hat{k}q(t))(\xi)|\mathrm{d}\xi\\ &\leq \|\hat{f}(t, q(t), \hat{k}q(t))\|_{L^2(\mathbb{R})}\|q(t)\|_{L^2(\mathbb{R})}\\ &\leq \sqrt{2}\alpha(t)(\|q(t)\|_{L^2(\mathbb{R})} + \|\hat{k}q(t)\|_{L^2(\mathbb{R})})\|q(t)\|_{L^2(\mathbb{R})}\\ &\leq \sqrt{2}\alpha(t)\|q(t)\|^2_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)M_1R\omega\|q(t)\|_{L^2(\mathbb{R})};\end{aligned}$$

- by using the Hölder inequality and (H_3),

$$\begin{aligned}\int_{\mathbb{R}} \lambda q(t)(\xi)\hat{K}(t, q(t))(\xi)\mathrm{d}\xi &= \int_{\mathbb{R}} \lambda q(t)(\xi)a(t)\int_{\mathbb{R}} K(\xi, \eta)q(t)(\eta)\mathrm{d}\eta\mathrm{d}\xi\\ &\leq \lambda a(t)\int_{\mathbb{R}} |q(t)(\xi)|\int_{\mathbb{R}} |K(\xi, \eta)q(t)(\eta)|\mathrm{d}\eta\mathrm{d}\xi\\ &\leq a(t)\int_{\mathbb{R}} |q(t)(\xi)|\|K(\xi, \cdot)\|_{L^2(\mathbb{R})}\|q(t)\|_{L^2(\mathbb{R})}\mathrm{d}\xi\\ &\leq a(t)\|q(t)\|^2_{L^2(\mathbb{R})};\end{aligned}$$

- recall that $\hat{g} \circ q(t) \in \hat{W}(q(t))$, from the definition of Q, condition (H_4), and the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}} \lambda q(t)(\xi)\hat{g}(q(t))(\xi)\mathrm{d}\xi &\le \lambda \int_{\mathbb{R}} |q(t)(\xi)\hat{g}(q(t))(\xi)|\mathrm{d}\xi \\ &\le \int_{\mathbb{R}} |q(t)(\xi)| \max_{i=1,2} \left| f_i\left(\xi, \int_{\mathbb{R}} \varphi(\eta)q(t)(\eta)\mathrm{d}\eta\right)\right| \mathrm{d}\xi \\ &\le \|q(t)\|_{L^2(\mathbb{R})}\|\psi(\cdot,R)\|_{L^2(\mathbb{R})},\end{aligned}$$

where we have used the fact that

$$\left|\int_{\mathbb{R}} \varphi(\eta)q(t)(\eta)\mathrm{d}\eta\right| \le \int_{\mathbb{R}} |\varphi(\eta)q(t)(\eta)|\mathrm{d}\eta \le \|q(t)\|_{L^2(\mathbb{R})} \le R.$$

Therefore, we deduce that for a.e. $t \in [0, \omega]$,

$$\begin{aligned}(q(t), q'(t)) &\le -s_1(t)\|q(t)\|^2_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)\|q(t)\|^2_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)M_1R\omega\|q(t)\|_{L^2(\mathbb{R})} + a(t)\|q(t)\|^2_{L^2(\mathbb{R})} \\ &\quad + \|q(t)\|_{L^2(\mathbb{R})}\|\psi(\cdot,R)\|_{L^2(\mathbb{R})} \\ &= \|q(t)\|^2_{L^2(\mathbb{R})}\left(-s_1(t) + \sqrt{2}\alpha(t) + a(t) + \frac{\|\psi(\cdot,R)\|_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)M_1R\omega}{\|q(t)\|_{L^2(\mathbb{R})}}\right).\end{aligned}$$

Since q is continuous, there exists $\varepsilon > 0$ such that

$$(\|\psi(\cdot,R)\|_{L^2(\mathbb{R})} + \sqrt{2}\alpha(t)M_1R\omega)\left(\frac{1}{\|q(t)\|_{L^2(\mathbb{R})}} - \frac{1}{R}\right) < \frac{\delta}{2}, \quad t \in (t_0 - \varepsilon, t_0), \tag{4.5}$$

with δ is given in (1.5). Therefore, by combining (4.5) with (1.5), we have that

$$(q(t), q'(t)) < 0, \quad \text{a.e. } t \in (t_0 - \varepsilon, t_0).$$

So, one sees that

$$\int_{t_0-\varepsilon}^{t_0} (q(t), q'(t))\mathrm{d}t < 0. \tag{4.6}$$

On the other hand, we claim that

$$\int_{t_0-\varepsilon}^{t_0} (q(t), q'(t))\mathrm{d}t = \frac{1}{2}(\|q(t_0)\|_{L^2(\mathbb{R})} - \|q(t_0 - \varepsilon)\|_{L^2(\mathbb{R})}) \ge 0,$$

which in contradiction with (4.6). Then we obtain the desired.

If there exists $q \in \partial Q$ such that $q = x_q^1$, then q is a solution of equation (4.1) with $x(0) = x(\omega)$.

A Schauder degree is well defined in $C([0, \omega]; L^2(\mathbb{R}))$ referring to condensing operator (see [12, p. 55]). If $q \ne x_q^1$ for all $q \in \partial Q$, then from the similar argument as above, T is a homotopy connecting the maps $T(\cdot,0)$ and $T(\cdot,1)$. Thus, by the normalization property of the degree, we have that

$$\deg(I - T(\cdot,1), Q) = \deg(I - T(\cdot,0), Q) = 1.$$

So by [12, Theorem 3.3.1, p. 60], there exists $q \in Q$ such that $q = x_q^1$.

In conclusion, there is at least one solution of equation (4.1) with $x(0) = x(\omega)$, which furnishes the solution of problem (1.1). □

5 Discussion

From the proof of Theorem 1.1, one can easily see that our methods can be used to study the following problems:

- the multipoint boundary value problem

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = f\left(t, u(t,\xi), \int\limits_0^t k(t,s)u(s,\xi)\mathrm{d}s\right) + a(t)\int\limits_{\mathbb{R}} K(x,\eta)u(t,\eta)\mathrm{d}\eta \\ \qquad + W\left(\xi, \int\limits_{\mathbb{R}} \varphi(\eta)u(t,\eta)\mathrm{d}\eta\right) - b(t,\xi)u(t,\xi), \quad t \in [0,\omega], \xi \in \mathbb{R}, \\ u(0,\xi) = \sum\limits_{i=1}^{p} \pi_i u(t_i,\xi), \quad \xi \in \mathbb{R}, \end{cases} \tag{5.1}$$

  where $\pi_i \in \mathbb{R}$ for $i = 1, 2, \ldots, p$, and $0 < t_1 < t_2 < \cdots < t_p \leq \omega$;
- the weighted mean value problem

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = f\left(t, u(t,\xi), \int\limits_0^t k(t,s)u(s,\xi)\mathrm{d}s\right) + a(t)\int\limits_{\mathbb{R}} K(x,\eta)u(t,\eta)\mathrm{d}\eta \\ \qquad + W\left(\xi, \int\limits_{\mathbb{R}} \varphi(\eta)u(t,\eta)\mathrm{d}\eta\right) - b(t,\xi)u(t,\xi), \quad t \in [0,\omega], \xi \in \mathbb{R}, \\ u(0,\xi) = \dfrac{1}{\omega}\int\limits_0^{\omega} \phi(t)u(t,\xi)\mathrm{d}t, \quad \xi \in \mathbb{R}, \end{cases} \tag{5.2}$$

  where $\phi : [0,\omega] \to \mathbb{R}$ is continuous.

We could obtain the following existence results.

Theorem 5.1. *Assume conditions* (H_1)–(H_5) *and* (1.5) *hold; suppose also that*

$$(M_1\omega + 1)e^{\|s_2\|_{L^1(0,\omega)}}\left(\|\sqrt{2}\alpha + a\|_{L^1(0,\omega)} + \sum_{i=1}^{p}|\pi_i|\right) < 1$$

is satisfied. Then, problem (5.1) *admits at least one solution u with* $\|u(t,\cdot)\| \leq R$, *for all* $t \in [0,\omega]$.

Theorem 5.2. *Assume conditions* (H_1)–(H_5) *and* (1.5) *hold; suppose also that*

$$(M_1\omega + 1)e^{\|s_2\|_{L^1(0,\omega)}}\left(\|\sqrt{2}\alpha + a\|_{L^1(0,\omega)} + \|\phi\|_{C([0,\omega])}\right) < 1$$

is satisfied. Then, problem (5.2) *admits at least one solution u with* $\|u(t,\cdot)\| \leq R$, *for all* $t \in [0,\omega]$.

The proofs deal with the boundary value problems associated with inclusion (2.1) equipped with boundary conditions

$$v(0) = \sum_{i=1}^{p} \pi_i v(t_i) \tag{5.3}$$

and

$$v(0) = \frac{1}{\omega}\int\limits_0^{\omega} \phi(t)v(t)\mathrm{d}t, \tag{5.4}$$

respectively. Then applying the similar topological argument as the one used in Theorem 1.1, one could obtain the solutions of problems (2.1), (5.3) and (2.1), (5.4). According to Section 2, these furnish the solutions of problems (5.1) and (5.2), respectively.

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