Research Article

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Admissible congruences on type B semigroups

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Abstract: The main aim of this article is to study admissible congruences on a type B semigroup. First, we give characterizations of the minimum admissible congruence whose trace is a normal congruence on a type B semigroup. After obtaining some properties of admissible congruences with the same trace on a type B semigroup, we introduce the notion of a normal subsemigroup and give characterizations of the minimum and maximum admissible congruences whose kernels are normal subsemigroups. Finally, the concept of a congruence pair of a type B semigroup is given, and two congruences associated with the congruence pair are obtained.

Keywords: admissible congruences, trace, type B semigroups, congruence pairs

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1 Introduction

Since the establishment of the algebraic theory of semigroups, regular semigroups have been an important research topic. In recent years, as a class of generalized regular semigroups, abundant semigroups have attracted more and more attention from semigroup scholars (see [1–14]). In 1979, the concept of a type B semigroup was introduced by Fountain in [15]. Recently, many semigroup scholars investigated type B semigroups and obtained some results (see [6–8,10,12,13]). Congruences play an important role in characterizations of properties of semigroups. The study of congruences of semigroups has been deeply developed (see [1–3,7,16–20]). The kernel trace method is very successful in the study of an inverse semigroup congruence. In 1954, the concept of a kernel normal system was proposed for the first time by Preston, and congruences on inverse semigroups are characterized by the kernel normal system. On the basis of a kernel normal system, Reilly and Scheiblich proposed concepts of kernels and traces of congruences on inverse semigroups and obtained a kernel trace method [21]. Petrich studied congruences on inverse semigroups by using the method of a kernel trace in [18]. El-Qalliali extends it to the study of congruences of ample semigroups in [1]. It is well known that ample semigroups and type B semigroups are generalizations of inverse semigroups in the range of abundant semigroups. Therefore, it is a natural thing to characterize congruences of a type B semigroup by using the above method. This article is committed to extending these results to type B semigroups.

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We proceed as follows: Section 2 provides some basic notions and properties of abundant semigroups. In particular, some properties of type B semigroups are given. In Section 3, we consider normal congruences of a type B semigroup S. It is shown that the minimum admissible congruence on S whose restriction to the idempotent set E of S is a normal congruence π and a congruence containing any admissible congruence whose restriction to E is a normal congruence π. In Section 4, we characterize admissible congruences with the same trace. In Section 5, we investigate kernels of congruences on a type B semigroup and obtain equivalent sets of kernels of σ_π and μ_π. In Section 6, we give the notion of a normal subsemigroup N of a type B semigroup and obtain some interesting results. In Section 7, we introduce the definition of a congruence pair on a type B semigroup. In Section 8, a conclusion is given.

2 Preliminaries

Throughout this article, we use notions and notations of [15,22–24]. For undefined concepts, the reader can refer to [24].

In the following, we first recall some basic properties of Green* relations $\mathcal{L}^*$ and $\mathcal{R}^*$. For convenience, $\mathcal{L}_a^*$ and $\mathcal{R}_a^*$ denote the $\mathcal{L}^*$-class and $\mathcal{R}^*$-class containing a, respectively; $E(S)$ denotes the set of idempotents of S; a^+ and a^* denote the typical idempotent of the $\mathcal{L}^*$-class and $\mathcal{R}^*$-class containing a, respectively.

Lemma 2.1. [23] *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent*:

(1) $a\mathcal{L}^*b$ $(a\mathcal{R}^*b)$;

(2) *for all* $x, y \in S^1$, $ax = ay$ $(xa = ya)$ *if and only if* $bx = by$ $(xb = yb)$.

Corollary 2.2. [23] *Let S be a semigroup and $a, e = e^2 \in S$. Then the following statements are equivalent*:

(1) $a\mathcal{L}^*e$ $(a\mathcal{R}^*e)$;

(2) $ae = a$ $(a = ea)$ *and for all* $x, y \in S^1$, $ax = ay$ $(xa = ya)$ *implies* $ex = ey$ $(xe = ye)$.

Obviously, the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ are right and left congruences on semigroup S, respectively.

Definition 2.1. [15] A semigroup S is rpp (resp., lpp) if each $\mathcal{L}^*$-class (resp., $\mathcal{R}^*$-class) of S contains an idempotent. A semigroup S is said to be abundant if it is both rpp and lpp.

Definition 2.2. [15] An rpp (resp., lpp) semigroup S is *right adequate* (resp., left adequate) if $E(S)$ is a semilattice. A semigroup is said to be adequate if it is both left and right adequate.

As in [15], if S is a right adequate semigroup, by Proposition 1.3 [15], any $\mathcal{L}^*$-class of S contains a unique idempotent. Dually, any $\mathcal{R}^*$-class of a left adequate semigroup S contains a unique idempotent.

Definition 2.3. [15] A right adequate semigroup S is right type B, if it satisfies the following conditions:

(B1) for all $e, f \in E(S^1)$, $a \in S$, $(efa)^* = (ea)^*(fa)^*$;

(B2) for all $a \in S$, $e \in E(S)$, if $e \leq a^*$, then there is $f \in E(S^1)$ such that $e = (fa)^*$.

Dually, a left adequate semigroup S is left type B, if it satisfies the following conditions:

(B1′) for all $e, f \in E(S^1)$, $a \in S$, $(aef)^+ = (ae)^+(af)^+$;

(B2′) for all $a \in S$, $e \in E(S)$, if $e \leq a^+$, then there is $f \in E(S^1)$ such that $e = (af)^+$.

A semigroup is said to be type B if it is both left and right type B.

Lemma 2.3. [15] *Let S be an adequate semigroup and $a, b \in S$. Then the following statements hold*:

(1) $a\mathcal{L}^*b$ *if and only if* $a^* = b^*$; $a\mathcal{R}^*b$ *if and only if* $a^+ = b^+$;

(2) $(ab)^* = (a^*b)^*$; $(ab)^+ = (ab^+)^+$;

(3) $(ab)^*b^* = (ab)^*$; $a^+(ab)^+ = (ab)^+$;
(4) $(ae)^* = a^*e$; $(ea)^+ = ea^+$.

Recall from [22] that a congruence ρ on an adequate semigroup S is said to be *admissible* if for all $a \in S, x, y \in S^1$,

$$ax\rho ay \;\Rightarrow\; a^*x\rho a^*y, \quad xa\rho ya \;\Rightarrow\; xa^+\rho ya^+.$$

Lemma 2.4. [22] *Let ρ be an admissible congruence on an adequate semigroup S. If a, b are two elements of S such that $a\rho b$, then $a^*\rho b^*$ and $a^+\rho b^+$.*

Lemma 2.5. [22] *Let ρ be a congruence on an abundant semigroup S. Then ρ is an admissible congruence on S if and only if $a\rho\mathcal{L}^*a^*\rho$ and $a\rho\mathcal{R}^*a^+\rho$ for all $a \in S$.*

Lemma 2.6. [14] *Let ρ be an admissible congruence on an adequate semigroup S. Then*
(1) *S/ρ is a type B semigroup if S is type B;*
(2) *for all $a \in S$, $(a\rho)^* = a^*\rho$ and $(a\rho)^+ = a^+\rho$.*

Let ρ be an admissible congruence on a type B semigroup S. We note that if $x\rho$ is an idempotent in S/ρ, then there exists an idempotent e in S such that $(x, e) \in \rho$.

Definition 2.4. [22] A homomorphism θ from an adequate semigroup S onto T is said to be an admissible homomorphism if

$$a\mathcal{L}^*(S)b \;\Rightarrow\; a\theta\mathcal{L}^*(T)b\theta; \quad a\mathcal{R}^*(S)b \;\Rightarrow\; a\theta\mathcal{R}^*(T)b\theta.$$

Definition 2.5. [15] Let S be an adequate semigroup. The relation μ is defined as follows:

$$(a, b) \in \mu \text{ if and only if for all } e \in E(S), (ea)^* = (eb)^* \text{ and } (ae)^+ = (be)^+.$$

As in [22], μ is the maximum congruence contained in $\mathcal{H}^*$. By [15], the congruence μ is an admissible congruence if any two elements of $E(S/\mu)$ commute. An adequate semigroup S is said to be *fundamental* if μ is the identity relation on S.

Lemma 2.7. [22] *Let S be an adequate semigroup. If $E(S/\mu)$ is a semilattice, then $E(S/\mu) = \{e\mu | e \in E(S)\}$.*

Lemma 2.8. [14] *Let S be a type B semigroup. If any two elements of $E(S/\mu)$ are commutative, then*
(1) *S/μ is a type B semigroup;*
(2) *S/μ is fundamental.*

Lemma 2.9. [6] *Let S be a type B semigroup. The relation σ is defined as follows:*

$$(a, b) \in \sigma \;\Leftrightarrow\; (\exists e \in E(S))\; eae = ebe.$$

Then σ is the least cancellative monoid congruence on S.

Lemma 2.10. *Let S be an adequate semigroup and $E(S/\mu)$ be a semilattice. Then S is a full subdirect product of a cancellative monoid and a fundamental type B semigroup if and only if S is a type B semigroup and $\sigma \bigcap \mu = \iota_S$, where ι_S is an identity relation on S.*

Proof. Let C be a cancellative monoid and T be a fundamental type B semigroup. Put $S = \{(a, t) \mid a \in C, t \in T\}$. Define a multiplication on S as follows:

$$(a, t)(b, x) = (ab, tx).$$

Clearly, S is a semigroup and $E(S) = \{(1, e) \in C \times T \mid e \in E(T)\}$ is a semilattice. Now, we prove that S is a type B semigroup. Let $(c, t) \in S$ such that $(c, t)(a, x) = (c, t)(b, y)$ for all $(a, x), (b, y) \in S^1$. Then $(ca, tx) = (cb, ty)$. Hence, $ca = cb$ and $tx = ty$. Note that C is a cancellative monoid and T is a fundamental type B semigroup. We have that $a = b$ and $t^*x = t^*y$. Furthermore, $(1, t^*)(a, x) = (1, t^*)(b, y)$. But $(c, t) = (c, t)(1, t^*)$. Therefore, $(c, t)\mathcal{L}^*(1, t^*)$. Dually, $(c, t)\mathcal{R}^*(1, t^+)$. Hence, S is an adequate semigroup. In other words, for all $(c, t) \in S$, we have that $(c, t)^* = (1, t^*)$ and $(c, t)^+ = (1, t^+)$. Let $(1, e), (1, f) \in E(S)$ and $(c, t) \in S$. Then

$$\begin{aligned}
[(1, e)(1, f)(c, t)]^* &= (c, eft)^* = (1, (eft)^*) \\
&= (1, (et)^*(ft)^*) \\
&= (1, (et)^*)(1, (ft)^*) \\
&= (c, et)^*(c, ft)^* \\
&= [(1, e)(c, t)]^*[(1, f)(c, t)]^*.
\end{aligned}$$

Hence, S satisfies Condition (**B1**). Let $(1, e) \in E(S)$, $(c, t) \in S$. Then

$$\begin{aligned}
(1, e) \le (c, t)^* &\Rightarrow (1, e) \le (1, t^*) \\
&\Rightarrow (1, e) = (1, e)(1, t^*) = (1, t^*)(1, e) \\
&\Rightarrow e = et^* = t^*e \\
&\Rightarrow e \le t^* \\
&= (\exists f \in E(T^1))e = (ft)^* \quad (\text{by Condition } (\mathbf{B2})) \\
&\Rightarrow (1, e) = (1, (ft)^*) = (c, (ft)^*) = [(1, f)(c, t)]^*,
\end{aligned}$$

where $(1, f) \in E(S)$. Hence, S satisfies Condition (**B2**) and S is a right type B semigroup. Dually, we can prove that S is a left type B semigroup. This shows that S is a type B semigroup.

Let $(c, t), (n, s) \in S$ such that $((c, t), (n, s)) \in \sigma \bigcap \mu$. Then there exists $(1, f) \in E(S)$ such that

$$(1, f)(c, t)(1, f) = (1, f)(n, s)(1, f),$$

where $f \in E(T)$. For any $(1, e) \in E(S)$, we have

$$[(1, e)(c, t)]^* = [(1, e)(n, s)]^* \text{ and } [(c, t)(1, e)]^+ = [(n, s)(1, e)]^+,$$

where $e \in E(T)$. This shows that $c = n$, $(1, (et)^*) = (1, (es)^*)$ and $(1, (te)^+) = (1, (se)^+)$. Therefore, $c = n$ and $(t, s) \in \mu(T)$. As T is fundamental, we have $t = s$. Thus, $\sigma \bigcap \mu = \iota_S$.

Conversely, suppose that S is a type B semigroup. Then S/σ is a cancellative monoid and S/μ is a fundamental type B semigroup. Define a mapping as follows:

$$\psi : S \to S/\sigma \times S/\mu, \quad x\psi = (x\sigma, x\mu).$$

It is easy to see that ψ is a homomorphism. Let $a\mu$ be an idempotent of S/μ. Then there exists $e \in E(S)$ such that $(e, a) \in \mu$. But $e\sigma$ is an identity of S/σ. Thus, $e\psi = (e\sigma, e\mu)$ and $Im\psi$ is a full subdirect product of $S/\sigma \times S/\mu$. By the hypothesis, $\sigma \bigcap \mu = \iota_S$. Therefore, ψ is one-to-one and $S \cong Im\psi$. □

3 Normal congruences of a type B semigroup

In this section, we shall consider a normal congruence on a type B semigroup. For convenience, we replace $E(S)$ by E in the remaining. As usual, for an arbitrary congruence ρ on a semigroup S, the restriction $\rho|_E$ of ρ on the idempotent set E is called a *trace* of ρ, denoted by trρ. Obviously, trρ is a congruence on the idempotent set E and if $e, f \in E$ with $e\rho f$ and $a \in S$, we have that $(ea, fa) \in \rho$ and $(ae, fe) \in \rho$. In particular, if ρ is an admissible, we have that $(ea)^*\rho(fa)^*$ and $(ae)^+\rho(af)^+$.

Definition 3.1. A congruence π on an idempotent set E of a type B semigroup S is said to be normal if for all $e, f \in E$ and $a \in S$,

$$e\pi f \;\Rightarrow\; (ea)^*\pi(fa)^* \text{ and } (ae)^+\pi(af)^+.$$

Definition 3.2. Let π be a normal congruence on E of a type B semigroup S. Define a relation on S as follows:

$$\sigma_\pi = \{(a, b) \in S \times S \mid (\exists e, f \in E) e\pi a^*\pi b^*, f\pi a^+\pi b^+, fae = fbe\}.$$

We now give the main theorem in this section.

Theorem 3.1. *Let S be a type B semigroup. Then σ_π is the minimum congruence on S whose restriction to E is π. Furthermore, σ_π is an admissible congruence on S.*

Proof. First, we prove that σ_π is an equivalence relation on S. Clearly, σ_π is reflexive and symmetric. Now, we show that σ_π is transitive. To see it, let $a, b, c \in S$ such that $(a, b) \in \sigma_\pi$ and $(b, c) \in \sigma_\pi$. Then there exist $e, f \in E$ such that $e\pi b^*\pi c^*$, $f\pi b^+\pi c^+$ and $fae = fbe$; and there exist $g, h \in E$ such that $g\pi b^*\pi c^*$, $h\pi b^+\pi c^+$ and $hbg = hcg$. Note that E is a semilattice. We have

$$fhaeg = hfaeg = hfbeg = fhbge = fhcge = fhceg.$$

Since $e\pi a^*\pi b^*$ and $g\pi b^*\pi c^*$, we have that $eg\pi a^*b^*\pi a^*a^* = a^*$ and $eg\pi b^*c^*\pi c^*c^* = c^*$. That is, there exists $eg \in E$ such that $eg\pi a^*\pi c^*$. Similarly, there exists $fh \in E$ such that $fh\pi a^+\pi c^+$. Therefore, σ_π is an equivalence relation on S.

Next, we prove that σ_π is a congruence on S. Let $a, b \in S$ such that $(a, b) \in \sigma_\pi$. Then $fae = fbe$ for some $e, f \in E$, $e\pi a^*\pi b^*$ and $f\pi a^+\pi b^+$. Hence, for all $c \in S$, $cfae = cfbe$ and $c^*fae = c^*fbe$. Note that $c^*f \leq c^*$. By Condition (**B2**), we have $c^*f = (gc)^*$ for some $g \in E^1$, and so $(gc)^*ae = (gc)^*be$. Therefore, $gcae = gcbe$. That is, $gc^+ca(ca)^*e = gc^+cb(cb)^*e$. Multiplying it on the left by c^*, we obtain that $c^*gc^+ca(ca)^*e = c^*gc^+cb(cb)^*e$. By the normality of π and Lemma 2.3(2), we have that

$$f\pi a^+ \Rightarrow (cf)^+\pi(ca^+)^+ = (ca)^+.$$

Note that $\mathcal{R}^*$ is a left congruence and S is $\mathcal{R}^*$-unipotent. We have that $gc^+ = (gc)^+$. Again since $c^*\mathcal{L}^*c$ and $\mathcal{L}^*$ is a right congruence, we have that $c^*(gc)^+\mathcal{L}^*c(gc)^+\mathcal{L}^*(c(gc)^+)^*$. Note that S is $\mathcal{L}^*$-unipotent. We have $c^*(gc^+) = (c(gc)^+)^*$. That is, $c^*gc^+\pi(c(gc)^+)^*$. Thus,

$$c^*gc^+\pi(c(gc)^*)^+\pi(cc^*f)^+ = (cf)^+.$$

Similarly, $(cb)^+\pi c^*gc^+$. Therefore, $c^*gc^+\pi(ca)^+\pi(cb)^+$. Clearly, $c^*gc^+ca\mathcal{L}^*(c^*gc^+ca)^*$. Again, since S is a right congruence, we have that

$$(c^*gc^+ca(ca)^*e)^*\mathcal{L}^*c^*gc^+ca(ca)^*e\mathcal{L}^*(c^*gc^+ca)^*(ca)^*e.$$

Hence, $(c^*gc^+ca(ca)^*e)^* = (c^*gc^+ca)^*(ca)^*e$ since S is $\mathcal{L}^*$-unipotent. Again since $e\pi a^*\pi b^*$, we obtain

$$(ca)^*e\pi(ca)^*a^*\pi(ca)^* \text{ and } (cb)^*e\pi(cb)^*b^*\pi(cb)^*.$$

Thus, $(c^*gc^+ca)^*(ca)^*e\pi(c^*gc^+ca)^*(ca)^*$. But, $c^*gc^+\pi(ca)^+$. By the normality of π, $(c^*gc^+ca)^*(ca)^*\pi((ca)^+ca)^*(ca)^*$. Therefore,

$$(c^*gc^+ca(ca)^*e)^*\pi((ca)^+ca)^*(ca)^* = (ca)^*.$$

Similarly, we can prove that $(c^*gc^+cb(cb)^*e)^*\pi(cb)^*$. Since $c^*gc^+ca(ca)^*e = c^*gc^+cb(cb)^*e$. We obtain that $(ca)^*\pi(cb)^*\pi(ca)^*e = e(ca)^*$. But, $gcae = gcbe$. Multiplying it on the left by c^* and on the right by $(ca)^+$, we have that $c^*gc^+cae(ca)^* = c^*gc^+cbe(ca)^*$. Thus, $(ca, cb) \in \sigma_\pi$.

On the other hand, let $a, b \in S$ such that $(a, b) \in \sigma_\pi$. Then $fae = fbe$ for some $e, f \in E$, $e\pi a^*\pi b^*$, $f\pi a^+\pi b^+$ and $faec = fbec$. Hence, $faec^+ = fbec^+$. Note that $ec^+ \leq c^+$. By Condition (**B2′**), there exists $h \in E^1$ such that $ec^+ = (ch)^+$, and so $fa(ch)^+ = fb(ch)^+$. Multiplying it on the right by ch, we obtain that $fach = fbch$. That is, $f(ac)^+achc^*c^+ = f(bc)^+bchc^*c^+$. By the normality of π and Lemma 2.3(2),

$$e\pi a^* \Rightarrow (ec)^*\pi(a^*c)^* = (ac)^*.$$

Again since $hc^* = c^*h\mathcal{L}^*ch\mathcal{L}^*(ch)^*$ and S is $\mathcal{L}^*$-unipotent, we have that $hc^* = (ch)^*$ and so $hc^*c^+\mathcal{R}^*(ch)^*c\mathcal{R}^*((ch)^*c)^+$. This gives that $hc^*c^+ = ((ch)^*c)^+$ since S is $\mathcal{R}^*$-unipotent. Thus,

$$hc^*c^+\pi((ch)^+c)^*\pi(ec^+c)^* = (ec)^*.$$

Therefore, $hc^*c^+\pi(ac)^*$. Similarly, we can prove that $hc^*c^+\pi(bc)^*$. Hence, $hc^*c^+\pi(ac)^*\pi(bc)^*$. Obviously, $achc^*c^+\mathcal{R}^*(achc^*c^+)^+$. Since $\mathcal{R}^*$ is a left congruence, we have that

$$(f(ac)^+achc^*c^+)^+\mathcal{R}^*f(ac)^+achc^*c^+\mathcal{R}^*f(ac)^+(achc^*c^+)^+.$$

Note that S is $\mathcal{R}^*$-unipotent. We have $(f(ac)^+achc^*c^+)^+ = f(ac)^+(achc^*c^+)^+$. But $f\pi a^+\pi b^+$, we have that

$$f(ac)^+\pi a^+(ac)^+\pi(ac)^+ \quad \text{and} \quad f(bc)^+\pi b^+(bc)^+\pi(bc)^+.$$

Thus, $f(ac)^+(achc^*c^+)^+\pi(ac)^+(achc^*c^+)^+$. Again $hc^*c^+\pi(ac)^*$, by the normality of π, $(ac)^+(achc^*c^+)^+\pi(ac)^+(ac(ac)^*)^+$. Therefore,

$$(f(ac)^+achc^*c^+)^+\pi(ac)^+(ac(ac)^*)^+ = (ac)^+.$$

Similarly, $(f(bc)^+bchc^*c^+)^+\pi(bc)^+$. But, $f(ac)^+achc^*c^+ = f(bc)^+bchc^*c^+$. We have

$$(ac)^+\pi(bc)^+\pi f(ac)^+.$$

As $fach = fbch$, so $f(ac)^+achc^*c^+ = f(ac)^+bchc^*c^+$. Thus, $(ac, bc) \in \sigma_\pi$.

Summing up the above arguments, we conclude that σ_π is a congruence on S.

Now, we prove that $\mathrm{tr}\sigma_\pi = \pi$. Clearly, $\mathrm{tr}\sigma_\pi \subseteq \pi$. Let $e, f \in E$ such that $e\pi f$. Then $fee = fe = ffe$. By the definition of σ_π, we have $e\sigma_\pi f$. Thus, $\mathrm{tr}\sigma_\pi = \pi$.

Next, we show that σ_π is the minimum congruence whose restriction to E is π. Let ρ be a congruence whose restriction to E is π. If $(a, b) \in \sigma_\pi$, then $fae = fbe$ for some $e, f \in E$, $e\pi a^*\pi b^*$ and $f\pi a^+\pi b^+$. Since $\mathrm{tr}\rho = \pi$, we have that $e\rho a^*\rho b^*$ and $f\rho a^+\rho b^+$. Hence,

$$a\rho = a^+aa^*\rho = a^+\rho a\rho a^*\rho = f\rho a\rho e\rho = (fae)\rho = (fbe)\rho = f\rho b\rho e\rho = b^+\rho b\rho b^*\rho = b^+bb^*\rho = b\rho.$$

Thus, $(a, b) \in \rho$. That is, $\sigma_\pi \subseteq \rho$. Therefore, σ_π is the minimum congruence whose restriction to E is π.

Finally, we prove that σ_π is admissible. Let $a \in S$, $s, t \in S^1$ such that $(as, at) \in \sigma_\pi$. Then $fase = fate$ for some $e, f \in E$, $e\pi(as)^*\pi(at)^*$ and $f\pi(as)^+\pi(at)^+$. By Lemma 2.3, $(as)^* = (a^*s)^*$ and $(at)^* = (a^*t)^*$, and so $e\pi(a^*s)^*\pi(a^*t)^*$. But

$$fase = fate \;\Rightarrow\; faa^*se = faa^*te \;\Rightarrow\; (fa)^*a^*se = (fa)^*a^*te.$$

Thus, $(a^*s, a^*t) \in \sigma_\pi$. Similarly, $(sa, ta) \in \sigma_\pi$ implies $(sa^+, ta^+) \in \sigma_\pi$. Therefore, σ_π is an admissible congruence. This completes the proof. □

Let π be an admissible congruence on S. By Theorem 3.1, we have that σ_π is the minimum admissible congruence on S whose restriction to E is π. As usual, a congruence ρ on a semigroup S is called *idempotent-separating* if $e\rho = f\rho$ implies $e = f$ for all $e, f \in E$. Similarly, we can define an idempotent-separating homomorphism.

Proposition 3.2. *Let ρ be an admissible congruence on a type B semigroup S whose restriction to E is π. Then S/ρ is an idempotent-separating homomorphic image of S/σ_π.*

Proof. Define a mapping ϕ as follows:

$$\phi : S/\sigma_\pi \to S/\rho, \quad (s\sigma_\pi)\phi = s\rho.$$

Then, it is easy to see that ϕ is a homomorphism of S/σ_π onto S/ρ. By Lemma 2.7, we have that

$$E(S/\sigma_\pi) = \{e\sigma_\pi \mid e \in E\}.$$

Let $e\sigma_\pi, f\sigma_\pi$ be two idempotents of S/σ_π, where $e, f \in E$. Then

$$(e\sigma_\pi)\phi = (f\sigma_\pi)\phi \;\Rightarrow\; e\rho = f\rho \;\Rightarrow\; (e, f) \in \rho \;\Rightarrow\; (e, f) \in \pi \quad (\mathrm{tr}\rho = \pi) \;\Rightarrow\; e\sigma_\pi = f\sigma_\pi.$$

This means that ϕ is idempotent-separating. This completes the proof. □

Definition 3.3. Let S be a type B semigroup and π be a normal congruence on E. Define a relation on S as follows:

$$\mu_\pi = \{(a, b) \in S \times S \mid (\forall e \in E)(ea)^* \pi (eb)^*, (ae)^+ \pi (be)^+\}.$$

Lemma 3.3. *Let S be a type B semigroup and π be a normal congruence on E. Then the following statements are equivalent*:
(1) $(a, b) \in \mu_\pi$;
(2) *for all* $e, f \in E$, $e\pi f$ *implies* $(ea)^*\pi(fb)^*$ *and* $(ae)^+\pi(bf)^+$;
(3) $(a\sigma_\pi, b\sigma_\pi) \in \mu(S/\sigma_\pi)$, *where* $\mu(S/\sigma_\pi)$ *denotes the relation* μ *on* S/σ_π.

Proof. (1) $\Rightarrow$ (2) is clear. Now, we show that (2) $\Rightarrow$ (1). For all $b \in S, e, f \in E$ with $e\pi f$, we have $(eb)^*\pi(fb)^*$. If $(a, b) \in \mu_\pi$, then $(ea)^*\pi(eb)^*$. Hence, $(ea)^*\pi(fb)^*$. On the other hand, for all $b \in S, e, f \in E$ with $e\pi f$, it follows that $(be)^+\pi(bf)^+$. If $(a, b) \in \mu_\pi$, then $(ae)^+\pi(be)^+$. Hence, $(ae)^+\pi(bf)^+$. That is, (2) holds.

(1) $\Leftrightarrow$ (3) For all $a, b \in S$, we have

$$\begin{aligned}(a, b) \in \mu_\pi &\Leftrightarrow (\forall e \in E)(ea)^*\pi(eb)^*, (ae)^+\pi(be)^+\\ &\Leftrightarrow (\forall e \in E)(ea)^*\sigma_\pi = (eb)^*\sigma_\pi, (ae)^+\sigma_\pi = (be)^+\sigma_\pi \ (\text{since } \operatorname{tr}\sigma_\pi = \pi)\\ &\Leftrightarrow (\forall e \in E)(e\sigma_\pi a\sigma_\pi)^* = (e\sigma_\pi b\sigma_\pi)^*\\ &\quad\ (a\sigma_\pi e\sigma_\pi)^+ = (b\sigma_\pi e\sigma_\pi)^+ \qquad (\text{since } \sigma_\pi \text{ is admissible})\\ &\Leftrightarrow (a\sigma_\pi, b\sigma_\pi) \in \mu(S/\sigma_\pi).\end{aligned}$$

□

Theorem 3.4. *Let S be a type B semigroup. Then* $\operatorname{tr}\mu_\pi = \pi$. *In particular, if ρ is an admissible congruence on S with* $\operatorname{tr}\rho = \pi$, *then* $\rho \subseteq \mu_\pi$.

Proof. Obviously, μ_π is an equivalence relation on S. Now, we prove that μ_π is a congruence on S. For any $a, b, c \in S, e \in E$, if $(a, b) \in \mu_\pi$, then $(ea)^*\pi(eb)^*$. By the normality of π and Lemma 2.3(2), we have that

$$(ea)^*\pi(eb)^* \ \Rightarrow\ ((ea)^*c)^*\pi((eb)^*c)^* \ \Rightarrow\ (eac)^* = (ea)^*c^*\pi((eb)^*c)^* = (ebc)^*.$$

Hence, $(eac)^*\pi(ebc)^*$. Since $(ce)^+ \in E$, we have $(a(ce)^+)^+\pi(b(ce)^+)^+$ from the definition of μ_π. By Lemma 2.3(2), $(a(ce)^+)^+ = (ace)^+$ and $(b(ce)^+)^+ = (bce)^+$. That is, $(ace)^+\pi(bce)^+$. Thus, $(ac, bc) \in \mu_\pi$. On the other hand, for all $a, b, c \in S, e \in E$, if $(a, b) \in \mu_\pi$, then $(ae)^+\pi(be)^+$. By the normality of π and Lemma 2.3(2), we have

$$(ae)^+\pi(be)^+ \ \Rightarrow\ (c(ae)^+)^+\pi(c(be)^+)^+ \ \Rightarrow\ (cae)^+\pi(cbe)^+.$$

Since $(ec)^* \in E$, we obtain $((ec)^*a)^*\pi((ec)^*b)^*$. But $((ec)^*a)^* = (eca)^*$ and $((ec)^*b)^* = (ecb)^*$. This shows that $(eca)^* = (ecb)^*$. Thus, $(ca, cb) \in \mu_\pi$. To sum up, μ_π is a congruence.

Next, we prove that $\operatorname{tr}\mu_\pi = \pi$. It is easy to see that $\pi \subseteq \mu_\pi$. For all $f, g \in E$ with $f\mu_\pi g$, then $ef\pi eg$ for all $e \in E$. Let $e = f$ and $e = g$, respectively. This gives that $f\pi fg$ and $gf\pi g$. As $fg = gf$. So $f\pi g$. Thus, $\operatorname{tr}\mu_\pi = \pi$.

Finally, we show that μ_π is the maximum admissible congruence whose trace is π. To see it, let ρ be any admissible congruence on S such that $\operatorname{tr}\rho = \pi$. If $(a, b) \in \rho$, then $(ea, eb) \in \rho$ and $(ae, be) \in \rho$ for all $e \in E$. By Lemma 2.4,

$$((ea)^*, (eb)^*) \in \rho \ \text{ and } \ ((ae)^+, (be)^+) \in \rho.$$

Hence, $(ea)^*\pi(eb)^*, (ae)^+\pi(be)^+$. That is, $(a, b) \in \mu_\pi$. This completes the proof. □

4 Congruences with same trace on a type B semigroup

In this section, we mainly describe the relationship between two admissible congruences ρ and τ on a type B semigroup S, which have the same trace. Clearly, if we restrict ρ to E, then $\operatorname{tr}\rho$ is normal, and

$\sigma_{\mathrm{tr}\rho}$, $\mu_{\mathrm{tr}\rho}$ are the minimum and the maximum admissible congruences on S, respectively. In particular, $\mathrm{tr}\sigma_{\mathrm{tr}\rho} = \mathrm{tr}\rho = \mathrm{tr}\mu_{\mathrm{tr}\rho}$, where

$$\sigma_{\mathrm{tr}\rho} = \{(a, b) \in S \times S \mid a^*\rho b^*, a^+\rho b^+, (\exists e \in a^*\rho \cap E, \exists f \in a^+\rho \cap E)fae = fbe\}$$

$$\mu_{\mathrm{tr}\rho} = \{(a, b) \in S \times S \mid (\forall e \in E)(ea)^*\pi(eb)^*, (ae)^+\pi(be)^+\}.$$

For convenience, we denote $\sigma_{\mathrm{tr}\rho}$ by σ_ρ and denote $\mu_{\mathrm{tr}\rho}$ by μ_ρ.

Theorem 4.1. *Let ρ be an arbitrary admissible congruence on a type B semigroup S. Then $\sigma_\rho \subseteq \rho \subseteq \mu_\rho$ and $\mathrm{tr}\sigma_\rho = \mathrm{tr}\rho = \mathrm{tr}\mu_\rho$.*

Proof. It follows directly from Theorems 3.1 and 3.4. □

Definition 4.1. Let ρ and τ be two congruences on a type B semigroup S with $\tau \subseteq \rho$. Define a congruence ρ/τ on S/τ as follows:

$$(a, b \in S)a\tau(\rho/\tau)b\tau \iff a\rho b.$$

Theorem 4.2. *Let ρ and τ be two admissible congruences on a type B semigroup S. Then the following statements are equivalent*:

(1) $\mathrm{tr}\rho = \mathrm{tr}\tau$;

(2) $\rho \subseteq \mu_\tau$ *and* $\mu_\tau/\rho = \mu(S/\rho)$;

(3) $(\forall a, b \in S)a\rho\mu(S/\rho)b\rho \iff a\tau\mu(S/\tau)b\tau$;

(4) $(\forall a, b \in S)a\rho\mathcal{H}^*(S/\rho)b\rho \iff a\tau\mathcal{H}^*(S/\tau)b\tau$;

(5) $\rho \cap \tau|_{e\rho}$ *and* $\rho \cap \tau|_{e\tau}$ *are cancellative congruences, where* $e \in E$;

(6) $\rho/\rho \cap \tau$ *and* $\tau/\rho \cap \tau$ *are congruences contained in* $\mathcal{H}^*(S/\rho \cap \tau)$.

Proof. (1) ⇒ (2) Since $\mu_{\mathrm{tr}\rho} = \mu_\rho, \mu_{\mathrm{tr}\tau} = \mu_\tau$ and $\mathrm{tr}\rho = \mathrm{tr}\tau$, we have $\mu_\rho = \mu_\tau$. Thus, $\rho \subseteq \mu_\tau$. For all $a, b \in S$, it follows that

$$a\rho(\mu_\tau/\rho)b\rho \iff a\rho(\mu_\rho/\rho)b\rho \iff a\mu_\rho b \iff (\forall e \in E)(ea)^*\rho(eb)^*, (ae)^+\rho(be)^+ \iff (a\rho, b\rho) \in \mu(S/\rho).$$

(2) ⇒ (1) It is easy to see that $\mathrm{tr}\rho \subseteq \mathrm{tr}\mu_\tau \subseteq \mathrm{tr}\tau$. For all $e, f \in E$, we have

$$e\tau f \Rightarrow e\mu_\tau f \Rightarrow e\rho(\mu_\tau/\rho)f\rho \Rightarrow e\rho = f\rho \Rightarrow e\rho f.$$

That is, $\mathrm{tr}\tau \subseteq \mathrm{tr}\rho$. Thus, $\mathrm{tr}\tau = \mathrm{tr}\rho$.

(1) ⇒ (3) For all $a, b \in S$, we have

$$\begin{aligned} a\rho\mu(S/\rho)b\rho &\iff (\forall e \in E)(ea)^*\rho = (eb)^*\rho,\ (ae)^+\rho = (be)^+\rho \\ &\iff (\forall e \in E)(ea)^*\tau = (eb)^*\tau,\ (ae)^+\tau = (be)^+\tau \\ &\iff a\tau\mu(S/\tau)b\tau. \end{aligned}$$

(3) ⇒ (1) For all $e, f \in E$, it follows that

$$e\rho f \iff e\rho = f\rho \iff e\rho\mu(S/\rho)f\rho \iff e\tau\mu(S/\tau)f\tau \iff e\tau = f\tau \iff e\tau f.$$

(1) ⇒ (4) For all $a, b \in S$ with $a\rho\mathcal{H}^*(S/\rho)b\rho$. We have $a\rho\mathcal{L}^*(S/\rho)b\rho$ and $a\rho\mathcal{R}^*(S/\rho)b\rho$. By the hypothesis, ρ is admissible. Thus, by Lemma 2.5, we have $a^*\rho\mathcal{L}^*(S/\rho)b^*\rho$ and $a^+\rho\mathcal{R}^*(S/\rho)b^+\rho$. By Lemma 2.6(1), S/ρ is a type B semigroup, it shows that S/ρ is $\mathcal{L}^*$-unipotent and $\mathcal{R}^*$-unipotent. Thus, $a^*\rho = b^*\rho$ and $a^+\rho = b^+\rho$. But, $\mathrm{tr}\rho = \mathrm{tr}\tau$, this shows that $a^*\tau = b^*\tau$ and $a^+\tau = b^+\tau$. Therefore, $a\tau\mathcal{H}^*(S/\tau)b\tau$. Similarly, $a\tau\mathcal{H}^*(S/\tau)b\tau$ implies $a\rho\mathcal{H}^*(S/\rho)b\rho$.

(4) ⇒ (1) For all $e, f \in E$ with $e\rho f$, we have $e\rho\mathcal{H}^*(S/\rho)f\rho$, and so $e\tau\mathcal{H}^*(S/\tau)f\tau$. Hence, $e\tau = f\tau$. That is, $e\tau f$. Similarly, $e\tau f$ implies $e\rho f$.

(1) ⇒ (5) Clearly, $e\rho$ is an adequate semigroup, where $e \in E$. Let $a, b, c \in e\rho$ and $(ab, ac) \in \rho \cap \tau$. By the hypothesis, ρ and τ are admissible congruences. Hence, $(a^*b, a^*c) \in \rho \cap \tau$. Since $e\rho$ is an adequate semigroup, we have $a^*, b^+ \in e\rho$. Thus, $(a^*, b^+) \in \rho \cap \tau$ and $(b, a^*b) \in \rho \cap \tau$. Similarly, we can show $(a^*, c^+) \in \rho \cap \tau$ such that $(a^*c, c) \in \rho \cap \tau$. According to transitivity, $(b, c) \in \rho \cap \tau$. Therefore, $\rho \cap \tau$ is left cancellative. Similarly, $\rho \cap \tau$ is right cancellative. Thus, $\rho \cap \tau|_{e\rho}$ is a cancellative congruence. Dually, we can prove that $\rho \cap \tau|_{e\tau}$ is a cancellative congruence.

(5) ⇒ (1) Let $g, h \in e\rho \cap E$. Clearly, $ggh = gh$, where $g, h, gh \in e\rho \cap E$. It follows that $(ggh, gh) \in \rho \cap \tau$. By the hypothesis, $\rho \cap \tau$ is a cancellative congruence. Hence, $(gh, h) \in \rho \cap \tau$. Similarly, we have $(hg, g) \in \rho \cap \tau$. As $gh = hg$, we have $(g, h) \in \rho \cap \tau$. In particular, for all $e, f \in E$,

$$e\rho f \;\Rightarrow\; f \in e\rho \;\Rightarrow\; (e, f) \in \rho \cap \tau \;\Rightarrow\; (e, f) \in \tau \;\Rightarrow\; e\tau f.$$

Similarly, $e\tau f \;\Rightarrow\; e\rho f$ is also true.

(1) ⇒ (6) For any $a, b \in S$, we obtain

$$\begin{aligned} a(\rho \cap \tau)\rho/\rho \cap \tau b(\rho \cap \tau) &\Rightarrow a\rho b \;\Rightarrow\; a^*\rho b^*, a^+\rho b^+ \\ &\Rightarrow a^*(\rho \cap \tau)\ \rho/\rho \cap \tau \quad b^*(\rho \cap \tau) \\ &\quad\ a^+(\rho \cap \tau)\ \rho/\rho \cap \tau \quad b^+(\rho \cap \tau) \\ &\Rightarrow a(\rho \cap \tau)\ \mathcal{L}^*(S/\rho \cap \tau)\ b(\rho \cap \tau) \\ &\quad\ a(\rho \cap \tau)\ \mathcal{R}^*(S/\rho \cap \tau)\ b(\rho \cap \tau) \\ &\Rightarrow a(\rho \cap \tau)\ \mathcal{H}^*(S/\rho \cap \tau)\ b(\rho \cap \tau). \end{aligned}$$

Hence, $\rho/\rho \cap \tau \subseteq \mathcal{H}^*(S/\rho \cap \tau)$. Similarly, $\tau/\rho \cap \tau \subseteq \mathcal{H}^*(S/\rho \cap \tau)$.

(6) ⇒ (1) For all $e, f \in E$, we have

$$e\rho f \;\Rightarrow\; e(\rho \cap \tau)\rho/\rho \cap \tau f(\rho \cap \tau) \;\Rightarrow\; e(\rho \cap \tau)\mathcal{H}^*(S/\rho \cap \tau)f(\rho \cap \tau) \;\Rightarrow\; e(\rho \cap \tau) = f(\rho \cap \tau) \;\Rightarrow\; e\tau f.$$

Similarly, $e\tau f$ implies $e\rho f$. This completes the proof. □

Corollary 4.3. *Let ρ be an admissible congruence on a type B semigroup S. Then $\rho = \mu_\rho$ if and only if S/ρ is fundamental.*

Proof. By Theorem 4.2,

$$\rho = \mu_\rho \Leftrightarrow \mu_\rho/\rho = \iota_S \Leftrightarrow \mu(S/\rho) = \iota_S \Leftrightarrow S/\rho \text{ is fundamental},$$

where ι_S is an identity relation on S. This completes the proof. □

5 Kernels of congruences on a type B semigroup

In this section, we investigate kernels of admissible congruences on a type B semigroup. Recall from the study of Petrich [19] that for any congruence ρ on a semigroup S, the kernel $\ker\rho$ of ρ is defined as follows:

$$\ker\rho = \{a \in S \mid (\exists e \in E)(e, a) \in \rho\}.$$

Proposition 5.1. *Let ρ be an admissible congruence on a type B semigroup S. If $a \in \ker\rho$, then $(a^+, a^*) \in \mathrm{tr}\rho$.*

Proof. Let $a \in \ker\rho$. Then there exists $e \in E$ such that $(e, a) \in \rho$. Since ρ is admissible, by Lemma 2.4, $(e, a^*) \in \rho$ and $(e, a^+) \in \rho$. Thus, $(a^*, a^+) \in \rho$. Clearly, $a^*, a^+ \in E$. Therefore, $(a^+, a^*) \in \mathrm{tr}\rho$. □

Proposition 5.2. *Let ρ be an admissible congruence on a type B semigroup S. Then $e\rho = e\mu_\rho \cap \ker\rho$ for all $e \in E$.*

Proof. Let $a \in S$, $e \in E$ such that $a \in e\mu_\rho \cap \ker\rho$. It follows that $a\mu_\rho e$ and there is $f \in E$ such that $a\rho f$. Since ρ is admissible, we have that $a^*\mu_\rho e$ and $a^*\rho f$. Again since $\mathrm{tr}\rho = \mathrm{tr}\mu_\rho$, we have $a^*\mu_\rho f$ and $a^*\rho e$. By transitivity, $e\rho f$ and $a\rho e$. Thus, $a \in e\rho$. That is, $e\mu_\rho \cap \ker\rho \subseteq e\rho$.

Conversely, if $a \in e\rho$, then $a \in \ker\rho$. Since $\rho \subseteq \mu_\rho$, we have $(a, e) \in \mu_\rho$. That is, $a \in e\mu_\rho$. Therefore, $a \in e\mu_\rho \cap \ker\rho$ and $e\rho \subseteq e\mu_\rho \cap \ker\rho$. □

Proposition 5.3. *Let ρ be an admissible congruence on a type B semigroup S. Then the following sets are the same*:
(1) $K_1 = \ker\sigma_\rho = \{a \in S \mid (\exists e \in E)(e, a) \in \sigma_\rho\}$;
(2) $K_2 = \{a \in S \mid (\exists e \in a^*\rho \cap E, f \in a^+\rho \cap E) fae = fe\}$.

Proof. For all $a \in S$, we have

$$\begin{aligned} a \in K_1 &\Rightarrow (\exists g \in E)(a, g) \in \sigma_\rho \\ &\Rightarrow a^*\rho g, a^+\rho g, (\exists e \in a^*\rho \cap E, f \in a^+\rho \cap E) fae = fge \\ &\Rightarrow (e \in a^*\rho \cap E, fg \in a^+\rho \cap E) fgae = fge \;\Rightarrow\; a \in K_2. \end{aligned}$$

On the other hand,

$$\begin{aligned} a \in K_2 &\Rightarrow (\exists e \in a^*\rho \cap E, f \in a^+\rho \cap E) fae = fe \\ &\Rightarrow fe = fae \quad \rho \quad a^+aa^* = a \\ &\Rightarrow fe\rho a^+, fe\rho a^* \quad (\text{since } \rho \text{ is admissible}). \end{aligned}$$

Again since $feafe = efaef = effeef = fe = fefefe, fe\rho a^+$ and $fe\rho a^*$, we have $(a, fe) \in \sigma_\rho$. That is, $a \in K_1$. This completes the proof. □

Corollaries directly from Proposition 5.3 are as follows.

Corollary 5.4. *Let ρ be an admissible congruence on a type B semigroup S. Then the following sets are the same*:
(1) $K_1 = \ker\sigma_\pi = \{a \in S \mid (\exists e \in E)(e, a) \in \sigma_\pi\}$;
(2) $K_2 = \{a \in S \mid (\exists e \in a^*\rho \cap E, f \in a^+\rho \cap E)\ fae = fe\}$

Corollary 5.5. *Let ρ be an admissible congruence on a type B semigroup S. Then the following sets are the same*:
(1) $K_1 = \ker\mu_\rho = \{a \in S \mid (\exists e \in E)(e, a) \in \mu_\rho\}$;
(2) $K_2 = \{a \in S \mid (\forall e \in E)\ ea\pi ae\}$.

Corollary 5.6. *Let ρ be an admissible congruence on a type B semigroup S. Then the following sets are the same*:
(1) $K_1 = \ker\mu_\pi = \{a \in S \mid (\exists e \in E)(e, a) \in \mu_\pi\}$;
(2) $K_2 = \{a \in S \mid (\forall e \in E)\ ea\pi ae\}$.

Proposition 5.7. *Let ρ be an admissible congruence on a type B semigroup S. Then the following sets are the same*:
(1) $\ker\sigma = \{a \in S \mid (\exists e \in E)\ eae = e\}$;
(2) $\ker\mu = \{a \in S \mid (\forall e \in E)\ ea = ae\}$.

Proof. Obviously,

$$a \in \ker\sigma \;\Rightarrow\; (\exists f \in E)\ (a, f) \in \sigma \;\Rightarrow\; (\exists e \in E)\ eae = efe \;\Rightarrow\; (\exists e \in E)\ feafe = feaef = fefef = fe.$$

That is, $\ker\sigma \subseteq \{a \in S \mid (\exists e \in E)\ eae = e\}$. Conversely, let $a \in S$ such that $eae = e$ for some $e \in E$. Then $(e, a) \in \sigma$, and so $a \in \ker\sigma$. That is, $\{a \in S \mid (\exists e \in E)\ eae = e\} \subseteq \ker\sigma$. Thus, $\ker\sigma = \{a \in S \mid (\exists e \in E)\ eae = e\}$. This completes the proof. □

The aforementioned corollaries show that the kernel of μ on S is the centralizer $E\xi$ of E (i.e., for all $e \in E, s \in E\xi, \quad es = se$).

Corollary 5.8. *Let S be a type B semigroup and $x \in S$. If $x \in E\xi$, then*

$$x^* = x^+, \quad x\mathcal{H}^*x^+.$$

Proof. For all $x \in S, e \in E$, if $x \in E\xi$, then $ex = xe$. Let $e = x^*$. Then $x = xx^* = x^*x$. By Lemma 2.1, $x^+ = x^*x^+$. Let $e = x^+$. Then $x = xx^+ = x^+x$. By Lemma 2.1, $x^* = x^*x^+$. To sum up, $x^* = x^+$. In other words, $x\mathcal{L}^*x^* = x^+$. Clearly, $x\mathcal{R}^*x^+$. Therefore, $x\mathcal{H}^*x^+$. □

6 Congruences with the same kernel on a type B semigroup

In this section, we shall extend the notion of normal subsemigroups in the class of inverse semigroups to the class of type B semigroups. By using the concept of a normal subsemigroup of a type B semigroup, some characterizations of congruences with the same kernel on a type B semigroup are given.

Definition 6.1. Let S be a type B semigroup and N be a full subsemigroup of S (i.e., $E(N) = E(S)$). Then N is said to be a *normal subsemigroup* of S if it satisfies the following conditions:
(a) $(\forall x, y \in S)(\forall\ n \in N)\ xy \in N \ \Rightarrow\ xny \in N$;
(b) $(\forall x, y \in S)(\forall\ n \in N)\ xny \in N \ \Rightarrow\ xn^*y \in N, xn^+y \in N$.

Remark 6.1.
(1) Obviously, the idempotent set E of S is a full subsemigroup of S and it satisfies Condition (**b**). If S is commutative, then Condition (**a**) holds for any subsemigroup of S. Thus, if S is commutative, then E is normal.
(2) There are full subsemigroups of commutative type B semigroups which do not satisfy Condition (**b**). For example, under the general multiplication, $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are commutative type B semigroups. Clearly, the set of non-negative integers is a full subsemigroup of $\mathbb{Z}$ and a normal subsemigroup of $\mathbb{Z}$. While $\mathbb{Z}$ is a full subsemigroup of $\mathbb{Q}$ and $\mathbb{Z}$ is not normal. Therefore, Condition (**a**) is not sufficient for N to be normal in S.
(3) When N and S are groups, for all $x, y \in S, n \in N, xy \in N$, we have $n^{-1} \in N$ and $xnn^{-1}y \in N, xn^{-1}ny \in N$. Hence, for all $x, y \in S, n \in N, xy \in N$ implies that $xny \in N \ \Leftrightarrow\ N$ is normal.
(4) There are non-normal subsemigroups which satisfy Condition (**b**). That is, Condition (**b**) is not sufficient for N to be normal in S.

Example 6.1. Let $\mathbb{R}$ be a real set. Put

$$S = \left\{\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \middle| \quad x, y \in \mathbb{R}, xy \neq 0\right\}, \quad T = \left\{\begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix} \middle| \quad x, y \in \mathbb{R}, xy \neq 0\right\}.$$

Let $G = S \cup T$. Then G is a group under the matrix multiplication. Clearly, G is a type B semigroup.

Let $N = \left\{\begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \middle| \quad x \neq 0\right\}$. Obviously, N is a subgroup of G. It is easy to see that N is a full subsemigroup of G. For all $A = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \in T, A^{-1} = \begin{bmatrix} 0 & b^{-1} \\ a^{-1} & 0 \end{bmatrix}$ and for all $M = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \in N, M^* = M^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$,

$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$. But $AMA^{-1} = \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \notin N$. This means that N is not a normal subsemigroup of G. Let $A, B \in G$, $C \in N$. Then $C^* = C^+ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in N$. Hence, $AC^*B = AC^+B = AB$. Note that

If $A, B \in S$ and $ACB \in N$, then $AB \in N$;
If $A \in S$ and $B \in T$, then $ACB \notin N$;
If $A \in T$ and $B \in S$, then $ACB \notin N$;
If $A, B \in T$ with $B = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$, $xy = 1$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $ACB \in N$ and so $AB \in N$.

From Example 6.1, Conditions (**a**) and (**b**) of normal semigroups are independent.

A non-empty subset A of a semigroup S is said to be *unitary* if for all $a \in A$, $s \in S$, $as, sa \in A$ imply that $s \in A$.

Theorem 6.1. *Let ρ be an admissible congruence on a type B semigroup S. Then*
(1) *If* $\ker\rho$ *is unitary, then* $\ker\rho$ *is a normal subsemigroup of S;*
(2) *If* $\ker\rho$ *is cancellative, then* $\ker\rho$ *is a normal subsemigroup of S.*

Proof. (1) By hypothesis, it is clear that $E \subseteq \ker\rho$ and $\ker\rho$ is a full subsemigroup of S. Let $x, y \in S$, $n \in \ker\rho$ and $xy \in \ker\rho$. Then there is $e \in E$ such that $(n, e) \in \rho$ and $(xny, xey) \in \rho$. Note that $ey^+ \leq y^+$. We have that there exists $f \in E^1$ such that $ey^+ = (yf)^+$ from Condition (**B2′**). Hence,

$$\begin{aligned} xeyf &= xey^+yf \\ &= x(yf)^+yf \quad (\text{since } ey^+ = (yf)^+) \\ &= xyf \in \ker\rho \, (\text{since } xy \in \ker\rho, f \in \ker\rho). \end{aligned}$$

Note that $f \in \ker\rho$ and $\ker\rho$ is unitary. We have that $xey \in \ker\rho$, and so $xny \in \ker\rho$. Thus, Condition (**a**) holds.

Since ρ is an admissible congruence on S and $(n, e) \in \rho$, we have $(n^*, e) \in \rho$. By transitivity, $(n, n^*) \in \rho$. Hence, for all $x, y \in S$, $(xny, xn^*y) \in \rho$. Thus, $xny \in \ker\rho$ implies $xn^*y \in \mathit{ker}\rho$. Similarly, $xny \in \ker\rho$ implies $xn^+y \in \mathit{ker}\rho$. This means that Condition (**b**) holds. To sum up, $\ker\rho$ is a normal subsemigroup of S.

(2) By the proof of (1), for all $x, y \in S$, $n \in \ker\rho$, there exist $e, f \in E$ such that $xeyf = xyf$. Since ρ is a congruence on S, we have $xeyf\rho xyf$. By the hypothesis, $xey\rho xy$. Note that $xy \in \ker\rho$, we have $xey \in \ker\rho$. Hence, $xny \in \ker\rho$. Thus, Condition (**a**) holds. By the proof of (1), Condition (**b**) holds. To sum up, $\ker\rho$ is a normal subsemigroup of S. □

Definition 6.2. Let N be any subsemigroup of a type B semigroup S. Define a syntactic congruence η_N of N as follows:

$$\eta_N = \{(a, b) \in S \times S \mid (\forall x, y \in S^1)\ xay \in N \iff xby \in N\}.$$

Obviously, η_N is a congruence and η_N is the maximum congruence. If $E(S) \subseteq N$, then $\ker\eta_N \subseteq N$. If S is a type B semigroup and N is normal, then we have that the maximum congruence on S whose kernel is N on S.

Proposition 6.2. *Let S be a type B semigroup and N be a normal subsemigroup of S. Then η_N is the maximum admissible congruence on S whose kernel is N.*

Proof. It is easy to see that η_N is a congruence on S. Now, we prove that $\ker\eta_N = N$. To see it, let $a \in S$. Then

$$a \in \ker\eta_N \Rightarrow (\exists e \in E)(e, a) \in \eta_N \Rightarrow (\forall x, y \in S^1)\ xay \in N \quad \text{if and only if} \quad xey \in N.$$

Let $x = a^+$ and $y = a^*$. We have that $a^+ea^* \in N$ since N is a full subsemigroup. Hence, $a^+aa^* \in N$. That is, $a \in N$. On the other hand, let $n \in N$. Then, for all $x, y \in S^1$, we have

$$xny \in N \Rightarrow xn^*y \in N \Rightarrow xnn^*y \in N \Rightarrow xny \in N.$$

Hence, $xny \in N$ if and only if $xn^*y \in N$. Thus, $(n, n^*) \in \eta_N$. This shows that $n \in \ker\eta_N$. Therefore, $\ker\eta_N = N$.

Next, we prove that η_N is an admissible congruence on S. For all $a \in S$, $m, n \in S^1$, if $am\eta_N \quad an$, then for all $x, y \in S^1$, $xamy \in N$ if and only if $xany \in N$. Since N is a normal semigroup of S, we obtain that $xa^*my \in N$ if and only if $xa^*ny \in N$. Thus, $a^*m\eta_N \; a^*n$. Similarly, for all $a \in S$, $m, n \in S^1$, $ma\eta_N \; na$ implies $ma^+\eta_N \; na^+$. Therefore, η_N is an admissible congruence.

Finally, we show that η_N is the maximum admissible congruence on S whose kernel is N. To see it, suppose ρ is an admissible congruence on S and $\ker\rho = N$. Let $(a, b) \in \rho$. Then for all $x, y \in S^1$, we have $(xay, xby) \in \rho$. Hence, $xay \in \ker \quad \rho$ if and only if $xby \in \ker\rho$. That is, $xay \in N$ if and only if $xby \in N$. Therefore, $(a, b) \in \eta_N$.

To sum up, η_N is the maximum admissible congruence on S whose kernel is N. □

Let S be a type B semigroup and N be a normal subsemigroup of S. In order to give the minimum admissible congruence on S whose kernel is N, we first give the following lemma:

Lemma 6.3. *Let S be a type B semigroup and N be a normal subsemigroup of S. Define a relation τ_N on S as follows*:

$$\tau_N = \{(xn_1y, xn_2y) \mid x, y \in S^1; n_1, n_2 \in N; n_1^+ = n_2^+\}.$$

Then the following statements hold:
(1) τ_N *is an equivalence relation on* S;
(2) $N = \{a \in S \mid (\exists e \in E)\ (a, e) \in \tau_N\}$;
(3) τ_N *is contained in an arbitrary admissible congruence on S whose kernel is N.*

Proof. (1) It is clear.

(2) Let $a \in N$. Clearly, $a = a^+aa^*$ and $a^+a^* = a^+a^+a^*$. Since $a, a^+ \in N$ and $a^+ = (a^+)^+ = a^+$, we have $(a, a^+a^*) \in \tau_N$. That is, there exists $e = a^+a^* \in E$ such that $(a, e) \in \tau_N$.

Conversely, let $a \in S$ and $(a, f) \in \tau_N$ for some $f \in E$. By the definition of τ_N, we have

$$a = xn_1y, \quad f = xn_2y; \quad n_1, n_2 \in N; \quad n_1^+ = n_2^+.$$

Note that $E \subseteq N$. We have $xn_2y = f \in N$. Since N is normal. We obtain $xn_1^+y = xn_2^+y \in N$. Hence, $xn_1y = xn_1^+n_1y \in N$. That is, $a \in N$ and (2) holds.

(3) Let ρ be any admissible congruence on S whose kernel is N. Let $(a, b) \in \tau_N$. By the definition of τ_N,

$$a = xn_1y, \quad b = xn_2y; \quad n_1, n_2 \in N; \quad n_1^+ = n_2^+.$$

Since $\ker\rho = N$, there are $e, f \in E$ such that $(n_1, e) \in \rho$, $(n_2, e) \in \rho$. Note that ρ is an admissible congruence on S. We have that $(n_1^+, e) \in \rho$ and $(n_2^+, f) \in \rho$. But $n_1^+ = n_2^+$, we have $(e, f) \in \rho$. Thus, $(xey, xfy) \in \rho$. It follows that:

$$(n_1, e) \in \rho, (n_2, f) \in \rho \Rightarrow (xn_1y, xey) \in \rho, (xn_2y, xfy) \in \rho \Rightarrow (xn_1y, xn_2y) \in \rho \Rightarrow (a, b) \in \rho.$$

Therefore, (3) holds. □

As usual, let S be an arbitrary relation on X. Define S^∞, the transitive closure of S, by $S^\infty = \bigcup\{S^n \mid n \geq 1\}$.

Proposition 6.4. *Let S be a type B semigroup and λ_N be a transitive closure of τ_N ($\lambda_N = \tau_N^t$). Then λ_N is the minimum admissible congruence on S whose kernel is N.*

Proof. Obviously, λ_N is a congruence on S. Now, we prove that $\ker\lambda_N = N$. To see it, let $n \in N$. Then there exists $e \in E$ such that $(n, e) \in \tau_N$. By Lemma 6.3, we have that $(n, e) \in \lambda_N$ for some $e \in E$ and $n \in \ker\lambda_N$.

Conversely, let $a \in \ker\lambda_N$. Then $(a, f) \in \lambda_N$ for some $f \in E$. By hypothesis, $\lambda_N = \tau_N^t$. Hence, there exist $a_1, a_2, \ldots, a_n \in S$ such that

$$(a, a_1) \in \tau_N, (a_1, a_2) \in \tau_N, \ldots, (a_{n-1}, a_n) \in \tau_N, (a_n, f) \in \tau_N.$$

Note that

$$(a_n, f) \in \tau_N \Rightarrow a_n = xn_1y, \quad f = xn_2y; \quad n_1, n_2, f \in N; \quad n_1^+ = n_2^+;$$

$$xn_2y = f \in N \Rightarrow xn_2^+y \in N \Rightarrow xn_1^+y \in N \Rightarrow xn_1y \in N \Rightarrow a_n \in N.$$

Similarly, $a_{n-1}, a_{n-2}, \ldots, a_1, a \in N$. That is, $a \in N$. Therefore, $\ker \lambda_N = N$.

Finally, we show that λ_N is the minimum admissible congruence on S whose kernel is N. Let ρ be any admissible congruence on S whose kernel is N. By Lemma 6.3, we have that $\tau_N \subseteq \rho$, and so $\lambda_N \subseteq \rho$. □

The following corollary can be obtained from Propositions 6.2 and 6.4.

Corollary 6.5. *Let S be a type B semigroup and ρ be an admissible congruence on S whose kernel is N. Then $\lambda_N \subseteq \rho \subseteq \eta_N$ and* $\ker\lambda_N = \ker\rho = \ker\eta_N$.

A congruence on S is said to be *idempotent-pure* if for all $a \in S$, $e \in E$ and $(a, e) \in \rho$ imply $a \in E$.

Proposition 6.6. *Let S be a type B semigroup and ρ be an admissible congruence. If λ_N is an admissible congruence on S, then S/ρ is an idempotent-pure image of S/λ_N.*

Proof. Define a mapping ϕ as follows:

$$\phi : S/\lambda_N \to S/\rho, \quad (a\lambda_N)\phi = a\rho.$$

It is easy to see that ϕ is a homomorphism from S/λ_N to S/ρ. Let $a \in S$, $e \in E$ such that $(a\lambda_N)\phi = (e\lambda_N)\phi$. Then $a\rho = e\rho$. That is, $a\rho e$. By Proposition 5.7, $\ker\lambda_N = N = \ker\rho$. Hence, $a\lambda_N e$. That is, $a\lambda_N = e\lambda_N$. Therefore, S/ρ is an idempotent-pure image of S/λ_N. □

7 Congruence pairs on a type B semigroup

In this section, we extend the concept of congruence pairs from inverse semigroup to type B semigroups.

Definition 7.1. Let N be a normal subsemigroup of a type B semigroup and π be a normal congruence on E. Then (π, N) is a congruence pair on S if the following statements hold:
(1) for all $n \in N$, $n^+\pi n^*$;
(2) for all $x, y \in S$, $e, f \in E$, $xey \in N$ and $e\pi f$ imply $xfy \in N$.

Lemma 7.1. *Let ρ be an admissible congruence on a type B semigroup and* $\ker\rho$ *is unitary. Then* $(\mathrm{tr}\rho, \ker\rho)$ *is a congruence pair of S.*

Proof. Obviously, $\mathrm{tr}\rho$ is a normal congruence. Since $\ker\rho$ is unitary, by Theorem 6.1, we have that $\ker\rho$ is a normal subsemigroup of S. Let $n \in \ker\rho$. By Proposition 5.1, $(n^+, n^*) \in \mathrm{tr}\rho$. Let $x, y \in S$, $e, f \in E$ such that $xey \in \ker\rho$ and $e\,\mathrm{tr}\rho f$. Then

$$e\,\mathrm{tr}\rho f \Rightarrow e\rho f \Rightarrow xey \quad \rho \quad xfy.$$

Since $xey \in \ker\rho$, we obtain $xfy \in \ker\rho$. Therefore, $(\mathrm{tr}\rho, \ker\rho)$ is a congruence pair of S. □

Let N be a normal subsemigroup of a type B semigroup and π be a normal congruence on E such that (π, N) is a congruence pair of S. A congruence ρ is said to be *associated with the congruence pair* (π, N) if $\ker\rho = N$, $\operatorname{tr}\rho = \pi$. The following theorem will show that a congruence associated with the congruence pair (π, N) can be constructed on S by using relations μ_π and η_N.

Theorem 7.2. *The relation $\mu_\pi \cap \eta_N$ is a congruence on a type B semigroup S associated with the congruence pair (π, N).*

Proof. Let $\rho = \mu_\pi \cap \eta_N$. Clearly, ρ is a congruence on S. Now, we prove that $\ker\rho = N$. Let $n \in N$. Then for all $e \in E \subseteq N$, we have $en \in N$. Since (π, N) is a congruence pair, we obtain that $(en)^+\pi(en)^*$, $n^+\pi n^*$ and $en^+\pi en^*$. Note that $en^* = (en^*)^*$ and $en^+ = (en)^+$. We have

$$(en)^*\pi(en)^+\pi en^+\pi en^*\pi(en^*)^*.$$

That is, $(en)^*\pi(en^*)^*$. Similarly, $(ne)^+\pi(n^*e)^+$. Thus, $(n, n^*) \in \mu_\pi$. By the proof of Proposition 6.2, $(n, n^*) \in \eta_N$. Thus, $(n, n^*) \in \rho$, $n \in \ker\rho$. That is, $N \subseteq \ker\rho$. On the other hand, it is easy to see that $\ker\rho \subseteq \ker\eta_N = N$. Therefore, $\ker\rho \subseteq N$ and $\ker\rho = N$.

Next, we show that $\operatorname{tr}\rho = \pi$. Obviously, $\rho \subseteq \mu_\pi$. For all $e, f \in E$ and $(e, f) \in \rho$, we have $(e, f) \in \mu_\pi$. Furthermore, $e\pi f$ and $\operatorname{tr}\rho \subseteq \pi$. On the other hand, let $e, f \in E$ with $e\pi f$. Then, by the definition of (π, N), we have that $xey \in N$ if and only if $xfy \in N$. Hence, $(e, f) \in \eta_N$. Since $(e, f) \in \mu_\pi$, we have that $(e, f) \in \rho$ and $\pi \subseteq \operatorname{tr}\rho$. Therefore, $\operatorname{tr}\rho = \pi$. ◻

Let ρ be an admissible congruence on a type B semigroup S. If $\ker\rho = N$ and $\operatorname{tr}\rho = \pi$, then $\rho \subseteq \mu_\pi$ from Theorem 3.4 and $\rho \subseteq \eta_N$ from Proposition 6.2. Thus, we have the following corollary:

Corollary 7.3. *Let ρ be an admissible congruence on a type B semigroup S. If* $\ker\rho = N$ *and* $\operatorname{tr}\rho = \pi$, *then $\rho \subseteq \mu_\pi \cap \eta_N$.*

Theorem 7.4. *The relation $\sigma_\pi \vee \lambda_N$ is a congruence on a type B semigroup S associated with the congruence pair (π, N).*

Proof. Let $\tau = \sigma_\pi \vee \lambda_N$. Then

$$\tau = \bigcap\{\rho \mid \rho \text{ is a congruence}, \sigma_\pi \subseteq \rho \text{ and } \lambda_N \subseteq \rho\},$$

and for all $a, b \in S$, $(a, b) \in \tau$ if and only if there exist $a_1, a_2, \ldots, a_n \in S$ such that

$$(a, a_1) \in \lambda_N, (a_1, a_2) \in \sigma_\pi, (a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi, (a_n, b) \in \lambda_N.$$

Therefore, τ is the minimum congruence containing both σ_π and λ_N.

Now, we prove $\ker\tau = N$. To see it, let $a \in \ker\tau$. Then there exists $e \in E$ such that $(a, e) \in \tau$. Hence, there exist $a_1, a_2, \ldots, a_n \in S$ such that

$$(a, a_1) \in \lambda_N, (a_1, a_2) \in \sigma_\pi, (a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi, (a_n, e) \in \lambda_N.$$

Since $(a_n, e) \in \lambda_N$, by Proposition 6.4, $a_n \in N$, $(a_{n-1}, a_n) \in \sigma_\pi$. By the definition of σ_π, there are $e, f \in E$ such that

$$e\pi a_n^*\pi a_{n-1}^*, f\pi a_n^+\pi a_{n-1}^+ \quad \text{and} \quad fa_n e = fa_{n-1}e.$$

Since N is normal, $a_n \in N$, $fa_n e \in N$ and $fa_{n-1}e \in N$, we have

$$fa_{n-1}ea_{n-1}^* \in N, \quad e\pi a_{n-1}^*.$$

By the definition of congruence pairs, we obtain

$$fa_{n-1} = fa_{n-1}a_{n-1}^*a_{n-1}^* \in N.$$

Thus, $a_{n-1}^+ f a_{n-1} \in N$. Again, since $f \pi a_{n-1}^+$, we have

$$a_{n-1} = a_{n-1}^+ a_{n-1}^+ a_{n-1} \in N.$$

Hence, $a_{n-1} \in N$ and $(a_{n-2}, a_{n-1}) \in \lambda_N$. By the proof of Theorem 6.4, $a_{n-2} \in N$. Since $(a_{n-3}, a_{n-2}) \in \sigma_\pi$, we have $a_{n-3} \in N$ from the above proof. The process will continue until we reach $a \in N$. Thus, $\ker\tau \subseteq N$. Conversely,

$$n \in N \;\Rightarrow\; n \in \ker\lambda_N \;\Rightarrow\; (\exists e \in E)(n, e) \in \lambda_N \;\Rightarrow\; (\exists e \in E)(n, e) \in \tau \;\Rightarrow\; n \in \ker\tau.$$

Hence, $N \subseteq \ker\tau$. Therefore, $\ker\tau = N$.

Next, we prove that $\mathrm{tr}\tau = \pi$. Let $e, f \in E$. Then

$$(e, f) \in \pi \;\Rightarrow\; (e, f) \in \sigma_\pi \;\Rightarrow\; (e, f) \in \tau.$$

This means that $\pi \subseteq \mathrm{tr}\,\tau$.

On the other hand, let $e, f \in E$ and $(e, f) \in \mathrm{tr}\tau$. That is, $(e, f) \in \tau$. Then there exist $a_1, a_2, \ldots, a_n \in S$ such that

$$(e, a_1) \in \lambda_N, (a_1, a_2) \in \sigma_\pi, (a_2, a_3) \in \lambda_N, \ldots, (a_{n-1}, a_n) \in \sigma_\pi, (a_n, f) \in \lambda_N.$$

Note that $(e, a_1) \in \lambda_N$. There are $b_1, b_2, \ldots, b_k \in S$ such that

$$(e, b_1) \in \tau_N, (b_1, b_2) \in \tau_N, \ldots, (b_{k-1}, b_k) \in \tau_N, (b_k, a_1) \in \tau_N.$$

Again $(e, b_1) \in \tau_N$, by the definition of τ_N,

$$e = xn_1y, b_1 = xn_2y; \quad n_1, n_2 \in N; \quad n_1^+ = n_2^+.$$

According to the definition of congruence pairs, for all $n \in N$, we have $n^* \pi n^+$. Thus, for all $x, y \in S$, $ny^+ \in N$, which follows $(ny^+)^+ \pi (ny^+)^*$. Since $(ny)^+ = (ny^+)^+$, we have $(ny)^+ \pi (ny^+)^*$. By the normality of π,

$$(x(ny)^+)^+ \pi (x(ny^+)^*)^+.$$

That is, $(xny)^+ \pi (x(ny^+)^*)^+$. In particular, since $n_1, n_2 \in N$, we have

$$(xn_1y)^+ \pi (x(n_1y^+)^*)^+, \quad (xn_2y)^+ \pi (x(n_2y^+)^*)^+.$$

Again since $n_1^* \pi n_1^+$, $n_1^+ = n_2^+$ and $n_2^* \pi n_2^+$, we obtain $n_1^* \pi n_2^*$, $n_1^* y^+ \pi n_2^* y^+$ and $(n_1y^+)^* \pi (n_2y^+)^*$. By the normality of π, $(x(n_1y^+)^*)^+ \pi (x(n_2y^+)^*)^+$. By transitivity of π, $(xn_1y)^+ \pi (xn_2y)^+$. That is, $e \pi b_1^+$. Note that $(b_1, b_2) \in \tau_N$. We have

$$b_1 = x_1n_3y_1, \quad b_2 = x_1n_4y_1; \quad n_3, n_4 \in N; \quad n_3^+ = n_4^+.$$

Similarly, we can prove $b_1^+ \pi b_2^+, b_2^+ \pi b_3^+, \ldots, b_k^+ \pi a_1^+$. By transitivity, we have $e \pi a_1^+$. Since $(a_1, a_2) \in \sigma_\pi$, we obtain $a_1^+ \pi a_2^+$. Note that $(a_2, a_3) \in \lambda_N$. According to the above procedure, $a_2^+ \pi a_3^+, a_3^+ \pi a_4^+, a_4^+ \pi a_5^+, \ldots, a_{n-1}^+ \pi a_n^+$ and $a_n^+ \pi f$. By transitivity, $e \pi f$. Therefore, $\mathrm{tr}\tau \subseteq \pi$.

To sum up, $\sigma_\pi \vee \lambda_N$ is a congruence on a type B semigroup associated with the congruence pair (π, N). This completes the proof. □

Let ρ be an admissible congruence on a type B semigroup. If $\ker\rho = N$, $\mathrm{tr}\rho = \pi$. By Theorem 3.1, we have $\sigma_\pi \subseteq \rho$. By Proposition 6.4, we obtain $\lambda_N \subseteq \rho$. Therefore, we obtain the following corollary:

Corollary 7.5. *Let ρ be an admissible congruence on a type B semigroup S. If $\ker\rho = N$ and $\mathrm{tr}\rho = \pi$, then $\sigma_\pi \vee \lambda_N \subseteq \rho$.*

Theorem 7.6. *Let ρ be an admissible congruence on a type B semigroup S. If $\ker\rho = N$ and $\mathrm{tr}\rho = \pi$, then $\mu_\pi \cap \eta_N$ and $\sigma_\pi \vee \lambda_N$ are congruences on S associated with the congruence pair (π, N). Furthermore, $\sigma_\pi \vee \lambda_N \subseteq \rho \subseteq \mu_\pi \cap \eta_N$.*

Proof. It follows from Theorems 7.2 and 7.4 and Corollary 7.5. □

8 Conclusion

As we know, abundant semigroups are generalized regular semigroups and type B semigroups are generalized inverse semigroups in the range of abundant semigroups. The kernel-trace approach consists in splitting the analysis of a congruence on a regular semigroup into two parts: the kernel and the trace. Here we develop the kernel-trace approach of inverse semigroups to the cases of type B semigroups. As a concrete application of the above approach, we introduce admissible congruences of type B semigroups. Admissible congruences introduced here could improve studies of the inverse semigroup theory.

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