DE GRUYTER Open Mathematics 2022; 20: 1819–1832

Research Article

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# On the connection between $S^p$-almost periodic functions defined on time scales and $\mathbb{R}$

https://doi.org/10.1515/math-2022-0550
received August 30, 2022; accepted December 13, 2022

**Abstract:** It is well known that a sufficient and necessary condition for a continuous function $g$ to be almost periodic on time scale $\mathbb{T}$ is the existence of an almost periodic function $f$ on $\mathbb{R}$ such that $f$ is an extension of $g$. The purpose of this article is to extend these results to $S^p$-almost periodic functions. We prove that the necessity is true, that is, an $S^p$-almost periodic function on $\mathbb{T}$ can be extended to an $S^p$-almost periodic function on $\mathbb{R}$. However, a counterexample is given to show that the sufficiency is not true in general. By introducing a concept of minor translation set and characterizing the almost periodicity on $\mathbb{T}$ in terms of this new concept, we obtain a condition to ensure the sufficiency. Moreover, we show the necessity of this condition by a counterexample.

**Keywords:** almost periodic, $S^p$-almost periodic, time scales

**MSC 2020:** 42A75, 26E70

## 1 Introduction

Periodic functions and almost periodic functions have numerous applications in the study of qualitative theory of differential equations, difference equations, and dynamic equations on time scales (see, e.g., [1–5]), and the papers for more details (see, e.g., [6,7]). The theory of time scales was initiated by Hilger in his Ph.D. thesis [8] in 1988 to unify continuous and discrete problems. The theory gives an effective mathematical technique to economics, biomathematics, quantum physics, etc. In 2011, Li and Wang introduced the almost periodic functions on time scales [9,10]. Since then, many concepts of almost periodic functions were introduced on time scales, such as pseudo almost periodicity, almost automorphy, $S^p$-almost periodicity, and others. Recently, Lizama and Mesquita have studied the connection between almost periodic functions defined on time scales and on the real line and obtained the following result.

**Proposition 1.1.** [11] *If $\mathbb{T}$ is invariant under translations, a necessary and sufficient condition for a continuous function $g : \mathbb{T} \to \mathbb{E}^n$ to be almost periodic on $\mathbb{T}$ is the existence of an almost periodic function $f : \mathbb{R} \to \mathbb{E}^n$ such that $f(t) = g(t)$ for every $t \in \mathbb{T}$, where $\mathbb{E}^n$ denotes the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$.*

Then Tang and Li extended this result to pseudo almost periodic functions and obtained the following results.

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**Proposition 1.2.** [12]
(*i*) *If* $\mathbb{T}$ *is invariant under translations and* $f \in \mathrm{PAP}(\mathbb{T}; \mathbb{E}^n)$, *then there exists* $g \in \mathrm{PAP}(\mathbb{R}; \mathbb{E}^n)$ *such that* $g(t) = f(t)$ *for* $t \in \mathbb{T}$;
(*ii*) *If* $\mathbb{T}$ *is invariant under translations and* $f \in \mathrm{PAP}(\mathbb{R}; \mathbb{E}^n)$ *is uniformly continuous, then the restriction function* $g = f|_{\mathbb{T}} : \mathbb{T} \to \mathbb{E}^n$ *is in* $\mathrm{PAP}(\mathbb{T}; \mathbb{E}^n)$,

*where* $\mathbb{E}^n$ *denotes the Euclidian space* $\mathbb{R}^n$ *or* $\mathbb{C}^n$.

The main purpose of this article is to extend these results to $S^p$-almost periodic functions on time scales. We prove that the necessity is true, that is, an $S^p$-almost periodic function on $\mathbb{T}$ can be extended to an $S^p$-almost periodic function on $\mathbb{R}$ (Theorem 3.1). However, a counterexample (Example 3.1) is given to show that the sufficiency is not true in general. Meanwhile, by introducing a concept of minor translation set of $\mathbb{T}$ and characterizing the almost periodicity on $\mathbb{T}$ in terms of this new concept (Theorems 3.2 and 3.3), we obtain a condition to ensure the sufficiency (Theorem 3.4). Moreover, under the condition "the interval $[\omega, \omega + \mathcal{K})_{\mathbb{T}} \cap \mathbb{T}'$ has finite elements," we obtain a sufficient and necessary condition for $g|_{\mathbb{T}} \in S^p\mathrm{AP}(\mathbb{T}; \mathbb{X})$. Then we give a counterexample to show that this condition is necessary (Example 3.2). Furthermore, we present a result on the connection between $S^p$-almost periodic functions defined on $\mathbb{T}_1$ and $\mathbb{T}_2$ from Theorem 3.4 (Corollary 3.1).

# 2 Preliminaries

The following concepts and results can be found in [8,13]. From now on, $\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}^+$ indicate the sets of nature numbers, positive integers, integers, real numbers, and nonnegative numbers, respectively. Let $\mathbb{E}^n$ be the Euclidian space $\mathbb{R}^n$ or $\mathbb{C}^n$ with Euclidian norm $|\cdot|$, and let $(\mathbb{X}, \|\cdot\|)$ be a Banach space.

## 2.1 Time scale

Let $\mathbb{T} \subset \mathbb{R}$ be a time scale, that is, $\mathbb{T} \neq \emptyset$ is closed. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

If $\sigma(t) > t$, we say that $t$ is right-scattered; otherwise, $t$ is right-dense. Similarly, if $\rho(t) < t$, we say that $t$ is left-scattered; otherwise $t$ is left-dense.

**Definition 2.1.** A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi := \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \quad t \in \mathbb{T}\} \neq \{0\}$$

and define

$$\mathcal{K} = \begin{cases} \inf\{|\tau| : \tau \in \Pi, \tau \neq 0\}, & \text{if } \mathbb{T} \neq \mathbb{R}; \\ 1, & \text{if } \mathbb{T} = \mathbb{R}. \end{cases}$$

In fact, if $\mathbb{T} \neq \mathbb{R}$, then we have $\mathcal{K} > 0$, and one can show that $\Pi = \mathcal{K}\mathbb{Z}$, and we say $\Pi$ the translation set of $\mathbb{T}$ (see, e.g., [12]).

In this article, we always assume that $\mathbb{T}$ is invariant under translations.

**Definition 2.2.** [8]
(i) A function $f : \mathbb{T} \to \mathbb{X}$ is continuous on $\mathbb{T}$ if $f$ is continuous at every right-dense point and at every left-dense point.

(ii) A function $f : \mathbb{T} \to \mathbb{X}$ is uniformly continuous on $\mathbb{T}$ if for $\varepsilon > 0$, there is a $\delta > 0$ such that $\|f(x_1) - f(x_2)\| < \varepsilon$ for $x_1, x_2 \in \mathbb{T}$ with $|x_1 - x_2| < \delta$.

Denote by $C(\mathbb{T}; \mathbb{X})$ the set of all continuous functions $g : \mathbb{T} \to \mathbb{X}$.

For $t, s \in \mathbb{T}, \quad t < s$, denote $(t, s)$, $[t, s]$, $(t, s]$, $[t, s)$ the standard intervals in $\mathbb{R}$, and use the following symbols:

$$(t,s)_{\mathbb{T}} = (t,s) \cap \mathbb{T}, \quad [t,s]_{\mathbb{T}} = [t,s] \cap \mathbb{T}, \quad (t,s]_{\mathbb{T}} = (t,s] \cap \mathbb{T}, \quad [t,s)_{\mathbb{T}} = [t,s) \cap \mathbb{T}.$$

**Proposition 2.1.** [13] *Fix a point $\omega \in \mathbb{T}$ and an interval $[\omega, \omega + \mathcal{K})_{\mathbb{T}}$, there are at most countably many right-scattered points $\{t_i\}_{i\in I}, \quad I \subseteq \mathbb{N}$ in this interval. If we denote $t_{ij} = t_i + j\mathcal{K}$, $i \in I$, $j \in \mathbb{Z}$, we obtain all the right-scattered points, and we have $\mu(t_{ij}) = \mu(t_i)$.*

Let $\mathcal{F}_1 = \{[t, s)_{\mathbb{T}} : t, s \in \mathbb{T} \quad \text{with } t \leqslant s\}$. Define a countably additive measure $m_1$ on $\mathcal{F}_1$ by assigning to every $[t, s)_{\mathbb{T}} \in \mathcal{F}_1$ its lengths, i.e.,

$$m_1([t,s)_{\mathbb{T}}) = s - t.$$

By using $m_1$, we can generate the outer measure $m_1^*$ on the power set $\mathcal{P}(\mathbb{T})$ of $\mathbb{T}$: for $E \in \mathcal{P}(\mathbb{T})$:

$$m_1^*(E) = \begin{cases} \inf\limits_{\mathcal{B}} \left\{ \sum\limits_{i\in I_{\mathcal{B}}} (s_i - t_i) \right\} \in \mathbb{R}^+, & \beta \notin E; \\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$ and

$$\mathcal{B} = \{\{[t_i, s_i) \in \mathcal{F}_1\}_{i\in I_{\mathcal{B}}} : I_{\mathcal{B}} \subset \mathbb{N}, E \subset \bigcup_{i\in I_{\mathcal{B}}} [t_i, s_i)_{\mathbb{T}}\}.$$

A set $A \subset \mathbb{T}$ is called $\Delta$-measurable if for $E \subset \mathbb{T}$, and we have

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let $\mathcal{M}(m_1^*) = \{A : A \text{ is a } \Delta - \text{measurable subset in } \mathbb{T}\}$. Restricting $m_1^*$ to $\mathcal{M}(m_1^*)$, we obtain the Lebesgue $\Delta$-measure, which is denoted by $\mu_\Delta$.

**Definition 2.3.** [13]

(i) A function $\mathcal{S} : \mathbb{T} - \mathbb{X}$ is said to be simple if $\mathcal{S}$ takes a finite number of values $c_1, c_2, \cdots, c_N$. Let $E_j = \{s \in \mathbb{T} : \mathcal{S}(s) = c_j\}$. Then

$$\mathcal{S} = \sum_{j=1}^{N} c_j \chi_{E_j},$$

where $\chi_{E_j}$ is the characteristic function of $E_j$, i.e.,

$$\chi_{E_j}(s) = \begin{cases} 1, & \text{if } s \in E_j; \\ 0, & \text{if } s \in \mathbb{T} \setminus E_j. \end{cases}$$

(ii) Assume that $E$ is a $\Delta$-measurable subset of $\mathbb{T}$ and $\mathcal{S} : \mathbb{T} \to \mathbb{X}$ is a $\Delta$-measurable simple function. Then the Lebesgue $\Delta$-integral of $\mathcal{S}$ on $E$ is defined as follows:

$$\int_E \mathcal{S}(s)\Delta s = \sum_{j=1}^{N} c_j \mu_\Delta(E_j \cap E).$$

(iii) A function $g : \mathbb{T} \to \mathbb{X}$ is a $\Delta$-integrable function if there exists a simple function sequence $\{g_k : k \in \mathbb{N}\}$ such that $g_k(s) \to g(s)$ a.e. in $\mathbb{T}$, then the integral of $g$ is defined as follows:

$$\int_{\mathbb{T}} g(s) = \lim_{k\to\infty} \int_{\mathbb{T}} g_k(s)\Delta s.$$

(iv) For $p \geqslant 1$, $g : \mathbb{T} \to \mathbb{X}$ is called locally $L^p\Delta$-integrable if $g$ is $\Delta$-measurable and for any compact $\Delta$-measurable set $E \subset \mathbb{T}$, the $\Delta$-integral

$$\int_E \|g(s)\|^p \Delta s < \infty.$$

The set of all $L^p\Delta$-integrable functions is denoted by $L^p_{\text{loc}}(\mathbb{T};\mathbb{X})$.

**Definition 2.4.** [14] Define $\|\cdot\|_{S^p} : L^p_{\text{loc}}(\mathbb{T};\mathbb{X}) \to \mathbb{R}^+$ as follows:

$$\|g\|_{S^p} := \sup_{s\in\mathbb{T}} \left( \frac{1}{\mathcal{K}} \int_s^{s+\mathcal{K}} \|g(r)\|^p \Delta r \right)^{\frac{1}{p}},$$

where $\mathcal{K}$ is defined in Definition 2.1. A function $g \in L^p_{\text{loc}}(\mathbb{T};\mathbb{X})$ is called $S^p$-bounded if $\|g\|_{S^p} < \infty$. The space of all $S^p$-bounded functions is denoted by $BS^p(\mathbb{T};\mathbb{X})$; if $\mathbb{T} = \mathbb{R}$, denote it by $BS^p(\mathbb{X})$.

**Lemma 2.1.** [15] *The norm $\|\cdot\|_{S^p}$ and $\|\cdot\|_{S^p_l}$ in $BS^p(X)$ given below are equivalent.*

$$\|g\|_{S^p} = \left( \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g(s)\|^p \mathrm{d}s \right)^{\frac{1}{p}},$$

$$\|g\|_{S^p_l} = \left( \sup_{t\in\mathbb{R}} \frac{1}{l} \int_t^{t+l} \|g(s)\|^p \mathrm{d}s \right)^{\frac{1}{p}}.$$

*In fact, we have*

$$\min\left(l^{\frac{1}{p}}, l^{-\frac{1}{p}}\right)\|g\|_{S^p} \leqslant \|g\|_{S^p_l} \leqslant \max\left(l^{\frac{1}{p}}, l^{-\frac{1}{p}}\right)\|g\|_{S^p}.$$

**Remark 2.1.** The almost periodic properties will be kept for equivalent norm of a Banach space.

## 2.2 $S^p$-almost periodic functions

**Definition 2.5.** [16] A set $A \subset \mathbb{T}$ is called relatively dense in $\mathbb{T}$ if there exists an $l > 0$ such that $[s, s+l]_{\mathbb{T}} \cap A \neq \varnothing$, $s \in \mathbb{T}$, and we call $l$ the inclusion length.

**Definition 2.6.** [9] A function $g \in C(\mathbb{T};\mathbb{X})$ is almost periodic on $\mathbb{T}$ if for $\varepsilon > 0$,

$$T(g,\varepsilon) = \{\tau \in \Pi : \|g(s+\tau) - g(s)\| < \varepsilon \text{ for } s \in \mathbb{T}\}$$

is a relatively dense set in $\Pi$. We call $T(g,\varepsilon)$ the $\varepsilon$-translation set of $g$ and $\tau$ the $\varepsilon$-translation period of $g$, and the set of all almost periodic functions on $\mathbb{T}$ is denoted by $\mathrm{AP}(\mathbb{T};\mathbb{X})$.

**Definition 2.7.** [10] A function $g : \mathbb{T} \to \mathbb{X}$ is said to be normal on $\Pi$ if for any sequence $\{\alpha_n'\} \subset \Pi$, and there is a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that $\{g(t+\alpha_n)\}$ converges uniformly for $t \in \mathbb{T}$.

**Lemma 2.2.** [10] *A function $g : \mathbb{T} \to \mathbb{X}$ is almost periodic on $\mathbb{T}$ if and only if it is normal on $\Pi$.*

**Definition 2.8.** [17,18]

(i) A function $g \in L^p_{\text{loc}}(\mathbb{R}; \mathbb{X})$ is said to be $S^p$-almost periodic if for $\varepsilon > 0$,

$$T(g, \varepsilon) = \left\{ \tau \in \mathbb{R} : \sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|g(s+\tau) - g(s)\|^p \mathrm{d}s \right)^{\frac{1}{p}} < \varepsilon \right\}$$

is relatively dense in $\mathbb{R}$.

(ii) ($S^p$-normality) A function $g \in L^p_{\text{loc}}(\mathbb{R}; \mathbb{X})$ is said $S^p$-normal if for any sequence $\{\alpha_n'\} \subset \mathbb{R}$, and there is a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that

$$\sup_{t \in \mathbb{R}} \left( \int_t^{t+1} \|g(s+\alpha_n) - g(s+\alpha_m)\|^p \mathrm{d}s \right)^{\frac{1}{p}} \to 0 \ (n, m \to \infty).$$

**Lemma 2.3.** [15] *A function $g$ is $S^p$-almost periodic on $\mathbb{R}$ if and only if $g$ is $S^p$-normal on $\mathbb{R}$.*

**Definition 2.9.** [14,19]

(i) A function $g \in L^p_{\text{loc}}(\mathbb{T}; \mathbb{X})$ is $S^p$-almost periodic on $\mathbb{T}$ if given $\varepsilon > 0$, the $\varepsilon$-translation set of $g$

$$T(g, \varepsilon) = \left\{ \tau \in \Pi : \|g(\cdot+\tau) - g(\cdot)\|_{S^p} < \varepsilon \right\}$$

is a relatively dense set in $\Pi$. The space of all these functions is denoted by $S^p AP(\mathbb{T}; \mathbb{X})$ with norm $\|\cdot\|_{S^p}$.

(ii) ($S^p$-normality on $\mathbb{T}$) Let $\Pi$ be the translation set of $\mathbb{T}$. A function $g \in L^p_{\text{loc}}(\mathbb{T}; \mathbb{X})$ is $S^p$-normal on $\Pi$ if for any sequence $\{\alpha_n'\} \subset \Pi$, there is a subsequence $\{\alpha_n\} \subset \{\alpha_n'\}$ such that

$$\|g(\cdot+\alpha_n) - g(\cdot+\alpha_m)\|_{S^p} \to 0 \ (n, m \to \infty).$$

**Lemma 2.4.** [14] *A function $g : \mathbb{T} \to \mathbb{X}$ is $S^p$-almost periodic on $\mathbb{T}$ if and only if $g$ is $S^p$-normal on $\Pi$.*

## 3 Main results

We first extend the necessity of Proposition 1.1 to $S^p$-almost periodic functions.

**Theorem 3.1.** *For any $g \in S^p AP(\mathbb{T}; \mathbb{X})$, there is $\hat{g} \in S^p AP(\mathbb{R}; \mathbb{X})$ such that $\hat{g}|_{\mathbb{T}} = g$.*

**Proof.** Let

$$\hat{g}(t) = \begin{cases} g(t), & t \in \mathbb{T} \\ g(t_{ij}), & t \in (t_{ij}, \sigma(t_{ij})), \end{cases} \tag{3.1}$$

where $t_{ij}$ is given for $\mathbb{T}$ in Proposition 2.1. For $t \in \mathbb{R} \backslash \mathbb{T}$, there is a right-scattered point $t_{ij}$ such that $t_{ij} < t < \sigma(t_{ij})$, and notice that $\sigma(t_{ij}) \leqslant t_{ij} + \mathcal{K}$, then we have

$$\begin{aligned} \sup_{t \in \mathbb{R} \backslash \mathbb{T}} \frac{1}{\mathcal{K}} \int_t^{t+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s &\leqslant \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{t_{ij}}^{\sigma(t_{ij})+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s \\ &= \sup_{t_{ij}} \frac{1}{\mathcal{K}} \left\{ \left( \int_{t_{ij}}^{\sigma(t_{ij})} + \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}} \right) \|\hat{g}(s)\|^p \mathrm{d}s \right\} \end{aligned}$$

$$\leqslant \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{t_{ij}}^{\sigma(t_{ij})} \|\hat{g}(s)\|^p \mathrm{d}s + \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s$$
$$= \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{t_{ij}}^{\sigma(t_{ij})} \|g(s)\|^p \Delta s + \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}} \|g(s)\|^p \Delta s$$
$$\leqslant \sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|g(s)\|^p \Delta s + \sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|g(s)\|^p \Delta s$$
$$= 2\|g\|_{S^p}^p < \infty.$$

Obviously,

$$\sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s = \sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|g(s)\|^p \Delta s = \|g\|_{S^p}^p < \infty.$$

Since

$$\|\hat{g}\|_{S_{\mathcal{K}}^p}^p = \sup_{t\in\mathbb{R}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s = \max\left\{\sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s,\ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s)\|^p \mathrm{d}s\right\},$$

we obtain

$$\|\hat{g}\|_{S_{\mathcal{K}}^p}^p \leqslant 2\|g\|_{S^p}^p. \tag{3.2}$$

Thus, $\hat{g} \in BS^p(\mathbb{X})$. Next we will show $\hat{g} \in S^p AP(\mathbb{R};\mathbb{X})$. For $\varepsilon > 0$, letting $\tau \in T(g,\varepsilon)$, we derive

$$\begin{aligned}\|\hat{g}(\cdot+\tau) - \hat{g}(\cdot)\|_{S_{\mathcal{K}}^p}^p &= \sup_{t\in\mathbb{R}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s\\ &= \max\left\{\sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s,\ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s\right\}.\end{aligned}$$

Since $\tau \in T(g,\varepsilon)$, by the same argument of the proof of (3.2), we have

$$\begin{aligned}&\sup_{t\in\mathbb{R}\setminus\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s\\
&\leqslant \sup_{t_{ij}} \frac{1}{\mathcal{K}}\left\{\left(\int_{t_{ij}}^{\sigma(t_{ij})} + \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}}\right)\|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s\right\}\\
&\leqslant \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{t_{ij}}^{\sigma(t_{ij})} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s + \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}} \|\hat{g}(s+\tau) - \hat{g}(s)\|^p \mathrm{d}s\\
&= \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{t_{ij}}^{\sigma(t_{ij})} \|g(s+\tau) - g(s)\|^p \Delta s + \sup_{t_{ij}} \frac{1}{\mathcal{K}} \int_{\sigma(t_{ij})}^{\sigma(t_{ij})+\mathcal{K}} \|g(s+\tau) - g(s)\|^p \Delta s\\
&\leqslant \sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|g(s+\tau) - g(s)\|^p \Delta s + \sup_{t\in\mathbb{T}} \frac{1}{\mathcal{K}} \int_{t}^{t+\mathcal{K}} \|g(s+\tau) - g(s)\|^p \Delta s\\
&< 2\varepsilon^p.\end{aligned}$$

Obviously,

$$\sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}\|\hat{g}(s+\tau)-\hat{g}(s)\|^p\mathrm{d}s=\sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}\|g(s+\tau)-g(s)\|^p\Delta s<\varepsilon^p.$$

Thus,

$$\|\hat{g}(\cdot+\tau)-\hat{g}(\cdot)\|_{S^p_{\mathcal{K}}}^p<2\varepsilon^p,$$

which means that $T(\hat{g},\varepsilon)$ is relatively dense in $\Pi$ by Lemma 2.1, and then relatively dense in $\mathbb{R}$. Therefore, we have $\hat{g}\in S^p\mathrm{AP}(\mathbb{R};\mathbb{X})$. □

The following example shows that the converse of Theorem 3.1 is not true in general.

**Example 3.1.** Let

$$g(t)=\begin{cases}\sin t, & t\in(n\pi,(n+1)\pi);\\ |n|, & t=n\pi.\end{cases}$$

$\mathbb{T}=\bigcup_{n\in\mathbb{Z}}[2n\pi,(2n+1)\pi]$. It is easy to see that $\Pi=2\pi\mathbb{Z}$ and

$$\sup_{t\in\mathbb{R}}\frac{1}{2\pi}\int_t^{t+2\pi}\|g(s)\|^p\mathrm{d}s\leqslant 1, \tag{3.3}$$

$$\sup_{t\in\mathbb{R}}\frac{1}{2\pi}\int_t^{t+2\pi}\|g(s+2k\pi)-g(s)\|^p\mathrm{d}s=0,\ (k\in\mathbb{Z}). \tag{3.4}$$

By (3.3), we have $g\in BS^p(\mathbb{R};\mathbb{R})$, and by (3.4), we can obtain $g\in S^p\mathrm{AP}(\mathbb{R};\mathbb{R})$. Let $\bar{g}=g\,|_{\mathbb{T}}$,

$$\begin{aligned}\|\bar{g}\|_{S^p}^p&=\sup_{t\in\mathbb{T}}\frac{1}{2\pi}\int_t^{t+2\pi}\|g(s)\|^p\Delta s\\&\geqslant\sup_{n\in\mathbb{Z}}\frac{1}{2\pi}\int_{2n\pi}^{(2n+2)\pi}\|g(s)\|^p\Delta s\\&\geqslant\sup_{n\in\mathbb{Z}}\frac{1}{2\pi}\left(\int_{[2n\pi,(2n+2)\pi]_{\mathbb{T}}}\|g(s)\|^p\mathrm{d}s+|2n+1|^p\pi\right)\\&\geqslant\sup_{n\in\mathbb{Z}}\frac{|2n+1|^p}{2}=+\infty,\end{aligned}$$

which implies $\bar{g}\notin BS^p(\mathbb{T};\mathbb{R})$, and then $\bar{g}\notin S^p\mathrm{AP}(\mathbb{T};\mathbb{R})$.

To study the converse of Theorem 3.1, we introduce the concept of minor translation set of $\mathbb{T}$ and the normality on minor translation set of $\mathbb{T}$.

**Definition 3.1.** Let $\Pi=\mathcal{K}\mathbb{Z}$ be the translation set of a time scale $\mathbb{T}$, and we say $\Pi_1=p\mathcal{K}\mathbb{Z}$ $(p\in\mathbb{N}^+)$ the minor translation sets of $\mathbb{T}$.

**Definition 3.2.** Let $\Pi_1\subset\Pi$ be a minor translation set of $\mathbb{T}$. A function $g:\mathbb{T}\to\mathbb{X}$ is said to be normal on $\Pi_1$ if for any sequence $\{\alpha_n'\}\subset\Pi_1$, and there is a subsequence $\{\alpha_n\}\subset\{\alpha_n'\}$ such that $\|g(t+\alpha_n)-g(t+\alpha_m)\|\to 0$ $(n,m\to\infty)$ uniformly for $t\in\mathbb{T}$.

Then we have the following result.

**Theorem 3.2.** *A function $g : \mathbb{T} \to \mathbb{X}$ is normal on $\Pi = \mathcal{K}\mathbb{Z}$ if and only if $g$ is normal on $\Pi_1$, where $\Pi_1 = p\mathcal{K}\mathbb{Z}$ is a minor translation set of $\mathbb{T}$.*

**Proof.** If $g$ is normal on $\Pi$ and $\Pi_1 \subset \Pi$, it is obvious that $g$ is normal on $\Pi_1$.

On the other hand, if $g$ is normal on $\Pi_1$, for $\{\alpha''_n\} \subset \Pi$, we have the decomposition of its element $\alpha''_n = \beta''_n + \tau''_n$ with $\beta''_n \in \Pi_1$, $\quad 0 \leqslant \tau''_n \leqslant (p-1)\mathcal{K}$. It is easy to see that $\{\tau''_n\}$ is bounded and finite, then there exists a convergent subsequence $\{\tau_n\}$ satisfying $\tau_n = q\mathcal{K}$, $0 \leqslant q \leqslant (p-1)$. So we can choose such a subsequence $\{\alpha'_n\} \subset \{\alpha''_n\}$ satisfying $\alpha'_n = \beta'_n + q\mathcal{K}$, where $\beta'_n \in \Pi_1$. Since $g$ is normal on $\Pi_1$, there is a subsequence $\{\beta_n\} \subset \{\beta'_n\}$ such that $\{g(t+\beta_n)\}$ converges uniformly for $t \in \mathbb{T}$. Then we can choose such a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ satisfying $\alpha_n = \beta_n + q\mathcal{K}$, and $\{g(t+\alpha_n)\} = \{g(t+q\mathcal{K}+\beta_n)\}$ converges uniformly for $t \in \mathbb{T}$. That is, $\{g(t+\alpha_n)\}$ converges uniformly for $t \in \mathbb{T}$, which means that $g$ is normal on $\Pi$. □

**Definition 3.3.** ($S^p$-normality on $\mathbb{T}$) Let $\Pi_1 \subset \Pi$ be a minor translation sets of $\mathbb{T}$. A function $g \in L^p_{\mathrm{loc}}(\mathbb{T};\mathbb{X})$ is $S^p$-normal on $\Pi_1$ if for any sequence $\{\alpha'_n\} \subset \Pi_1$, there is a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$\|g(\cdot+\alpha_n) - g(\cdot+\alpha_m)\|_{S^p} \to 0 \ (n, m \to \infty).$$

By using the same argument proof of Theorem 3.2, we can prove the corresponding result for $S^p$-almost periodic function, and we omit the details.

**Theorem 3.3.** *A function $g : \mathbb{T} \to \mathbb{X}$ is $S^p$-normal on $\Pi$ if and only if $g$ is $S^p$-normal on $\Pi_1$, where $\Pi_1 = p\mathcal{K}\mathbb{Z}$ is a minor translation set of $\mathbb{T}$.*

For convenience, let $\mathbb{T}' = \mathrm{cl}\{\text{all right-scattered points of } \mathbb{T}\}$. Then $\mathbb{T}'$ is a time scale. Let $\Pi' = \mathcal{K}'\mathbb{Z}$ be its translation set. Let $f \in S^p AP(\mathbb{R};\mathbb{X})$ and denote $\bar{f} = f|_{\mathbb{T}}$, $\check{f} = f|_{\mathbb{T}'}$.

**Lemma 3.1.** $\Pi$ *is a minor translation set of* $\mathbb{T}'$.

**Proof.** By Proposition 2.1, we have $\Pi = \mathcal{K}\mathbb{Z} \subset \Pi' = \mathcal{K}'\mathbb{Z}$, and then $\mathcal{K} \in \Pi'$. This implies that $\mathcal{K} = p\mathcal{K}'$ for some positive integer $p$, that is, $\Pi$ is a minor translation set of $\mathbb{T}'$. □

Now we are in the position to present the following results, where the first part gives a sufficient condition to ensure the converse of Theorem 3.1.

**Theorem 3.4.** *Let $f \in S^p AP(\mathbb{R};\mathbb{X})$ and $\mathbb{T} \neq \mathbb{R}$ be invariant under translations.*
(i) *If $\check{f} \in AP(\mathbb{T}';\mathbb{X})$, then $\bar{f} \in S^p AP(\mathbb{T};\mathbb{X})$.*
(ii) *Suppose $[\omega, \omega+\mathcal{K})_{\mathbb{T}} \cap \mathbb{T}'(\omega \in \mathbb{T})$ has finite elements. If $\bar{f} \in S^p AP(\mathbb{T};\mathbb{X})$, then $\check{f} \in AP(\mathbb{T}';\mathbb{X})$.*

**Proof.** (i) Since $\check{f} \in AP(\mathbb{T}';\mathbb{X})$, $\check{f}$ is normal on $\Pi'$ by Lemma 2.2 and is normal on $\Pi$ by Theorem 3.2. For $\{\alpha''_n\} \subset \Pi$, and we can extract $\{\alpha'_n\} \subset \{\alpha''_n\}$ such that

$$\|\check{f}(t+\alpha'_n) - \check{f}(t+\alpha')\| \to 0 \ (n', m' \to \infty) \text{ uniformly for } t \in \mathbb{T}'. \tag{3.5}$$

Since $f \in S^p AP(\mathbb{R};\mathbb{X})$, it is normal on $\Pi$ by Lemma 2.4, and we can extract $\{\alpha_n\} \subset \{\alpha'_n\}$ such that

$$\|f(\cdot+\alpha_n) - f(\cdot+\alpha_m)\|_{S^p} \to 0 \ (n, m \to \infty). \tag{3.6}$$

By (3.5) and (3.6), we have

$$
\begin{aligned}
\|\bar f(\cdot+\alpha_n)-\bar f(\cdot+\alpha_m)\|_{S^p}^p &= \sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}\|\bar f(s+\alpha_n)-\bar f(s+\alpha_m)\|^p\Delta s\\
&= \sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\Bigg(\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}\|f(s+\alpha_n)-f(s+\alpha_m)\|^p\mathrm{d}s+\sum_{t_{ij}\in\mathbb{T}'\cap[t,t+\mathcal{K}]}\|f(t_{ij}+\alpha_n)\\
&\quad -f(t_{ij}+\alpha_m)\|^p\mu(t_{ij})\Bigg)\\
&\leqslant \sup_{t\in\mathbb{R}}\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}\|f(s+\alpha_n)-f(s+\alpha_m)\|^p\mathrm{d}s+\sup_{t\in\mathbb{T}'}\|f(t+\alpha_n)\\
&\quad -f(t+\alpha_m)\|^p\\
&= \|f(\cdot+\alpha_n)-f(\cdot+\alpha_m)\|_{S^p}+\sup_{t\in\mathbb{T}'}\|\check f(t+\alpha_n)-\check f(t+\alpha_m)\|^p\\
&\quad \to\ 0\ (n,m\to\infty).
\end{aligned}
$$

It shows that $\bar f:\mathbb{T}\to\mathbb{X}$ is $S^p$-normal on $\Pi$, by Lemma 2.4, we have $\bar f\in S^pAP(\mathbb{T};\mathbb{X})$.

(ii) Since $[\omega,\omega+\mathcal{K})\cap\mathbb{T}'$ $(\omega\in\mathbb{T})$ has finite elements, for these right-scattered point $t_i$, we have $\mu(t_i)>0$ with minimum $h>0$. Suppose that $\check f\notin AP(\mathbb{T}';\mathbb{X})$. Then $\check f$ is not normal on $\Pi'$ and is not normal on $\Pi$ by Theorem 3.2. Hence, there is a sequence $\{\beta_n\}\subset\Pi$ and $\varepsilon_0>0$, for any $N>0$, there are $n_0',m_0'>N$ and $t_0'\in\mathbb{T}'$ such that $\|\check f(t_0'+\beta_{n_0'})-\check f(t_0'+\beta_{m_0'})\|>\varepsilon_0$. Since $\bar f\in S^pAP(\mathbb{T};\mathbb{X})$, by Lemma 2.4, we can extract $\{\alpha_n\}\subset\{\beta_n\}$ such that

$$\|\bar f(\cdot+\alpha_n)-\bar f(\cdot+\alpha_m)\|_{S^p}\to 0\ (n,m\to\infty).$$

Choose $N$ large enough such that for $n,m>N$,

$$\|\bar f(\cdot+\alpha_n)-\bar f(\cdot+\alpha_m)\|_{S^p}^p<\frac{\varepsilon_0^p h}{2\mathcal{K}}. \tag{3.7}$$

On the other hand, there exist $n_0,m_0>N$ and $t_0\in\mathbb{T}'$ such that

$$\|\check f\left(t_0+\alpha_{n_0}\right)-\check f\left(t_0+\alpha_{m_0}\right)\|>\varepsilon_0. \tag{3.8}$$

By (3.8), we have

$$
\begin{aligned}
\|\bar f\left(\cdot+\alpha_{n_0}\right)-\bar f\left(\cdot+\alpha_{m_0}\right)\|_{S^p}^p &= \sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\int_t^{t+\mathcal{K}}\|\bar f\left(s+\alpha_{n_0}\right)-\bar f\left(s+\alpha_{m_0}\right)\|^p\Delta s\\
&= \sup_{t\in\mathbb{T}}\frac{1}{\mathcal{K}}\Bigg(\int_{[t,t+\mathcal{K}]_{\mathbb{T}}}\|f\left(s+\alpha_{n_0}\right)-f\left(s+\alpha_{m_0}\right)\|^p\mathrm{d}s+\sum_{t_{ij}\in[t,t+\mathcal{K}]_{\mathbb{T}}\cap\mathbb{T}'}\|f\left(t_{ij}+\alpha_{n_0}\right)\\
&\quad -f\left(t_{ij}+\alpha_{m_0}\right)\|^p\mu(t_{ij})\Bigg)\\
&\geqslant \frac{1}{\mathcal{K}}\Bigg(\int_{[t_0,t_0+\mathcal{K}]_{\mathbb{T}}}\|f\left(s+\alpha_{n_0}\right)-f\left(s+\alpha_{m_0}\right)\|^p\mathrm{d}s+\|\check f\left(t_0+\alpha_{n_0}\right)-\check f\left(t_0+\alpha_{m_0}\right)\|^p\mu(t_0)\Bigg)\\
&\geqslant \|\check f\left(t_0+\alpha_{n_0}\right)-\check f\left(t_0+\alpha_{m_0}\right)\|^p\frac{\mu(t_0)}{\mathcal{K}}\geqslant\frac{\varepsilon_0^p h}{\mathcal{K}}>\frac{\varepsilon_0^p h}{2\mathcal{K}}.
\end{aligned}
$$

This contradicts (3.7). Thus, $\check f\in AP(\mathbb{T}';\mathbb{X})$. □

**Remark 3.1.** The following example shows that the condition "the interval $[\omega, \omega + \mathcal{K})_{\mathbb{T}} \cap \mathbb{T}'$ has finite elements" in Theorem 3.4 is necessary.

**Example 3.2.** Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$g(t) = \begin{cases} n, & t \in \left[10^n \cdot l + \frac{1}{2^{10^n}}, 10^n \cdot l + \frac{1}{2^{10^n - 1}}\right), (l \text{ is odd}, \ n \geqslant 2); \\ 1, & \text{otherwise}. \end{cases}$$

Let $\mathbb{T} = \left\{k, k + \frac{1}{2^n}\right\}, (k \in \mathbb{Z}, n \in \mathbb{N}^+)$. Then $\Pi = \mathbb{Z}$, $\mathbb{T} \backslash \mathbb{Z}$ is the set of all right-scattered points of $\mathbb{T}$, and $\mathbb{T} = \mathbb{T}' = \mathrm{cl}(\mathbb{T} \backslash \mathbb{Z})$. Let $\bar{g} = g|_{\mathbb{T}}$ and $\hat{g}$ be the extension of $\bar{g}$ as (3.1), and one has $g(t) = \hat{g}(t), (t \in \mathbb{R})$. It is easy to see $\bar{g} = g|_{\mathbb{T}'} = \breve{g}$ since $\mathbb{T} = \mathbb{T}'$. We will prove that $\bar{g} \in S^1AP(\mathbb{T}; \mathbb{R})$, but $\bar{g} \notin AP(\mathbb{T}; \mathbb{R})$ and $\breve{g} \notin AP(\mathbb{T}'; \mathbb{R})$.

Indeed, if we have

$$\|\bar{g}(\cdot + m \cdot 10^n) - \bar{g}(\cdot)\|_{S^1} = \sup_{t \in \mathbb{T}} \int_t^{t+1} |g(s + m \cdot 10^n) - g(s)| \Delta s \to 0 \ (n \to \infty) \tag{3.9}$$

uniformly for $m \in \mathbb{Z}$, then for any $\varepsilon > 0$, there exists $N > 0$ such that for any $n > N$,

$$\sup_{t \in \mathbb{T}} \int_t^{t+1} |g(s + m \cdot 10^n) - g(s)| \Delta s < \varepsilon \ \text{ uniformly for } \ m \in \mathbb{Z}.$$

This implies that: for a fixed $n > N$, $\{m \cdot 10^n\}_{m \in \mathbb{Z}}$ is relatively dense in $\mathbb{Z}$ and $T(\bar{g}, \varepsilon)$ is relatively dense in $\Pi$; thus, $\bar{g} \in S^1AP(\mathbb{T}; \mathbb{R})$. By Theorem 3.1, we have $g \in S^1AP(\mathbb{R}; \mathbb{R})$. On the other hand, by the construction of $g$, we obtain $g$ is unbounded on $\mathbb{T}$ and on $\mathbb{T}'$, which means $\bar{g} \notin AP(\mathbb{T}; \mathbb{R})$ and $\breve{g} \notin AP(\mathbb{T}'; \mathbb{R})$.

Now we need only to prove that (3.9) holds. Let $t \in \mathbb{T}$, if there is not a number $10^k \cdot h + \frac{1}{2^{10^{k_1}}} \in [t, t+1]$, $(10 \nmid h)$ for some positive integers $k, k_1$ and integer $h$, we have $g(s + m \cdot 10^n) = g(s) = 1$ for all $s \in [t, t+1]$, and $\int_t^{t+1} |g(s + m \cdot 10^n) - g(s)| \Delta s = 0$. It means that the supremum of $\int_t^{t+1} |g(s + m \cdot 10^n) - g(s)| \Delta s$ will be obtained when $10^k \cdot h + \frac{1}{2^{10^{n_1}}} \in [t, t+1]$ for some positive integers $k, n_1$ and integer $h$. So we have three cases: $t = 10^k \cdot h$, $t = 10^k \cdot h + \frac{1}{2^{n_1}}$, or $t = 10^k \cdot h - \frac{1}{2^{n_1}}$ for some positive integer $n_1$ and integer $h$.

Case 1: $t = 10^k \cdot h$. We have the following cases:

(i) If $k > n$, we have

$$\begin{aligned} \int_{10^k \cdot h}^{10^k \cdot h + 1} |g(s + m \cdot 10^n) - g(s)| \Delta s &= \left| g\left(10^k \cdot h + m \cdot 10^n + \frac{1}{2^{10^n}}\right) - g\left(10^k \cdot h + \frac{1}{2^{10^n}}\right) \right| \cdot \frac{1}{2^{10^n}} \\ &\quad + \left| g\left(10^k \cdot h + m \cdot 10^n + \frac{1}{2^{10^k}}\right) - g\left(10^k \cdot h + \frac{1}{2^{10^k}}\right) \right| \cdot \frac{1}{2^{10^k}} \\ &= \begin{cases} 0, & \text{if } h, m \text{ are even}; \\ \frac{n-1}{2^{10^n}}, & \text{if } h \text{ is even and } m \text{ is odd}; \\ \frac{k-1}{2^{10^k}}, & \text{if } h \text{ is odd and } m \text{ is even}; \\ \frac{n-1}{2^{10^n}} + \frac{k-1}{2^{10^k}}, & \text{if } h, m \text{ are odd}. \end{cases} \end{aligned}$$

(ii) If $k = n$, we have the following cases:

(a) There exists $k_1 \in \mathbb{N}^+$ such that $10^{k_1} | (h + m)$, and we have

$$\int_{10^k\cdot h}^{10^k\cdot h+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right) - g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right| \cdot \frac{1}{2^{10^n}}$$

$$+ \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^{n+k_1}}}\right) - g\left(10^n\cdot h+\frac{1}{2^{10^{n+k_1}}}\right)\right| \cdot \frac{1}{2^{10^{n+k_1}}}$$

$$= \begin{cases} 0, & \text{if } m, \dfrac{h+m}{10^{k_1}} \text{ are even;} \\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd and } \dfrac{h+m}{10^{k_1}} \text{ is even;} \\ \dfrac{n+k_1-1}{2^{10^{n+k_1}}}, & \text{if } m \text{ is even and } \dfrac{h+m}{10^{k_1}} \text{ is odd;} \\ \dfrac{n-1}{2^{10^n}} + \dfrac{n+k_1-1}{2^{10^{n+k_1}}}, & \text{if } m, \dfrac{h+m}{10^{k_1}} \text{ are odd.} \end{cases}$$

(b) If $10\nmid(h+m)$, we have

$$\int_{10^k\cdot h}^{10^k\cdot h+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right) - g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right| \cdot \frac{1}{2^{10^n}}$$

$$= \begin{cases} 0, & \text{if } m \text{ is odd;} \\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is even.} \end{cases}$$

(iii) If $k < n$, we have

$$\int_{10^k\cdot h}^{10^k\cdot h+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^k\cdot h+m\cdot 10^n+\frac{1}{2^{10^k}}\right) - g\left(10^k\cdot h+\frac{1}{2^{10^k}}\right)\right| \cdot \frac{1}{2^{10^k}} = 0.$$

Case 2: $t = 10^k\cdot h + \frac{1}{2^{n_1}}$. We have the following cases:

(i) If $k > n$, we have the following cases:

(a) If $n_1 < 10^n < 10^k$, then

$$\int_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = 0,$$

since $g(s+m\cdot 10^n) = g(s) = 1$ for $s \in \left[10^k\cdot h+\frac{1}{2^{n_1}}, 10^k\cdot h+\frac{1}{2^{n_1}}+1\right]$.

(b) If $10^n \leqslant n_1 < 10^k$, then

$$\int_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s$$

$$= \left| g\left(10^k\cdot h+m\cdot 10^n+\frac{1}{2^{10^n}}\right) - g\left(10^k\cdot h+\frac{1}{2^{10^n}}\right)\right| \cdot \frac{1}{2^{10^n}}$$

$$= \begin{cases} 0, & \text{if } m \text{ is even;} \\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd.} \end{cases}$$

(c) If $10^n < 10^k \leqslant n_1$, then

$$\int\limits_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^k\cdot h+m\cdot 10^n+\frac{1}{2^{10^n}}\right)-g\left(10^k\cdot h+\frac{1}{2^{10^n}}\right)\right|\cdot\frac{1}{2^{10^n}}$$
$$+\left| g\left(10^k\cdot h+m\cdot 10^n+\frac{1}{2^{10^k}}\right)-g\left(10^k\cdot h+\frac{1}{2^{10^k}}\right)\right|\cdot\frac{1}{2^{10^k}}$$
$$=\begin{cases} 0, & \text{if } h, m \text{ are even;}\\ \dfrac{n-1}{2^{10^n}}, & \text{if } h \text{ is odd and } m \text{ is even;}\\ \dfrac{k-1}{2^{10^k}}, & \text{if } h \text{ is even and } m \text{ is odd;}\\ \dfrac{n-1}{2^{10^n}}+\dfrac{k-1}{2^{10^k}}, & \text{if } h, m \text{ are odd.}\end{cases}$$

(ii) If $k = n$, we have the following cases:

(a) If $10^k = 10^n > n_1$, then

$$\int\limits_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = 0,$$

since $g(s+m\cdot 10^n) = g(s)$ for $s \in \left[10^k\cdot h+\frac{1}{2^{n_1}}, 10^k\cdot h+\frac{1}{2^{n_1}}+1\right]$.

(b) If $10^k = 10^n = n_1$, then

$$\int\limits_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right)-g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right|\cdot\frac{1}{2^{10^n}}$$
$$=\begin{cases} 0, & \text{if } m \text{ is even;}\\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd.}\end{cases}$$

(c) If $10^k = 10^n < n_1 < 10^{n+1}$ or $10^k = 10^n < 10^{n+1} \leqslant n_1$, but $10 \nmid (h+m)$, we have

$$\int\limits_{10^n\cdot h+\frac{1}{2^{n_1}}}^{10^n\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right)-g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right|\cdot\frac{1}{2^{10^n}}$$
$$=\begin{cases} 0, & \text{if } m \text{ is even;}\\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd.}\end{cases}$$

(d) If $10^k = 10^n < 10^{n+k_1} \leqslant n_1 < 10^{n+k_1+1}$ and $10^{k_2}|(h+m)$, $10^{k_2+1}\nmid(h+m)$ for some $k_1, k_2 \in \mathbb{N}^+$, $k_1 < k_2$, we have

$$\int\limits_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^n\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right)-g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right|\cdot\frac{1}{2^{10^n}}$$
$$=\begin{cases} 0, & \text{if } m \text{ is even;}\\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd.}\end{cases}$$

(e) If $10^k = 10^n < 10^{n+k_1} \leqslant n_1 < 10^{n+k_1+1}$ and $10^{k_2}|(h+m)$, $10^{k_2+1}\nmid(h+m)$ for some $k_1, k_2 \in \mathbb{N}^+$, $k_1 \geqslant k_2$, we have

$$\int_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^n\cdot h+\frac{1}{2^{n_1}}+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^n}}\right) - g\left(10^n\cdot h+\frac{1}{2^{10^n}}\right)\right| \cdot \frac{1}{2^{10^n}}$$
$$+\left| g\left(10^n\cdot(h+m)+\frac{1}{2^{10^{n+k_2}}}\right) - g\left(10^n\cdot h+\frac{1}{2^{10^{n+k_2}}}\right)\right| \cdot \frac{1}{2^{10^{n+k_2}}}$$
$$=\begin{cases} 0, & \text{if } m, \dfrac{h+m}{10^{k_2}} \text{ are even;} \\ \dfrac{n-1}{2^{10^n}}, & \text{if } m \text{ is odd and } \dfrac{h+m}{10^{k_2}} \text{ is even;} \\ \dfrac{n+k_2-1}{2^{10^{n+k_2}}}, & \text{if } m \text{ is even and } \dfrac{h+m}{10^{k_2}} \text{ is odd;} \\ \dfrac{n-1}{2^{10^n}}+\dfrac{n+k_2-1}{2^{10^{n+k_2}}}, & \text{if } m, \dfrac{h+m}{10^{k_2}} \text{ are odd.} \end{cases}$$

(iii) If $k < n$, we have

$$\int_{10^k\cdot h+\frac{1}{2^{n_1}}}^{10^k\cdot h+\frac{1}{2^{n_1}}+1} |g(s+10^n)-g(s)|\Delta s = 0,$$

since $g(s+10^n) = g(s)$ for $s \in \left[10^k\cdot h+\frac{1}{2^{n_1}}, 10^k\cdot h+\frac{1}{2^{n_1}}+1\right]$.

Case 3: $t = 10^k\cdot h - \frac{1}{2^{n_1}}$. We can discuss similarly as Case: 2 $t = 10^k\cdot h+\frac{1}{2^{n_1}}$ and we can obtain

$$\sup_{t\in\mathbb{T}} \int_t^{t+1} |g(s+m\cdot 10^n)-g(s)|\Delta s = \frac{n-1}{2^{10^n}} + \frac{n}{2^{10^{n+1}}} \to 0 \ (n\to\infty).$$

Overall, we obtain (3.9) for three cases.

To complete this work, we give the following results for any two time scales $\mathbb{T}_1$ and $\mathbb{T}_2$ from Theorem 3.4.

**Corollary 3.1.** *Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two invariant time scales under translations, and $\Pi_1 = \mathcal{K}_1\mathbb{Z}$ be translation set of $\mathbb{T}_1$. Any interval $[\omega, \omega+\mathcal{K}_1)_{\mathbb{T}_1} \cap \mathbb{T}_1'$ ($\omega \in \mathbb{T}_1$) has finite elements. Suppose $\mathbb{T}_1 \subset \mathbb{T}_2$, $f \in S^p\mathrm{AP}(\mathbb{T}_2;\mathbb{X})$. Then $f|_{\mathbb{T}_1} \in S^p\mathrm{AP}(\mathbb{T}_1;\mathbb{X})$ if and only if $f|_{\mathbb{T}_1'} \in \mathrm{AP}(\mathbb{T}_1';\mathbb{X})$.*

**Proof.** By Theorem 3.1, we know that there is $g \in S^p\mathrm{AP}(\mathbb{R};\mathbb{X})$ such that $g|_{\mathbb{T}_2} = f$. Then we have $g|_{\mathbb{T}_1} = f|_{\mathbb{T}_1}$ and $g|_{\mathbb{T}_1'} = f|_{\mathbb{T}_1'}$ since $\mathbb{T}_1 \subset \mathbb{T}_2$.

If $g|_{\mathbb{T}_1} = f|_{\mathbb{T}_1} \in S^p\mathrm{AP}(\mathbb{T}_1;\mathbb{X})$, by Theorem 3.4 (ii), we have $f|_{\mathbb{T}_1'} = g|_{\mathbb{T}_1'} \in \mathrm{AP}(\mathbb{T}_1';\mathbb{X})$. On the other hand, if $g|_{\mathbb{T}_1'} = f|_{\mathbb{T}_1'} \in \mathrm{AP}(\mathbb{T}_1';\mathbb{X})$, by Theorem 3.4 (i), we have $f|_{\mathbb{T}_1} = g|_{\mathbb{T}_1} \in S^p\mathrm{AP}(\mathbb{T}_1;\mathbb{X})$. □

**Acknowledgement:** The authors would like to thank the anonymous referees for their valuable comments, which can improve the presentation and the quality of the earlier version of this article.

**Funding information**: This article was supported by the National Natural Science Foundation of China (No. 11971329).

**Author contributions:** Hao Yang: Conceptualization, Writing-Original Draft; Hong-Xu Li: Funding Acquisition, Supervision, Resources, Writing-Review and Editing.

**Conflict of interest**: The authors state no conflict of interest.

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