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Research Article

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On regular subgroup functors of finite groups

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Abstract: A subgroup functor τ is said Φ-regular if for all primitive groups G, whenever $H \in \tau(G)$ is a p-subgroup and N is a minimal normal subgroup of G, then $|G : N_G(H \cap N)| = p^d$ for some integer d. In this article, we investigate groups in which some primary subgroups are τ-subgroups for a Φ-regular subgroup functor τ, and we obtain new criteria for the supersolubility or p-nilpotency of a group.

Keywords: finite group, subgroup functor, primary subgroup, $\mathfrak{F}$-hypercenter

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1 Introduction

Groups in this article are all finite. The reader is referred to [1,2] for unexplained notations and terminologies. Let G be a group and p a prime. Then G_p denotes a Sylow p-subgroup of G and $\pi(G)$ is the set of all primes dividing $|G|$. Recall that τ is said a subgroup functor if $1 \in \tau(G)$ is a subset of subgroups of G and $\theta(\tau(G)) = \tau(\theta(G))$ for every isomorphism $\theta : G \longrightarrow G^*$. If $H \in \tau(G)$, then H is said a τ-subgroup of G [2]. The sets of all Sylow subgroups, maximal subgroups, subnormal subgroups, permutable subgroups, and so on are some known subgroup functors. Investigating the influence of subgroup functor, the notions of regular and Φ-regular subgroup functors are proposed in [2], and some interesting results were obtained in [2,3].

Definition 1.1. [2] Let τ be a subgroup functor. Then τ is said:
(1) regular if for any group G, whenever $H \in \tau(G)$ is a p-subgroup and N is a minimal normal subgroup of G, then $|G : N_G(H \cap N)|$ is a power of p.
(2) Φ-regular if for any primitive group G, whenever $H \in \tau(G)$ is a p-subgroup and N is a minimal normal subgroup of G, then $|G : N_G(H \cap N)|$ is a power of p.

Let $\tau(G)$ be the set of all normal subgroups, permutable subgroups, or s-permutable subgroups. Then τ is regular. Recall that a *primitive group* is a group in which some maximal subgroups have trivial core. Clearly, if a subgroup functor is regular, then it must be Φ-regular, but the converse is not true in general. For example, let $\tau(G) = \{H | H \leq \Phi(G)\}$ for all group G. Then $\tau(G)$ is Φ-regular, but it is not regular. Generally, if τ_1 is regular and $\tau(G) = \{HK | H \leq \Phi(G), K \in \tau_1(G)$ and $HK = KH\}$ for all groups G, then τ is Φ-regular.

Let H/K be a chief factor of a group G and $\mathfrak{F}$ a class of groups. Recall that H/K is said $\mathfrak{F}$-central if $(H/K) \rtimes (G/C_G(H/K)) \in \mathfrak{F}$. The largest normal subgroup of G whose G-chief factors are all $\mathfrak{F}$-central is $\mathfrak{F}$-*hypercenter* and is denoted by $Z_{\mathfrak{F}}(G)$. Furthermore, the largest normal subgroup of G, in which all non-

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Frattini G-chief factors are $\mathfrak{F}$-central in G, is called the $\mathfrak{F}\Phi$-*hypercenter* (compared with [4]), and denoted by $Z_{\mathfrak{F}\Phi}(G)$. Let $\mathfrak{U}$ be the formation of all supersoluble groups. Then $Z_{\mathfrak{U}}(G)$ is the largest normal subgroup of G in which all G-chief factors are cyclic, and $Z_{\mathfrak{U}\Phi}(G)$ is the largest normal subgroup in which all non-Frattini G-chief factors are cyclic. By investigating on subgroups in which some primary subgroups (i.e., subgroups of prime-power order) are Φ-regular, we obtain the following result.

Theorem 1.2. *Let τ be a Φ-regular subgroup functor and $E \trianglelefteq G$. If for all $p \in \pi(E)$, there is a p-subgroup $D(p)$ with $|\Phi(G) \cap O_p(E)| < |D(p)| < |E_p|$ such that all subgroups of E with order $|D(p)|$ are contained in $\tau(G)$, then $E \leq Z_{\mathfrak{U}\Phi}(G)$.*

Fixing a prime p, we have

Theorem 1.3. *Let τ be a Φ-regular subgroup functor and $E \trianglelefteq G$. Let $p \in \pi(E)$ and assume that $N_G(E_p)$ is p-nilpotent. If there is a p-subgroup D with $|\Phi(G) \cap O_p(E)| < |D| < |E_p|$ such that all subgroups of E with order $|D|$ are contained in $\tau(G)$, then $E \leq Z_{\mathfrak{F}\Phi}(G)$, where $\mathfrak{F}$ is the formation of all p-nilpotent groups.*

2 Preliminary results

The following results are known, and we list them here as lemmas.

Lemma 2.1. [5, III, Lemma 3.3] *Let G be a group, $N \trianglelefteq G$ and $U \leq G$. If $N \leq \Phi(U)$, then $N \leq \Phi(G)$.*

Lemma 2.2. [6, Lemma 1.8.16] *Let N be a nilpotent normal subgroup of G. If $N \cap \Phi(G) = 1$, then N is complemented in G.*

Recall that the generalized Fitting subgroup of a group G is the largest normal quasinilpotent subgroup (see [7]). Let $F_p^*(G)/O_{p'}(G) = F^*(G/O_{p'}(G))$. A *formation* is a class of groups $\mathfrak{F}$ that is closed under subdirect products and epimorphic images, and that a formation $\mathfrak{F}$ is saturated when $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$, where $\Phi(G)$ denotes the Frattini subgroup of G. We have the following lemma.

Lemma 2.3. *Let $E \trianglelefteq G$. Assume that all G-chief factors H/K are cyclic whenever $H \leq F_p^*(E)$ and $p||H/K|$. Then all G-chief factors L/M are cyclic whenever $L \leq E$ and $p||L/M|$.*

Proof. Let G act on E by conjugation. By choosing $f(p) = \mathfrak{A}(p-1)$, the class of all abelian groups with exponents dividing $p-1$, it follows directly from [8, Corollary 4.3] that the lemma holds. □

The following result is a corollary of Lemma 2.3 and can also be found in [9].

Lemma 2.4. [9, Lemma 2.17] *Let E be a normal subgroup of G. If $F^*(E) \leq Z_{\mathfrak{U}}(G)$, then $E \leq Z_{\mathfrak{U}}(G)$.*

As we know, $Z_{\mathfrak{U}\Phi}(G) = Z_{\mathfrak{U}}(G)$ does not hold in general (see Example 1.2 in [4]). But we have the following result.

Lemma 2.5. *Let X be a normal subgroup of G. If $X/\Phi(X) \leq Z_{\mathfrak{U}}(G/\Phi(X))$, then $X \leq Z_{\mathfrak{U}}(G)$.*

Proof. If $\Phi(X) = 1$, the assertion is clear. Assume that $\Phi(X) \neq 1$ and let $N \leq \Phi(X)$ be a minimal normal subgroup of G. Then the hypotheses hold on $(G/N, X/N)$, and so, $X/N \leq Z_{\mathfrak{U}}(G/N)$ by induction on $|X|$. Thus, it is enough to show that N is cyclic. If X possesses a minimal normal subgroup L different to N, then $NL/L \leq \Phi(X/L)$, and hence, $(X/L)/\Phi(X/L) \leq Z_{\mathfrak{U}}((G/L)/\Phi(X/L))$. Thus, $X/L \leq Z_{\mathfrak{U}}(G/L)$ by induction on $|X|$.

It follows that $N \cong NL/L$ is cyclic. Assume that N is the unique minimal normal subgroup of G contained in X. Since $X/\Phi(X) \le Z_{\mathfrak{U}}(G/\Phi(X))$, X is supersoluble by [9, Lemma 2.16]. Let p be the largest prime divisor of $|X|$. Then $X_p \trianglelefteq X$ and $N \le X_p$. If $\Phi(X_p) \neq 1$ then $N \le \Phi(X_p)$. Since $X_p/N \le X/N \le Z_{\mathfrak{U}}(G/N)$, it follows directly from [10, Lemma 2.8] that $X_p \le Z_{\mathfrak{U}}(G)$, and hence, N is cyclic. Assume that $\Phi(X_p) = 1$. Then by Maschke's theorem, N has an X-invariant complement in X_p. This is nonsense for $N \le \Phi(X)$. Hence, $X \le Z_{\mathfrak{U}}(G)$, and the lemma holds. □

Lemma 2.6. *Let $E \trianglelefteq G$. If $E \cap \Phi(G) = \Phi(E)$, then $E \le Z_{\mathfrak{U}\Phi}(G)$ if and only if $E \le Z_{\mathfrak{U}}(G)$.*

Proof. The "if" part is clear, and we only prove the "only if" part. Assume $E \cap \Phi(G) = \Phi(E) \neq 1$ and let $N \le \Phi(E)$ be a minimal normal subgroup of G. Then $E/N \cap \Phi(G/N) = \Phi(E/N)$ and $E/N \le Z_{\mathfrak{U}\Phi}(G/N)$ hold. By induction on $|E|$, we can obtain that $E/N \le Z_{\mathfrak{U}}(G/N)$, and it follows that $E \le Z_{\mathfrak{U}}(G)$ by Lemma 2.5.

Assume $E \cap \Phi(G) = \Phi(E) = 1$. Let N be a minimal normal subgroup of G contained in E. Then N is cyclic since $N \le E \le Z_{\mathfrak{U}\Phi}(G)$ and $N \nleq \Phi(G)$. Clearly, E is soluble. By Lemma 2.2, we see that $F(E) = F^*(E)$ is the product of some minimal normal subgroups of G. Hence, $F(E) \le Z_{\mathfrak{U}}(G)$ and so $E \le Z_{\mathfrak{U}}(G)$ by Lemma 2.4. This completes the proof of the lemma. □

By [4, Proposition 5.2] and [11, Theorem A], we have the following lemmas.

Lemma 2.7. [4] *Let $\mathfrak{F}$ be a saturated formation and E a normal subgroup with $G/E \in \mathfrak{F}$. If $E \le Z_{\mathfrak{F}\Phi}(G)$, then $G \in \mathfrak{F}$.*

Lemma 2.8. [11] *Let E be a normal subgroup of G. If $F(E) \le Z_{\mathfrak{U}\Phi}(G)$ and E is soluble, then $E \le Z_{\mathfrak{U}\Phi}(G)$.*

3 Proofs of Theorems 1.2 and 1.3

Lemma 3.1. *Let all minimal normal subgroups of a group G be cyclic and $\Phi(G) = 1$. Then G is supersoluble.*

Proof. Since $\Phi(G) = 1$ and all minimal normal subgroups of G are cyclic, $F(G) = \mathrm{Soc}(G)$ is the product of all minimal normal subgroups of G. Assume that $p||F(G)|$ and let $P = O_p(G)$. By [1, A (10.6)], there exists a subgroup M of G such that $G = F(G) \rtimes M$. Let $F(G) = N_1 \times N_2 \times \cdots \times N_r$. Since $F(G)$ is abelian, $F(G) \le C_G(F(G))$ and $G/C_G(F(G)) = MC_G(F(G))/C_G(F(G)) \cong M/C_M(F(G)) = M/\bigcap_{i=1}^{r} C_M(N_i)$ is isomorphic to a subgroup of the group $M/C_M(N_1) \times M/C_M(N_2) \times \cdots \times M/C_M(N_r)$, that is abelian because all N_i are cyclic. Let $C = C_M(F(G))$. We have that C is a normal subgroup of M. Assume that $C \neq 1$, and let R be a minimal normal subgroup of M contained in C. Then R is normal in $F(G)M = G$, and so R is a minimal normal subgroup of G. By hypothesis, R is cyclic, and so $R \le F(G) \cap M = 1$. This contradiction shows that $C = 1$ and, since $M \cong G/F(G)$ is abelian and $F(G)$ is the product of cyclic minimal normal subgroups of G and G is supersoluble. □

Proof of Theorem 1.2. Assume that the theorem does not hold, and let G be a counterexample of minimal order. We prove the theorem via the following steps.

(1) *All minimal normal subgroups of G are contained in E.*

Let N be a minimal normal subgroup of G. Suppose that $N \nleq E$ and H/N in EN/N is a subgroup of order $|D(p)|$ for some prime p. Then $|H \cap E| = |D(p)|$ and hence $H \cap E \in \tau(G)$. Let $\theta : G \longrightarrow G/N$ be the natural epimorphism. Then $H/N = (H \cap E)N/N = \theta(H \cap E) \in \theta(\tau(G)) = \tau(\theta(G)) = \tau(G/N)$. $|\Phi(G/N) \cap O_p(EN/N)| < |D(p)| < |E_pN/N|$ is clear. Thus, the hypotheses hold on G/N and hence $EN/N \le Z_{\mathfrak{U}\Phi}(G/N)$ since $|G/N| < |G|$. It follows that $E \le Z_{\mathfrak{U}\Phi}(G)$, a contradiction, and hence (1) holds.

(2) $\Phi(G) = 1$.

Assume that $\Phi(G) \neq 1$, and let $N \le \Phi(G)$ be a minimal normal subgroup of G and $q||N|$. In $\overline{G} = G/N$, choose $|\overline{D}(p)| = |D(p)|$ if $p \neq q$ and $|\overline{D}(p)| = |D(p)/N|$ if $p = q$. Then $|\Phi(\overline{G}) \cap O_p(\overline{E})| < |\overline{D}(p)| < |\overline{E}_p|$, and by a similar argument as (1), we can obtain that all subgroups in $\overline{E}$ of order $|\overline{D}(p)|$ are τ-subgroups. Thus, the hypotheses hold on G/N, and so, $E/N \leq Z_{\mathfrak{U}\Phi}(G/N)$. Hence, $E \leq Z_{\mathfrak{U}\Phi}(G)$. This contradicts the choice of G and (2) holds.

(3) *If N is a minimal normal subgroup of G, then N is cyclic.*

Since $\Phi(G) = 1$ by (2), there is a maximal subgroup M such that $G = NM$. Then G/M_G is primitive and NM_G/M_G is a minimal normal subgroup of G/M_G. Let p be a prime divisor of N and H be a subgroup of order $D(p)$. Then, $H \in \tau(G)$, and hence, $HM_G/M_G \in \tau(G/M_G)$ by choosing θ to be the natural epimorphism of G to G/M_G. Thus, $|G/M_G : N_{G/M_G}((HM_G \cap NM_G)/M_G)|$ is a p-number. It follows that $G/M_G = (G/M_G)_p N_{G/M_G}((HM_G \cap NM_G)/M_G)$, where $(G/M_G)_p$ is a Sylow p-subgroup of G/M_G containing $(HM_G \cap NM_G)/M_G$. Then $((HM_G \cap NM_G)/M_G)^{G/M_G} \leq (G/M_G)_p$ is a p-subgroup and so $(HM_G \cap NM_G)/M_G \leq O_p(G/M_G)$. Since $H \cap N \neq 1$, we see that $(NM_G/M_G) \cap O_p(G/M_G) \neq 1$ and so $N \cong NM_G/M_G$ is a p-group. Furthermore, we have $N \cap M = 1$ and $G = N \rtimes M$. Thus, $G_p = N \rtimes M_p$. If $|D(p)| < |N|$, then let H_1 be a normal subgroup of G_p contained in N with $|H_1| = |D(p)|$ and let $H_2 = 1$; if $|D(p)| \geq |N|$, then let H_1 be a normal subgroup of G_p and maximal in N, and let H_2 be a subgroup of M_p of order $|D(p)|/|H_1|$. Let $H = H_1H_2$. Then $|H| = |D(p)|$ and $H_1 = H \cap N \trianglelefteq G_p$. Moreover, it holds that $HM_G \cap NM_G = (HM_G \cap N)M_G \leq (HM_p \cap N)M_G = (H_1M_p \cap N)M_G = H_1M_G$. Since $H_1M_G \leq HM_G \cap NM_G$ is clear, $HM_G \cap NM_G = H_1M_G \trianglelefteq G_pM_G$. Since H is Φ-regular in G, we see that $|G : N_G(HM_G \cap NM_G)| = |G/M_G : N_{G/M_G}((HM_G \cap NM_G)/M_G)|$ is a p-number. Thus, $HM_G \cap NM_G \trianglelefteq G = G_pN_G(HM_G \cap NM_G)$. It follows that $H_1 = N \cap H_1M_G \trianglelefteq G$. But N is minimal normal in G, so $H_1 = 1$ and N is cyclic.

(4) *The final contradiction*

By Lemma 3.1 and step (3), we see that G is supersoluble, and hence, $E \leq Z_{\mathfrak{U}\Phi}(G)$, which contradicts the choice of G. This is the final contradiction and the theorem holds. □

Proof of Theorem 1.3. Assume that the theorem is not true, and let G be a counterexample of minimal order. Since the hypotheses still hold on $G/\Phi(G)$ and $G/O_{p'}(E)$, $\Phi(G) = O_{p'}(E) = 1$ by the minimality of G. Let N be a minimal normal subgroup of G. If $N \nleq E$, then $EN/N \leq Z_{\mathfrak{F}\Phi}(G/N)$. It follows that $E \leq Z_{\mathfrak{F}\Phi}(G)$, a contradiction. Thus, $N \leq E$. Choose H to be a subgroup of order $|D|$, $H \cap N$ is maximal in N and is normal in a Sylow p-subgroup of G. Let M be a complement of N in G. Then $|G : N_G(HM_G \cap NM_G)| = |G/M_G : N_{G/M_G}((HM_G \cap NM_G)/M_G)|$ is a p-number. It follows that $HM_G \cap NM_G \trianglelefteq G = G_pN_G(HM_G \cap NM_G)$ and $H_1 = N \cap HM_G \trianglelefteq G$. But N is minimal normal in G, so $H \cap N = 1$ and N is cyclic. By Lemma 3.1, E is supersoluble and $E_p \trianglelefteq E$. It follows that $E_p \trianglelefteq G$ and $G = N_G(E_p)$ is p-nilpotent by the hypotheses. Thus, $E \leq Z_{\mathfrak{F}\Phi}(G)$, where $\mathfrak{F}$ is the formation of all p-nilpotent groups. □

4 Some remarks, examples and applications

1. The following example shows that $|\Phi(G) \cap O_p(E)| < |D(p)|$ is necessary in Theorem 1.2.

Example 4.1. Let $A = \langle a | a^{p^2} = 1\rangle$ be a cyclic group of order p^2 and $B = \langle b | b^{q^2} = 1\rangle$ a cyclic group of order q^2 and $q \nmid (p-1)$. Assume that $K = A_1 \times A_2 \times \cdots \times A_q$, where $A_i \cong A$ and B acts on K with $(a_1, a_2, \cdots, a_q)^b = (a_q, a_1, \cdots, a_{q-1})$. Then $G = K \rtimes B$ is non-supersoluble. Let $\tau(G) = \{H | H \leq \Phi(G)\}$. Then τ is Φ-regular, and all minimal subgroups of G are τ-subgroups since they are contained in $\Phi(G)$. Let $E = G$. Then $E \leq Z_{\mathfrak{U}\Phi}(G)$ does not hold.

A similar example shows that $|\Phi(G) \cap O_p(E)| < |D|$ is necessary in Theorem 1.3.

Example 4.2. Let $A = \langle a | a^{p^2} = 1\rangle$ be a cyclic group of order p^2, $B = \langle b | b^q = 1\rangle$ a cyclic group of order q and $M = A \wr B$ be the regular wreath product of A and B. Let $T = \langle x | x^{p^2} = 1\rangle$. Assume that $R = M_1 \times M_2 \times \cdots \times M_p$, where $M_i \cong M$ and T acts on R with $(a_1, a_2, \cdots, a_p)^x = (a_p, a_1, \cdots, a_{p-1})$. Then $G = R \rtimes T$ is non-p-nilpotent. Let $\tau(G) = \{H | H \leq \Phi(G)\}$. Then τ is Φ-regular, and all minimal subgroups of order p of G are τ-subgroups since they are contained in $\Phi(G)$. Let $E = G$. Then $E \le Z_{\mathfrak{F}\Phi}(G)$ does not hold, where $\mathfrak{F}$ is the formation of all p-nilpotent group.

2. In Theorem 1.2, $E \le Z_{\mathfrak{U}}(G)$ does not hold in general (by choosing $\overline{E}$ to be the $\mathfrak{U}\Phi$-hypercenter of $\overline{G} = G/N$ in [4, Example 1.2]). But, by applying Lemma 2.6, we see that $E \le Z_{\mathfrak{U}}(G)$ if $\Phi(G) \cap E = \Phi(E)$.

3. By Theorems 1.2 and 1.3, we have the following results.

Theorem 4.3. *Let E be a normal subgroup of a group G and G/E supersoluble. Assume that τ is a Φ-regular subgroup functor. If for every prime divisor p of $|E|$, there is a p-subgroup $D(p)$ with $|\Phi(G) \cap O_p(E)| < |D(p)| < |E_p|$ such that all subgroups in E of order $|D(p)|$ are contained in $\tau(G)$, then G is supersoluble.*

Proof. By Theorem 1.2, we see that $E \le Z_{\mathfrak{U}\Phi}(G)$, and it follows directly from Lemma 2.7 that G is supersoluble. □

Theorem 4.4. *Let E be a soluble normal subgroup of a group G with G/E supersoluble. Assume that τ is a Φ-regular subgroup functor. If for every prime divisor p of $F^*(E)$, there is a p-subgroup $D(p)$ with $|\Phi(G) \cap O_p(F^*(E))| < |D(p)| < |(F^*(E))_p|$ such that all subgroups in E of order $|D(p)|$ are contained in $\tau(G)$, then G is supersoluble.*

Proof. By Theorem 1.2 $F^*(E) \le Z_{\mathfrak{U}\Phi}(G)$, and hence, $E \le Z_{\mathfrak{U}\Phi}(G)$ by Lemma 2.8. It follows from Lemma 2.7 that G is supersoluble. □

Theorem 4.5. *Let E be a normal subgroup of a group G with p-nilpotent quotient and τ a Φ-regular subgroup functor. Let P be a Sylow p subgroup of E. If $N_G(P)$ is p-nilpotent, and there is a p-subgroup D with $|\Phi(G) \cap O_p(E)| < |D| < |P|$ such that all subgroups in E of order $|D|$ are contained in $\tau(G)$, then G is p-nilpotent.*

Proof. By Theorem 1.3, $E \le Z_{\mathfrak{F}\Phi}(G)$, where $\mathfrak{F}$ is the formation of all p-nilpotent groups, and hence, G is p-nilpotent by Lemma 2.7. □

Theorem 4.6. *Let E be a normal subgroup of a group G with p-nilpotent quotient and τ a Φ-regular subgroup functor. Let P be a Sylow p subgroup of $F_p^*(E)$. If $N_G(P)$ is p-nilpotent, and there is a p-subgroup D with $|\Phi(G) \cap O_p(E)| < |D| < |P|$ such that all subgroups in $F_p^*(E)$ of order $|D|$ are contained in $\tau(G)$, then G is p-nilpotent.*

Proof. Let $\mathfrak{F}$ be the formation of all p-nilpotent groups. By Theorem 1.3, $F_p^*(E) \le Z_{\mathfrak{F}\Phi}(G)$ is p-nilpotent and hence $F_p^*(E) = O_{p'}(E)P$. Since $N_G(P)$ is p-nilpotent, by the Frattini Argument, $G = O_{p'}(E)N_G(P)$ is p-nilpotent and the theorem holds. □

4. It can be verified that in Theorems 1.3, 4.5 and 4.6 "$N_G(P)$ is p-nilpotent" is not necessary if p is the minimal divisor of $|G|$.

5. Let τ be a group functor. If $\tau(G)$ is the set of all normal subgroups; permutable subgroups; s-permutable subgroups; c-normal subgroups; c-semipermtable subgroups (if G is soluble) or $\mathfrak{Z}$-permutable subgroups of G, then the subgroup functor τ is Φ-regular (compared with [2]). By choosing $D(p)$ to be a maximal subgroup of a Sylow p-subgroup of E, our main results uniform a lot of known results (see some theorems in [13–21] and so on).

6. An s-permutably embedded subgroup (compared with [22], etc) is subgroup in which every Sylow subgroup is also a Sylow subgroup of some s-permutable subgroup. Let $\tau(G) = \{H | H$ is s-permutably embedded in $G\}$. Then τ is not regular in the universe of all finite groups, but it is regular in the universe of all soluble groups (compared with [2, III, Example 1.9]). Thus, many works about s-permutably embedded subgroups can be generalized by regular subgroup functor.

7. We also observed that in the literature, there were some group functors that do not correspond to a regular subgroup functor. A subgroup H of G is said $\mathcal{M}$-supplemented in G if there exists a subgroup B of G such that $G = HB$ and $H_1B < G$ for every maximal subgroup H_1 of H (compared with [23], etc.). Let $\tau(G) = \{H|H$ is $\mathcal{M}$-supplemented in $G\}$. Since $H \in \tau(G)$ whenever H is a minimal supplement of a proper normal subgroup of G, $\tau(G)$ does not correspond to a regular subgroup functor in general. For example, Let $G = S_5$ be the symmetric group of degree 5, and $H = \langle(1\ 2\ 3\ 4)\rangle$. Let $B = A_5$. Then, $G = HB$ and $H_1B < G$ for all proper subgroups H_1 of H. Thus, H is $\mathcal{M}$-supplemented in G. Clearly, $|G : N_G(H \cap A_5|$ is not a 2-number.

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