Research Article

Wanlin Wang and Pengfei Guo*

Finite groups whose maximal subgroups of even order are MSN-groups

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Abstract: A finite group G is called an MSN-group if all maximal subgroups of the Sylow subgroups of G are subnormal in G. In this article, we investigate the structure of finite groups G such that G is a non-MSN-group of even order in which every maximal subgroup of even order is an MSN-group. In addition, we determine the minimal simple groups all of whose second maximal subgroups are MSN-groups.

Keywords: MSN-groups, maximal subgroups, solvable groups, simple groups

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1 Introduction

In this article, only finite groups are considered, and our notation is standard.

The study of the structure of groups that have some kind of property has attracted much attention in group theory, and many meaningful results about this topic have been obtained. Subnormal subgroups play a key role in the study of the structure of groups. Srinivasan [1] proved that the supersolvability of a group G has the property that the maximal subgroups of Sylow subgroups are normal, and proved that it has the Sylow tower property for some ordering of the primes in $\pi(G)$ but not necessarily supersolvable if the maximal subgroups of Sylow subgroups are subnormal. Guo and Guo [2] called groups in which all maximal subgroups of the Sylow subgroups are subnormal MSN-groups and investigated the structure of minimal non-MSN-groups (those groups that are not MSN-groups but whose proper subgroups are all MSN-groups).

Recently, Meng et al. [3] studied the structure of groups all of whose maximal subgroups of even order are MS-groups. (A group G is called an MS-group if all minimal subgroups of G permute with every Sylow subgroup of G.) Meng and Lu [4] investigated the structure of groups in which all maximal subgroups of even order are supersolvable groups and determined the non-abelian simple groups all of whose second maximal subgroups of even order are supersolvable groups.

The aim of this article is to investigate groups all of whose maximal subgroups of even order are MSN-groups. Furthermore, we determine the minimal simple groups all of whose second maximal subgroups are MSN-groups.

*** Corresponding author: Pengfei Guo,** School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, P. R. China, e-mail: guopf999@163.com
Wanlin Wang: School of Mathematics and Statistics, Hainan Normal University, Haikou 571158, P. R. China, e-mail: wangwanlin888@163.com

2 Preliminary results

We collect some lemmas that will be frequently used in the sequel.

Lemma 2.1. [1, Theorem 3] *Let G be an MSN-group. Then G has the Sylow tower property for some ordering of the primes in $\pi(G)$, and hence, G is solvable.*

Lemma 2.2. *A subgroup of an MSN-group must be an MSN-group.*

Proof. By the definition of MSN-groups and the property of subnormal subgroups, the lemma is true. □

Lemma 2.3. [5, Theorem 2.7] *Let G be a group. Then the following conditions are equivalent:*
(*i*) *Maximal subgroups of Sylow subgroups of G are subnormal in G;*
(*ii*) $G = H \rtimes K$, *where* (1) *H is a nilpotent normal Hall subgroup of G, and K is a group whose Sylow subgroups are cyclic and the maximal subgroups of its Sylow subgroups are normal in K;* (2) *a generator x of any Sylow p-subgroup of K induces an automorphism of order 1 or p on H.*

Remark 2.4. [2, Remark] In Lemma 2.3, the normality of maximal subgroups of p-Sylow subgroups of K can be extended to G.

Lemma 2.5. [2, Lemma 2.8] *Let G be a solvable minimal non-MSN-group. Then $|G| = p^a q^b$, where p and q are distinct primes and at least one of a and b is more than* 1.

Lemma 2.6. [6, Theorem 10.1.4] *If a group G has a fixed-point-free automorphism of order* 2, *then G is abelian.*

Lemma 2.7. [7, Corollary 1] *Every minimal simple group is isomorphic to one of the following groups:*
(*i*) $\mathrm{PSL}(3,3)$;
(*ii*) *The Suzuki group* $\mathrm{Sz}(2^q)$, *where q is an odd prime;*
(*iii*) $\mathrm{PSL}(2,p)$, *where p is a prime with $p > 3$ and $p^2 \not\equiv 1 (\mathrm{mod}\ 5)$;*
(*iv*) $\mathrm{PSL}(2,2^q)$, *where q is a prime;*
(*v*) $\mathrm{PSL}(2,3^q)$, *where q is an odd prime.*

3 Main results

In this section, we give the classification of solvable groups all of whose maximal subgroups of even order are MSN-groups and give the structure of non-abelian simple groups all of whose maximal subgroups of even order are MSN-groups. Furthermore, we determine minimal simple groups all of whose second maximal subgroups are MSN-groups.

Theorem 3.1. *Let G be a solvable non-MSN-group of even order. If all maximal subgroups of G of even order are MSN-groups, then $|\pi(G)| \le 3$.*

Proof. Let $\pi(G) = \{p_1, p_2, \dots, p_s\}$ with $p_1 = 2$ and $\{P_1, P_2, \dots, P_s\}$ be a Sylow basis of G. If G is a minimal non-MSN-group, then $|\pi(G)| = 2$ by Lemma 2.5. So the conclusion holds.

Now we assume that G is not a minimal non-MSN-group. By hypothesis, G possesses a maximal subgroup M of odd order which is not an MSN-group. Without loss of generality, let $M = P_2 \cdots P_s$.

Since M is not an MSN-group, then there exist a positive integer j and a maximal subgroup P^* of P_j such that P^* is not subnormal in M. Without loss of generality, we can let $j = s$. If $s \ge 4$, then

$P_1P_iP_s$ $(i = 2, \ldots, s-1)$ is a proper subgroup of G of even order, and hence, it is an MSN-group by hypothesis and Lemma 2.2. If P_s is non-cyclic, then P_s is normal in $P_1P_iP_s$ by Lemma 2.3, and so P_s is normal in $G = P_1P_2 \cdots P_s$. Therefore, P^* is subnormal in G, a contradiction. Hence, P_s is cyclic and P^* is normal in $P_1P_iP_s$ by Remark 2.4. So P^* is normal in $G = P_1P_2 \cdots P_s$, a contradiction. Thus, $|\pi(G)| \leq 3$. □

We first solve the case of the solvable non-MSN-group G having two prime divisors.

Theorem 3.2. *Let G be a solvable non-MSN-group with $\pi(G) = \{2, p\}$. Then all maximal subgroups of G of even order are MSN-groups if and only if G is a minimal non-MSN-group.*

Proof. Clearly, the maximal subgroup of G of odd order (if exists) is a Sylow subgroup, and so it is an MSN-group. By Lemma 2.2, the rest is clear. □

We next solve the case of the solvable non-MSN-group G having three prime divisors.

Theorem 3.3. *Let G be a solvable non-MSN-group of even order with $|\pi(G)| = 3$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$, and $R \in \mathrm{Syl}_2(G)$ with $p > q > r = 2$. Suppose that all maximal subgroups of G of even order are MSN-groups. Then one of the following statements holds:*

- (I) *$G = M \times R$, where M is a minimal non-MSN-group and R is a cyclic group of order 2;*
- (II) *$G = \langle a, b | a^p = b^{2q^n} = 1, b^{-1}ab = a^i \rangle$, where $i^{2q} \not\equiv 1 (\mathrm{mod}\ p)$, $i^{2q^2} \equiv 1 (\mathrm{mod}\ p)$, $n \geq 2$, $0 < i < p$;*
- (III) *$G = M \rtimes R = (P \rtimes Q) \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with P non-cyclic and $Q = \langle b \rangle$ of order q^n, $n \geq 2$, b induces an automorphism of order q^2 on P and R is a cyclic group of order 2;*
- (IV) *$G = M \rtimes R = (Q \rtimes P) \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with P and Q non-cyclic, $P\Phi(Q)$ is nilpotent, and R is a cyclic group of order 2;*
- (V) *$G = M \rtimes R = (Q \rtimes P) \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with Q non-cyclic and $P = \langle a \rangle$ of order p^m, $m \geq 2$, a induces an automorphism of order p^2 on Q, and R is a cyclic group of order 2;*
- (VI) *$G = M \rtimes R = (P \rtimes Q) \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with P and Q non-cyclic, $\Phi(P)Q$ is nilpotent, and R is a cyclic group of order 2;*
- (VII) *$G = M \rtimes R = PQ \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with P cyclic and Q non-cyclic, R is a cyclic group of order 2;*
- (VIII) *$G = M \rtimes R = (P \rtimes Q) \rtimes R$ with $1 < C_M(R) < M$, where M is a non-MSN-group with P of order p and Q non-cyclic, R is a cyclic group of order 2;*
- (IX) *$G = R \rtimes M = R \rtimes (P \rtimes Q)$, where $M = P \rtimes Q = \langle a, b | a^p = b^{q^n} = 1, b^{-1}ab = a^i \rangle$, $i^q \not\equiv 1 (\mathrm{mod}\ p)$, $i^{q^2} \equiv 1 (\mathrm{mod}\ p)$, $n \geq 2$, $0 < i < p$, PR is nilpotent and QR is a minimal non-abelian group, and R is elementary abelian with $|R| \geq 4$;*
- (X) *$G = R \rtimes M = R \rtimes (P \rtimes Q)$, where $M = P \rtimes Q = \langle a, b | a^p = b^{q^n} = 1, b^{-1}ab = a^i \rangle$, $i^q \not\equiv 1 (\mathrm{mod}\ p)$, $i^{q^2} \equiv 1 (\mathrm{mod}\ p)$, $n \geq 2$, $0 < i < p$, QR is nilpotent and PR is a minimal non-abelian group, and R is elementary abelian with $|R| \geq 4$;*
- (XI) *$G = R \rtimes M = R \rtimes (P \rtimes Q)$, where $M = P \rtimes Q = \langle a, b | a^p = b^{q^n} = 1, b^{-1}ab = a^i \rangle$, $i^q \not\equiv 1 (\mathrm{mod}\ p)$, $i^{q^2} \equiv 1 (\mathrm{mod}\ p)$, $n \geq 2$, $0 < i < p$, b induces an automorphism of order q on R and PR is non-nilpotent, and R is elementary abelian with $|R| \geq 4$;*
- (XII) *$G = R \rtimes M = R \rtimes PQ$, where M is a non-MSN-group with P cyclic and Q non-cyclic, QR is nilpotent and PR is a minimal non-abelian group, and R is elementary abelian with $|R| \geq 4$;*
- (XIII) *$G = R \rtimes M = R \rtimes (P \rtimes Q)$, where M is a non-MSN-group with P non-cyclic and $Q = \langle b \rangle$ of order q^n, $n \geq 2$, b induces an automorphism of order q^2 on P, PR is nilpotent and QR is a minimal non-abelian group, and R is elementary abelian with $|R| \geq 4$.*

Proof. Since $|\pi(G)| = 3$, G is not a minimal non-MSN-group by Lemma 2.5. Then there exists a maximal subgroup M of G of order odd such that M is not an MSN-group by hypothesis. The solvability of G implies that we can let $\pi(G) = \{p, q, r\}$ with $p > q > r = 2$ and $\{P, Q, R\}$ be a Sylow basis of G. Furthermore, we can let M be a Hall $2'$-subgroup of G and $G = MR$, where $M = PQ$.

By hypothesis and Lemma 2.2, both PR and QR are MSN-groups. We first prove that R is either of order 2 or elementary abelian from two cases as follows.

Case 1 R is cyclic.

Since PR and QR are MSN-groups, then $\Phi(R)$ is normal in PR and QR by Remark 2.4, and so $\Phi(R)$ is normal in G. If $\Phi(R) \neq 1$, then $M < M\Phi(R) < MR = G$, which contradicts that M is a maximal subgroup of G, so $\Phi(R) = 1$ and R is of order 2.

Case 2 R is non-cyclic.

Since PR and QR are MSN-groups, then R is normal in PR and QR by Lemma 2.3, and so R is normal in G. We have that $\Phi(R)$ is normal in G as $\Phi(R)$ char $R \trianglelefteq G$. If $\Phi(R) \neq 1$, then $M < M\Phi(R) < MR = G$, which contradicts that M is a maximal subgroup of G, so $\Phi(R) = 1$ and R is elementary abelian.

We next complete the rest of the proof as follows.

(1) R is of order 2.

Clearly, M is the normal 2-complement.

Suppose $C_M(R) = 1$. Then an automorphism of R acting on M is both of order 2 and fixed-point-free. It follows from Lemma 2.6 that M is abelian, and so M is an MSN-group, a contradiction. Thus, $C_M(R) > 1$.

If $C_M(R) = M$, then $MR = M \times R$, and M_1R is an MSN-group by hypothesis for any maximal subgroup M_1 of M. By Lemma 2.2, M_1 is an MSN-group, and so M is a minimal non-MSN-group. Therefore, G is of type (I).

Now we consider that $1 < C_M(R) < M$. Let $1 \trianglelefteq \cdots \trianglelefteq K \trianglelefteq PQ \trianglelefteq G$ be a chief series of G. Since G is solvable, then one of P and Q is contained in K. If $Q \leq K$, then KR is an MSN-group by hypothesis, and Q is either cyclic or normal in K by Lemma 2.3. If Q is normal in K, then Q is normal in G as Q char $K \trianglelefteq G$. Similarly, if $P \leq K$, then P is either cyclic or normal in G. Obviously, Q cannot be both cyclic and normal in G, so we discuss from the five cases as follows.

(1-a) Q is cyclic but not normal in G.

Clearly, $P \trianglelefteq G$. Then $P\Phi(Q)R$ is an MSN-group by hypothesis, and so $P\Phi(Q)$ is an MSN-group by Lemma 2.2. Similarly, since $\Phi(P)QR$ is contained in some maximal subgroup of G of even order, then $\Phi(P)Q$ is an MSN-group by Lemma 2.2.

Let $Q = \langle z \rangle$ with $|z| = q^n$, $n \geq 2$. If P is cyclic, then let $P = \langle a \rangle$ with $|a| = p^m$. As $\Phi(P)Q$ is an MSN-group, $\Phi(P)\Phi(Q)$ is nilpotnet by Remark 2.4. If $\Phi(P) \neq 1$, then $\Omega_1(P) \leq \Phi(P)$, and hence, $P\Phi(Q)$ is nilpotnet by a result in [8]. Therefore, $M = PQ$ is an MSN-group by Lemma 2.3, a contradiction. Thus, $\Phi(P) = 1$ and P is cyclic of order p. Since $1 < C_M(R) < M$ and G is non-abelian with all Sylow subgroups cyclic, we have that $G = P \rtimes (Q \times R)$ by [9, 10.1.10]. Let $Q \times R = \langle b \rangle$ and $b^{-1}ab = a^i$. It follows that $(b^{2q})^{-1}ab^{2q} = a^{i^{2q}} \neq a$ from $\langle b^{2q} \rangle$ is not subnormal in PQ. Therefore, $i^{2q} \not\equiv 1 (\mathrm{mod}\ p)$. Since $P\Phi(Q)$ is an MSN-group, $\langle b^{2q^2} \rangle$ is normal in $P\Phi(Q)$ by Remark 2.4. Hence, $(b^{2q^2})^{-1}ab^{2q^2} = a^{i^{2q^2}} = a$, $i^{2q^2} \equiv 1 (\mathrm{mod}\ p)$, and G is of type (II). If P is non-cyclic, using similar arguments as mentioned earlier, $\langle z^{q^2} \rangle$ is normal in $P\Phi(Q)$. Hence, G is of type (III).

(1-b) Q is non-cyclic and $Q \trianglelefteq G$.

Since $P\Phi(Q)R$ is contained in some maximal subgroup of G of even order, $P\Phi(Q)$ is an MSN-group by hypothesis and Lemma 2.2. Similarly, $\Phi(P)Q$ is an MSN-group. If P is non-cyclic, then $P\Phi(Q)$ is nilpotent by Lemma 2.3, so G is of type (IV). If P is cyclic, then by Lemma 2.3 let $P = \langle a \rangle$ with $|a| = p^m$ and $m \geq 2$. Since $\Phi(P)Q$ is an MSN-group, $\langle a^{p^2} \rangle$ is normal in $\Phi(P)Q$ by Remark 2.4. So G is of type (V).

(1-c) P is non-cyclic and $P \trianglelefteq G$.

If Q is cyclic, then G is of type (III) by the same arguments as in (1-a). If Q is non-cyclic, then $\Phi(P)QR$ is contained in some maximal subgroup of G of even order, so $\Phi(P)Q$ is an MSN-group by hypothesis and Lemma 2.2. Therefore, $\Phi(P)Q$ is nilpotent by Lemma 2.3, and G is of type (VI).

(1-d) P is cyclic but not noraml in G.

Clearly, Q is non-cyclic, so G is of type (VII).

(1-e) P is cyclic and $P \trianglelefteq G$.

If Q is cyclic, then G is of type (II) by the same arguments as in (1-a). If Q is non-cyclic, we have that P is of order p. Otherwise, $\Omega_1(P) \le \Phi(P)$ and $\Phi(P)Q$ is an MSN-group by hypothesis and Lemma 2.2. By Lemma 2.3, $\Phi(P)Q$ is nilpotent, and so PQ is nilpotent by a result in [8], a contradiction. Thus, G is of type (VIII).

(2) R is normal in G, and it is elementary abelian.

Assume that neither P nor Q is cyclic. Since PR and QR are MSN-groups, PR and QR are nilpotent by Lemma 2.3. Therefore, $MR = M \times R$. By hypothesis and Lemma 2.2, MR_1 is an MSN-group, where $R_1 < R$. By Lemma 2.2 again, M is an MSN-group, a contradiction. Hence, either P or Q is cyclic.

(2-a) Both P and Q are cyclic.

By hypothesis and Lemma 2.2, $P\Phi(Q)$ and $\Phi(P)Q$ are MSN-groups, and so PQ is a minimal non-MSN-group. Let $P = \langle a\rangle$ and $Q = \langle b\rangle$ with $|a| = p^m$ and $|b| = q^n$. Then by examining [2, Theorem 3.2], we conclude that $M = PQ = \langle a, b | a^p = b^{q^n} = 1, b^{-1}ab = a^i\rangle$, $i^q \not\equiv 1(\mathrm{mod}\ p)$, $i^{q^2} \equiv 1(\mathrm{mod}\ p)$, $n \ge 2$, $0 < i < p$. Clearly, PR and QR are not all nilpotent. If PR is nilpotent, then there does not exist a non-trivial subgroup R_1 of R such that $R_1Q = QR_1$. Otherwise, if $R_1Q = QR_1$, where $R_1 < R$, we have that R_1PQ is a proper subgroup of G of even order. Then R_1PQ is an MSN-group by hypothesis and Lemma 2.2, and so PQ is an MSN-group by Lemma 2.2, a contradiction. Since QR is an MSN-group, then $\Phi(Q)R$ is nilpotent by Remark 2.4. Thus, QR is a minimal non-abelian group, and G is of type (IX). If QR is nilpotent, then by a similar argument as earlier, there does not exist a non-trivial subgroup R_1 of R such that $R_1P = PR_1$. Therefore, PR is a minimal non-abelian group, and G is of type (X). If both PR and QR are non-nilpotent, then b induces an automorphism of order q on R by Lemma 2.3, and so G is of type (XI).

(2-b) P is cyclic and Q is non-cyclic.

Since PR and QR are MSN-groups, $\Phi(P)R$ and QR are nilpotent by Remark 2.4 and Lemma 2.3, respectively. By the similar arguments as in (2-a), there does not exist a non-trivial subgroup R_1 of R such that $R_1P = PR_1$. Hence, PR is a minimal non-abelian group since $\Phi(P)R$ is nilpotent. Therefore, G is of type (XII).

(2-c) P is non-cyclic and Q is cyclic.

Using similar arguments as in (2-b), $\Phi(Q)R$ and PR are nilpotent, and so P is normal in G. Then $P\Phi(Q)R$ is an MSN-group by hypothesis, so $P\Phi(Q)$ is an MSN-group by Lemma 2.2. Let $Q = \langle b\rangle$ with $|b| = q^n$, $n \ge 2$. Then $\langle b^{q^2}\rangle$ is normal in $P\Phi(Q)$ by Remark 2.4. By the similar arguments as in (2-a), there does not exist a non-trivial subgroup R_1 of R such that $R_1Q = QR_1$. We have that QR is a minimal non-abelian group since $\Phi(Q)R$ is nilpotent. Thus, G is of type (XIII). □

Corollary 3.4. *Let G be a solvable non-MSN-group of even order and suppose that all maximal subgroups of G of even order are MSN-groups. If $4 \| |G|$, then G is a minimal non-MSN-group or $|\pi(G)| = 3$ and G possesses a normal Sylow 2-subgroup which is elementary abelian.*

Proof. It is obvious by Theorems 3.2 and 3.3. □

Theorem 3.5. *Let G be a non-abelian simple group. If all maximal subgroups of G of even order are MSN-groups, then G is isomorphic to either A_5 or* PSL(2, 2^q), *where A_5 is the alternating group of degree 5 and $2^q - 1$ is square-free for an odd prime q.*

Proof. Let M be a maximal subgroup of G. If M is a group of odd order, then M is solvable by Feit-Thompson Theorem [10] on the solvability of group of odd order. If M is a group of even order, then M is an MSN-group by hypothesis. By applying Lemma 2.1, M is solvable. So G is a minimal simple group. Using similar arguments as the proof in [2, Theorem 3.1], the required result holds. □

Theorem 3.6. *Let G be a minimal simple group all of whose second maximal subgroups are MSN-groups. Then G is isomorphic to one of the following types*:

(*I*) A_5;

(*II*) *The Suzuki group* $\mathrm{Sz}(2^3)$;

(*III*) PSL(2, p), *where* p *is an odd prime with* $p > 5$, $5 \nmid p^2 - 1$, *only one of* $(p + 1)/4$ *and* $(p - 1)/4$ *is an odd prime if* $p \geq 13$, *and* $(p - 1)/2$ *is square-free or* r^2 *for an odd prime* r *if* $(p + 1)/4$ *is an odd prime*;
(*IV*) PSL(2, 2^q), *where* q *is an odd prime and* $2^q - 1$ *is square-free or* r^2 *for an odd prime* r;
(*V*) PSL(2, 3^q), *where* q *and* $(3^q + 1)/4$ *are odd primes*, $(3^q - 1)/2$ *is square-free or* r^2 *for an odd prime* r.

Proof. By Lemma 2.7, we have that G is isomorphic to one of the following simple groups.
(*i*) PSL(3, 3);
(*ii*) The Suzuki group Sz(2^q), where q is an odd prime;
(*iii*) PSL(2, p), where p is a prime with $p > 3$ and $p^2 \not\equiv 1 (\text{mod } 5)$;
(*iv*) PSL(2, 2^q), where q is a prime;
(*v*) PSL(2, 3^q), where q is an odd prime.

Case 1 $G \ncong$ PSL(3, 3).
Suppose $G \cong$ PSL(3, 3). Then G contains a maximal subgroup which is isomorphic to the Hesse group A of order $2^4 3^3$. By hypothesis, A is either an MSN-group or a minimal non-MSN-group. Clearly, all Sylow subgroups of A are non-cyclic, so at least one of the Sylow subgroups of A is normal in A by Lemma 2.3 and a result in [2, Theorem 3.2]. However, the Sylow subgroups of A are all non-normal. Hence $G \ncong$ PSL(3, 3).
Case 2 $G \cong$ Sz(2^3).
Suppose $G \cong$ Sz(2^q). Then by [11, Theorem 9], G has maximal subgroups: the Frobenius group M with a cyclic complement H of order $2^q - 1$ and kernel K of order 2^{2q}; the dihedral group D of order $2(2^q - 1)$ and the Frobenius group F of order $4(2^q \pm 2^{\frac{q+1}{2}} + 1)$. Clearly, the dihedral group D of order $2(2^q - 1)$ is an MSN-group. The Frobenius group M is either an MSN-group or a minimal non-MSN-group by hypothesis. If M is a minimal non-MSN-group, then $2^q - 1$ is p^l by Lemma 2.5 for an odd prime p and a positive integer l with $l > 1$. Since K is non-abelian, then $\Phi(K)H$ is an MSN-group with $\Phi(H) \neq 1$, and so $\Phi(K)\Phi(H) = \Phi(K) \times \Phi(H)$ by Remark 2.4, which contradicts the fact that M is a Frobenius group. Thus, M must be an MSN-group, and we have that $2^q - 1$ is square-free. Otherwise, by Remark 2.4, there exists a non-trivial maximal subgroup H_1 of the Sylow subgroups of H such that $KH_1 = K \times H_1$, which contradicts the fact M is a Frobenius group. By hypothesis, the Frobenius group F is either an MSN-group or a minimal non-MSN-group. Clearly, the Sylow 2-subgroup of F is neither cyclic nor normal, so F is a minimal non-MSN-group. By the similar arguments as given earlier, $2^q \pm 2^{\frac{q+1}{2}} + 1$ is r^t for an odd prime r and a positive integer t. If $t > 1$, then there exists a non-trivial subgroup F_1 of F such that the Sylow 2-subgroup of F_1 is neither cyclic nor normal, which contradicts that F is a minimal non-MSN-group. Therefore, $2^q \pm 2^{\frac{q+1}{2}} + 1$ are odd primes. By the computation, we easily have that at least one of $2^q + 2^{\frac{q+1}{2}} + 1$ and $2^q - 2^{\frac{q+1}{2}} + 1$ is a multiple of 5 for any odd prime q with $q \geq 5$, a contradiction. Thus, only odd prime 3 satisfies all of the aforementioned conditions, and so G is of type (II).
Case 3 $G \cong A_5$ or $G \cong$ PSL(2, p), where p is an odd prime with $p > 5$, $5 \nmid p^2 - 1$, only one of $(p + 1)/4$ and $(p - 1)/4$ is an odd prime if $p \geq 13$, and $(p - 1)/2$ is square-free or r^2 for an odd prime r if $(p + 1)/4$ is an odd prime.
Suppose $G \cong$ PSL(2, p). By [12, Corollary 2.2], G has maximal subgroups: the alternating group A_4 of degree 4 when $p \equiv \pm 3 (\text{mod } 8)$; the symmetric group S_4 of degree 4 when $p^2 \equiv 1 (\text{mod } 16)$; the dihedral groups of order $p \pm 1$; and the Frobenius group M with a cyclic complement H of order $(p - 1)/2$ and kernel K of order p. $G \cong A_5$ if $p = 5$, so G is of type (I). Obviously, the prime 7 satisfies the condition, so we only consider the case when $p \geq 13$. Furthermore, 4 must divide the order of either D_{p-1} or D_{p+1} if $p \geq 13$.
Suppose $p \geq 13$ and $4 \| |D_{p-1}|$. Clearly, the Sylow 2-subgroup of D_{p-1} is non-cyclic. Since D_{p-1} is not an MSN-group, we have that D_{p-1} is a minimal non-MSN-group by hypothesis. Note $2 \nmid (p - 1)/4$. Otherwise, there exists a maximal subgroup M_1 of D_{p-1} such that the Sylow 2-subgroup of M_1 is neither cyclic nor normal, which contradicts D_{p-1} is a minimal non-MSN-group. Then by examining [2, Theorem 3.2], we have that $(p - 1)/4$ is an odd prime, say q. Then $p + 1 = 2(2q + 1)$, and so D_{p+1} is an MSN-group, as desired. And $(p - 1)/2 = 2q$, and hence, the Frobenius group M of order $p(p - 1)/2$ is an MSN-group, as desired. Suppose $p \geq 13$ and $4 \| |D_{p+1}|$. By the similar arguments as given earlier, D_{p+1} must be a minimal non-

MSN-group and $2 \nmid (p+1)/4$. Then by examining [2, Theorem 3.2], we have that $(p+1)/4$ is an odd prime, say t. Then $p-1=2(2t-1)$, and so D_{p-1} is an MSN-group, as desired. And $(p-1)/2=2t-1$ is odd, using similar arguments as in Case 2, $(p-1)/2$ is square-free if M is an MSN-group. If M is a minimal non-MSN-group, then $(p-1)/2$ is r^m by Lemma 2.5 for an odd prime r and a positive integer m with $m>1$. We have $m=2$, if not, there exists a maximal subgroup H_1 of H such that KH_1 is an MSN-group with $\Phi(H_1) \neq 1$, and so $K\Phi(H_1) = K \times \Phi(H_1)$ by Remark 2.4, which contradicts the fact M is a Frobenius group. Now we have that G is of type (III).

Case 4 $G \cong A_5$ or $G \cong \mathrm{PSL}(2, 2^q)$, where q is an odd prime and $2^q - 1$ is square-free or r^2 for an odd prime r.

If $G \cong \mathrm{PSL}(2, 2^q)$, then by [12, Corollary 2.2], G has maximal subgroups: the dihedral groups of order $2(2^q \pm 1)$; the Frobenius group M with a cyclic complement H of order $2^q - 1$ and kernel K is elementary abelian of order 2^q; and the alternating group A_4 of degree 4 when $q=2$. Clearly, $G \cong A_5$ if $q=2$, so G is of type (I). Now we consider that $q>2$. By the similar arguments as in Case 2, 2^q-1 is square-free if M is an MSN-group. If M is a minimal non-MSN-group, then 2^q-1 is r^m by Lemma 2.5 for an odd prime r and a positive integer m with $m>1$. We prove that there does not exist a non-trivial subgroup K_1 of K such that $K_1 \trianglelefteq M$. If not, then K_1H is an MSN-group, and so $K_1\Phi(H) = K_1 \times \Phi(H)$ by Remark 2.4, which contradicts the fact M is a Frobenius group. Now we prove that $m=2$. If not, then by the similar arguments as the proof in Case 3, there exists a maximal subgroup H_1 of H with $\Phi(H_1) \neq 1$ such that $K\Phi(H_1) = K \times \Phi(H_1)$, which contradicts the fact M is a Frobenius group. Hence, G is of type (IV).

Case 5 $G \cong \mathrm{PSL}(2, 3^q)$, where q and $(3^q+1)/4$ are odd primes, $(3^q-1)/2$ is square-free or r^2 for an odd prime r.

If $G \cong \mathrm{PSL}(2, 3^q)$, then by [12, Corollary 2.2], G has maximal subgroups: the dihedral groups of order $3^q \pm 1$; the Frobenius group M with a cyclic complement H of order $(3^q-1)/2$ and kernel K is elementary abelian of order 3^q; and the alternating group A_4 of degree 4. Clearly, $4 | 3^q + 1$, and the dihedral group of order 3^q-1 is an MSN-group, so we only consider the Frobenius group M and the dihedral group D of order 3^q+1. Using similar arguments as in Case 4, $(3^q-1)/2$ is square-free if M is an MSN-group, and $(3^q-1)/2$ is r^2 for an odd prime r if M is a minimal non-MSN-group. Using similar arguments as in Case 3, D must be a minimal non-MSN-group and $2 \nmid (3^q+1)/4$. By examining [2, Theorem 3.2], $(3^q+1)/4$ is an odd prime. So G is of type (V). □

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References

[1] S. Srinivasan, *Two sufficient conditions for supersolvability of finite groups*, Israel J. Math. **35** (1980), no. 3, 210–214, DOI: https://doi.org/10.1007/BF02761191.

[2] P. F. Guo and X. Y. Guo, On minimal non-MSN-groups, Front. Math. China **6** (2011), no. 5, 847–854, DOI: https://doi.org/10.1007/s11464-011-0115-z.

[3] W. Meng, Y. Deng, and J. K. Lu, *Finite groups with S-quasinormal subgroups*, Comm. Algebra **49** (2021), no. 6, 1–9, DOI: https://doi.org/10.1080/00927872.2021.1876082.

[4] W. Meng and J. K. Lu, *Finite groups with supersolvable subgroups of even order*, Ric. Mat. (2021), DOI: https://doi.org/10.1007/s11587-021-00656-3.

[5] Q. H. Zhang, L. F. Wang, and P. F. Guo, *On the structure of some finite groups*, Southeast Asian Bull. Math. **30** (2006), 995–1002.

[6] D. Gorenstein, *Finite Groups*, New York: Harper & Row Publishers; 1968.

[7] J. G. Thompson, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. **74** (1968), no. 3, 383–437, DOI: https://doi.org/10.1090/s0002-9904-1968-11953-6.
[8] T. J. Laffey, A Lemma *on finite p-group and some consequences*, Math. Proc. Camb. Phil. Soc. **75** (1974), no. 2, 133–137, DOI: https://doi.org/10.1017/S0305004100048350.
[9] D. J. S. Robinson, *A Course in the Theory of Groups*, New York-Heidelberg-Berlin: Springer-Verlag, 1980.
[10] W. Feit and J. G. Thompson, *Solvability of groups of odd order*, Pacific J. Math. **13** (1963), no. 3, 775–1027, DOI: https://doi.org/10.2140/pjm.1963.13.775.
[11] M. Suzuki, *On a class of doubly transitive groups*, Ann. Math. **75** (1962), no. 1, 105–145, DOI: https://doi.org/10.2307/1970423.
[12] O. H. King, *The subgroup structure of finite classical groups in terms of geometric configurations*, in: B. S. Webb (eds.), Surveys in Combinatorics, London Math. Soc. Lecture Note Ser. vol. 327, Cambridge: Cambridge University Press, 2005, pp. 29–56.