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Research Article

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On modules related to McCoy modules

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Abstract: In this paper, we first investigate the relationships between the McCoy module and related modules based on their relationships in rings. After that, we improve some properties of McCoy modules and introduce ZPZC modules which extend the notion of McCoy modules. We observe the structure of ZPZC modules providing a number of examples of problems that arise naturally in the process. Finally, answers to some open questions related to the ZPZC condition are provided.

Keywords: McCoy ring, McCoy module, duo module, ZPZC module, ZPZC ring

MSC 2020: 13C13, 16D70, 16D80, 16U99

Dedicated to Ilwon Kang for his retirement from Kyung Hee University.

1 Introduction

All rings in this paper are associative with identity, and modules are unitary right modules unless we specify otherwise. Let R be a ring and M be the right R-module. We denote $R[x]$ for the polynomial ring over R, and $M[x]$ for the polynomial module over M.

For a commutative ring R, McCoy [1, Theorem 2] proved that if a polynomial $f(x)$ is a zero-divisor in $R[x]$, then $f(x)c = 0$ for some nonzero c in R. This result does not hold for noncommutative rings according to [2]. Following [3], the rings which meet the above condition are called the *right McCoy rings*. The *left McCoy rings* are defined analogously, and McCoy rings are both left and right McCoy. There are several ways in which McCoy theorem can be generalized by weakening the commutativity condition. For instance, all reversible rings (i.e., $ab = 0$ implies $ba = 0$ for any $a, b \in R$) are McCoy [4, Theorem 2]. As another class of examples, according to [5, Theorem 8.2], right duo rings are right McCoy, where a ring is *right duo* if every right ideal is also a left ideal. The *left duo rings* are defined symmetrically, and rings with both conditions are called the *duo rings*. There are other natural conditions that imply the McCoy property, not directly related to commutativity. For instance, reduced rings are reversible; therefore, they are McCoy rings. Furthermore, reduced rings are Armendariz rings. Here, a ring R is called *Armendariz* if whenever polynomials $f(x) = \sum_{i=0}^{m} a_i x^i$ and $g(x) = \sum_{j=0}^{n} b_j x^j$ in $R[x]$ satisfy $f(x)g(x) = 0$, we have $a_i b_j = 0$ for every i and j [3]. Thus, McCoy rings are shown to be a unifying generalization of reversible, Armendariz, and duo rings. Recently, the author introduced the class of ZPZC rings which generalizes the class of McCoy rings (ZPZC is short for “zero-divisor polynomials have zero-divisor coefficients”). An example of a ZPZC ring that is not a McCoy ring was provided in [6, Example 2.7] and a condition for the right ZPZC rings to be right McCoy is given in [6, Proposition 2.15].

However, it is natural to extend the notion of rings to one of the modules. For instance, Cui and Chen [7] introduced the class of McCoy modules, extending the notion of McCoy rings: $m(x)g(x) = 0$, where

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$m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]\backslash\{0\}$, implies that there exists a nonzero element $r \in R$ such that $m(x)r = 0$. They established some properties of the class of McCoy modules and presented some equivalent conditions for McCoy modules. Not only the concepts of reduced, reversible, Armendariz, and right duo rings but also the concepts of rings related to them are also naturally extended to modules. For instance, the right R-module M is called:

(1) *reduced* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mR \bigcap Ma = 0$ [8],
(2) *symmetric* if $mab = 0$ implies $mba = 0$, for any $m \in M$ and $a, b \in R$ [9],
(3) *Armendariz* if whenever polynomials $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]$ satisfy $m(x)g(x) = 0$, we have $m_i b_j = 0$ for every i and j [10],
(4) *duo* if for any R-submodule N of M and any R-endomorphism f of M, $f(N) \subseteq N$ [11],
(5) *semicommutative* if for any $m \in M$ and any $a \in R$, $ma = 0$ implies $mRa = 0$ [10].

We note that symmetric modules are the notion of module corresponding to reversible rings, and as stated in [11], the right R-module R, which is a duo, is a right duo ring. All Armendariz modules are obviously McCoy modules, and according to [8, Lemma 1.5], reduced modules are McCoy. It follows from [7, Proposition 2.4] that semicommutative modules over reduced rings are McCoy modules. Recall that a ring R is semicommutative if $ab = 0$ entails $aRb = 0$ for any $a, b \in R$. Generally, semicommutative rings are not McCoy, as stated by [4, Section 3]. For more details on the McCoy condition, the readers can refer to [4,5,7, 12–20].

The purposes of this paper are as follows:

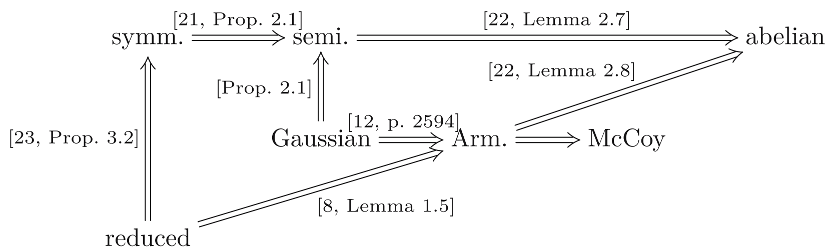
(1) to complete the implication diagram among modules related to McCoy modules,
(2) to improve some known results of McCoy modules,
(3) to introduce ZPZC modules which extend McCoy modules,
(4) to answer the question [6, Question 2.8(2)] that is “is there a ring which is left McCoy and right ZPZC, but not right McCoy?”, suggested by the referee of [6].

Thus, the implication diagram and various related examples are provided in Section 2. Among other things, an unpredictable example for which duo modules may not be McCoy is constructed (Example 2.5). In Section 3, one improves some known properties of McCoy modules. Precisely, we prove that every right module over a right π-duo reduced ring is McCoy (Theorem 3.3), which is an extended result of [7, Proposition 2.4]. It is also proven that if a ring R is right uniform, then the direct sum of ZPZC modules as a right R-module is McCoy (Theorem 3.9), which makes [7, Proposition 2.9] a corollary. In Section 4, we observe some properties of ZPZC modules with several examples. In particular, we show that for every cyclic module M over a right duo ring R, $\mathbb{T}_n(M)$ is a ZPZC right $\mathbb{T}_n(R)$-module (Theorem 4.12) from which one can find many examples of ZPZC modules. In Section 5, one can see some counterexamples for answers to questions of [6]. Especially, we construct a left McCoy and right ZPZC ring but not right McCoy (Example 5.1) for giving a negative answer of [6, Question 2.8(2)].

For $m \in M$, $\mathrm{ann}_R(m)$ is used for the set of all $r \in R$ such that $mr = 0$. For fixed integer $n \geq 1$, $\mathbb{M}_n(A)$ and $\mathbb{T}_n(A)$ stand for the set of all $n \times n$ full matrices over a set A and the set of all $n \times n$ upper triangular matrices over a set A, respectively, and $E_{i,j}$ means the matrix unit which is the matrix with (i, j)-entry 1 and elsewhere 0.

2 McCoy and duo modules

We here study the relationships between McCoy and related modules, as stated in Section 1. We in advance show the diagram that contains all implications among related modules (with no other implications holding, except by transitivity). In [5, p. 615], one can see implications immediately in the case of $M = R$.



In the above diagram, we cannot find duo modules even though the right duo rings are both semicommutative and right McCoy. To explain the reason, we construct several examples in this section.

An R-module M is called *abelian* if for any $m \in M$ and $a \in R$, any idempotent $e \in R$, we have $mae = mea$ [22]. Recall that a ring R is *abelian* if every idempotent in R is central. Following [22, Lemma 2.7], semicommutative modules are abelian modules. An R-module M is called *Gaussian* [12] if for any $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$, $g(x) = \sum_{j=0}^{q} b_j x^j \in R[x]$ and $m(x)g(x) = \sum_{k=0}^{p+q} c_k x^k$, we have $\sum_{i=0}^{p}\sum_{j=0}^{q} m_i R b_j R = \sum_{k=0}^{p+q} c_k R$. It was shown that Gaussian modules are Armendariz [12, p. 2594].

Proposition 2.1. *If an R-module M is Gaussian, then M is a semicommutative module.*

Proof. Let $m \in M$ and $a \in R$ such that $ma = 0$. Since M is Gaussian, $mRa \subseteq mRaR = maR = \{0\}$. Therefore, M is semicommutative. □

Remark 2.2.
(1) Note that $R = \mathbb{Z}_4$ is a local ring with $J(R)^2 = \{0\}$, where $J(R)$ is the Jacobson radical of R. Thus, the commutativity of R guarantees that R is Gaussian according to [24, Proposition 1.8]. But R is not reduced. Hence, Gaussian modules may not be reduced.
(2) As stated in [25, p. 2268], the polynomial ring over $\mathbb{Z}$, $\mathbb{Z}[x]$ is not Gaussian since $\mathbb{Z}[x]$ is not Prüfer by [26, Theorem 7.7]. However, $\mathbb{Z}[x]$ is a reduced module over itself. Thus, reduced modules do not imply Gaussian in general.
(3) Referring to [25, Example 11], symmetric modules need not be Gaussian modules.
(4) There exists a duo module which is not Gaussian by (2).

According to [24, Lemma 1.4], every right Gaussian ring is a right duo ring. However, this implication does not hold in modules anymore.

Example 2.3. Let $R = \mathbb{Z}_2$ and $M = m_1R \oplus m_2R$ be the free R-module with basis $\{m_1, m_2\}$. One can easily check that M is Gaussian. Consider the R-module endomorphism h of M induced by $h(m_1) = m_2$ and $h(m_2) = m_1$. Then, $h(m_1R) = m_2R \nsubseteq m_1R$. Thus, M is not duo.

Right duo rings are semicommutative rings and thus abelian rings. However, duo modules need not be abelian modules.

Example 2.4. Let $E = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra with identity and noncommuting indeterminates a, b over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the relation $ba = 1$. We denote $r = r + I$ for simplicity. Now consider the cyclic free R-module $L = mR$ and the R-submodule $N = maR$ of L. Then, the factor module $M = L/N$ is isomorphic to R/aR as an R-module. Note that each element $r \in R$ can be written in the following form:

$$r = \alpha + g_0(a)a + h(b)b + g_1(a)ab + g_2(a)ab^2 + \cdots,$$

where $\alpha \in \mathbb{Z}_2$ and $g_0(x), \dots, h(x) \in \mathbb{Z}_2[x]$. If $h(x) = \sum_{j=0}^{t} \beta_j x^j$ with $\beta_t \neq 0$, then

$$ra^{t+1} = \alpha a^{t+1} + g_0(a)a^{t+2} + \left(\sum_{j=0}^{t-1}\beta_j b^j\right)ba^{t+1} + \beta_t + \cdots.$$

Therefore, every R-submodule of R/aR contains 1. Thus, $M \cong R/aR$ is a simple R-module, which forces that M is a duo module. Finally, we claim that M is not abelian. Let $e = ab$ in R. Then, $e^2 = e$ in R and $\overline{m}eb = (\overline{m}a)b^2 = \overline{0}$ in M. But $\overline{m}be = \overline{m}(ba)b = \overline{m}b \neq \overline{0}$ in M. Hence, M is not abelian, which implies that M is not semicommutative by [22, Lemma 2.7].

Camillo and Nielsen proved that all right duo rings are right McCoy in [5, Theorem 8.2]. However the next example shows that there is a duo module which is not McCoy.

Example 2.5. Let $E = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, c\rangle$ be the free algebra with identity and noncommuting indeterminates as labeled above over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the following relations: for each $0 \leq i \leq 2$, $0 \leq j, k \leq 1$,

$$\begin{aligned}&a_0b_0 = 0, \quad a_0b_1 + a_1b_0 = 0, \quad a_1b_1 + a_2b_0 = 0, \quad a_2b_1 = 0,\\ &a_0a_i = a_2a_i = b_ja_i = b_jb_k = ca_i = cb_j = 0.\end{aligned}$$

We denote $r = r + I$ for simplicity. Applying Bergman's diamond lemma [27], we can write each $r \in R$ in the following unique reduced form:

$$\begin{aligned}r = \alpha &+ f_0(a_1)a_0 + f_1(a_1)a_1 + f_2(a_1)a_2 + f_3(a_1)b_0 + f_4(a_1)b_1 + g(c)c\\ &+ h_0^{(1)}(a_1)a_0c + h_1^{(1)}(a_1)a_1c + h_2^{(1)}(a_1)a_2c + h_3^{(1)}(a_1)b_0c + h_4^{(1)}(a_1)b_1c\\ &+ h_0^{(2)}(a_1)a_0c^2 + h_1^{(2)}(a_1)a_1c^2 + h_2^{(2)}(a_1)a_2c^2 + h_3^{(2)}(a_1)b_0c^2 + h_4^{(2)}(a_1)b_1c^2 + \cdots,\end{aligned}$$

where $\alpha \in \mathbb{Z}_2$ and $f_0(x), \ldots, f_4(x), g(x), h_0^{(p)}(x), \ldots, h_4^{(p)}(x) \in \mathbb{Z}_2[x]$ for each $p \in \mathbb{N}$. Note that if $rc \in cR$ for some $r \in R$, then r must be zero or identity.

Let $L = yR$ be the cyclic free R-module. Set $M = L/N$, where $N = \{0, yc, yc^2, \ldots\} = ycR$ is the R-submodule of L generated by the relation $yc = 0$. Then, we can write each $\overline{m} \in M$ uniquely in the following form:

$$\begin{aligned}\overline{m} = \overline{y}\alpha &+ \overline{y}f_0(a_1)a_0 + \overline{y}f_1(a_1)a_1 + \overline{y}f_2(a_1)a_2 + \overline{y}f_3(a_1)b_0 + \overline{y}f_4(a_1)b_1\\ &+ \overline{y}h_0^{(1)}(a_1)a_0c + \overline{y}h_1^{(1)}(a_1)a_1c + \overline{y}h_2^{(1)}(a_1)a_2c + \overline{y}h_3^{(1)}(a_1)b_0c + \overline{y}h_4^{(1)}(a_1)b_1c\\ &+ \overline{y}h_0^{(2)}(a_1)a_0c^2 + \overline{y}h_1^{(2)}(a_1)a_1c^2 + \overline{y}h_2^{(2)}(a_1)a_2c^2 + \overline{y}h_3^{(2)}(a_1)b_0c^2 + \overline{y}h_4^{(2)}(a_1)b_1c^2 + \cdots.\end{aligned}$$

First, we claim that an R-module M is not McCoy. Consider the following nonzero polynomials $m(x) = \overline{y}a_0 + \overline{y}a_1x + \overline{y}a_2x^2 \in M[x]$ and $r(x) = b_0 + b_1x \in R[x]$. Then, $m(x)r(x) = 0$ by the first row in the relations. But there is no nonzero element $r \in R$ such that $\overline{y}a_1r = 0$ by the relations. Thus, M is not McCoy.

Finally, we claim that M is a duo R-module. Referring to [11, Lemma 1.1], we will find all R-endomorphism of M. Since M is still a cycle R-module, for any R-endomorphism, we need to examine the image of $\overline{y}$. Consider the correspondence $h : M \to M$ defined by $h(\overline{y}r) = \overline{y}a_0r$ for any $r \in R$. Note that $h(\overline{y}0) = \overline{y}a_00 = \overline{0} \neq \overline{y}a_0c = h(\overline{y}c)$ despite $0 + N = yc + N$. Therefore, h is not well-defined. From the fact that if $rc \in cR$ for some $r \in R$, then r must be zero or identity in R, we can notice that if h' is an R-endomorphism of M induced by $h'(\overline{y}) = \overline{y}r'$ for some $r' \in R$, then r' must be zero or identity in R by the well-definedness. This means that if h' is an R-endomorphism of M, then h' must be either the zero or the identity endomorphism. Thus, M must be duo by [11, Lemma 1.1].

Remark 2.6. Based on Example 2.5, one may suspect that every duo module is cyclic. However, the example in [28, Section 3] eliminates the possibility of the suspicion.

Even though reversible (and so symmetric) rings are McCoy rings [4, Theorem 2], symmetric modules need not be McCoy modules. We construct a symmetric R-module that is not McCoy.

Example 2.7. Let $E = \mathbb{Z}_2\langle a_0, a_1\rangle$ be the free algebra with identity and indeterminates a_0, a_1 over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the following relations:

$$a_i a_j = 0\ (0 \le i, j \le 1).$$

We simply identify $r = r + I$ in R. Applying the diamond lemma [27], we can write each element $r \in R$ in the following reduced form:

$$r = \alpha + \alpha_0 a_0 + \alpha_1 a_1,$$

where $\alpha, \alpha_0, \alpha_1 \in \mathbb{Z}_2$. Now, let $L = m_0R \oplus m_1R \oplus m_2R$ be the free R-module with basis $\{m_0, m_1, m_2\}$. Set $M = L/N$ where N is the submodule of L generated by the following relations:

$$m_0a_0 = 0,\ m_0a_1 + m_1a_0 = 0,\ m_1a_1 + m_2a_0 = 0,\ m_2a_1 = 0.$$

We also denote $m = m + N$ in M for simplicity. Then, we can write each element $m \in M$ uniquely in the following form:

$$m = m_0\beta_0 + m_1\beta_1 + m_2\beta_2 + m_1\gamma_0a_0 + m_1\gamma_1a_1,$$

where $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1 \in \mathbb{Z}_2$. Since R is commutative, M is symmetric. Next, we claim that M is not McCoy. Consider the nonzero polynomials $m(x) = m_0 + m_1x + m_2x^2 \in M[x]$ and $r(x) = a_0 + a_1x \in R[x]$. Then, $m(x)r(x) = 0$ by the relations. It suffices to show that $\text{ann}_R(m_1) = \{0\}$. Suppose that $m_1r = 0$ for some $r \in R$ and write $r = \alpha + \alpha_0a_0 + \alpha_1a_1$. Then, $m_1r = m_1(\alpha + \alpha_0a_0 + \alpha_1a_1) = 0$ implies that $\alpha = \alpha_0 = \alpha_1 = 0$, indicating that $r = 0$ as required.

Recall that an R-module M is *Dedekind finite* (or *directly finite*) if $M \cong M \oplus N$ for some R-module N implies that $N = \{0\}$. From [29, Exercise 1.8], M is Dedekind finite if and only if the endomorphism ring $End(M)$ is a Dedekind finite ring (i.e., for any $f, g \in End(M)$, $fg = 1$ implies $gf = 1$). Both right McCoy and abelian rings are Dedekind finite rings. However, the implications do not hold for modules.

Example 2.8. Let $R = \mathbb{Z}_2$ and let $M = m_1R \oplus m_2R \oplus \cdots$ be the infinite free module over R. Note that the R-module M is reduced and Gaussian. However, by [30, Exercise 4.3.3], M is injective. This implies that M is not a Dedekind finite module by [31, Proposition 5.7].

3 McCoy modules and ZPZC modules

In [6], the class of ZPZC rings that contains all McCoy rings is introduced. In this section, we extend the notion of (right) ZPZC rings to the one of the modules. Then, we provide new properties of McCoy modules by improving known results, and offer some conditions for which ZPZC modules are McCoy.

Definition 3.1. Let M be a right module over a ring R and $M[x]$ be the corresponding polynomial module over $R[x]$. M is referred to as *ZPZC* if $\text{ann}_{R[x]}(m(x)) \neq \{0\}$ implies $\text{ann}_R(m_i) \neq \{0\}$ for each m_i, where $m(x) = \sum_{i=0}^{p} m_ix^i \in M[x]$.

All McCoy modules are obviously ZPZC modules. But the converse does not hold by [6, Example 2.7]. Here, we construct a nontrivial ZPZC module which is not McCoy.

Example 3.2. Let $E = \mathbb{Z}_2\langle a_0, a_1, b_0, b_1, b_2\rangle$ be the free algebra over $\mathbb{Z}_2$ with identity and commuting indeterminates a_0, a_1, b_0, b_1, b_2. Set $R = E/I$ where I is the ideal of E generated by the following relations: for each $i, j \in \{0, 1\}$ and $k, \ell \in \{0, 1, 2\}$,

$$a_ia_j = a_ib_k = b_kb_\ell = 0.$$

We identify $r = r + I$ for simple expression. Then, we can write each element $r \in R$ uniquely in the following reduced form:

$$r = \alpha + \alpha_0 a_0 + \alpha_1 a_1 + \beta_0 b_0 + \beta_1 b_1 + \beta_2 b_2,$$

where $\alpha, \alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2 \in \mathbb{Z}_2$. Now let $L = m_0R \oplus m_1R \oplus n_0R \oplus n_1R \oplus n_2R$ be the free R-module with basis $\{m_0, m_1, n_0, n_1, n_2\}$. Set $M = L/N$ where N is the R-submodule of L generated by the following relations:

$$\begin{aligned}
&m_0a_0 = m_0a_1 = m_0b_0 = m_0b_1 = m_0b_2 = 0,\\
&m_1a_0 = m_1a_1 = m_1b_1 = 0, m_1b_2 + n_1b_2 = 0,\\
&n_0a_0 = 0, n_0a_1 + n_1a_0 = 0, n_0b_0 = n_0b_1 = n_0b_2 = 0,\\
&n_1a_1 + n_2a_0 = 0, n_1b_0 = 0,\\
&n_2a_1 = n_2b_0 = n_2b_1 = n_2b_2 = 0.
\end{aligned}$$

For every $m \in M$, we also identify $m = m + N$ for simple expression. Then, we can write each element $m \in M$ uniquely in the following form:

$$m = m_0\gamma_0 + m_1\gamma_1 + m_1\gamma_2 b_0 + n_0\delta_0 + n_1\delta_1 + n_2\delta_2 + n_1\delta_3 a_0 + n_1\delta_4 a_1 + n_1\delta_5 b_1 + n_1\delta_6 b_2,$$

where $\gamma_0, \gamma_1, \gamma_2, \delta_0, \dots, \delta_6 \in \mathbb{Z}_2$. For any $m \in M$, we can easily check that m must be annihilated by one of the elements in $\{b_0, b_1, b_2\}$. Thus, M is ZPZC. Finally, consider the nonzero polynomials $m(x) = m_0 + m_1x + n_0x^2 + n_1x^3 + n_2x^3 \in M[x]$ and $r(x) = a_0 + a_1x \in R[x]$. Then, $m(x)r(x) = 0$. But since $\operatorname{ann}_R(m_1) \bigcap \operatorname{ann}_R(n_1) = \{0\}$, we can conclude that M is not McCoy.

From Example 2.7, we obtain that a module M over a commutative ring may not be ZPZC. Thus, we will determine some condition that makes a module over a commutative ring implies ZPZC. A ring R is called *right π-duo* if for any $a \in R$, there is a positive integer n such that $Ra^n \subseteq aR$ (see [32]). Of course, the notion of right π-duo rings is an extended notion of semicommutative modules. Thus we extend [7, Proposition 2.4] by the same method.

Theorem 3.3. (cf. [7, Proposition 2.4]). *If a ring R is reduced and right π-duo, then every right R-module M is McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]\backslash\{0\}$ with $m(x)r(x) = 0$ and $r_0 \neq 0$. Since R is right π-duo, $r_1r_0^n = r_0r_1'$ for some $n \in \mathbb{N}$ and $r_1' \in R$. From $m_0r_1 + m_1r_0 = 0$, we obtain that

$$m_0r_1r_0^n + m_1r_0^{n+1} = m_0r_0r_1' + m_1r_0^{n+1} = m_1r_0^{n+1} = 0.$$

Similarly, from $r_2r_0^{n^2+n} = r_0^{n+1}r_2'$ and $r_1r_0^{n^2+n} = r_0^{n+1}r_1''$ for some $r_2', r_1'' \in R$, we obtain

$$m_0r_2r_0^{n^2+n} + m_1r_1r_0^{n^2+n} + m_2r_0^{n^2+n+1} = m_0r_0^{n+1}r_2' + m_1r_0^{n+1}r_1'' + m_2r_0^{n^2+n+1} = m_2r_0^{n^2+n+1} = 0.$$

Continuing in this manner, we arrive at the conclusion that $m(x)r_0^k = 0$ for some $k \in \mathbb{N}$. □

Remark 3.4. According to [32, Proposition 1.9(4)], right π-duo rings are abelian. From the fact that every reduced ring is abelian, the converse of Theorem 3.3 does not hold true in general by [5, Theorem 7.1]. Moreover, the hypothesis of Theorem 3.3 is not superfluous by the following examples.

(1) In Example 2.7, the R-module M is not McCoy where the ring R is right π-duo but not reduced.

(2) Let $R = \mathbb{Z}_2\langle a_0, a_1\rangle$ be the free algebra with identity and noncommuting indeterminates a_0, a_1 over $\mathbb{Z}_2$. Clearly, R is a domain and so reduced. Note that R is not right π-duo. Let $L = m_0R \oplus m_1R \oplus m_2R$ be the free R-module with basis $\{m_0, m_1, m_2\}$. Set $M = L/N$ where N is the submodule of L generated by the following relations:

$$m_0a_0 = 0, m_0a_1 + m_1a_0 = 0, m_1a_1 + m_2a_0 = 0, m_2a_1 = 0.$$

After suppressing the bar notation in M, we consider the nonzero polynomials $m(x) = m_0 + m_1x + m_2x^2 \in M[x]$ and $r(x) = a_0 + a_1x \in R[x]$. Then, $m(x)r(x) = 0$ but $\operatorname{ann}_R(m_1) = \{0\}$. Hence, M is not McCoy.

A ring R is *semiprime* if the zero ideal is semiprime, and *right Gaussian* if R_R is Gaussian. Recall that a ring R is called *weakly right duo* if for each $a \in R$, there exists a positive integer n such that a^nR is a two-sided ideal of R (see [32] or [33]). According to [24, Lemma 1.4], every right Gaussian ring is right duo. It is clear that every right duo ring is weakly right duo, and every weakly right duo ring is right π-duo but not conversely in each case. Note that every semiprime right duo ring is reduced. Thus by Theorem 3.3 and [32, Proposition 2.2(1)], we obtain:

Corollary 3.5. *Let $J(R)$ be the Jacobson radical of a ring R and M be a right module over R. Then, M is McCoy if one of the following conditions holds*:
(1) *R is right π-duo with $J(R) = \{0\}$.*
(2) *R is weakly right duo with $J(R) = \{0\}$.*
(3) *R is semiprime and right duo.*
(4) *R is semiprime and right Gaussian.*
(5) *R is commutative with $J(R) = \{0\}$.*
(6) *R is commutative semiprime.*
(7) *R is a commutative domain.*

Let R be a commutative ring and S a multiplicative subset of R. We say that an ideal A of R is *S-principal* if $As \subseteq aR \subseteq A$ for some $s \in S$ and $a \in A$ (see [34, Definition 1]). We say that R is an *S-principal ideal ring* if each ideal of R is S-principal. Clearly, every commutative principal ideal ring is an S-principal ideal ring for any S of R. According to [35, Corollary 2.2], every commutative domain D is always an S-principal ideal ring, where $S = D\backslash\{0\}$. Based on the fact that every module over a commutative domain or a commutative principal ideal ring is McCoy (see [7, Proposition 2.15]), one may suspect that every module over a commutative S-principal ideal ring is McCoy or ZPZC. But the following example eliminates the possibility of suspicion.

Example 3.6. Let $E = \mathbb{Z}_2\langle a_0, a_1, s\rangle$ be the free algebra with identity and commuting indeterminates a_0, a_1, s over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the relations:

$$a_0^2 = a_0a_1 = a_1^2 = a_0s = a_1s = 0, s^2 = s.$$

We denote $r = r + I$ for simplicity. Then, we can write each $r \in R$ in the following reduced form:

$$r = \alpha + \alpha_0a_0 + \alpha_1a_1 + \beta s,$$

where $\alpha, \alpha_0, \alpha_1, \beta \in \mathbb{Z}_2$. One can easily show that if an ideal A of R is nonprincipal, then either $As \subseteq \{0\}$ or $As \subseteq sR \subseteq A$. Therefore, R is a commutative S-principal ideal ring, where $S = \{1, s\}$. Now let $L = m_0R \oplus m_1R \oplus m_2R$ be the free R-module with basis $\{m_0, m_1, m_2\}$. Set $M = L/N$ where N is the R-submodule of L generated by the following relations:

$$m_0a_0 = 0, m_0a_1 + m_1a_0 = 0, m_1a_1 + m_2a_0 = 0, m_2a_1 = 0.$$

Consider the following nonzero polynomials $m(x) = \overline{m}_0 + \overline{m}_1x + \overline{m}_2x^2 \in M[x]$ and $r(x) = a_0 + a_1x \in R[x]$. Then, $m(x)r(x) = 0$ by the relations. However, $\mathrm{ann}_R(\overline{m}_1) = \{0\}$. Hence, M_R is not ZPZC.

We consider some conditions under which ZPZC modules are McCoy. Recall that a ring R is called *right chain* [36] if the lattice of right ideals of R is linearly ordered. Right chain rings can also be referred to as right uniserial rings.

Proposition 3.7. (see Example 4.4(3)). *If R is a right uniserial ring and M is a ZPZC right R-module, then M is McCoy.*

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]\backslash\{0\}$ with $m(x)r(x) = 0$. Then, the ZPZC condition guarantees that each $\mathrm{ann}_R(m_i)$ is nonzero. Since R is right uniserial, we can find a nonzero element c_k such that $c_k \in \mathrm{ann}_R(m_k) \subseteq \mathrm{ann}_R(m_i)$ for all i. Hence, $m_i c_k = 0$ for all i, which implies that M is McCoy. □

From the opposite ring R^{op} of the ring R in [4, Section 3], one can notice easily that the converse of Proposition 3.7 does not hold true in general. From Example 3.2, one can also notice that there exists a ZPZC R-module but not McCoy, when the ring R is not right uniserial. In fact, Proposition 3.7 was started from the suspicion whether every module over a right uniserial ring is McCoy or not. The suspicion is very reasonable because every right uniserial ring is McCoy [37, Corollary 6.3]. Thus, it is worthwhile exploring the suspicion. However, we failed to solve it and so leave it as an open question:

Question 3.8. Can we delete the condition that M is ZPZC in Proposition 3.7?

From [29, p. 84], A module is called *uniform* if the intersection of any two nonzero submodules is nonzero. Recall that a submodule N of a module M is *essential* if for every submodule H of M, $H \bigcap N = \{0\}$ implies that $H = \{0\}$. A module M is uniform if and only if every nonzero submodule of M is essential. A ring is *right uniform* if it is uniform as a right module over itself. According to [29, p. 506], a module M is *finitely cogenerated* if for any family of submodules $\{M_k \mid k \in K\}$ in M, $\bigcap_{k\in K} M_k = \{0\}$ implies $\bigcap_{k'\in K'} M_{k'} = \{0\}$ for some finite subset K' of K. The following theorem makes [7, Proposition 2.9] as a corollary.

Theorem 3.9. (cf. [7, Proposition 2.9] and see Example 4.2). *Let K be an index set and let M_k be a ZPZC right R-module for each $k \in K$.*
(1) *If R is right uniform, then the direct sum of M_k as a right R-module, $M = \bigoplus_{k\in K} M_k$ is McCoy.*
(2) *For an infinite index set K, if R_R is uniform and finitely cogenerated, then the direct product of M_k as a right R-module, $M = \prod_{k\in K} M_k$ is McCoy.*

Proof.
(1) Consider nonzero polynomials $m(x) = \sum_{i=0}^{p} (m_{ki}) x^i \in M[x]$ and $r(x) \in R[x]$ with $m(x)r(x) = 0$. Fix $k \in K$ and let $m_k(x) = \sum_{i=0}^{p} m_{ki} x^i \in M_k[x]$ where m_{ki} is the kth component of the coefficient (m_{ki}) of $m(x)$ for each i. Note that the set $K' = \{k \in K | m_k(x) \neq 0\}$ is finite. From $m_k(x)r(x) = 0$ in $M_k[x]$ and M_k is ZPZC for each k, we obtain $\mathrm{ann}_R(m_{k'i}) \neq \{0\}$ for each i and $k' \in K'$. The uniform condition guarantees that there exists a nonzero $c \in \bigcap_{k'\in K', 0\le i\le p} \mathrm{ann}_R(m_{k'i})$. Thus, $m(x)c = 0$ which shows that M is McCoy.
(2) Consider nonzero polynomials $m(x) = \sum_{i=0}^{p} (m_{ki}) x^i \in M[x]$ and $r(x) \in R[x]$ with $m(x)r(x) = 0$. Fix $k \in K$ and let $m_k(x) = \sum_{i=0}^{p} m_{ki} x^i \in M_k[x]$ where m_{ki} is the kth component of the coefficient (m_{ki}) of $m(x)$ for each i. Then, for any finite subset K' of K, $\bigcap_{k'\in K', 0\le i\le p} \mathrm{ann}_R(m_{k'i}) \neq \{0\}$ by the uniform condition and ZPZC condition. Since R is finitely cogenerated, there exists a nonzero $c \in \bigcap_{k\in K, 0\le i\le p} \mathrm{ann}_R(m_{ki}) \neq \{0\}$. Hence, c annihilates $m(x)$, completing the proof. □

Corollary 3.10. [6, Proposition 2.15] *If R is a right uniform and right ZPZC ring, then R is necessarily right McCoy.*

Remark 3.11. The converse of Theorem 3.9(1) does not be true (see Remark 5.2(5)). Additionally, for right ideals $\mathrm{ann}_R(m_1)$ and $\mathrm{ann}_R(n_1)$ of the ring R in Example 3.2, $\mathrm{ann}_R(m_1) \bigcap \mathrm{ann}_R(n_1) = \{0\}$ illustrates that R is not right uniform. Thus, we cannot drop the condition that R is right uniform in Theorem 3.9(1).

4 Properties of ZPZC modules

In this section, we focus on the structure of ZPZC modules. Various examples and practical properties of ZPZC modules are provided. First, we examine the direct products of ZPZC modules.

Proposition 4.1. *Let R_i be a ring and M_i be an R_i-module for each $i \in \Gamma$. Then, each M_i is ZPZC if and only if the direct product of M_i, $\prod_{i\in\Gamma} M_i$ is a ZPZC ($\prod_{i\in\Gamma} R_i$)-module.*

Proof. This proof is almost the same as the proof of [6, Proposition 2.6]. □

The ZPZC condition does not pass to direct products of ZPZC R-modules as an R-module.

Example 4.2. (cf. [12, Example 3.3]). We use the ring R and module M in Example 2.7. Note that R is local with a unique maximal ideal $J = \{0, a_0, a_1, a_0 + a_1\}$. Set $S = R[y, z]/K$, where K is the ideal of the polynomial ring $R[y, z]$ generated by the following relations:

$$a_0 y = a_0 z = 0,\ a_1 y = a_1 z = 0,\ y^2 = yz = z^2 = 0.$$

Then, $\overline{M} = M[y, z]/M[y, z]K$ is an S-module. Put $M_1 = \overline{M}/\overline{M}y$ and $M_2 = \overline{M}/\overline{M}z$. Because $y \in \mathrm{ann}_S(M_1)$ and $z \in \mathrm{ann}_S(M_2)$, M_1 and M_2 are McCoy (and so ZPZC) as S-modules. After suppressing the bar notation for simplicity, we have

$$(m(x), m(x))r(x) = (0, 0) \in (M_1[x] \oplus M_2[x])_{S[x]} \cong (M_1 \oplus M_2)[x]_{S[x]},$$

where $m(x) = m_0 + m_1 x + m_2 x^2$ in $M_1[x]$ and $M_2[x]$, and $r(x) = a_0 + a_1 x \in S[x]$. But, $\mathrm{ann}_S((m_1, m_1)) = \{0\}$. Thus, $M_1 \oplus M_2$ is not a ZPZC S-module.

Next, we introduce some basic properties of ZPZC modules.

Proposition 4.3. *Let R be a ring and M be a right R-module.*
(1) (cf. [6, Example 2.11(1)]). *Every submodule of a ZPZC module is ZPZC.*
(2) (see Example 4.5). *R is right ZPZC if and only if every cyclic free right R-module is ZPZC.*
(3) (cf. [7, Proposition 2.3(2)]). *M is ZPZC if and only if every finitely generated submodule of M is ZPZC.*

Proof.
(1) This is obvious.
(2) This follows from $M = mR \cong R$ as a right R-module.
(3) The forward direction is clear by (1) and so we only deal with the reverse direction. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) \in R[x]\setminus\{0\}$ with $m(x)r(x) = 0$. Consider the submodule N of M generated by all m_i. Then, $m(x) \in N[x]$. By hypothesis, N is ZPZC. Therefore, for each m_i, there exists a nonzero $c_i \in R$ such that $m_i c_i = 0$, which shows that M is ZPZC. □

We exemplify the results of Proposition 4.3.

Example 4.4.
(1) The author showed that the class of right ZPZC rings is not closed under subrings in [6, Example 2.11(1)]. However Proposition 4.3(1) illuminates that every right ideal of a right ZPZC ring must be right ZPZC (without identity).
(2) Let $M = m\mathbb{T}_2(\mathbb{Z}_6)$ be the free right module over $\mathbb{T}_2(\mathbb{Z}_6)$. Since $\mathbb{T}_2(\mathbb{Z}_6)$ is a right ZPZC ring but not right McCoy by [5, Proposition 10.2] and [6, Theorem 2.2], M is ZPZC but not McCoy by Proposition 4.3(2).

In view of Proposition 4.3(2), one may suspect that if M_R is either cyclic or free over a right ZPZC ring R, then M is ZPZC. But the following example eliminates the possibility of each suspicion.

Example 4.5.
(1) Let $R = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1\rangle$ be the free algebra with identity and noncommuting indeterminates a_0, a_1, a_2, b_0, b_1 over $\mathbb{Z}_2$. Obviously, R is a domain and so a right ZPZC ring. Let $L = mR$ be the free R-module and N be an R-submodule of L generated by the following relations:

$$ma_0b_0 = 0,\ ma_0b_1 + ma_1b_0 = 0,\ ma_1b_1 + ma_2b_0 = 0,\ ma_2b_1 = 0.$$

Then, $M = L/N$ is a cyclic R-module. After suppressing the bar notation in M, consider the nonzero polynomials $m(x) = ma_0 + ma_1x + ma_2x^2 \in M[x]$ and $r(x) = b_0 + b_1x \in R[x]$. Then $m(x)r(x) = 0$ by the relations. But, $\mathrm{ann}_R(ma_1) = \{0\}$. Thus, M is not a ZPZC R-module.

(2) Let $R = \mathbb{T}_2(\mathbb{Z}_6)$ be the right ZPZC ring and $M = m_1R \oplus m_2R$ be the free R-module with basis $\{m_1, m_2\}$. After suppressing the bar notation in $\mathbb{Z}_6$, consider the following nonzero polynomials:

$$m(x) = \left(m_1\begin{bmatrix}0 & 4\\ 0 & 0\end{bmatrix}, m_2\begin{bmatrix}0 & 3\\ 0 & 0\end{bmatrix}\right) + \left(m_1\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}, m_2\begin{bmatrix}3 & 1\\ 0 & 0\end{bmatrix}\right)x + \left(m_1\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}, m_2\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\right)x^2 \in M[x],$$

$$r(x) = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}x \in R[x].$$

Then, $m(x)r(x) = 0$. However, an easy calculation shows that if $r \in R$ annihilates the middle coefficient of $m(x)$ on the right side, then $r = 0$. This means that the free R-module M is not ZPZC.

Remark 4.6.

(1) (see Remark 4.23(3)). As well-known results (see [29, p. 21, Proposition 4.3, and p. 127]), the following irreversible implications hold:

$$\text{free} \Rightarrow \text{projective} \Rightarrow \text{flat} \Rightarrow \text{torsionfree}.$$

Thus, Example 4.5(2) shows that a projective or flat or torsionfree module over a right ZPZC ring may not be ZPZC.

(2) Based on Example 2.8, one may suspect that every injective module over a right ZPZC ring is ZPZC. For a counterexample, let D be any division ring, $R = \mathbb{T}_2(D)$, and $E = \mathbb{M}_2(D)$. Then, D is certainly a semisimple ring and thus E_R is injective by [29, Example 3.43], when R is a right ZPZC ring by [6, Corollary 2.3]. From [6, Remark 3.9], we can conclude that E_R is not a ZPZC R-module.

(3) (cf. [7, Proposition 2.10]). In Example 4.5(2), each cyclic free R-module m_1R and m_2R is ZPZC by Proposition 4.3(2). Additionally, m_1R is R-isomorphic to m_2R. Thus, a direct sum of copies of a ZPZC module does not have to be ZPZC.

(4) (cf. [7, Proposition 2.15]). Note that $\mathbb{T}_2(\mathbb{Z}_6)$ is a principal right ideal ring. Thus, Example 4.5(2) shows that the commutative condition must be required in [7, Proposition 2.15]. Additionally, $\mathbb{T}_2(\mathbb{Z}_6)$ is also an $\mathbb{I}_2(S)$-principal right ideal ring for any multiplicative subset S of $\mathbb{Z}_6$ (see [35, Theorem 2.14]). Hence, there is a free module that is not a ZPZC module over an S-principal right ideal ring.

Recall that a module is called *Bézout* if every finitely generated submodule is cyclic. We provide a sufficient condition under which a right module over a right ZPZC ring is ZPZC.

Proposition 4.7. (cf. [7, Proposition 2.7]). *Let R be a right ZPZC ring. If M is a cyclic submodule of a flat right R-module, then M is ZPZC.*

Proof. Let $M = mR$ and consider $m(x) = \sum_{i=0}^{p} m_ix^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_jx^j \in R[x]\setminus\{0\}$ with $m(x)r(x) = 0$. Then for each m_i, we can write $m_i = ma_i$ for some $a_i \in R$. By [29, Theorem 4.24], the flat condition guarantees that $a(x)r(x) = 0$ where $a(x) = \sum_{i=0}^{p} a_ix^i \in R[x]$. Hence, $m_ic_i = ma_ic_i = 0$ for some nonzero $c_i \in R$ and each i, as desired. □

The hypothesis “Bézout and flat” of the following corollary is not superfluous by Example 4.5.

Corollary 4.8. *Let R be a right ZPZC ring. If a right R-module M is Bézout and flat, then M is ZPZC.*

Generally, although some R-submodule is ZPZC of an R-module M, M may not be ZPZC. We find a type of R-submodule that the ZPZC condition can be lifted up to R-overmodule.

Proposition 4.9. (cf. [6, Proposition 2.12]). *Let M be a right module over a right ZPZC ring R and e be a nontrivial central idempotent such that $r \neq er \neq 0$ for all nonzero $r \in R$ with $1 - e \neq r \neq e$. If MeR is a ZPZC eR-module and $M(1 - e)R$ is a ZPZC $(1 - e)R$-module, then M_R is ZPZC.*

Proof. Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]\backslash\{0\}$ with $m(x)r(x) = 0$. Then, $m(x)r(x)e = me(x)er(x) = 0$ and $m(x)r(x)(1 - e) = m(1 - e)(x)(1 - e)r(x) = 0$. Because one of $er(x)$ and $(1 - e)r(x)$ is nonzero, there exists a nonzero element $ec_i \in eR$ or $(1 - e)d_i \in (1 - e)R$ such that $m_i ec_i = 0$ or $m_i(1 - e)d_i = 0$ for each i. Thus, $\mathrm{ann}_R(m_i) \neq \{0\}$ for each i. □

Question 4.10. Is the converse of Proposition 4.9 true?

Next, we consider matrix modules. According to [7, Proposition 2.6], every cyclic module over a right duo ring is McCoy. Based on this fact, we can extend [6, Lemma 2.1 and Theorem 2.2] to cyclic modules by the same arguments.

Lemma 4.11. (cf. [6, Lemma 2.1]). *Let R be a right duo ring and $M = mR$ be a cyclic right R-module. If $A = [m_{i,j}] \in \mathbb{T}_n(M)$ and some $\mathrm{ann}_R(m_{i,i}) \neq \{0\}$ for some $i \geq 1$, then $\mathrm{ann}_{\mathbb{T}_n(R)}(A) \neq \{0\}$.*

Proof. We first write $m_{i,j} = mr_{i,j}$ for some $r_{i,j} \in R$ and each i, j. Without loss of generality, we assume that $i \geq 1$ is the smallest index such that $\mathrm{ann}_R(m_{i,i}) \neq \{0\}$. Fix $\beta_i \in R\backslash\{0\}$ such that $m_{i,i}\beta_i = 0$. Consider the matrix

$$B = (\beta_1)E_{1,i} + (r_{1,1}\beta_2)E_{2,i} + (r_{2,2}r_{1,1}\beta_3)E_{3,i} + \cdots + (r_{i-2,i-2} \cdots r_{1,1}\beta_{i-1})E_{i-1,i} + (r_{i-1,i-1} \cdots r_{1,1}\beta_i)E_{i,i}.$$

Then, the upper triangular matrix B is nonzero. We will choose the β's so that $AB = 0$. Note that the equation $AB = 0$ is equivalent to the set of equations

$$\begin{aligned}
0 &= m_{1,1}\beta_1 + m_{1,2}(r_{1,1}\beta_2) + \cdots + m_{1,i}(r_{i-1,i-1} \cdots r_{1,1}\beta_i),\\
0 &= m_{2,2}(r_{1,1}\beta_2) + m_{2,3}(r_{2,2}r_{1,1}\beta_3) + \cdots + m_{2,i}(r_{i-1,i-1} \cdots r_{1,1}\beta_i),\\
&\vdots\\
0 &= m_{i-1,i-1}(r_{i-2,i-2} \cdots r_{2,2}r_{1,1}\beta_{i-1}) + m_{i-1,i}(r_{i-1,i-1} \cdots r_{2,2}r_{1,1}\beta_i),\\
0 &= m_{i,i}(r_{i-1,i-1} \cdots r_{2,2}r_{1,1}\beta_i).
\end{aligned}$$

Now notice that $\mathrm{ann}_r(m_{i,i})$ is a two-sided ideal of R since R is right duo. Therefore,

$$0 \neq r_{i-1,i-1} \cdots r_{2,2}r_{1,1}\beta_i \in \mathrm{ann}_R(m_{i,i})$$

from our choice β_i. Thus, the last equation holds. Working upwards, we will recursively choose $\beta_{i-1}, \beta_{i-2}, \cdots, \beta_1 \in R$ such that the equations hold. For each $i - 1 \geq k \geq 1$, we can find β_k such that

$$(r_{k,k} \cdots r_{1,1})\beta_k = (-r_{k,k+1})(r_{k,k} \cdots r_{1,1})\beta_{k+1} + \cdots + (-r_{k,i})(r_{i-1,i-1} \cdots r_{1,1})\beta_i = -\sum_{s=k+1}^{i} r_{k,s}\left(\prod_{t=s-1}^{1} r_{t,t}\right)\beta_s \in R(r_{k,k} \cdots r_{1,1})R$$

from the fact that $aR = RaR$ for every $a \in R$. Hence, $0 \neq B \in \mathrm{ann}_{\mathbb{T}_n(R)}(A)$. □

Theorem 4.12. (cf. [6, Theorem 2.2]). *Fix a positive integer n. If R is a right duo ring and M is a cyclic right R-module, then $\mathbb{T}_n(M)$ is a ZPZC right $\mathbb{T}_n(R)$-module.*

Proof. Let $A(x) \in \mathbb{T}_n(M)[x]$ and $B(x) \in \mathbb{T}_n(R)[x]\backslash\{0\}$ such that $A(x)B(x) = 0$. Since there is the natural identifications $\mathbb{T}_n(M)[x] = \mathbb{T}_n(M[x])$ and $\mathbb{T}_n(R)[x] = \mathbb{T}_n(R[x])$, we can write $A(x) = [a_{i,j}(x)]$ and $B(x) = [b_{i,j}(x)]$ where $a_{i,j}(x) \in M[x]$ and $b_{i,j}(x) \in R[x]$. Because $B(x) \neq 0$, we have $b_{i,j}(x) \neq 0$ for the maximal i, and then the maximal j. In particular, $a_{i,i}(x)b_{i,j}(x) = 0$. Since R is right McCoy by [5, Theorem 8.2], we obtain that for each coefficient $a_{i,i,k}$ of $a_{i,i}(x)$, $\mathrm{ann}_R(a_{i,i,k}) \neq \{0\}$. Now, the previous lemma applies to show that each right annihilator set of each coefficient of $A(x)$ is nonzero in $\mathbb{T}_n(R)$. □

From Theorem 4.12, we obtain many examples of ZPZC modules. For instance, if R is a division ring, then R is clearly (right) duo and so (right) McCoy. Thus, a cyclic module over a division ring is McCoy by [7, Proposition 2.6]. This implies that if M is a cyclic module over a division ring R, then $\mathbb{T}_n(M)$ is a ZPZC $\mathbb{T}_n(R)$-module. Further, from the diagrams in [5] and Section 2, we obtain the following:

Corollary 4.13. *Fix an integer $n \geq 1$, and let M be a cyclic module over a ring R. Then, $\mathbb{T}_n(M)$ is a ZPZC $\mathbb{T}_n(R)$-module if one of the following assertions holds true*:
(1) *R is (right) duo and M is McCoy.*
(2) *R is (right) Gaussian and M is Gaussian.*
(3) *R is commutative and M is Armendariz.*
(4) (see Remark 5.4). *R is a division ring and M is reduced.*

In Theorem 4.12, the hypothesis that M is cyclic is not superfluous by the following example or Example 5.5.

Example 4.14. Let $R = \mathbb{Z}_6$ be the right ZPZC ring and $M = m_1R \oplus m_2R$ be the free R-module with basis $\{m_1, m_2\}$. After suppressing the bar notation in $\mathbb{Z}_6$, consider the following nonzero polynomials:

$$A(x) = \begin{bmatrix} 0 & m_1 4 + m_2 3 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_1 2 + m_2 3 & m_1 + m_2 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} m_1 5 + m_2 5 & 0 \\ 0 & 0 \end{bmatrix} x^2 \in \mathbb{T}_2(M)[x],$$
$$B(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x \in \mathbb{T}_2(R)[x].$$

Then, $A(x)B(x) = 0$. However, an easy calculation shows that if $C \in \mathbb{T}_2(R)$ annihilates the middle coefficient of $A(x)$ on the right side, then $C = 0$. Thus, the right $\mathbb{T}_2(R)$-module $\mathbb{T}_2(M)$ is not ZPZC.

Given an R-module M and $n \geq 2$, we denote $\mathbb{B}_n(M) = \{[m_{ij}] \in \mathbb{T}_n(M) | m_{11} = \cdots = m_{nn}\}$ and $\mathbb{H}_n(M) = \{[m_{ij}] \in \mathbb{B}_n(M) | m_{st} = m_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1\}$. Then, $\mathbb{B}_n(M)$ is a $\mathbb{B}_n(R)$-module and $\mathbb{H}_n(M)$ is a $\mathbb{H}_n(R)$-module.

Proposition 4.15. (cf. [7, Proposition 2.12]). *Fix $n \geq 1$. Then, the following statements are equivalent*:
(1) *M is a ZPZC R-module.*
(2) *$\mathbb{B}_n(M)$ is a ZPZC $\mathbb{B}_n(R)$-module.*
(3) *$\mathbb{H}_n(M)$ is a ZPZC $\mathbb{H}_n(R)$-module.*

Proof. This proof is nearly identical to the proof of [6, Theorem 3.5]. □

If we consider $\mathbb{M}_n(M)$, $\mathbb{T}_n(M)$, $\mathbb{B}_n(M)$, and $\mathbb{H}_n(M)$ for $n \geq 2$ as R-modules under the usual scalar multiplication, the above equivalences will be broken even when R is commutative.

Example 4.16. We use the ring R and module M in Example 3.2. Take $m(x) = m_0 + m_1x$, $n(x) = n_0 + n_1x + n_2x^2$ in $M[x]$, and $r(x) = a_0 + a_1x \in R[x]$. Then $m(x)r(x) = 0 = n(x)r(x)$. Now consider the nonzero polynomial $H(x) = \begin{bmatrix} m_0 & n_0 \\ 0 & m_0 \end{bmatrix} + \begin{bmatrix} m_1 & n_1 \\ 0 & m_1 \end{bmatrix} x + \begin{bmatrix} 0 & n_2 \\ 0 & 0 \end{bmatrix} x^2 \in \mathbb{H}_2(M)[x] \subseteq \mathbb{B}_2(M)[x] \subset \mathbb{T}_2(M)[x] \subset \mathbb{M}_2(M)[x]$. Then $H(x)r(x) = 0$. Since $\mathrm{ann}_R(m_1) \bigcap \mathrm{ann}_R(n_1) = \{0\}$, $\begin{bmatrix} m_1 & n_1 \\ 0 & m_1 \end{bmatrix} r = 0$ implies $r = 0$. Thus, $\mathbb{M}_2(M)$, $\mathbb{T}_2(M)$, $\mathbb{B}_2(M)$, and $\mathbb{H}_2(M)$ are not ZPZC R-modules.

Remark 4.17. If M is a McCoy R-module, then $\mathbb{M}_n(M)$, $\mathbb{T}_n(M)$, $\mathbb{B}_n(M)$, and $\mathbb{H}_n(M)$ are McCoy as R-modules. Indeed, if $m(x)r(x) = 0$ and $n(x)r(x) = 0$ for some nonzero polynomials $m(x), n(x) \in M[x]$ and $r(x) \in R[x]$, then $(m(x) + n(x)x^k)r(x) = 0$, where $k = \deg(m(x)) + 1$. Therefore, there exists a nonzero $r \in R$ such that $m(x)r = n(x)r = 0$. Thus, $\begin{bmatrix} m(x) & n(x) \\ 0 & m(x) \end{bmatrix} r(x) = 0$ implies $\begin{bmatrix} m(x) & n(x) \\ 0 & m(x) \end{bmatrix} r = 0$. From this idea, one can easily show that $\mathbb{M}_n(M)$, $\mathbb{T}_n(M)$, $\mathbb{B}_n(M)$ and $\mathbb{H}_n(M)$ are McCoy as R-modules for each positive integer n, when M is McCoy.

According to [29, Theorem 10.6], a multiplicative subset T of a ring R is a right denominator set if and only if the right ring of fractions with respect to T, RT^{-1} exists. A right R-module M is called *T-torsionfree* if for $m \in M$ and $t \in T$, $mt = 0$ implies $m = 0$. According to [38, Proposition 10.11(a)], every right RT^{-1}-module is T-torsionfree as a right R-module.

Theorem 4.18. *Let T be a right denominator set in a ring R. If M is a right RT^{-1}-module, then M is a right ZPZC R-module if and only if M is a right ZPZC RT^{-1}-module.*

Proof. ($\Rightarrow$) Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in RT^{-1}[x]\backslash\{0\}$ with $m(x)r(x) = 0$. Then, we can write that for each j, $r_j = b_j t^{-1}$, where $t \in T$. Take $b(x) = \sum_{j=0}^{q} b_j x^j \in R[x]\backslash\{0\}$. From the fact that

$$m(x)r(x)\frac{t}{1} = m(x)\left(\frac{b_0}{t} + \frac{b_1}{t}x + \cdots + \frac{b_q}{t}x^q\right)\frac{t}{1} = 0,$$

we obtain that $m(x)b(x) = 0$. Since M_R is ZPZC, there exists a nonzero $c_i \in R$ such that $m_i c_i = 0$ for each i. Therefore, $m_i c_i 1^{-1} = 0$ for each i. Hence, we can conclude that M is a ZPZC RT^{-1}-module.

($\Leftarrow$) Let $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} b_j x^j \in R[x]\backslash\{0\}$ with $m(x)r(x) = 0$. If we let $b(x) = \sum_{j=0}^{q} b_j 1^{-1} x^j \in RT^{-1}[x]\backslash\{0\}$, then $m(x)b(x) = 0$. Since $M_{RT^{-1}}$ is ZPZC, there exists a nonzero $c_i = r_i t_i^{-1} \in RT^{-1}$ such that $m_i c_i = 0$. From the fact that

$$(m_i c_i)\frac{t_i}{1} = m_i\left(\frac{r_i}{t_i} \cdot \frac{t_i}{1}\right) = m_i \frac{r_i}{1} = 0,$$

for each i, we obtain that $m_i r_i = 0$ for each i. Hence, we are done. □

According to [29, (10.17)], a ring R is right Ore if and only if the classical right quotient ring of R, $Q_{cl}^r(R)$ exists.

Corollary 4.19. (cf. [7, Theorem 2.17]). *Let R be a right Ore ring. If M is a $Q_{cl}^r(R)$-module. then M is a ZPZC R-module if and only if M is a ZPZC $Q_{cl}^r(R)$-module.*

Following the same approach in the proof of Theorem 4.18, we partially extend [6, Theorem 3.12].

Corollary 4.20. *If T is a right denominator set in a ring R, then R is right ZPZC if and only if RT^{-1} is right ZPZC.*

The proof of the next theorem is almost the same as the proof of [6, Theorem 3.15], but we insert it for the sake of completeness.

Theorem 4.21. (cf. [15, Proposition 2.6(1)]). *The class of ZPZC R-modules is closed under direct limits.*

Proof. Let $D = \langle M_u, \alpha_{u,v} \rangle$ be a direct system of right ZPZC R-modules M_u for $u \in U$ and R-module homomorphisms $\alpha_{u,v} : M_u \to M_v$ for each $u \le v$ and $m \in M_u$, satisfying $\alpha_{u,v}(m) = m$, where U is a directed partially ordered set. Set $M = \varinjlim M_u$ to be the direct limit of D with $\iota_u : M_u \to M$ and $\iota_v \circ \alpha_{u,v} = \iota_u$. Let a and $b \in M$. Then, $a = \iota_u(a_u)$, $b = \iota_v(b_v)$ for some $u, v \in U$, and there is $w \in U$ such that $u \le w$, $v \le w$. Define $a + b = \iota_w(\alpha_{uw}(a_u) + \alpha_{vw}(b_v))$ and $ar = \iota_w(\alpha_{uw}(a_u r))$, where $r \in R$. Under the above operations, M forms a right

R-module. Now consider polynomials $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]\backslash\{0\}$ with $m(x)r(x) = 0$. Then, there exists $w \in U$ such that $m(x) \in M_w[x]$. Therefore, we get $m(x)r(x) = 0$ in $M_w[x]$. Thus for each nonzero m_i, there exists nonzero $c_i \in R$ such that $m_i c_i = 0$ in M_w since M_w is a ZPZC R-module. Hence, for each nonzero m_i, $m_i c_i = 0$ in M, completing the proof. □

For an index set I of a chain A, a subset J of I is said to be *dense* in I if between any two distinct elements $x, y \in I$ with $x < y$, there exists at least one element $z \in J$ with $x \leq z \leq y$ (see [39, 2.6.6]).

Corollary 4.22. (cf. [15, Proposition 2.6(2)]). *Let A be a chain of R-modules M_i and J be a subset of I. If M_j is ZPZC for all $j \in J$ and J is dense in I, then $M = \bigcup_{i \in I} M_i$ is a ZPZC R-module.*

Proof. Note that $M = \bigcup_{i \in I} M_i$ is a direct limit of ZPZC R-modules. Thus, Theorem 4.21 applies. □

Remark 4.23.
(1) In [6, Proposition 3.15], the condition "with injective maps" must be required. However, the condition does not be required in Theorem 4.21.
(2) By a similar approach as the proof of Theorem 4.21, the class of McCoy R-modules is closed under direct limits. Of course, the condition "with injective maps" does not be required.
(3) Due to [29, Theorem 4.34], a right R-module M is flat if any only if M is a direct limit of finitely generated free R-modules. Thus, if a right module M over a right McCoy ring is flat, then M is McCoy by (2). This fact also follows from [29, Theorem 4.24] and extends [7, Corollary 2.11] by weakening the projective condition to the flat condition.

In view of Remark 4.6(1) and Remark 4.23(3), it is of interest to provide an explicit example of a right module M over a right McCoy ring that is torsionfree but not McCoy. However, the author did not find any clue for this question so far. Thus, we leave it as an open question:

Question 4.24. Over a right McCoy ring, is there a torsionfree module but not McCoy?

We end this section with the following remarks.

Remark 4.25. In the diagram of Section 2, we can add the implication "McCoy ⇒ ZPZC". The other implications (except the transitivity) do not hold according to Examples 2.5 and 2.7.

Remark 4.26. Anderson and Chun [12] defined dual McCoy, content McCoy and dual content McCoy modules. We can extend these concepts to ZPZC modules. A right R-module M is called *dual ZPZC* if for nonzero polynomials $m(x) \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]$, $m(x)r(x) = 0$ implies that there exists a nonzero $n_j \in M$ such that $n_j r_j = 0$ for each j. A right R-module M is said to be *content ZPZC* (resp., *dual-content ZPZC*), if for nonzero polynomials $m(x) = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $r(x) = \sum_{j=0}^{q} r_j x^j \in R[x]$, $m(x)r(x) = 0$ implies that there exists a nonzero $c_i \in C(r(x))$ (resp., $n_j \in S(m(x))$) such that $m_i c_i = 0$ for each i (resp., $n_j r_j = 0$ for each j), where $C(r(x))$ is the right ideal of R generated by the coefficients of $r(x)$ (resp., $S(m(x))$ is the R-submodule of M generated by the coefficients of $m(x)$).
(1) Even though M is a ZPZC R-module, M need not be dual ZPZC when we consider the opposite ring of the ring in [4, Section 3]. Conversely, the ring in [4, Section 3] shows that dual ZPZC modules need not be ZPZC.
(2) ZPZC modules do not imply content ZPZC modules. We use once again the ring R and the R-module M in Example 3.2. Consider the nonzero polynomials $n(x) = n_0 + n_1 x + n_2 x^2 \in M[x]$ and $r(x) = a_0 + a_1 x \in R[x]$ with $n(x)r(x) = 0$. Note that $C(r(x)) = a_0 R + a_1 R = \{0, a_0, a_1, a_0 + a_1\}$ and $\mathrm{ann}_R(n_1) = \{0, b_0\}$. Thus, the ZPZC module M is not content ZPZC.

(3) There is a ZPZC module but not dual content ZPZC. Let $R = \mathbb{Z}_2\langle b_0, b_1, b_2, c\rangle$ be the free algebra with identity and noncommuting indeterminates b_0, b_1, b_2, c over $\mathbb{Z}_2$. Then, it is clear that R is a domain. Let $L = m_0R \oplus m_1R$ be the free R-module with basis $\{m_0, m_1\}$. Set $M = L/N$, where N is the R-submodule of L generated by the following relations: for each $r \in R$,

$$m_0b_0 = 0, m_0b_1 + m_1b_0 = 0, m_0b_2 + m_1b_1 = 0, m_1b_2 = 0, m_0rc = m_1rc = 0.$$

We simply denote $m = m + N$. One can easily notice that M is a McCoy R-module since $mc = 0$ for every $m \in M$. Now consider the following nonzero polynomials $m(x) = m_0 + m_1x \in M[x]$ and $r(x) = b_0 + b_1x + b_2x^2 \in R[x]$. Then, $m(x)r(x) = 0$ by the relations. But there is no nonzero element $n \in M$ such that $nb_1 = 0$. Thus, the ZPZC R-module M is not dual content ZPZC.

(4) To show that there is a dual content ZPZC module but not ZPZC, we use the ring R and the module M in Example 2.7. Consider nonzero polynomials $u(x) = \sum_{i=0}^{p} u_ix^i \in M[x]$ and $v(x) = \sum_{j=0}^{q} v_jx^j \in R[x]$ with $u(x)v(x) = 0$. Assume that there exists a coefficient v_ℓ of $v(x)$ with 1 in its support and ℓ is minimally chosen. If the element m_0 appears in the support of u_k and k is minimally chosen, then m_0 must appear in the support of the $(k + \ell)$-degree coefficient of $u(x)v(x)$, which is a contradiction. Therefore, m_0 must not appear in the support of any coefficient of $u(x)$. By the same reason, the elements m_1 and m_2 must not appear in the support of any coefficient of $u(x)$ and thus the elements m_1a_0 and m_1a_1 must not appear in the support of any coefficient of $u(x)$. This means that $u(x) = 0$ which contradicts to the hypothesis that $u(x)$ is a nonzero polynomial in $M[x]$. Hence, there is no coefficient of $v(x)$ with 1 in its support. Now note that one of the following three elements $m_0a_1 = m_1a_0$, $m_1a_1 = m_2a_0$, and $m_1a_0 + m_1a_1$ must be in the submodule $S(u(x))$ of M. Since there is no coefficient of $v(x)$ with 1 in its support, we obtain that $m_0a_1v(x) = 0$ or $m_1a_1v(x) = 0$ or $(m_0a_1 + m_1a_1)v(x) = 0$ by the relations, which implies that M is a dual content McCoy module. Consequently, there exists a dual content ZPZC module but not ZPZC.

5 Three solved problems on ZPZC rings

We devote this section to providing answers to some questions raised in [6], which are related to ZPZC rings. First, a ring that is left McCoy and right ZPZC, but not right McCoy is constructed for giving a negative answer of [6, Question 2.8(2)].

Example 5.1. Let $E = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, c\rangle$ be the free algebra with identity and six noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, c$ over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the following relations:

$$\begin{aligned}
&a_0b_0 = 0, a_0b_1 = a_1b_0, a_1b_1 = 0, a_2b_i = 0(0 \le i \le 1),\\
&a_ia_j = a_ic = 0(0 \le i \le 1, 0 \le j \le 2), a_2a_j = a_j(0 \le j \le 2), a_2c = ca_2 = c,\\
&b_ia_j = b_ib_k = b_ic = ca_i = cb_i = c^2 = 0(0 \le i, k \le 1, 0 \le j \le 2).
\end{aligned}$$

One can check, via the diamond lemma [27], that these relations form a reduced system. We identify $a = a + I$ in R for simplicity. Then, we can write each element $r \in R$ uniquely in the following form:

$$r = \alpha + \alpha_0a_0 + \alpha_1a_1 + \alpha_2a_2 + (\beta_0 + \beta_1a_1)b_0 + \beta_2b_1 + \gamma c,$$

where $\alpha, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \gamma \in \mathbb{Z}_2$.

Claim 1. The ring R is not right McCoy.

Proof. Let $a(x) = a_0 + a_1x + a_2x^2$ and $b(x) = b_0 + b_1x$ in $R[x]$. Then, $a(x)b(x) = 0$ by the first row of the relations. It suffices to show that if $a(x)r = 0$ for some $r \in R$, then $r = 0$. Write r in the unique form. From $a_0r = 0$, we obtain $\alpha = \beta_2 = 0$. Similarly, $a_1r = 0$ implies $\beta_0 = 0$. Finally, since $\alpha = \beta_0 = \beta_2 = 0$, we have that $a_2r = 0$ implies $\alpha_0 = \alpha_1 = \alpha_2 = \beta_1 = \gamma = 0$. Thus, $r = 0$, as desired. □

Claim 2. The ring R is right ZPZC.

Proof. Let $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j$ be nonzero polynomials with $p(x)q(x) = 0$ in $R[x]$. First, observe that if a nonzero element r in the unique form has zero constant term, then r can be annihilated by one of the following elements in $\{b_0, b_1, b_0 + b_1\}$ on the right side. Thus, we may assume that there exists a coefficient of $p(x)$ with 1 in its support. Let k be the smallest index such that p_k has a nonzero constant term. Now we will use a similar argument as in the proof of [5, Proposition 3.2]. Assume that q_ℓ has a nonzero constant term in its support and ℓ is minimally chosen. If we compute the $(k + \ell)$-degree coefficient of $p(x)q(x)$, then, by the minimality of k and ℓ, we can see that the $(k + \ell)$-degree coefficient of $p(x)q(x)$ must have a nonzero constant term. This contradicts to the fact that $p(x)q(x) = 0$. Thus, every nonzero coefficient of $q(x)$ must have zero constant term. Under the situation that the nonzero constant element 1 appears in the support of p_k with the smallest index k, and 1 does not appear in the support of any coefficient of $q(x)$, we have the following two cases.

Case 1. a_2 is not in the support of p_k.

Suppose that there exists a coefficient q_ℓ of $q(x)$ with a_0 in its support and ℓ is minimally chosen. If a_2 appears in the support of some coefficient among $\{p_0, \dots, p_{k-1}\}$, then we can find the smallest index $0 \le s \le k - 1$ such that a_2 is in the support of p_s and a_2 is not in the support of any $p_0, \dots, p_{s-1}$. Since 1 does not appear in the support of any $p_0, \dots, p_{k-1}$ and a_2 does not appear in the support of any $p_0, \dots, p_{s-1}$, we obtain that the nonzero element $a_2 a_0 = a_0$ must be in the support of the $(s + \ell)$-degree coefficient of $p(x)q(x)$. This is a contradiction. Therefore, a_2 must not appear in the support of any coefficient among $\{p_0, \dots, p_{k-1}\}$. This implies that a_0 must be in the support of the $(k + \ell)$-degree coefficient of $p(x)q(x)$. This is also a contradiction. Thus, a_0 cannot be in the support of any coefficients of $q(x)$. Similarly, a_1, a_2, b_0, b_1, c and so $a_1 b_0$ do not appear in the support of any coefficients of $q(x)$. Thus, $q(x) = 0$, which is a contradiction.

Case 2. a_2 is in the support of p_k.

Since 1 and a_2 are in the support of p_k, we easily see that $p_k c = 0$ by the relations.

Hence, we can reach that if there exists the smallest index k such that 1 is in the support of p_k, then 1 must not be in the support of any coefficient of $q(x)$ and a_2 must be in the support of p_k. Under this result, we finally assume that there exists another coefficient of $p(x)$ with 1 in its support among $\{p_{k+1}, \dots, p_m\}$. Let u be the smallest index such that p_u has a nonzero constant term for $k + 1 \le u \le m$. If a_2 is not in the support of p_u, then we can obtain a contradiction by the same argument of Case 1. Thus, a_2 must be in the support of p_u. Repeating this argument, we obtain that for each coefficient p_i of $p(x)$, if 1 appears in the support of p_i, then a_2 must be also in the support of p_i. Combining the above results, we can conclude that for each i,

$$\begin{cases} p_i b_0 = 0 \text{ or } p_i b_1 = 0 \text{ or } p_i(b_0 + b_1) = 0, & \text{if 1 does not appear in the support of } p_i \\ p_i c = 0, & \text{if 1 appears in the support of } p_i, \end{cases}$$

which confirms that the ring R is right ZPZC. □

Claim 3. The ring R is left McCoy.

Proof. Let $p(x) = \sum_{i=0}^{m} p_i x^i$ and $q(x) = \sum_{j=0}^{n} q_j x^j$ be nonzero polynomials with $p(x)q(x) = 0$ in $R[x]$. If $q(x)$ has no coefficient with 1 in its support, then $b_0 q(x) = 0$. Thus, we may assume that there exists a coefficient with 1 in its support. Let k be the smallest index such that q_k has this property. Then, we can show that each coefficient p_i of $p(x)$ must have zero constant term. Following a similar approach in Case 1 of Claim 2, we observe that a_2 must appear in the support of q_k. Following the same method as the last paragraph of Claim 2, we can also observe that a_2 must be in the support of q_v for each coefficient q_v with 1 of $q(x)$. Furthermore, if there is a coefficient q_s of $q(x)$ such that a_2 is in the support of q_s with zero constant term (under the case that a_2 is in the support of q_v for each coefficient q_v with 1 of $q(x)$), we can also obtain a contradiction. Thus, we can reach that a_2 must be in the support of each coefficient with 1 of $q(x)$, and a_2 cannot be in the support of each coefficient with zero constant term of $q(x)$. Hence, $cq(x) = 0$, leading to the conclusion that R is left McCoy. □

Capitalizing on Example 5.1, we have some remarks.

Remark 5.2.

(1) From Example 5.1, one may ask that if an (S, T)-bimodule M is McCoy as an S-module, then M is ZPZC as a T-module. As mentioned in [6, Example 2.9 and Remark 2.10(1)], the ring R in [4, Section 3] provides a negative answer.

(2) If we think $M = R_R$ in Example 5.1, then Example 5.1 shows that there exists a dual McCoy and ZPZC module but not McCoy. Further, note that M is both content and dual content ZPZC.

(3) Referring [13], we note that the ring R in Example 5.1 is left outer McCoy. Of course, the ring R is left Camillo. But there is no nonzero element $r \in R$ such that $rf(x) = rg(x) = 0$, for some nonzero polynomials $f(x), g(x)$ with $f(x)g(x) = 0$ in $R[x]$. To show this, consider the two nonzero polynomials $p(x) = (1 + a_2)$ and $q(x) = a_2$ with $p(x)q(x) = 0$. An easy calculation shows that if $rp(x) = rq(x) = 0$, then $r = 0$.

(4) The ring R in Example 5.1 is not right duo since $Rb_0 = \{0, b_0, a_1b_0, b_0 + a_1b_0\} \nsubseteq b_0R = \{0, b_0\}$ (or by [5, Proposition 8.2]). Thus, there exists a right ZPZC ring which is neither right McCoy nor right duo.

(5) For the ring R in Example 5.1, R^{op} is not right uniform since $a_0R^{op} \bigcap b_0R^{op} = \{0\}$. Thus, the converse of Theorem 3.9(1) does not be true in general by Example 5.1.

In [6, Questions 2.5 and 3.10], the author asked whether $\mathbb{T}_n(R)$ is a right ZPZC ring or not, where R is one of a domain, a reduced, a right ZPZC ring. We provide answers to these questions negatively.

Example 5.3. Let $E = \mathbb{Z}_2\langle a_0, a_1, a_2, b_1, c_0, c_1\rangle$ be the free algebra with identity and noncommuting indeterminates $a_0, a_1, a_2, b_1, c_0, c_1$ over $\mathbb{Z}_2$. Set $R = E/I$ where I is the ideal of E generated by the following relations:

$$a_0c_1 = b_1c_0, \; a_1c_1 = b_1c_1 + a_2c_0.$$

We simply denote $a = a + I$ in R. Note that R is a domain (and thus it is both reduced and ZPZC). To show that $\mathbb{T}_2(R)$ is not right ZPZC, consider the following nonzero polynomials in $\mathbb{T}_2(R)[x]$:

$$A(x) = \begin{bmatrix} a_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} a_2 & a_2 \\ 0 & 0 \end{bmatrix}x^2 \quad \text{and} \quad B(x) = \begin{bmatrix} 0 & 0 \\ 0 & c_0 \end{bmatrix} + \begin{bmatrix} 0 & c_1 \\ 0 & c_1 \end{bmatrix}x.$$

Then, $A(x)B(x) = 0$ by the relations. Suppose that $\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p & q \\ 0 & r \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{T}_2(R)$. Then, we obtain $p = q = r = 0$ in R. Thus, $\begin{bmatrix} a_1 & b_1 \\ 0 & 0 \end{bmatrix}$ is not a left zero-divisor in $\mathbb{T}_2(R)$, as desired.

Remark 5.4. Let M be an R-module. By Example 5.3, $\mathbb{T}_n(M)$ need not be a ZPZC $\mathbb{T}_n(R)$-module even when M is a reduced module over a domain R.

Every division ring is duo. Thus, if R is a division ring, then, $\mathbb{T}_n(R)$ is a right ZPZC ring by [6, Theorem 2.2]. Based on this fact, one may suspect that if M is a module over a division ring R, then $\mathbb{T}_n(M)$ is a ZPZC $\mathbb{T}_n(R)$-module for any positive integer n. But the following example eliminates the possibility of the suspicion.

Example 5.5. We use the ring R and module M in Example 2.3. Then, M is a ZPZC R-module. Consider the following nonzero polynomials

$$A(x) = \begin{bmatrix} m_2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix}x + \begin{bmatrix} m_1 + m_2 & m_1 + m_2 \\ 0 & 0 \end{bmatrix}x^2 \in \mathbb{T}_2(M)[x], \; B(x) = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{1} \end{bmatrix} + \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{0} & \bar{1} \end{bmatrix}x \in \mathbb{T}_2(R)[x].$$

Then, $A(x)B(x) = 0$. Suppose that $\begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \overline{p} & \overline{q} \\ \overline{0} & \overline{r} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ in $\mathbb{T}_2(M)$. Then, we obtain $\overline{p} = \overline{q} = \overline{r} = \overline{0}$ in R. Thus, $\mathrm{ann}_{\mathbb{T}_2(R)}\left(\begin{bmatrix} m_1 & m_2 \\ 0 & 0 \end{bmatrix}\right) = \left\{\begin{bmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{bmatrix}\right\}$, as needed.

Remark 5.6.

(1) Let $\mathbb{S}_2(R)$ be the set of all 2×2 scalar matrices over R in Example 5.5. Then, $\mathbb{S}_2(R)$ is a subring of $\mathbb{T}_2(R)$ with the same identity. Clearly, $\mathbb{T}_2(M)$ is a right $\mathbb{S}_2(R)$-module. Since R is (ring) isomorphic to $\mathbb{S}_2(R)$, $\mathbb{T}_2(M)$ is a McCoy $\mathbb{S}_2(R)$-module. Thus, we obtain that a McCoy R_1-module may not be ZPZC R_2-module, where R_1 is a subring of a ring R_2 with the same identity.

(2) We consider the rings $R_1 = \begin{bmatrix} \mathbb{Z}[x] & \mathbb{Z}[x] \\ 0 & \mathbb{Z} \end{bmatrix}$, $R_2 = \mathbb{T}_2(\mathbb{Z}[x])$, and the nonzero polynomials

$$F(y) = \begin{bmatrix} 0 & -x \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}y + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}y^2,\ G(y) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}y \in R_1[y]$$

in [6, Example 2.11(1)]. Then, R_1 is a subring of a right ZPZC ring R_2 with the same identity. Since the middle coefficient of $F(y)$ is a left zero-divisor in R_2 but not in R_1, $M = R_2$ is a ZPZC R_2-module but not a ZPZC R_1-module.

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References

[1] N. H. McCoy, *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942), no. 5, 286–295, https://doi.org/10.1080/00029890.1942.11991226.

[2] L. Weiner, *Concerning a theorem of McCoy: Solution to problem 4419*, Amer. Math. Monthly **59** (1952), no. 5, 336–337, DOI: https://doi.org/10.2307/2307526.

[3] M. B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), no. 1, 14–17, DOI: https://doi.org/10.3792/pjaa.73.14.

[4] P. P. Nielsen, *Semi-commutativity and the McCoy condition*, J. Algebra **298** (2006), no. 1, 134–141, DOI: https://doi.org/10.1016/j.jalgebra.2005.10.008.

[5] V. Camillo and P. P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), no. 3, 599–615, DOI: https://doi.org/10.1016/j.jpaa.2007.06.010.

[6] J. Baeck, *The rings where zero-divisor polynomials have zero-divisor coefficients*, Rocky Mountain J. Math. **51** (2021), no. 3, 771–785, DOI: http://doi.org/10.1216/rmj.2021.51.771.

[7] J. Cui and J. Chen, *On McCoy modules*, Bull. Korean Math. Soc. **48** (2011), no. 1, 23–33, DOI: https://doi.org/10.4134/BKMS.2011.48.1.023.

[8] T. K. Lee and Y. Zhou, *Reduced modules*, in: Rings, Modules, Algebras, and Abelian Groups, Lecture Notes in Pure Appllied Mathematics, Vol. 236, Dekker, New York, 2004, pp. 365–377.

[9] J. Lambek, *On the representations of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), no. 3, 359–368, DOI: https://doi.org/10.4153/CMB-1971-065-1.

[10] A. M. Buhphang and M. B. Rege, *Semi-commutative modules and Armendariz modules*, Arab. J. Math. Sci. **8** (2002), no. 1, 53–65.

[11] A. Ç. Özcan, A. Harmanci, and P. F. Smith, *Duo modules*, Glasg. Math. J. **48** (2006), no. 3, 533–545, DOI: https://doi.org/10.1017/S0017089506003260.

[12] D. D. Anderson and S. Chun, *McCoy modules and related modules over commutative rings*, Comm. Algebra **45** (2017), no. 6, 2593–2601, DOI: https://doi.org/10.1080/00927872.2016.1233218.

[13] J. Baeck, N. K. Kim, Y. Lee, and P. P. Nielsen, *Zero-divisor placement, a condition of Camillo, and the McCoy property*, J. Pure Appl. Algebra **224** (2020), no. 12, 106432, DOI: https://doi.org/10.1016/j.jpaa.2020.106432.

[14] V. Camillo, T. K. Kwak, and Y. Lee, *On a generalization of McCoy rings*, J. Korean Math. Soc. **50** (2013), no. 5, 959–972, DOI: https://doi.org/10.4134/JKMS.2013.50.5.959.

[15] C. Y. Hong, Y. C. Jeon, N. K. Kim, and Y. Lee, *The McCoy condition on noncommutative rings*, Comm. Algebra **39** (2011), no. 5, 1809–1825, DOI: https://doi.org/10.1080/00927872.2010.480952.

[16] M. T. Koşan, *Extensions of rings having McCoy condition*, Canad. Math. Bull. **52** (2009), no. 2, 267–272, DOI: https://doi.org/10.4153/CMB-2009-029-5.

[17] Z. Lei, J. Chen, and Z. Ying, *A question on McCoy rings*, Bull. Aust. Math. Soc. **76** (2007), no. 1, 137–141, DOI: https://doi.org/10.1017/S0004972700039526.

[18] R. Mazurek and M. Ziembowski, *On right McCoy rings and right McCoy rings relative to u.p.-monoids*, Commun. Contemp. Math. **17** (2015), no. 4, 15500496, DOI: https://doi.org/10.1142/S0219199715500492.

[19] M. Zahiri, A. Moussavi, and R. Mohammadi, *On a skew McCoy ring*, Comm. Algebra **47** (2019), no. 10, 4061–4065, DOI: https://doi.org/10.1080/00927872.2019.1576188.

[20] R. Zhao and Z. Liu, *Extensions of McCoy rings*, Algebra Colloq. **16** (2009), no. 3, 495–502, DOI: https://doi.org/10.1142/S1005386709000467.

[21] N. J. Groenewald and D. Ssevviiri, *Classical completely prime submodules*, Hacet. J. Math. Stat. **45** (2016), no. 3, 717–729, DOI: http://doi.org/10.15672/HJMS.20164513096.

[22] N. Agayev, G. Güngöroğlu, A. Harmanci, and S. Halicioğlu, *Abelian modules*, Acta Math. Univ. Comenian. **78** (2009), no. 2, 235–244.

[23] M. B. Rege and A. M. Buhphang, *On reduced modules and rings*, Int. Electron. J. Algebra **3** (2008), 58–74.

[24] R. Mazurek and M. Ziembowski, *Right Gaussian rings and skew power series rings*, J. Algebra **330** (2011), no. 1, 130–146, DOI: https://doi.org/10.1016/j.jalgebra.2010.11.014.

[25] D. D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), no. 7, 2265–2272, DOI: https://doi.org/10.1080/00927879808826274.

[26] W. Narkiewicz, *Polynomial Mappings*, Springer, Berlin, 1995.

[27] G. M. Bergman, *The diamond lemma for ring theory*, Adv. Math. **29** (1978), no. 2, 178–218, DOI: https://doi.org/10.1016/0001-8708(78)90010-5.

[28] J. Castro Pérez, M. Medina Bárcenas, J. RíosMontes, and A. Zaldívar Corichi, *On semiprime Goldie modules*, Comm. Algebra **44** (2016), no. 11, 4749–4768, DOI: https://doi.org/10.1080/00927872.2015.1113290.

[29] T. Y. Lam, Lectures on modules and rings, *Graduate Texts in Mathematics* vol. 189, Springer-Verlag, New York, 1999.

[30] T. W. Hungerford, *Algebra*, Springer, New York, 1974.

[31] K. R. Goodearl, *Von Neumann Regular Rings*, Pitman, London-San Francisco-Melbourne, 1979.

[32] N. K. Kim, T. K. Kwak, and Y. Lee, *On a generalization of right duo rings*, Bull. Korean Math. Soc. **53** (2016), no. 3, 925–942, DOI: https://doi.org/10.4134/BKMS.b150441.

[33] X. Yao, *Weakly right duo rings*, Pure Appl. Math. Sci. **21** (1985), 19–24.

[34] D. D. Anderson and T. Dumitrescu, *S-Noetherian rings*, Comm. Algebra **30** (2002), no. 9, 4407–4416, DOI: http://doi.org/10.1081/AGB-120013328.

[35] J. Baeck, *On S-principal right ideal rings*, AIMS Math. **7** (2022), no. 7, 12106–12122, DOI: http://doi.org/10.3934/math.2022673.

[36] C. Faith, *Rings and things and a fine array of twentieth century associative algebra*, Mathematical Surveys and Monographs, Vol. 65, American Mathematical Society, Providence, 1999.

[37] G. Marks, R. Mazurek, and M. Ziembowski, *A unified approach to various generalizations of Armendariz rings*, Bull. Aust. Math. Soc. **81** (2010), no. 3, 361–397, DOI: https://doi.org/10.1017/S0004972709001178.

[38] K. R. Goodearl and R. B. Warfield Jr., *An introduction to noncommutative Noetherian rings*, London Mathematical Society Student Texts, Vol. 61, 2nd ed., Cambridge University Press, Cambridge, 2004.

[39] R. Fraïssé, *Theory of relations*, Stud. Logic Found. Math, Vol. 145, North-Holland, Amsterdam, 2000.