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Research Article

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Inverse eigenvalue problems for rank one perturbations of the Sturm-Liouville operator

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Abstract: This article is concerned with the inverse eigenvalue problem for rank one perturbations of the Sturm-Liouville operator. I obtain the relationship between the spectra of the Sturm-Liouville operator and its rank one perturbations, and from the spectra I reconstruct the perturbed operators, provided that the potential is known *a priori*.

Keywords: inverse problem, Sturm-Liouville operator, rank one perturbation

MSC 2020: 34L05, 34A55, 34L40

1 Introduction

Consider the boundary problem

$$Ly \coloneqq -y'' + q(x)y + \alpha c(x)\int_0^1 \overline{c(x)}y\mathrm{d}x = \lambda y \tag{1.1}$$

on $(0, 1)$ with

$$y(0) = y(1) = 0, \tag{1.2}$$

where $q(x), c(x) \in L_2(0, 1)$, $q(x) = \overline{q(x)}$, and $\alpha = \pm 1$. It is known [1] that the operator L is a self-adjoint operator in $L^2(0, 1)$ and its spectrum consists of discrete eigenvalues. This problem is related to the boundary problem

$$L_0 y \coloneqq -y'' + q(x)y = \lambda y \tag{1.3}$$

on $(0, 1)$ with (1.2). The operator L_0 is also self-adjoint in $L_2(0, 1)$ and it has a discrete spectrum consisting of simple real eigenvalues, see [2]. The operator L is a rank one perturbation of the Sturm-Liouville operator L_0.

The goal of this article is to deal with the inverse problem of recovering the operator L from the spectra of L_0 and L, by applying the work in [1] and the perturbation theory for linear operators [3].

Such operator L appears not only in electronics but also in other areas such as the theory of diffusion processes, see [4]. Some spectral and inverse spectral problems for rank one perturbations of the Sturm-Liouville operator have been investigated in [1,5–20]. In particular, in [17] we deal with this problem by solving the initial value problem (1.1) with the initial conditions

$$y(0) = 0, y'(0) = h, h \in \mathbb{R},$$

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which may have non-unique solutions. In this article, we obtain the relationship of spectra of L_0, L by the method in [1] and the perturbation theory [3], which provides a necessary preliminary for treating with the inverse eigenvalue problem. Actually, the proof of the result is more brief and the approach we use can also solve inverse problems for the other operators, see [18].

The main result in this article is as follows:

Theorem 1.1. *Let $\{\gamma_i\}_{i=1}^{\infty}$, $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of the operators L_0, L, respectively, then $c(x)$ in* (1.1) *can be recovered by $\{\gamma_i\}_{i=1}^{\infty}$, $\{\lambda_i\}_{i=1}^{\infty}$.*

We will prove it in the next section.

2 The proof of the main theorem

We describe some preliminaries which will be needed subsequently, based on [1, p. 8–10]. Since the operator L_0 is bounded from below (see [2]), for simplicity, we suppose that $L_0 \geq 0$. One can define the scale of spaces $H_{\pm 1}(L_0)$ associated with L_0 as follows. The space $H_{+1}(L_0)$ is $D(L_0^{1/2})$ with the norm

$$\|\varphi\|_{H_{+1}} = \|(L_0 + 1)^{1/2}\varphi\|_{L^2(0,1)}$$

in which $H_{+1}(L_0)$ is easily seen to be complete. For $H_{-1}(L_0)$, take $L^2(0, 1)$ with the norm given by

$$\|\varphi\|_{H_{-1}} = \|(L_0 + 1)^{-1/2}\varphi\|_{L^2(0,1)}$$

and complete it. Note that $H_{+1}(L_0)$ and $H_{-1}(L_0)$ are dual in such a way that $\varphi \in H_{-1}(L_0)$ is associated with the function η given by $l_\varphi(\eta) = (\varphi(x), \eta(x))$. Clearly, $c(x)$ lies in $H_{-1}(L_0)$. Hence, one can define $F(z)$ by

$$F(z) = \int_0^1 (c(x), (L_0 - z)^{-1}c(x))\mathrm{d}x. \tag{2.1}$$

Combining with Section 2 in [1, p. 115], one can find a spectral measure $\mathrm{d}\mu > 0$ such that

$$F(z) = \int_0^1 \frac{\mathrm{d}\mu(x)}{x - z}. \tag{2.2}$$

The following preliminaries are due to [3, pp. 245–250]. By (2.1), $\alpha F(z) + 1$ is the Weinstein-Aronszajn determinant associated with L_0 and $(L - L_0)$. All zeros of $\alpha F(z) + 1$ are the eigenvalues of the operator L. The multiplicity index for $\alpha F(z) + 1$ is given by

$$\nu(\zeta;\, \alpha F(z) + 1) = \begin{cases} k, & \text{if } \zeta \text{ is a zero of } \alpha F(z) + 1 \text{ of order } k, \\ -k, & \text{if } \zeta \text{ is a pole of } \alpha F(z) + 1 \text{ of order } k, \\ 0, & \text{for all other } \zeta \in \mathbb{C}. \end{cases}$$

The multiplicity function for a closed operator T is defined by

$$\tilde{\nu}(\zeta;\, T) = \begin{cases} 0, & \text{if } \zeta \text{ belongs to the resolvent set}, \\ m, & \text{if } \zeta \text{ is an eigenvalue of } T \text{ with multiplicity } m, \\ +\infty, & \text{for all other } \zeta \in \mathbb{C}. \end{cases}$$

Let γ_i be the eigenvalue of L_0, $v_i(x)$ be the normalized eigenfunction corresponding to the eigenvalue γ_i, $i = 1, 2, \ldots$. Then

$$\tilde{\nu}(\zeta;\, L_0) = \begin{cases} 1, & \zeta \in \{\gamma_i\}_{i=1}^{\infty}, \\ 0, & \zeta \in \mathbb{C} \backslash \{\gamma_i\}_{i=1}^{\infty}. \end{cases}$$

By Theorem 6.2 in [3], there is

$$\tilde{\nu}(\zeta;\, L) = \nu(\zeta;\, \alpha F(z) + 1) + \tilde{\nu}(\zeta;\, L_0) = \begin{cases} 1 + \nu(\zeta;\, \alpha F(z) + 1),\ \zeta \in \{\gamma_i\}_{i=1}^{\infty}, \\ \nu(\zeta;\, \alpha F(z) + 1),\ \zeta \in \mathbb{C}\backslash\{\gamma_i\}_{i=1}^{\infty}. \end{cases} \tag{2.3}$$

In the following proposition, we give the relationship of the spectra of L_0 and L.

Proposition 2.1. *Let $\{\gamma_i\}_{i=1}^{\infty}$, $\{\lambda_i\}_{i=1}^{\infty}$ be the eigenvalues of the operators L_0, L, respectively, then the following relation holds*:

$$\gamma_i \le \lambda_i \le \gamma_{i+1} \tag{2.4}$$

or

$$\lambda_i \le \gamma_i \le \lambda_{i+1}. \tag{2.5}$$

Proof. By (2.1) and [2, p. 15],

$$F(z) = \int_0^1\int_0^1 G(x, y, z)c(x)\overline{c(y)}\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{\infty} \frac{|c_i|^2}{\lambda_i - z}, \tag{2.6}$$

where $G(x, y, z)$ is the Green's function for L_0 and $c_i = \int_0^1 c(x)v_i(x)\mathrm{d}x$. Thus, $F(z)$ is meromorphic with simple poles in the points $z = \gamma_{n_i}$, $n_i \in \{j \in \mathbb{N}^+ | c_j \ne 0\}$. Moreover, for $x \in \mathbb{R}\backslash\{\gamma_{n_i}\}_{i=1}^{\infty}$, it follows

$$F(x + i0) = F(x)$$

and

$$\frac{\mathrm{d}F(x + i0)}{\mathrm{d}x} = \int \frac{\mathrm{d}\mu(y)}{(x - y)^2} < \infty. \tag{2.7}$$

Let $\{a_i\}_{i=1}^{\infty}$ be the zero set of $\alpha F(x + i0) + 1$. In the case $\alpha = 1$, by (2.6) and (2.7),

$$\lim_{x\to\gamma_{n_i}^-} F(x + i0) = +\infty, \tag{2.8}$$

$$\lim_{x\to\gamma_{n_i}^+} F(x + i0) = -\infty, \tag{2.9}$$

and

$$\lim_{x\to-\infty} F(x + i0) = 0. \tag{2.10}$$

Then there is no zero of $F(x + i0) + 1$ in $(-\infty, \gamma_{n_1})$ and exactly one zero in $(\gamma_{n_i}, \gamma_{n_{i+1}})$, that is,

$$\gamma_{n_i} < a_i < \gamma_{n_{i+1}}. \tag{2.11}$$

All zeros of $F(z) + 1$ are the eigenvalues of the operator L, thus the zeros of $F(z) + 1$ are real and consist of the zeros of $F(x + i0)$. Then we have the multiplicity function

$$\nu(\zeta;\, F(z) + 1) = \begin{cases} 1, & \zeta \in \{a_i\}_{i=1}^{\infty}, \\ -1, & \zeta \in \left\{\gamma_{n_i}\right\}_{i=1}^{\infty}, \\ 0, & \text{for all other } \zeta \in \mathbb{C}. \end{cases}$$

Using (2.3), it follows

$$\{\lambda_i\}_{i=1}^{\infty} = \{a_i\}_{i=1}^{\infty} \cup \left(\{\gamma_i\}_{i=1}^{\infty}\backslash\left\{\gamma_{n_i}\right\}_{i=1}^{\infty}\right). \tag{2.12}$$

Combining with (2.11), we have (2.4). Similarly, in the case $\alpha = -1$, (2.5) holds. The proof is complete. □

Now we prove the main result.

Proof of Theorem 1.1. By (2.12), there is

$$\left\{\gamma_{n_i}\right\}_{i=1}^{\infty} = \{\gamma_i\}_{i=1}^{\infty}\backslash\{\lambda_i\}_{i=1}^{\infty},\ \{a_i\}_{i=1}^{\infty} = \{\lambda_i\}_{i=1}^{\infty}\backslash\left(\{\gamma_i\}_{i=1}^{\infty}\backslash\left\{\gamma_{n_i}\right\}_{i=1}^{\infty}\right). \tag{2.13}$$

Let

$$\xi(\lambda) = \alpha\frac{1}{\pi}Arg(1 + \alpha F(\lambda + i0)).$$

In the case $\alpha = 1$, by (2.8)–(2.10), we have

$$F(x + i0) > -1 \text{ for } x \in \left(-\infty, \gamma_{n_1}\right),$$
$$F(x + i0) < -1 \text{ for } x \in \left(\gamma_{n_i}, a_i\right),$$

and

$$F(x + i0) > -1 \text{ for } x \in \left(a_i, \gamma_{n_{i+1}}\right),$$

then

$$\xi(\lambda) = \begin{cases} 0, & \lambda \in \left(-\infty, \gamma_{n_1}\right), \\ 1, & \lambda \in \left(\gamma_{n_i}, a_i\right), \\ 0, & \lambda \in \left(a_i, \gamma_{n_{i+1}}\right). \end{cases}$$

Similarly, in the case $\alpha = -1$,

$$\xi(\lambda) = \begin{cases} 0, & \lambda \in (-\infty, a_1), \\ -1, & \lambda \in \left(a_i, \gamma_{n_i}\right), \\ 0, & \lambda \in \left(\gamma_{n_i}, a_{i+1}\right). \end{cases}$$

By (I.16) and (I.19) in [1],

$$\int \frac{\xi(\lambda)}{\lambda - z}\mathrm{d}\lambda = \ln(1 + \alpha F(z)),$$

that is,

$$F(z) = \alpha\prod_{i=1}^{\infty}\frac{a_i - z}{\gamma_{n_i} - z} - \alpha. \tag{2.14}$$

Combining with (2.6), $|c_i|^2 = \lim_{z\to\gamma_i}(\gamma_i - z)F(z)$. Thus, we can construct

$$c(x) = \sum_{i=1}^{\infty} c_i v_i(x), \tag{2.15}$$

where $v_i(x)$ is the normalized eigenfunction corresponding to the eigenvalue γ_i, and the proof is therefore complete. □

The boundary value problem (1.1)–(1.2) can be constructed by the following algorithm:

Algorithm 2.2.
(i) From the given sequences $\{\gamma_n, \lambda_n\}_{n=1}^{\infty}$, we construct the functions $F(z)$ by (2.13) and (2.14);
(ii) Find the number $|c_i|$ by $|c_i|^2 = \lim_{z\to\lambda_i}(\lambda_i - z)F(z)$;
(iii) Calculate $c(x)$ by (2.15).

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References

[1] B. Simon, *Spectral analysis of rank one perturbations and applications*, CRM Proc. Lecture Notes **8** (1995), 109–149.
[2] G. Freiling and V. Yurko, *Inverse Sturm-Liouville Problems and Their Applications*, NOVA Science Publishers, New York, 2001.
[3] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1980.
[4] A. D. Wentzell, *On boundary conditions for multidimensional diffusion processes*, Teor. Veroyatn. Primen. **4** (1959), 172–85.
[5] S. Albeverio and L. P. Nizhnik, *Schrödinger operators with nonlocal point interactions*, J. Math. Anal. Appl. **332** (2007), no. 2, 884–895, DOI: https://doi.org/10.1016/j.jmaa.2006.10.070.
[6] S. Albeverio, R. O. Hryniv, and L. P. Nizhnik, *Inverse spectral problems for non-local Sturm-Liouville operators*, Inverse Problems **23** (2007), no. 2, 523–535, DOI: https://doi.org/10.1088/0266-5611/23/2/005.
[7] M. H. Annaby, H. A. Hassan, and O. H. El-Haddad, *Perturbed discrete Sturm-Liouville problems and associated sampling theorems*, Rocky Mountain J. Math. **39** (2009), no. 6, 1781–1807, DOI: https://doi.org/10.1216/RMJ-2009-39-6-1781.
[8] O. Bourget, V. Cortes, R. del Rio, and C. Fernandez, *Resonances under rank-one perturbations*, J. Math. Phys. **58** (2017), no. 9, 093502, DOI: https://doi.org/10.1063/1.4989882.
[9] P. Freitas, *A nonlocal Sturm-Liouville eigenvalue problem*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), no. 1, 169–188, DOI: https://doi.org/10.1017/S0308210500029279.
[10] F. Gesztesy and B. Simon, *Rank one perturbations at infinite coupling*, J. Funct. Anal. **128** (1995), no. 1, 245–252, DOI: https://doi.org/10.1006/jfan.1995.1030.
[11] V. V. Kravchenko, *Direct and Inverse Sturm-Liouville Problems: A Method of Solution*, Springer, Cham, 2020.
[12] V. A. Marchenko, *Sturm-Liouville Operators and Applications: Revised Edition*, AMS Chelsea Publishing, Rhode Island, 2011.
[13] L. P. Nizhnik, *Inverse eigenvalue problems for nonlocal Sturm-Liouville operators*, Methods Funct. Anal. Topology **15** (2009), no. 1, 41–47.
[14] L. P. Nizhnik, *Inverse nonlocal Sturm-Liouville problem*, Inverse Problems **26** (2010), 635–684, DOI: https://doi.org/10.1088/0266-5611/26/12/125006.
[15] L. P. Nizhnik, *Inverse spectral nonlocal problem for the first order ordinary differential equation*, Tamkang J. Math **42** (2011), no. 3, 385–394, DOI: https://doi.org/10.5556/j.tkjm.42.2011.881.
[16] L. P. Nizhnik, *Inverse eigenvalue problems for nonlocal Sturm-Liouville operators on a star graph*, Methods Funct. Anal. Topology **18** (2012), no. 1, 68–78.
[17] X. Wu, P. Niu, and G. Wei, *An inverse eigenvalue problem for a nonlocal Sturm-Liouville operator*, J. Math. Anal. Appl. **494** (2021), no. 2, 124661, DOI: https://doi.org/10.1016/j.jmaa.2020.124661.
[18] X. Wu, *Inverse eigenvalue problems for singular rank one perturbations of a Sturm-Liouville operator*, Adv. Math. Phys. **2021** (2021), 1–5, DOI: https://doi.org/10.1155/2021/1681873.
[19] V. A. Zolotarev, *Direct and inverse problems for an operator with nonlocal potential*, Mat. Sb. **203** (2012), no. 2, 1785–1807, DOI: https://doi.org/10.1070/SM2012v203n12ABEH004287.
[20] V. A. Zolotarev, *Inverse spectral problem for the operators with non-local potential*, Math. Nachr. **292** (2019), no. 3, 1502–1523, DOI: https://doi.org/10.1002/mana.201700029.