DE GRUYTER Open Mathematics 2022; 20: 1715–1722

Research Article

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A Billingsley type theorem for Bowen topological entropy of nonautonomous dynamical systems

https://doi.org/10.1515/math-2022-0541
received July 19, 2022; accepted December 4, 2022

Abstract: This article is devoted to the study of the Bowen topological entropy for nonautonomous dynamical systems, which is an extension of the classical definition of Bowen topological entropy. We show that the Bowen topological entropy can be determined by the local entropies of measures for nonautonomous dynamical systems, which extends Ma and Wen's result.

Keywords: Bowen topological entropy, measure-theoretical lower entropy, nonautonomous dynamical system

MSC 2020: 54H20, 37B20, 37B40

1 Introduction

In 1958, Kolmogorov applied the notion of entropy from information theory to ergodic theory. Since then, the concepts of entropy, in particular the topological entropy and measure-theoretic entropy, were useful for studying topological and measure-theoretic structures of dynamical systems, that is, topological entropy (see [1–3]) and measure-theoretic entropy (see [4,5]). In 1973, Bowen [6] introduced the topological entropy for any set in a topological dynamical system by resembling the Hausdorff dimension. Bowen's topological entropy plays a key role in topological dynamics and dimension theory [7]. In 2012, Feng and Huang [8] defined the measure-theoretic entropy for Borel probability measures from the idea of Brin and Katok [9] and showed that there is a certain variational principle between Bowen topological entropy and measure-theoretic entropy for an arbitrary noninvariant compact set. The Hausdorff dimension can be determined with the help of Billingsley's theorem [10]. For the Bowen topological entropy of the autonomous discrete dynamical system, Ma and Wen [11] gave an analog Billingsley's theorem.

In contrast with the autonomous discrete dynamical systems, Kolyada and Snoha [12] introduced the topological entropy of nonautonomous discrete dynamical systems. Since then, the topological and measurable theory entropies of the nonautonomous systems were developed by many authors (see [13–22]). Recently, Xu and Zhou [23] introduced the measure-theoretic entropy for arbitrary Borel probability measure in nonautonomous dynamical systems (NADS) and gave the variational principles for entropies in nonautonomous cases. Liu and Zhao [24] studied the Bowen polynomial entropy for NADS and established a variational principle for polynomial entropy on compact subsets in the context of NADS. In this article, we extend the result of Ma and Wen [11] and show that the Bowen topological entropy can be determined via

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the local entropies of measures for the NADS, which can be considered as an analog of Billingsley's theorem for the Hausdorff dimension.

2 Preliminaries

Let $\mathbb{N}$ and $\mathbb{N}_+$ denote the sets of all nonnegative integers and all positive integers, respectively. We call (X, ϕ) is NADS if (X, d) is a compact metric space and $\phi : [0, +\infty) \times X \to X$ is a continuous map with $\phi(0, x) = x$ for every $x \in X$. Given a NADS (X, ϕ). For all $t \in [0, +\infty)$, we define

$$d_t^{\phi}(x, y) = \max_{0 \le s \le t}\{d(\phi(s, x), \phi(s, y))\}. \tag{1}$$

For all $t \in [0, +\infty)$, $\varepsilon > 0$, and $x \in X$, we set

$$B_t^{\phi}(x, \varepsilon) = \{y \in X : d_t^{\phi}(x, y) < \varepsilon\}. \tag{2}$$

Remark 2.1. From (1) and (2), if $t_2 \ge t_1 \ge 0$, we have $B_{t_2}^{\phi}(x, \varepsilon) \subseteq B_{t_1}^{\phi}(x, \varepsilon)$.

Let $M(X)$ denote the set of all Borel probability measures on X. Xu and Zhou [23] give the following definition from the idea of Brin and Katok [9].

Definition 2.1. [23] Let (X, ϕ) be a NADS and $\mu \in M(X)$. Then we call

$$\underline{h}_\mu(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to +\infty} -\frac{1}{t} \log \mu(B_t^{\phi}(x, \varepsilon))$$

the measure-theoretical lower local entropy of μ at the point $x \in X$.

Definition 2.2. [23] Let (X, ϕ) be a NADS and $\mu \in M(X)$. The measure-theoretical lower entropy of μ is defined by

$$\underline{h}_\mu(\phi) = \int \underline{h}_\mu(\phi, x) \mathrm{d}\mu(x).$$

Given a NADS (X, ϕ). We denote $\phi^i(x) := \phi(i, x)$ for $i \in \mathbb{N}$ and $x \in X$. Let $\Phi = \{\phi^i\}_{i=0}^{\infty}$. For any $n \in \mathbb{N}_+$ and $x, y \in X$, we define

$$d_n^{\Phi}(x, y) = \max\{d(\phi^i(x), \phi^i(y)) : 0 \le i < n\}. \tag{3}$$

For all $n \in \mathbb{N}$, $\varepsilon > 0$ and $x \in X$, set

$$B_n^{\Phi}(x, \varepsilon) = \{y \in X : d_n^{\Phi}(x, y) < \varepsilon\}. \tag{4}$$

Definition 2.3. [23] Let $\mu \in M(X)$. The measure-theoretical lower entropy of μ is defined by

$$\underline{h}_\mu(\Phi) = \int \underline{h}_\mu(\Phi, x) \mathrm{d}\mu(x),$$

where

$$\underline{h}_\mu(\Phi, x) = \lim_{\varepsilon \to 0} \liminf_{n \to +\infty} -\frac{1}{n} \log \mu(B_n^{\Phi}(x, \varepsilon)).$$

Next, we give the definition and some basic properties of Bowen's topological entropy for NADSs.

Let (X, ϕ) be a NADS and $Z \subseteq X$ a subset of X. Fix $\varepsilon > 0$, we call $\Gamma = \{B^{\phi}_{t_i}(x_i, \varepsilon)\}_i$ a cover of Z if $Z \subseteq \bigcup_i B^{\phi}_{t_i}(x_i, \varepsilon)$. For $\Gamma = \{B^{\phi}_{t_i}(x_i, \varepsilon)\}_i$, we denote $t(\Gamma) = \inf_i\{t_i\}$. Given $s \geq 0$, $N \in \mathbb{N}_+$, and $\varepsilon > 0$, we define

$$\mathcal{M}(Z, \phi, s, N, \varepsilon) = \inf_{\Gamma} \sum_i \exp(-st_i),$$

where the infimum is taken over all covers Γ of Z with $t(\Gamma) \geq N$. The quantity $\mathcal{M}(Z, \phi, s, N, \varepsilon)$ does not decrease as N increases, and hence, the following limit exists:

$$\mathcal{M}(Z, \phi, s, \varepsilon) = \lim_{N\to\infty} \mathcal{M}(Z, \phi, s, N, \varepsilon).$$

We can easy to prove that there exists a critical value,

$$h^B_{\text{top}}(\phi, Z, \varepsilon) = \inf\{s : \mathcal{M}(Z, \phi, s, \varepsilon) = 0\} = \sup\{s : \mathcal{M}(Z, \phi, s, \varepsilon) = +\infty\}.$$

Definition 2.4. The quantity

$$h^B_{\text{top}}(\phi, Z) = \lim_{\varepsilon\to 0} h^B_{\text{top}}(\phi, Z, \varepsilon)$$

is called the Bowen topological entropy of Z (with respect to ϕ).

Remark 2.2. From the Definition 2.4, we have the following results.
(1) For $Z_1 \subseteq Z_2 \subseteq X$, we have $h^B_{\text{top}}(\phi, Z_1) \leq h^B_{\text{top}}(\phi, Z_2)$.
(2) For $Z \subseteq \bigcup_{i=1}^{\infty} Z_i$, $s \geq 0$, and $\varepsilon > 0$, we have $\mathcal{M}(Z, \phi, s, \varepsilon) \leq \sum_{i=1}^{\infty} \mathcal{M}(Z_i, \phi, s, \varepsilon)$ and $h^B_{\text{top}}(\phi, Z) \leq \sup_{i\geq 1} h^B_{\text{top}}(\phi, Z_i)$.

Proof. The first statement follows directly from the definitions of Bowen topological entropy. We shall prove the second statement. Let $Z \subseteq \bigcup_{i=1}^{\infty} Z_i$, $s \geq 0$ and $\varepsilon > 0$. Given $\delta > 0$ and $i \in \mathbb{N}_+$, we can take $N_i \in \mathbb{N}_+$ and a cover $\Gamma_i = \{B^{\phi}_{t_{i_j}}(x_j, \varepsilon)\}_j$ of the set Z_i with $t(\Gamma_i) \geq N_i$ such that

$$\left| \mathcal{M}(Z_i, \phi, s, \varepsilon) - \sum_{j\geq 1} \exp\left(-st_{i_j}\right) \right| \leq \frac{\delta}{2^i}.$$

Since $Z \subseteq \bigcup_{i=1}^{\infty} Z_i$, it follows that the collection $\Gamma = \{B^{\phi}_{t_{i_j}}(x_j, \varepsilon), i \geq 1, j \geq 1\}$ is a cover of Z. Now we take $N = \inf_i\{N_i\}$, then $t(\Gamma) \geq N$. Hence, we have

$$\mathcal{M}(Z, \phi, s, N, \varepsilon) \leq \sum_{i\geq 1, j\geq 1} \exp\left(-st_{i_j}\right) \leq 2\delta + \sum_{i=1}^{\infty} \mathcal{M}(Z_i, \phi, s, \varepsilon). \tag{5}$$

Letting $N \to \infty$ and δ is arbitrarily small in (5), we obtain

$$\mathcal{M}(Z, \phi, s, \varepsilon) \leq \sum_{i=1}^{\infty} \mathcal{M}(Z_i, \phi, s, \varepsilon). \tag{6}$$

Next, we will prove that $h^B_{\text{top}}(\phi, Z) \leq \sup_{i\geq 1} h^B_{\text{top}}(\phi, Z_i)$. Let $s \geq 0$ and $\varepsilon > 0$. Assume that $h^B_{\text{top}}(\phi, Z_i) < s$ for $i = 1, 2, \ldots$. It follows that $\mathcal{M}(Z_i, \phi, s, \varepsilon) = 0$, and hence, by (6), we have $\mathcal{M}(Z, \phi, s, \varepsilon) = 0$, which implies that $h^B_{\text{top}}(\phi, Z, \varepsilon) \leq s$. Therefore, $h^B_{\text{top}}(\phi, Z) \leq \sup_{i\geq 1} h^B_{\text{top}}(\phi, Z_i)$. □

Let(X, ϕ) be a NADS and $\Phi = \{\phi^i\}_{i=0}^{\infty}$. For $Z \subseteq X$, $s \geq 0$, $n \in \mathbb{N}_+$, and $\varepsilon > 0$, we define

$$\mathcal{M}(Z, \Phi, s, N, \varepsilon) = \inf_{\Gamma} \sum_i \exp(-sn_i),$$

where the infimum is taken over all covers $\Gamma = \{B^{\Phi}_{n_i}(x_i, \varepsilon)\}_i$ of Z with $n(\Gamma) = \min_i n_i \geq N$. The quantity $\mathcal{M}(Z, \Phi, s, N, \varepsilon)$ does not decrease as N increases, and hence, we let

$$\mathcal{M}(Z, \Phi, s, \varepsilon) = \lim_{N\to\infty} \mathcal{M}(Z, \Phi, s, N, \varepsilon).$$

We denote the critical value

$$h^B_{\mathrm{top}}(\Phi, Z, \varepsilon) = \inf\{s : \mathcal{M}(Z, \Phi, s, \varepsilon) = 0\} = \sup\{s : \mathcal{M}(Z, \Phi, s, \varepsilon) = +\infty\}.$$

Definition 2.5. We call the quantity

$$h^B_{\mathrm{top}}(\Phi, Z) = \lim_{\varepsilon\to 0} h^B_{\mathrm{top}}(\Phi, Z, \varepsilon)$$

the Bowen topological entropy of Z (with respect to Φ).

Remark 2.3. From Definition 2.5, we have the following results.
(1) For $Z_1 \subseteq Z_2 \subseteq X$, we have $h^B_{\mathrm{top}}(\Phi, Z_1) \le h^B_{\mathrm{top}}(\Phi, Z_2)$.
(2) For $Z \subseteq \bigcup_{i=1}^{\infty} Z_i$, $s \ge 0$, and $\varepsilon > 0$, we have $\mathcal{M}(Z, \Phi, s, \varepsilon) \le \sum_{i=1}^{\infty} \mathcal{M}(Z_i, \Phi, s, \varepsilon)$ and $h^B_{\mathrm{top}}(\Phi, Z) \le \sup_{i\ge 1} h^B_{\mathrm{top}}(\Phi, Z_i)$.

3 Main results

In this section, we show in this note that an analogue of the Billingsley's Theorem does exist for Bowen's topological entropy of the NADS.

Lemma 3.1. *Let (X, ϕ) be a NADS with a metric d. Suppose $\varepsilon > 0$ and $\mathcal{B} = \{B^{\phi}_{t_i}(x_i, \varepsilon)\}_{i\in\mathcal{I}}$. Then for any $\mathcal{F} \subseteq \mathcal{B}$, there exists a finite or countable subfamily $\mathcal{G} = \{B^{\phi}_{t_i}(x_i, \varepsilon)\}_{i\in\mathcal{I}'}$ of pairwise disjoint balls in $\mathcal{B}$ such that*

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{i\in\mathcal{I}'} B^{\phi}_{t_i}(x_i, 3\varepsilon).$$

Proof. Let Ω denote the partially ordered (by inclusion) set consisting of all subfamilies ω of $\mathcal{F}$, which satisfies the following properties:
(1) ω consists of disjoint balls from $\mathcal{F}$.
(2) If a ball $B^{\phi}_{t_i}(x_i, \varepsilon) \in \mathcal{F}$ meets some ball from ω, then there exists $B^{\phi}_{t_j}(x_j, \varepsilon) \in \omega$ such that $t_j \le t_i$ and $B^{\phi}_{t_i}(x_i, \varepsilon) \cap B^{\phi}_{t_j}(x_j, \varepsilon) \ne \emptyset$.

If we take $t_x = \min\{t : B^{\phi}_{t}(x, \varepsilon) \in \mathcal{F}\}$, then $\omega_x = \{B^{\phi}_{t_x}(x, \varepsilon)\} \in \Omega$. This implies that $\Omega \ne \emptyset$. Let $C \subset \Omega$ be a chain. Then $\bigcup_{\omega\in C}\omega$ belongs to Ω and is an upper bound of C. Furthermore, there exists a maximal element $\mathcal{G}$ in Ω by Zorn's lemma.

We claim that each ball in $\mathcal{F}$ dose meet some ball in $\mathcal{G}$. Suppose this conclusion is not true. We take $t = \min\{t_i : B^{\phi}_{t_i}(x, \varepsilon) \cap B = \emptyset,\ \text{ for any } B \in \mathcal{G}\}$. It is easy to verify that the family $\mathcal{G} \cup B^{\phi}_{t}(x, \varepsilon)$ satisfies (1) and (2), and hence, $\mathcal{G} \cup B^{\phi}_{t}(x, \varepsilon) \in \Omega$. This contradicts the maximality of $\mathcal{G}$. For any $B^{\phi}_{t_i}(x_i, \varepsilon) \in \mathcal{F}$, there exists $t_j \le t_i$ and $B^{\phi}_{t_j}(x_j, \varepsilon) \in \mathcal{G}$ such that $B^{\phi}_{t_i}(x_i, \varepsilon) \cap B^{\phi}_{t_j}(x_j, \varepsilon) \ne \emptyset$. Since $B^{\phi}_{t_i}(x_i, \varepsilon) \subseteq B^{\phi}_{t_j}(x_j, \varepsilon)$, we have

$$B^{\phi}_{t_i}(x_i, \varepsilon) \subseteq B^{\phi}_{t_j}(x_j, \varepsilon) \subseteq B^{\phi}_{t_j}(x_j, 3\varepsilon)$$

by the triangle inequality. This finishes the proof of the lemma. □

Theorem 3.1. *Let (X, ϕ) be a NADS, $\mu \in M(X)$, $Z \subseteq X$ a Borel subset, and $s > 0$. If $\underline{h}_{\mu}(\phi, x) \le s$ for all $x \in Z$, then $h^B_{\mathrm{top}}(\phi, Z) \le s$.*

Proof. Let $Z \subseteq X$ be a Borel subset and let $s > 0$. For a fixed $r > 0$, we denote

$$Z_k = \left\{x \in Z : \liminf_{t\to+\infty} -\frac{1}{t}\log\mu(B_t^{\phi}(x,\varepsilon)) < s + r \text{ for all } \varepsilon \in \left(0, \frac{1}{k}\right)\right\}.$$

As $\underline{h}_\mu(\phi, x) \le s$ for all $x \in Z$, we have $Z = \bigcup_{k=1}^{\infty} Z_k$.

Now fix $k \in \mathbb{N}_+$ and $0 < \varepsilon < \frac{1}{5k}$. For each $x \in Z_k$, there exists a strictly increasing sequence $\{t_j(x)\}_{j=1}^{\infty}$ (where $t_{j+1}(x) - t_j(x) \ge 1$ for $j \in \mathbb{N}_+$) such that

$$\mu\left(B^{\phi}_{t_j(x)}(x,\varepsilon)\right) \ge \exp(-t_j(x)(s+r)) \text{ for all } j \in \mathbb{N}_+.$$

We take $\gamma_j(x) \in [0,1)$ such that $t_j(x) - \gamma_j(x) = n_j(x) \in \mathbb{N}^+$. Hence, $\{n_j(x)\}_{j=1}^{\infty}$ is a strictly increasing sequence and $B^{\phi}_{t_j(x)}(x,\varepsilon) \subseteq B^{\phi}_{n_j(x)}(x,\varepsilon)$, and we have

$$\mu\left(B^{\phi}_{n_j(x)}(x,\varepsilon)\right) \ge \exp(-(n_j(x)+1)(s+r)) \text{ for all } j \in \mathbb{N}_+.$$

Next, we use t_j and n_j instead of $t_j(x)$ and $n_j(x)$, respectively, for simplicity of notions if there is no confusion caused.

For any $N \ge 1$, the set Z_k is contained in the union of the sets in the family:

$$\mathcal{F}' = \left\{B^{\phi}_{t_j}(x,\varepsilon) : x \in Z_k,\ t_j \ge N\right\}.$$

Furthermore, the set Z_k is contained in the union of the sets in the family

$$\mathcal{F} = \left\{B^{\phi}_{n_j}(x,\varepsilon) : x \in Z_k,\ n_j \ge N\right\}.$$

Using Lemma 3.1, there exists a finite or countable subfamily $\mathcal{G} = \{B^{\phi}_{n_i}(x_i,\varepsilon)\}_{i\in\mathcal{I}} \subseteq \mathcal{F}$ of pairwise disjoint balls such that

$$Z_k \subseteq \bigcup_{i\in\mathcal{I}} B^{\phi}_{n_i}(x_i, 3\varepsilon).$$

Note that

$$\mu\left(B^{\phi}_{n_j}(x,\varepsilon)\right) \ge \exp(-(s+r))\exp(-n_j(s+r)) \text{ for all } j \in \mathbb{N}_+.$$

Since $\{B^{\phi}_{n_i}(x_i,\varepsilon)\}_{i\in\mathcal{I}}$ is the family of pairwise disjoint balls, it follows that

$$\mathcal{M}(Z_k,\phi,s+r,N,3\varepsilon) \le \sum_{i\in\mathcal{I}} \exp(-n_i(s+r)) \le \exp(s+r)\sum_{i\in\mathcal{I}}\mu\left(B^{\phi}_{n_i}(x_i,\varepsilon)\right) \le \exp(s+r),$$

where

$$\mathcal{M}(Z,\phi,s,N,\varepsilon) = \inf_{\Gamma}\sum_i \exp(-st_i),$$

and the infimum is taken over all covers Γ of Z with $t(\Gamma) \ge N$. Furthermore, we have

$$\mathcal{M}(Z_k,\phi,s+r,3\varepsilon) = \lim_{N\to\infty}\mathcal{M}(Z_k,\phi,s+r,N,3\varepsilon) \le \exp(s+r) \le +\infty,$$

which implies that $h^B_{\mathrm{top}}(\phi, Z_k, 3\varepsilon) \le s + r$ for all $0 < \varepsilon < \frac{1}{5k}$. Since ε is arbitrary, it follows that

$$h^B_{\mathrm{top}}(\phi, Z_k) \le s + r \text{ for all } k \in \mathbb{N}_+.$$

Moreover,

$$h^B_{\mathrm{top}}(\phi, Z) = h^B_{\mathrm{top}}\left(\phi, \bigcup_{k=1}^{\infty} Z_k\right) \le \sup_{k\ge1} h^B_{\mathrm{top}}(\phi, Z_k) \le s + r.$$

Therefore, $h^B_{\mathrm{top}}(\phi, Z) \le s$ for the arbitrariness of r. □

Theorem 3.2. *Let (X, ϕ) be a NADS, $\mu \in M(X)$, $Z \subseteq X$ a Borel subset, and $s > 0$. If $\underline{h}_\mu(\phi, x) \geq s$ for all $x \in Z$ and $\mu(Z) > 0$, then $h^B_{\mathrm{top}}(\phi, Z) \geq s$.*

Proof. Let $Z \subseteq X$ be a Borel subset and let $s > 0$. Fix an $r > 0$. For each $k \in \mathbb{N}_+$, set

$$Z_k = \left\{ x \in Z : \liminf_{t\to+\infty} -\frac{1}{t}\log\mu(B^\phi_t(x,\varepsilon)) > s - r \ \text{ for all } \varepsilon \in \left(0, \frac{1}{k}\right] \right\}.$$

Since $-\frac{1}{t}\log\mu(B^\phi_t(x,\varepsilon))$ increases as ε decreases, we have

$$Z_k = \left\{ x \in Z : \liminf_{t\to+\infty} -\frac{1}{t}\log\mu(B^\phi_t(x,\varepsilon)) > s - r, \varepsilon = \frac{1}{k} \right\}.$$

Moreover, since $\underline{h}_\mu(\phi, x) \geq s$ for all $x \in Z$, it follows that $Z_k \subseteq Z_{k+1}$ and $Z = \bigcup_{k=1}^\infty Z_k$. So by the continuity of the measure [25], then $\lim_{k\to\infty}\mu(Z_k) = \mu(Z) > 0$. Fix a sufficiently large $k \geq 1$ with $\mu(Z_k) > \frac{1}{2}\mu(Z) > 0$. For all $N \in \mathbb{N}_+$, set

$$\begin{aligned} Z_{k,N} &= \left\{ x \in Z : \liminf_{t\to+\infty} -\frac{1}{t}\log\mu(B^\phi_t(x,\varepsilon)) > s - r \ \text{ for all } t \geq N \text{ and } \varepsilon \in \left(0, \frac{1}{k}\right] \right\} \\ &= \left\{ x \in Z : \liminf_{t\to+\infty} -\frac{1}{t}\log\mu(B^\phi_t(x,\varepsilon)) > s - r, \ \varepsilon = \frac{1}{k} \right\}. \end{aligned}$$

Clearly, $Z_{k,N} \subseteq Z_{k,N+1}$ and $\bigcup_{N=1}^\infty Z_{k,N} = Z_k$. Moreover, we can take an $\bar{N} \in \mathbb{N}_+$ such that $\mu(Z_{k,\bar{N}}) > \frac{1}{2}\mu(Z_k) > 0$. For simplicity of notation, let $\bar{Z} = Z_{k,\bar{N}}$ and $\bar{\varepsilon} = \frac{1}{k}$. By the choice of $\bar{Z}$, we have

$$\mu(B^\phi_t(x,\varepsilon)) \leq \exp(-t(s - r)) \ \text{ for all } x \in \bar{Z}, 0 < \varepsilon < \bar{\varepsilon} \text{ and } t \geq N. \tag{7}$$

Fix a sufficiently large $N > \bar{N}$. For each cover, $\mathcal{F} = \left\{B^\phi_{t_i}\left(y_i, \frac{\varepsilon}{2}\right)\right\}_{i\in\mathcal{I}}$ of $\bar{Z}$ with $0 < \varepsilon < \bar{\varepsilon}$ and $t_i \geq N \geq \bar{N}$ for each $i \in \mathcal{I}$. Without the loss of generality, assume that $\bar{Z} \cap B^\phi_{t_i}\left(y_i, \frac{\varepsilon}{2}\right) \neq \emptyset$ for all $i \in \mathcal{I}$. Thus, there exists $x_i \in \bar{Z} \cap B^\phi_{t_i}\left(y_i, \frac{\varepsilon}{2}\right)$ for all $i \in \mathcal{I}$. Moreover, we have

$$B^\phi_{t_i}\left(y_i, \frac{\varepsilon}{2}\right) \subseteq B^\phi_{t_i}(x_i, \varepsilon) \ \text{ for all } i \in \mathcal{I}.$$

In combination with (7), which implies that

$$\sum_{i\in\mathcal{I}} \exp(-t_i(s - r)) \geq \sum_{i\in\mathcal{I}} \mu\left(B^\phi_{t_i}(x_i, \varepsilon)\right) \geq \mu(\bar{Z}) > 0.$$

Therefore,

$$\mathcal{M}\left(\bar{Z}, \phi, s - r, N, \frac{\varepsilon}{2}\right) \geq \mu(\bar{Z}) > 0.$$

Consequently, we have

$$\mathcal{M}\left(\bar{Z}, \phi, s - r, \frac{\varepsilon}{2}\right) = \lim_{N\to\infty} \mathcal{M}\left(\bar{Z}, \phi, s - r, N, \frac{\varepsilon}{2}\right) \geq \mu(\bar{Z}) > 0,$$

which implies that $h^B_{\mathrm{top}}\left(\phi, \bar{Z}, \frac{\varepsilon}{2}\right) \geq s - r$. Letting $\varepsilon \to 0$, we have $h^B_{\mathrm{top}}(\phi, \bar{Z}) \geq s - r$. Since $h^B_{\mathrm{top}}(\phi, Z) \geq h^B_{\mathrm{top}}(\phi, \bar{Z})$, it follows that $h^B_{\mathrm{top}}(\phi, Z) \geq s - r$. This shows that $h^B_{\mathrm{top}}(\phi, Z) \geq s$ because r is arbitrary. □

By using the same method of Theorems 3.1 and 3.2, we can prove the following result.

Theorem 3.3. *Let (X, φ) be a NADS, $\Phi = \{\phi^i\}_{i=0}^\infty$, $\mu \in M(X)$, $Z \subseteq X$ a Borel subset, and $s \geq 0$.*
(1) *If $\underline{h}_\mu(\Phi, x) \leq s$ for all $x \in Z$, then $h^B_{\mathrm{top}}(\Phi, Z) \leq s$.*
(2) *If $\underline{h}_\mu(\Phi, x) \geq s$ for all $x \in Z$ and $\mu(Z) > 0$, then $h^B_{\mathrm{top}}(\Phi, Z) \geq s$.*

As application of Theorems 3.1 and 3.2, we give an alternative proof of the following result of Bowen [6], which generalizes one-half of the well-known variational principle for entropy.

Theorem 3.4. *Let* (X, ϕ) *be a NADS and* $\mu \in M(X)$*. Then* $\underline{h}_\mu(\phi) \leq h^B_{\mathrm{top}}(\phi, Z)$ *for all* $Z \subseteq X$ *with* $\mu(Z) = 1$.

Proof. Since $\underline{h}_\mu(\phi) = \int \underline{h}_\mu(\phi, x)\mathrm{d}\mu(x)$ and $\mu(Z) = 1$, it follows that

$$Z_\delta = \{x \in Z : \underline{h}_\mu(\phi, x) \geq \underline{h}_\mu(\phi) - \delta\}$$

has positive μ-measure for all $\delta > 0$, that is, $\mu(Z_\delta) > 0$. Moreover, by Theorem 3.2, we have $h^B_{\mathrm{top}}(\phi, Z_\delta) \geq \underline{h}_\mu(\phi) - \delta$. Again, $Z_\delta \subseteq Z$ for all $\delta > 0$, which implies that $h^B_{\mathrm{top}}(\phi, Z) \geq h^B_{\mathrm{top}}(\phi, Z_\delta)$ and $h^B_{\mathrm{top}}(\phi, Z) \geq \underline{h}_\mu(\phi) - \delta$. Therefore, $\underline{h}_\mu(\phi) \leq h^B_{\mathrm{top}}(\phi, Z)$ for the arbitrariness of $\delta > 0$. □

By using the same method of Theorem 3.4, we can prove the next result.

Theorem 3.5. *Let* (X, ϕ) *be a NADS,* $\Phi = \{\phi^i\}_{i=0}^\infty$ *and* $\mu \in M(X)$*. Then* $\underline{h}_\mu(\Phi) \leq h^B_{\mathrm{top}}(\Phi, Z)$ *for all* $Z \subseteq X$ *with* $\mu(Z) = 1$.

Acknowledgements: The authors thank the referees for their valuable suggestions and comments.

Funding information: The work was supported by the Foundation in Higher Education institutions of Henan Province, PR China (No. 23A110020).

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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