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Research Article

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Some new characterizations of finite *p*-nilpotent groups

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Abstract: In this article, some new sufficient conditions of p-nilpotency of finite groups are obtained by using c-normality and Φ-supplementary of the maximal or the 2-maximal subgroups of the Sylow p-subgroups.

Keywords: c-normal subgroups, Φ-supplemented subgroups, 2-maximal subgroups, p-nilpotency

MSC 2020: 20D05, 20D10, 20D20

1 Introduction

All groups considered are finite.

For a group G, we denote by $\Phi(G)$ the intersection of all maximal subgroups of G. Let H be a subgroup of G. H is said to be supplemented in G provided that there exists a subgroup T of G such that $G = HT$. The supplemented subgroups have a significant influence on the structure of finite groups. It was proved by Kegel in [1,2] that a group G is soluble if every maximal subgroup of G has a cyclic supplement in G or if some nilpotent subgroup of G has a nilpotent supplement in G. In [3], Wang introduced the concept of c-normality of subgroups. H is said to be c-normal in G [3] if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \le H_G$. Furthermore, Yu [4] studied the relationship between Φ-supplemented subgroups and the structure of finite groups. We say that H is Φ-supplemented in G [4] if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \le \Phi(H)$. By using these special supplemented subgroups, many authors have obtained a series of interesting results (see [3–9]). We further carried out this study and obtained some new criteria for the p-nilpotency of finite groups in terms of c-normality and Φ-supplementary of the maximal or 2-maximal subgroups of the Sylow p-subgroups.

All other unexplained notions and terminology are standard and the reader is referred to [10].

2 Preliminaries

In this section, we recall some facts, which will be used in this article.

Lemma 2.1. *Suppose that H is c-normal in G. Then, the following statements hold*:
(1) *If $H \le M \le G$, then H is c-normal in M.*

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(2) *If* $N \trianglelefteq G$ and $N \leq H$, *then* H/N *is c-normal in* G/N.
(3) *If* $N \trianglelefteq G$ and $(|H|, |N|) = 1$, *then* HN/N *is c-normal in* G/N.

Proof. See [3, Lemma 2.1]. □

Lemma 2.2. *Suppose that H is* Φ*-supplemented in G. Then, the following statements* (1)–(3) *hold*:
(1) *If* $H \leq M \leq G$, *then H is* Φ*-supplemented in M.*
(2) *If* $N \trianglelefteq G$ *and* $N \leq H$, *then* H/N *is* Φ*-supplemented in* G/N.
(3) *If* $N \trianglelefteq G$ *and* $(|H|, |N|) = 1$, *then* HN/N *is* Φ*-supplemented in* G/N.

Proof. See [4]. □

Lemma 2.3. *Suppose that N is normal in G and G/N is a p-nilpotent group, where p is a prime divisor of* $|G|$ *and* $(|G|, p-1) = 1$. *If* $|N| = p$, *then G is p-nilpotent.*

Proof. Since $|N| = p$, $|\mathrm{Aut}(N)| = p - 1$ and $N \leq C_G(N)$. Because N is normal in G and $(|G|, p-1) = 1$, $(|N_G(N)/C_G(N)|, p-1) = 1$. Since $N_G(N)/C_G(N)$ is isomorphic to some subgroup of $\mathrm{Aut}(N)$, $N_G(N) = C_G(N)$, that is, $N \leq Z(G)$. Hence, G is p-nilpotent by G/N is p-nilpotent. □

Lemma 2.4. *Let G be* A_4*-free and p be prime divisor of* $|G|$ *with* $(|G|, p-1) = 1$. *If* $p^3 \nmid |G|$, *then G is p-nilpotent.*

Proof. See [11, Lemma 2.8]. □

Lemma 2.5. *If P is a Sylow p-subgroup of G, where p is a prime divisor of* $|G|$, *and* $N \trianglelefteq G$ *such that* $P \cap N \leq \Phi(P)$, *then N is p-nilpotent.*

Proof. See [10, Chapter 4, Theorem 4.7]. □

3 Main results

Theorem 3.1. *Suppose that P is a Sylow p-subgroup of a group G, where p is a prime divisor of* $|G|$ *and* $(|G|, p-1) = 1$. *If every maximal subgroup of P is c-normal or* Φ*-supplemented in G, then G is p-nilpotent.*

Proof. Suppose that the statement is not true, and let G be a counterexample of minimal order. Then, we have the following steps.

(1) There exists a unique minimal normal subgroup N in G. Moreover, G/N is p-nilpotent.

We pick a minimal normal subgroup of G, say N. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. Let M/N be a maximal subgroup of PN/N and set $H = M \cap P$. Then, $M = M \bigcap PN = (M \cap P)N = HN$ and $H \leq P$. Therefore,

$$|P : H| = |P : M \cap P| = \frac{|P|}{|M||P|/|MP|} = \frac{|MP|}{|M|} = \frac{|NP/N|}{|M/N|} = p,$$

that is, H is a maximal subgroup of P. By the hypotheses, there exists a normal subgroup T of G such that $H \cap T \leq H_G$ or $H \cap T \leq \Phi(H)$. Suppose that N is not a subgroup of T. Then, $N \cap T = 1$. Since both N and T are normal in G, $|NT| = |N||T|$ and $|NT|\,||G|$. Because, $|HT/T| = |H/H \cap T|$ is a power of p, N is an abelian subgroup of G and $N < P$. Thus, M is the maximal subgroup of P. By the hypotheses, M is c-normal or Φ-supplemented in G. It follows that M/N is c-normal or Φ-supplemented in G/N by Lemmas 2.1 and 2.2. Now, we assume that $N \leq T$. Since $G = HT$ and T is normal in G, $G/N = HT/N = (M/N)(T/N)$ and T/N is normal in G/N. If $H \cap T \leq H_G$, then $(M/N) \cap (T/N) = (HN \cap T)/N = (H \cap T)N/N \leq H_G N/N \leq$

$(HN/N)_{(G/N)} = (M/N)_{(G/N)}$. If $H \cap T \leq \Phi(H)$, then $(M/N) \cap (T/N) = (HN \cap T)/N = (H \cap T)N/N \leq \Phi(H)N/N \leq \Phi(HN/N) = \Phi(M/N)$. Therefore, M/N is c-normal or Φ-supplemented in G/N. Obviously, $(|G/N|, p-1) = 1$. Hence, G/N satisfies the hypotheses. By the choice of G, G/N is p-nilpotent. Because the class of all p-nilpotent groups forms a saturated formation, we deduce that N is the only minimal normal subgroup in G.

(2) N is not p-nilpotent.

Assume that N is p-nilpotent. Let L be the normal p-complement of N. Because L char N and N is normal in G, L is normal in G. The minimal normality of N shows that $L = 1$, that is, N is a p-subgroup. Since G/N is p-nilpotent, $\Phi(G) = 1$. Let M be a maximal subgroup of G with $G = [N]M$. Suppose that K is a Sylow p-subgroup of M such that $P = [N]K$. Let A be a maximal subgroup of N and A is normal in P. Set $H = AK$. Then, H is a maximal subgroup of P. By the hypotheses, there exists a normal subgroup T of G such that $H \cap T \leq H_G$ or $H \cap T \leq \Phi(H)$. Because N is the unique minimal normal subgroup of G, $N \leq T$. If $H \cap T \leq \Phi(H)$, then $H = H \cap P = H \cap NK = (H \cap N)K \leq (H \cap T)K \leq \Phi(H)K$. Since $H = AK$, $H = K$. Because $P = [N]H$ and H is a maximal subgroup of P, $|N| = p$. By (1) and Lemma 2.3, G is p-nilpotent. This contradiction shows that $H \cap T \leq H_G$ and $H_G \neq 1$. Then, $N \leq H_G$ by (1). Thus, $P = [N]K \leq H$, a contradiction.

(3) The finial contradiction.

If $NP < G$, then NP satisfies the hypotheses. The choice of G yields that NP is p-nilpotent, and so N is p-nilpotent, a contradiction by Step (2). Therefore, $NP = G$. Since $G/N = NP/N$ is p-subgroup, there exists a normal subgroup M/N of G/N such that $|G : M| = p$. Because P is a Sylow p-subgroup of G, $G = PM$. Then, $|P : P \cap M| = |PM : M| = p$, that is, $P \cap M$ is a maximal subgroup of P. Set $H = P \cap M$. By the hypotheses, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$ or $H \cap T \leq \Phi(H)$. Because N is the unique minimal normal subgroup of G, $N \leq T$ and $N \leq M$. Suppose first that $H \cap T \leq \Phi(H)$. Since H is normal in P, $\Phi(H) \leq \Phi(P)$. Then, $P \cap N \leq P \cap (M \cap T) = (P \cap M) \cap T = H \cap T \leq \Phi(H) \leq \Phi(P)$. By Lemma 2.5, N is p-nilpotent, a contradiction by Step (2). If $H \cap T \leq H_G$ and $H_G \neq 1$, then, $N \leq H_G$ by the unique minimal normality of N. Therefore, N is p-nilpotent. This is the final contradiction and the proof is completed. □

Corollary 3.1. *Assume that P is a Sylow p-subgroup of G, where p is the smallest prime divisor of $|G|$. Suppose that every maximal subgroup of P is c-normal or Φ-supplemented in G. Then, G is p-nilpotent.*

Corollary 3.2. *Suppose that every maximal subgroup of any Sylow subgroup of a group is c-normal or Φ-supplemented in G. Then, G is a Sylow tower group of supersolvable type.*

Proof. Let p be the smallest prime dividing $|G|$ and P be a Sylow p-subgroup of G. By Corollary 3.1, G is p-nilpotent. Let K be the normal p-complement of G. By Lemmas 2.1 and 2.2, K satisfies the hypothesis of the corollary. It follows that K is a Sylow tower group of supersolvable type by induction, which implies that G is also a Sylow tower group of supersolvable type. □

Corollary 3.3. *Assume that P is a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ and $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is c-normal in G. Then, G is p-nilpotent.*

Corollary 3.4. *Assume that P is a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ and $(|G|, p-1) = 1$. Suppose that every maximal subgroup of P is Φ-supplemented in G. Then, G is p-nilpotent.*

Corollary 3.5. *Let p be a prime dividing the order of G with $(|G|, p-1) = 1$ and E be a normal subgroup of G such that G/E is p-nilpotent. Suppose that P is a Sylow p-subgroup of E and every maximal subgroup of P is c-normal or Φ-supplemented in G. Then, G is p-nilpotent.*

Proof. By Lemmas 2.1 and 2.2, every maximal subgroup of P is c-normal or Φ-supplemented in E. Obviously, $(|E|, p-1) = 1$. By Theorem 3.1, E is p-nilpotent. Let T be the normal p-complement of E, then T is normal in G. Suppose that $T \neq 1$. Then, by Lemmas 2.1 and 2.2, the factor group G/T and its normal subgroup E/T satisfy the hypotheses. Thus, by induction, we have that G/T is p-nilpotent. It follows that G is p-nilpotent, as expected. Now, we suppose that $T = 1$. Then, $P = E$. Let K/P be the normal p-complement of G/P. Then, K is normal in G and G/K is p-group. It is easy to see that K satisfies the hypotheses of Theorem 3.1. Hence, K is p-nilpotent. Let S be the normal p-complement of K. Because G/K is p-group, S is the normal p-complement of G, which implies that G is p-nilpotent. □

Theorem 3.2. *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ and $(|G|, p-1) = 1$. Suppose that G is A_4-free and every 2-maximal subgroup of P is c-normal or Φ-supplemented in G. Then, G is p-nilpotent.*

Proof. Suppose that the assertion is not true, and let G be a counterexample with minimal order. By Lemma 2.4, $p^3 \| |G|$. We proceed via the following steps.

(1) G contains a unique minimal normal subgroup N with G/N p-nilpotent.

Let N be a minimal normal subgroup of G. Since P is a Sylow p-subgroup of G, PN/N is a Sylow p-subgroup of G/N. Let M/N be a 2-maximal subgroup of PN/N and set $H = M \cap P$. Then, $M = M \cap PN = (M \cap P)N = HN$ and $H \le P$. Therefore,

$$|P : H| = |P : M \cap P| = \frac{|P|}{|M||P|/|MP|} = \frac{|MP|}{|M|} = \frac{|NP/N|}{|M/N|} = p^2,$$

that is, H is a 2-maximal subgroup of P. By the hypotheses, there exists a normal subgroup T of G such that $H \cap T \le H_G$ or $H \cap T \le \Phi(H)$. Since N is a minimal normal subgroup of G, we have that $N \cap T = 1$ or N. Suppose first that $N \cap T = 1$. Then, $|NT| = |N||T|$ and $|NT|\,||G|$. Because, $|HT/T| = |H/H \cap T|$ is a power of p, N is an abelian subgroup of G and $N < P$. Thus, $M = H$ is a 2-maximal subgroup of P and M is c-normal or Φ-supplemented in G. It follows that M/N is c-normal or Φ-supplemented in G/N by Lemmas 2.1 and 2.2. Now, we assume that $N \le T$. Since $G = HT$ and T is normal in G, $G/N = HT/N = (M/N)(T/N)$ and T/N is normal in G/N. If $H \cap T \le H_G$, then $(M/N) \cap (T/N) = (HN \cap T)/N = (H \cap T)N/N \le H_G N/N \le (HN/N)_{(G/N)} = (M/N)_{(G/N)}$. If $H \cap T \le \Phi(H)$, then $(M/N) \cap (T/N) = (HN \cap T)/N = (H \cap T)N/N \le \Phi(H)N/N \le \Phi(HN/N) = \Phi(M/N)$. Therefore, M/N is c-normal or Φ-supplemented in G/N. Obviously, $(|G/N|, p-1) = 1$ and G/N is A_4-free. Hence, G/N satisfies the hypotheses. By the choice of G, G/N is p-nilpotent. Because all p-nilpotent groups form a saturated formation, N is unique in G.

(2) $O_p(G) \ne 1$.

Suppose that $O_p(G) = 1$. Let H be a 2-maximal subgroup of P and H is normal in P. Then, $\Phi(H) \le \Phi(P)$. By the hypotheses, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \le H_G$ or $H \cap T \le \Phi(H)$. Since $O_p(G) = 1$ and H is a p-subgroup, $H_G = 1$. Therefore, $H \cap T \le \Phi(H)$. Since $\Phi(H) < H$, $T < G$. Because $G/T = HT/T \cong H/H \cap T$ is p-subgroup, G/T is p-subgroup. We can take a maximal normal subgroup M/T of G/T such that $|G : M| = p$. Set $K = M \cap P$ and L is a maximal subgroup of K. Since $G = HT$ and $T \le M$, $G = HM$. Because $p = |K : L| = |(M \cap P) : L| = \frac{|M||P|}{|MP||L|} = \frac{|M||P|}{|G||L|} = \frac{|P|}{p|L|}$, L is a 2-maximal subgroup of P. By the hypotheses, L is c-normal or Φ-supplemented in G. It follows that L is c-normal or Φ-supplemented in M by Lemmas 2.1 and 2.2. Since $(|G|, p-1) = 1$ and $M < G$, $(|M|, p-1) = 1$. The foregoing arguments show that M satisfies the hypotheses. By the choice of G, M is p-nilpotent. Let S be the normal p-complement of M. Because G/M is p-group, S is the normal p-complement of G. This contradiction shows that $O_p(G) \ne 1$.

(3) The final contradiction.

By (1) and (2), N is the unique minimal normal subgroup of G and $N \le O_p(G)$. Let H be a 2-maximal subgroup of P. By the hypotheses, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \le H_G$ or $H \cap T \le \Phi(H)$. If $T < G$, discussing as in Step (2), one can prove that G is p-nilpotent, a contradiction. Thus, $T = G$. It follows that $H = H \cap T = H_G$ is normal in G. By (1) and (2), N is the unique minimal normal subgroup of G, $N \le O_p(G)$, and there exists a maximal subgroup M of G such that $G = [N]M$. Let K be a Sylow p-subgroup of M such that $P = [N]K$. Let A be a maximal subgroup of N and A be normal in P. Let B be a maximal subgroup of K. Thus, $AB \le P$ and AB is a 2-maximal subgroup of P. The choice of N shows that $N \le AB$ since AB is normal in G by previous arguments, a final contradiction. □

Corollary 3.6. *Let p be a prime dividing the order of G with $(|G|, p-1) = 1$ and E be a normal subgroup of G such that G/E is p-nilpotent. Suppose that P is a Sylow p-subgroup of E and every 2-maximal subgroup of P is c-normal or Φ-supplemented in G. If G is A_4-free, then G is p-nilpotent.*

Proof. By arguments similar to those used in the proof of Corollary 3.5, one can prove this result.

Theorem 3.3. *Assume that P is a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Suppose that $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal or Φ-supplemented in G. Then, G is p-nilpotent.*

Proof. If $p = 2$, then by Theorem 3.1, G is p-nilpotent. Now we prove the theorem for the case of odd prime p. Suppose that the statement is not true, and let G be a counterexample of minimal order. If $p^3 \nmid |G|$, then P is abelian. Let K be the normal p-complement of $N_G(P)$, then $N_G(P) = P \times K$. Thus, $[P, H] = 1$. It follows that $C_G(P) = P \times K = N_G(P)$. By the famous theorem of Burnside, G is p-nilpotent. Thus, $p^3 || G|$. We proceed via the following steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, by Lemmas 2.1 and 2.2, $G/O_{p'}(G)$ satisfies the hypotheses. The choice of G yields that $G/O_{p'}(G)$ is p-nilpotent. Consequently, G is p-nilpotent, a contradiction. Hence, $O_{p'}(G) = 1$.

(2) If M is a proper subgroup of G with $P \leq M$, then M is p-nilpotent.

Since $N_M(P) = N_G(P) \cap M$ and $N_G(P)$ is p-nilpotent, $N_M(P)$ is p-nilpotent. By Lemmas 2.1 and 2.2, M satisfies the hypotheses. The choice of G yields that M is p-nilpotent.

(3) G is not a non-abelian simple group and G has unique minimal normal subgroup N. Moreover, G/N is p-nilpotent and $\Phi(G) = 1$.

Let H be a maximal subgroup of P. By the hypotheses, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$ or $H \cap T \leq \Phi(H)$. If $T = G$, then $H \cap T = H_G$ is normal in G. Otherwise, $H = 1$ and $|P| = p$, which contradicts the fact that $p^3 || G|$ by previous argument. If $T \neq G$, then T is a proper subgroup of G and $T \trianglelefteq G$. Therefore, G is not a non-abelian simple group. By arguments similar to those used in the proof of Theorem 3.1, one can see that the remaining assertions hold.

(4) $G = PQ$ is solvable, where Q is a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by [12, Corollary], there exists a characteristic subgroup L of P such that $N_G(L)$ is not p-nilpotent. By (2), $N_G(L) = G$. This leads to $N \leq L$. By (3), G is p-solvable. Then, for any $q \in \pi(G)$ and $q \neq p$, there exists a Sylow q-subgroup Q of G such that $K = PQ$ is a subgroup of G. If $K \neq G$, then by (2), K is p-nilpotent. By [13, Theorem 9.3.1], $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus, $G = K = PQ$ is solvable.

(5) The final contradiction.

By (1) and (2), N is the unique minimal normal subgroup of G and $N \leq O_p(G)$. By Step (3), there exists a maximal subgroup M of G such that $G = MN$ and $M \cap N = 1$. Since N is an elementary abelian p-group, $N \leq C_G(N)$ and $C_G(N) \cap M \trianglelefteq G$. By the uniqueness of N, we have $C_G(N) \cap M = 1$ and $N = C_G(N)$. But $N \leq O_p(G) \leq F(G) \leq C_G(N)$, hence $N = O_p(G) = C_G(N)$. If $P \cap M = P$, then $N \leq P \leq M$, a contradiction. Thus, we take a maximal subgroup H of P such that $P \cap M \leq H$. If $P \cap M = 1$, then $P = N$. It follows that $N_G(P) = G$ is p-nilpotent, a contradiction. Therefore, $P \cap M \neq 1$. By the hypotheses, there exists a normal subgroup T of G such that $H \cap T \leq H_G$ or $H \cap T \leq \Phi(H)$. By the uniqueness of N, $N \leq T$. We assert that $|N| = p$.

If $H \cap T \leq H_G$, then $H \cap N = H \cap T \cap N \leq H_G \cap N \leq H \cap N$. Consequently, we have that $H \cap N = H_G \cap N$ is normal in G, and therefore, $H \cap N = N$ or $H \cap N = 1$. Assume that $H \cap N = N$. Then, $N \leq H$. Since $P = P \cap NM = N(P \cap M)$ and $P \cap M \leq H$, $P = H$. This contradiction shows that $H \cap N = 1$. Since $P = P \cap NM = N(P \cap M) = NH$ and $|N : H \cap N| = |NH : H| = |P : H| = p$, $|N| = p$.

If $H \cap T \leq \Phi(H)$, then $H = H \cap P = H \cap N(P \cap M) \leq (H \cap N)(P \cap M) \leq (H \cap T)H \leq \Phi(H)H = H$. Thus, $P \cap M = H$ and $|N| = p$.

Since $|N| = p$, $\mathrm{Aut}(N)$ is cyclic of order $p - 1$. If $q > p$, then HQ is p-nilpotent, and thus $Q \leq C_G(N) = N$, a contradiction. On the other hand, if $q < p$, then $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$ because $N = C_G(N)$. Hence, M and, in particular, Q are cyclic. Since Q is a cyclic group and $q < p$, we know that G is q-nilpotent. It follows that P is normal in G. This implies that $N_G(P) = G$ is p-nilpotent, a final contradiction. □

Corollary 3.7. *Let E be a normal subgroup of G such that G/E is p-nilpotent, where p is a prime divisor of $|G|$. Suppose that P is a Sylow p-subgroup of E, $N_G(P)$ is p-nilpotent, and every maximal subgroup of P is c-normal or Φ-supplemented in G. Then, G is p-nilpotent.*

Proof. Since $N_E(P) = E \cap N_G(P)$ and $N_G(P)$ is p-nilpotent, $N_E(P)$ is p-nilpotent. By Lemmas 2.1 and 2.2, every maximal subgroup of P is c-normal or Φ-supplemented in E. By Theorem 3.3, E is p-nilpotent. Let T be the normal p-complement of E. Then, T is normal in G. By using the arguments used in the proof of Corollary 3.5, we may assume that $T = 1$ and $E = P$ is a p-group. In this case, by our hypotheses, $N_G(P) = G$ is p-nilpotent. □

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