Research Article

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Symmetric results of a Hénon-type elliptic system with coupled linear part

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Abstract: In this article, we study the elliptic system:

$$\begin{cases} -\Delta u + \mu_1 u = |x|^\alpha u^3 + \lambda v, & x \in \Omega \\ -\Delta v + \mu_2 v = |x|^\alpha v^3 + \lambda u, & x \in \Omega \\ u, v > 0, x \in \Omega, u = v = 0, x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is the unit ball. By the variational method, we prove that if α is sufficiently small, the ground state solutions of the system are radial symmetric, and if $\alpha > 0$ is sufficiently large, the ground state solutions are nonradial; however, the solutions are Schwarz symmetry.

Keywords: elliptic system, variational method, ground state solutions, symmetry

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1 Introduction

In this article, we study the following elliptic system:

$$\begin{cases} -\Delta u + \mu_1 u = |x|^\alpha u^3 + \lambda v, & x \in \Omega \\ -\Delta v + \mu_2 v = |x|^\alpha v^3 + \lambda u, & x \in \Omega \\ u, v > 0, x \in \Omega, u = v = 0, x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is the unit ball, $\mu_1, \mu_2 > 0$, $\alpha > 0$. The parameter $\lambda > 0$ satisfies several conditions.

System (1.1) is the nonautonomous case of the following elliptic system:

$$\begin{cases} -\Delta u + \mu_1 u = u^3 + \lambda v, & x \in \Omega \\ -\Delta v + \mu_2 v = v^3 + \lambda u, & x \in \Omega \\ u, v > 0, x \in \Omega, u = v = 0, x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$) is an open domain. (1.2) arises in a binary mixture of the Bose-Einstein condensates with a coupled linear part. In recent years, many researchers have been concerned with system (1.2). They have proved the existence, multiplicity, and limit problems of the solutions of system (1.2), and see [1–4] and the references therein.

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Gidas et al. [5] studied the elliptic equation:

$$-\Delta u = f(|x|, u), \quad u > 0, \quad x \in \Omega, \quad u = 0, \quad x \in \partial\Omega. \tag{1.3}$$

They proved that if Ω is a ball and $f(|x|, u) = f(u) \in C^1$, then the solutions are radially symmetric. After that, many researchers studied the symmetric properties of elliptic equations, and see [6–9] and the references therein. In [7], they pointed out that if the domain cannot be convex or $f(|x|, u)$ is increasing by $|x|$, one can have the nonradial positive solutions. Smets et al. [9] studied the Hénon equation:

$$-\Delta u = |x|^\alpha u^p, \quad u > 0, \quad x \in \Omega, \quad u = 0 \quad x \in \partial\Omega, \tag{1.4}$$

where Ω is the unit ball and $\alpha > 0$, $1 < p < 2^* - 1$. They proved that if $\alpha > 0$ is large enough, the ground state solutions of (1.4) is nonradial symmetry, and Boheure et al. [6] studied the symmetry-breaking results of the Hénon-Lane-Emden system, and see also [7] and the references therein.

Following these ideas, we prove the following.

Theorem 1.1. *Let $\lambda > 0$ be small. Then there exists $\bar{\alpha} > 0$ such that for all $\alpha > \bar{\alpha}$ the ground state solution is nonradial. However, the solutions are foliated Schwarz symmetric.*

In [8,10,11], Kajikiya obtained the multiple nonradial solutions of equation (1.4) by the group critical theory. By the unique and nondegenerate results, they proved, if α is sufficiently small, the ground state solutions of (1.4) is radially symmetric. It is an interesting result; since for the Hénon equation, the moving plane method cannot be used, one also prove the radial results. In [12,13], the authors also considered the nonradial solutions. In this article, we first obtain a result of unicity, then we prove the following:

Theorem 1.2. *For all $\lambda > 0$ be sufficiently small, the system* (1.2) *has a unique positive radial solution. If this solution is nondegenerate, then for α small, the ground state solution of system* (1.1) *should be radially symmetric.*

This article is organized as follows. In Section 2, we present several preliminary results, and in Sections 3 and 4, we present the proof of the main results.

2 Some preliminaries and lemmas

Let $H_0^1(\Omega)$ be the classical Sobolev space and $\mathcal{H} = H_0^1(\Omega) \times H_0^1(\Omega)$ with the norm:

$$\|(u, v)\| = \left(\int_\Omega |\nabla u|^2 + \mu_1 u^2 + |\nabla v|^2 + \mu_2 v^2 \mathrm{d}x \right)^{\frac{1}{2}}.$$

We assume $0 < \lambda < \min\{\mu_1, \mu_2\}$.

The energy functionals of (1.1) and (1.2) are

$$I_\alpha(u, v) = \frac{1}{2}\|(u, v)\|^2 - \frac{1}{4}\int_\Omega |x|^\alpha (u^4 + v^4)\mathrm{d}x - \lambda \int_\Omega uv\mathrm{d}x$$

and

$$I(u, v) = \frac{1}{2}\|(u, v)\|^2 - \frac{1}{4}\int_\Omega (u^4 + v^4)\mathrm{d}x - \lambda \int_\Omega uv\mathrm{d}x.$$

The Nehari manifolds are

$$\mathcal{N}_\alpha = \{(u, v) \in \mathcal{H}\backslash\{(0, 0)\} : I'_\alpha(u, v)(u, v) = 0\}$$

and

$$\mathcal{N} = \{(u, v) \in \mathcal{H}\backslash\{(0, 0)\} : I'(u, v)(u, v) = 0\}.$$

Definition 2.1. $(\bar{u}, \bar{v})$ is the ground state solution of (1.1), if it satisfies

$$c_\alpha = I_\alpha(\bar{u}, \bar{v}) = \inf_{\mathcal{N}_\alpha} I_\alpha(u, v).$$

If the infimum is obtained from the radial functions, we denote it as c_α^r.

By the moving plane method, one can prove that the solutions of system (1.2), denoted (ω, h), are radial. If

$$\begin{cases} -\Delta\phi + \mu_1\phi = 3\omega^2\phi + \lambda\psi, \\ -\Delta\psi + \mu_2\psi = 3h^2\psi + \lambda\phi \end{cases} \tag{2.1}$$

has only trivial solutions, in this case (ω, h) is nondegenerate. One can check Damascelli and Pacella [14] and Troy [15] for the radially symmetric results, see also Dancer et al. [16], Sato and Wang [17], and Tavares and Weth [18]. For unique and nondegenerate results of the elliptic system, one can see [19–21].

3 The Proof of Theorem 1.1

Lemma 3.1. *For any $(u, v) \in \mathcal{H}$, there exists a unique $t > 0$ such that $(tu, tv) \in \mathcal{N}_\alpha$.*

Proof. For any $(u, v) \in H$, by calculating directly, we have

$$I'_\alpha(tu, tv)(tu, tv) = t^2\|(u, v)\|^2 - 2t^2\lambda \int_\Omega uv\mathrm{d}x - t^4 \int_\Omega |x|^\alpha(u^4 + v^4)\mathrm{d}x, \tag{3.1}$$

and thus,

$$t^2 = \frac{\|(u, v)\|^2 - 2\lambda\int_\Omega uv\mathrm{d}x}{\int_\Omega |x|^\alpha(u^4 + v^4)\mathrm{d}x}.$$

We have completed the proof. □

Lemma 3.2. *c_α is achieved, and if $\alpha > 0$ is large enough, we have that*

$$c_\alpha \le C\alpha,$$

where C is a constant independent of α.

Proof. By the variational method, we can prove that c_α and c_α^r are achieved, and $c_\alpha \le c_\alpha^r$. Now we give the estimate of c_α.

Let $\varphi, \psi \in C_0^\infty(\Omega)$ and $x_0 = \left(1 - \frac{1}{\alpha}, 0, 0\right)$, and

$$\varphi_\alpha(x) = \varphi(\alpha(x - x_0)), \quad \psi_\alpha(x) = \psi(\alpha(x - x_0)).$$

When α is large, one can prove $\mathrm{supp}\varphi_\alpha \subseteq \Omega$, $\mathrm{supp}\psi_\alpha \subseteq \Omega$. Since $C_0^\infty(\Omega) \subseteq H_0^1(\Omega)$, then

$$c_\alpha \le I_\alpha(t\varphi, t\psi),$$

where t satisfies $(t\varphi, t\psi) \in \mathcal{N}_\alpha$. Then we have

$$I_\alpha(t\varphi_\alpha, t\psi_\alpha) = \frac{1}{4}t^4 \int\limits_\Omega |x|^\alpha(\varphi_\alpha^4 + \psi_\alpha^4)\mathrm{d}x$$

and

$$t^2 = \frac{\|(\varphi_\alpha, \psi_\alpha)\|^2 - 2\lambda\int_\Omega \varphi_\alpha\psi_\alpha \mathrm{d}x}{\int_\Omega |x|^4(\varphi_\alpha^4 + \psi_\alpha^4)\mathrm{d}x}.$$

By combining these results, we have that

$$I_\alpha(t\varphi_\alpha, t\psi_\alpha) = \frac{1}{4}\frac{\left(\|(\varphi_\alpha, \psi_\alpha)\|^2 - 2\lambda\int_\Omega \varphi_\alpha\psi_\alpha \mathrm{d}x\right)^2}{\int_\Omega |x|^4(\varphi_\alpha^4 + \psi_\alpha^4)\mathrm{d}x}.$$

Let $y = \alpha(x - x_0)$, then $\mathrm{d}x = \frac{1}{\alpha^3}\mathrm{d}y$. Because of $\mathrm{supp}\varphi_\alpha \subset \Omega$, then we have $y \in \Omega$, so $x - x_\alpha \in \Omega_{\frac{1}{\alpha}}$, which is a ball with $r = \frac{1}{\alpha}$, and then we have that

$$\begin{aligned}\int\limits_\Omega |x\,|^\alpha(\varphi_\alpha^4 + \psi_\alpha^4)\mathrm{d}x &\geq \left(1 - \frac{2}{\alpha}\right)^\alpha \int\limits_\Omega (\varphi_\alpha^4 + \psi_\alpha^4)\mathrm{d}x \\ &= \left(1 - \frac{2}{\alpha}\right)^\alpha \int\limits_\Omega (\varphi^4 + \psi^4)\mathrm{d}x \\ &= \left(1 - \frac{2}{\alpha}\right)^\alpha \int\limits_\Omega \frac{1}{\alpha^3}(\varphi^4 + \psi^4)\mathrm{d}y.\end{aligned} \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}\int\limits_\Omega (|\nabla\varphi_\alpha|^2 + \varphi_\alpha^2)\mathrm{d}x &= \int\limits_\Omega \alpha^{-3}(|\nabla\varphi|^2 + \varphi^2)\mathrm{d}y \\ &= \alpha^{-1}\int\limits_\Omega (\alpha^{-2}|\nabla\varphi|^2 + \alpha^{-2}\varphi^2)\mathrm{d}y \\ &\leq \alpha^{-1}\int\limits_\Omega (|\nabla\varphi|^2 + \alpha^{-2}\varphi^2)\mathrm{d}y,\end{aligned} \tag{3.3}$$

and similarly, we have that

$$\int\limits_\Omega (|\nabla\psi_\alpha|^2 + \psi_\alpha^2)\mathrm{d}x \leq \alpha^{-1}\int\limits_\Omega (|\nabla\psi|^2 + \alpha^{-2}\psi^2)\mathrm{d}y. \tag{3.4}$$

Combining the previous inequalities (3.2)–(3.4), we have that

$$c_\alpha \leq C\alpha,$$

where C is a positive constant independent of α. □

Lemma 3.3. *Let (u_α, v_α) be a radial ground state solution of* (1.1) *and c_α^r is the energy value. Then we have that*

$$c_\alpha^r \geq C\alpha^3,$$

where C is a constant independent of α.

Proof. For convenience, we replace (u, v) of (u_α, v_α). Define

$$Y(x) = (\nabla u, x)\nabla u + (\nabla v, x)\nabla v - \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2)x + \frac{1}{4}|x|^\alpha(u^4 + v^4)x + \lambda uvx - \frac{1}{2}(\mu_1 u^2 + \mu_2 v^2)x,$$

where $x \in \mathbb{R}^3$. Let

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

then we have

$$(\nabla u, x)\nabla u = (u_r')^2(x_1, x_2, x_3),\ (\nabla v, x)\nabla v = (v_r')^2(x_1, x_2, x_3),$$

and

$$\begin{aligned}&\operatorname{div}\left((\nabla u, x)\nabla u - \frac{1}{2}|\nabla u|^2 x + \frac{1}{4}|x|^\alpha u^4 x - \frac{1}{2}u^2 x\right)\\ &= \frac{3}{2}|\nabla u|^2 + \frac{3+\alpha}{4}|x|^\alpha u^4 - \frac{3}{2}u^2 + ru_r'(u_r'' + u^3|x|^\alpha - u),\end{aligned} \tag{3.5}$$

and

$$\operatorname{div}\left((\nabla v, x)\nabla v - \frac{1}{2}|\nabla v|^2 x + \frac{1}{4}|x|^\alpha v^4 x - \frac{1}{2}v^2 x\right) = \frac{3}{2}|\nabla v|^2 + \frac{3+\alpha}{4}|x|^\alpha v^4 - \frac{3}{2}v^2 + rv_r'(v_r'' + v^3|x|^\alpha - v), \tag{3.6}$$

and

$$\operatorname{div}(\lambda uvx) = 3\lambda uv + \lambda ruv_r' + \lambda rvu_r'. \tag{3.7}$$

By combining the previous equalities, we obtain the following:

$$\begin{aligned}\operatorname{div} Y(x) &= \frac{3}{2}(|\nabla u|^2 + |\nabla v|^2) + \frac{3+\alpha}{4}|x|^\alpha(u^4 + v^4) + 3\lambda uv - \frac{3}{2}(\mu_1 u^2 + \mu_2 v^2)\\ &\quad + ru_r'(u_r'' + u^3|x|^\alpha - \mu_1 u + \lambda v) + rv_r'(v_r'' + v^3|x|^\alpha - \mu_2 v + \lambda u).\end{aligned} \tag{3.8}$$

Since (u, v) is a radial solution,

$$u_r'' = \Delta u - \frac{2u_r'}{r}, \quad v_r'' = \Delta v - \frac{2v_r'}{r},$$

$$ru_r'(u_r'' + u^3|x|^\alpha - \mu_1 u + \lambda v) = ru_r'\left(\Delta u - \frac{2u_r'}{r} + |x|^\alpha u^3 - \mu_1 u + \lambda v\right) = ru_r' \cdot \frac{-2u_r'}{r} = -2(u_r')^2 = -2|\nabla u|^2,$$

$$-2|\nabla v|^2 = rv_r'(v_r'' + v^3|x|^\alpha - \mu_2 v + \lambda u).$$

Thus,

$$\operatorname{div} Y(x) = \frac{3+\alpha}{4}|x|^\alpha(u^4 + v^4) + 3\lambda uv - \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) - \frac{3}{2}(\mu_1 u^2 + \mu_2 v^2).$$

By divergence theorem, we obtain that , i.e.,

$$\int_\Omega \operatorname{div} Y(x)\mathrm{d}x = \int_{\partial\Omega} Y(x)\mathbf{n}\mathrm{d}S$$

$$\begin{aligned}&\int_{\partial\Omega}\left\{(\nabla u, x)\nabla u \cdot n + (\nabla v, x)\nabla v \cdot n - \frac{1}{2}(|\nabla u|^2 \cdot x + |\nabla v| \cdot x) \cdot n\right\}\mathrm{d}S\\ &= \int_\Omega\left\{\frac{3+\alpha}{4}|x|^\alpha(u^4 + v^4) + 3\lambda(|\nabla u|^2 + |\nabla v|^2) - \frac{3}{2}(\mu_1 u^2 + \mu_2 v^2)\right\}\mathrm{d}x.\end{aligned}$$

Again since (u, v) is a solution, we have that

$$\begin{aligned}
&\int_{\partial\Omega} \left\{(\nabla u, x)\frac{\partial u}{\partial n} + (\nabla v, x)\frac{\partial v}{\partial n} - \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2)(x \cdot n)\right\}\mathrm{d}S \\
&= \frac{\alpha - 3}{4}\int_{\Omega} |x|^{\alpha}(u^4 + v^4)\mathrm{d}x + \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2)\mathrm{d}x \\
&\geq \frac{\alpha - 3}{4}\int_{\Omega} |x|^{\alpha}(u^4 + v^4)\mathrm{d}x.
\end{aligned}$$

Since u and v are radial functions, following pp. 72 of [6], we have

$$(\nabla u, x)(\nabla v, x) = (\nabla u, \nabla v)|x|^2 \tag{3.9}$$

and

$$\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 \mathrm{d}S = C\left|\int_{\partial\Omega} \frac{\partial u}{\partial n}\mathrm{d}S\right|^2, \quad \int_{\partial\Omega} \left|\frac{\partial v}{\partial n}\right|^2 \mathrm{d}S = C\left|\int_{\partial\Omega} \frac{\partial v}{\partial n}\mathrm{d}S\right|^2.$$

Since

$$c_{\alpha} = \frac{1}{4}\int_{\Omega} |x|^{\alpha}(u^4 + v^4)\mathrm{d}x$$

and

$$\int_{\partial\Omega} \left\{(\nabla u, x)\frac{\partial u}{\partial n} + (\nabla v, x)\frac{\partial v}{\partial n} - \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2)(x \cdot n)\right\}\mathrm{d}S = \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \left|\frac{\partial v}{\partial n}\right|^2 \mathrm{d}S,$$

and then we obtain

$$(\alpha - 3)c_{\alpha} \leq \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 + \left|\frac{\partial v}{\partial n}\right|^2 \mathrm{d}S. \tag{3.10}$$

By the divergence theorem, we have $\int_{\Omega} \Delta u \mathrm{d}x = \int_{\partial\Omega} \frac{\partial u}{\partial n}\mathrm{d}S$. Notice that

$$-\Delta u + \mu_1 u = |x|^{\alpha}u^3 + \lambda v,$$

and

$$\begin{aligned}
\int_{\Omega} |x|^{\alpha}u^3\mathrm{d}x &= \int_{\Omega} |x|^{\frac{\alpha}{4}} \cdot |x|^{\frac{3\alpha}{4}}u^3\mathrm{d}x \\
&\leq \left(\int_{\Omega} |x|^{\alpha}\mathrm{d}x\right)^{\frac{1}{4}}\left(\int_{\Omega} |x|^{\alpha}u^4\mathrm{d}x\right)^{\frac{3}{4}} \\
&= C\left(\int_0^1 r^{\alpha+2}dr\right)^{\frac{1}{4}}\left(\int_{\Omega} |x|^{\alpha}u^4\mathrm{d}x\right)^{\frac{3}{4}} \\
&= \frac{C}{(\alpha + 3)^{\frac{1}{4}}}(\int_{\Omega} |x|^{\alpha}u^4\mathrm{d}x)^{\frac{3}{4}},
\end{aligned} \tag{3.11}$$

and

$$\left|\int_\Omega \lambda v + \mu_1 u \mathrm{d}x\right| \le C\left\{\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right\} \le 2C\left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}}. \tag{3.12}$$

Thus, we obtain that

$$\left|\int_\Omega \Delta u \mathrm{d}x\right| \le \frac{1}{(\alpha+3)^{\frac{1}{4}}}\left(\int_\Omega |x|^\alpha u^4 \mathrm{d}x\right)^{\frac{3}{4}} + 2C\left(\int_\Omega |\nabla u|^2 + |\nabla v|^2 \mathrm{d}x\right)^{\frac{1}{2}} \le \frac{1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{3}{4}} + C c_\alpha^{\frac{1}{2}} \tag{3.13}$$

and

$$\left|\int_\Omega \Delta v \mathrm{d}x\right| \le \frac{1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{3}{4}} + C(\lambda) c_\alpha^{\frac{1}{2}}. \tag{3.14}$$

We also obtain that

$$\int_{\partial\Omega}\left(\left|\frac{\partial u}{\partial n}\right|^2 + \left|\frac{\partial v}{\partial n}\right|^2\right)\mathrm{d}S = C\left\{\left|\int_\Omega \Delta u \mathrm{d}x\right|^2 + \left|\int_\Omega \Delta v \mathrm{d}x\right|^2\right\},$$

and by the previous estimations, one can obtain that

$$\int_{\partial\Omega}\left(\left|\frac{\partial u}{\partial n}\right|^2 + \left|\frac{\partial v}{\partial n}\right|^2\right)\mathrm{d}S \le C\left\{\frac{1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{3}{4}} + C c_\alpha^{\frac{1}{2}}\right\}^2. \tag{3.15}$$

From (3.10), we obtain

$$(\alpha - 3)c_\alpha \le \left\{\frac{1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{3}{4}} + C c_\alpha^{\frac{1}{2}}\right\}^2,$$

and then we have

$$(\alpha - 3)^{1/2} c_\alpha^{1/2} \le \frac{C_1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{3}{4}} + C_2 c_\alpha^{\frac{1}{2}}$$

and

$$(\alpha - 3)^{1/2} \le \frac{C_1}{(\alpha+3)^{\frac{1}{4}}} c_\alpha^{\frac{1}{4}} + C_2.$$

If $\alpha > 0$ is large enough, we obtain $\alpha^{\frac{3}{4}} \le C c_\alpha^{\frac{1}{4}}$, i.e.,

$$c_\alpha \ge C\alpha^3,$$

where C, C_1, and C_2 are positive constants independently of α. In conclusion, we complete the proof. □

Now we list the proof of Theorem 1.1.

The proof of Theorem 1.1. By Lemmas 3.2 and 3.3, we can prove that if $\alpha > 0$ is large enough, the ground state solutions cannot be radial. Let

$$f_1(|x|, u, v) = |x|^\alpha u^3 + \lambda v, \quad f_2(|x|, u, v) = |x|^\alpha v^3 + \lambda u,$$

then $f_1(|x|, u, v)$ and $f_2(|x|, u, v)$ are convex with respect to u and v. Since (u, v) is a ground state solution, then the Morse index $M(u, v) \le 1 < N = 3$, and then by [14], we have that (u, v) is foliated Schwarz symmetry, and one can also follow the idea of Wang and Willem [23].

4 The Proof of Theorem 1.2

By using the idea of Wei and Yao [20], we obtain the following results.

Lemma 4.1. *Let $\lambda > 0$ be small enough. Then the positive solution of system* (1.2) *is radially symmetric, and it is unique.*

Proof. Following the idea of Gidas et al. [5], Damascelli and Pacella [14], or Troy [15], we prove that the positive solutions are radially symmetric.

Let ω_i be the unique positive solution of the following scalar equation with the subcritical exponent:

$$-\Delta u + \mu_i u = u^3 x \in \Omega, \quad u = 0 \quad x \in \partial\Omega,$$

where $i = 1, 2$.

Let $\Phi(\lambda, u, v) = I'_\alpha(u, v)$, one can prove that $\Phi(0, \omega_1, \omega_2) = 0$, and we have

$$\Phi'(0, \omega_1, \omega_2)(\phi, \psi) = \int_\Omega |\nabla\phi|^2 + \mu_1\phi^2 + |\nabla\psi|^2 + \psi^2 \mathrm{d}x - 3\int_\Omega \omega_1^2\phi^2 + \omega_2\psi^2 \mathrm{d}x.$$

By the compact theorem, one can prove that $\Phi'(0, \omega_1, \omega_2)$ is invertible, and then by implicity function theorem, if $\lambda > 0$ is small enough, the solution of system is uniqueness. □

Next we present the proof of Theorem 1.2.

By contradiction, the ground state solutions of (1.1) are nonradial. By the variational method, we prove that (1.1) has a nontrivial radial solution, one can see [22], denoting $(\tilde{u}_\alpha, \tilde{v}_\alpha)$, and another nonradial solution, denoting (u_α, v_α). Now we give the L^∞ estimate of (u_α, v_α) and $(\tilde{u}_\alpha, \tilde{v}_\alpha)$, for convenience, we replace (u_α, v_α) by (u, v). Since (u, v) is a solution of (1.1), let u^{2q-1} be a test function, then we have

$$\int_\Omega -\Delta u u^{2q-1} + \mu_1 u^{2q} \mathrm{d}x = \int_\Omega |x|^\alpha u^{2q+2} \mathrm{d}x + \lambda \int_\Omega u^{2q-1} v \mathrm{d}x,$$

where $2q - 1 > 1$. By Green's formula, we obtain that

$$\text{LHS} = \int_\Omega \nabla u \nabla(u^{2q-1}) + \mu_1 u^{2q} \mathrm{d}x = (2q-1)\int_\Omega |\nabla u|^2 u^{2q-2} \mathrm{d}x + \mu_1 \int_\Omega u^{2q} \mathrm{d}x = \frac{2q-1}{q^2}\int_\Omega |\nabla u^q|^2 \mathrm{d}x + \mu_1 \int_\Omega |u^q|^2 \mathrm{d}x.$$

By Hölder's and Young's inequality, we have that

$$\begin{aligned}\int_\Omega v u^{2q-1} \mathrm{d}x &\le \left(\int_\Omega u^{2q+2} \mathrm{d}x\right)^{\frac{2q-1}{2q+2}} \left(\int_\Omega v^{\frac{2q+2}{3}} \mathrm{d}x\right)^{\frac{3}{2q+2}} \\ &\le \frac{2q-1}{2q+2}\int_\Omega u^{2q+2} \mathrm{d}x + \frac{3}{2q+2}\int_\Omega v^{\frac{2q+2}{3}} \mathrm{d}x \\ &\le \frac{2q-1}{2q+2}\int_\Omega u^{2q+2} \mathrm{d}x + \frac{3}{2q+2}\left(\int_\Omega v^{2q+2}\right)^{1/3}\left(\int_\Omega 1 \mathrm{d}x\right)^{2/3} \\ &\le \int_\Omega u^{2q+2} \mathrm{d}x + C \le C\int_\Omega u^{2q+2} \mathrm{d}x,\end{aligned}$$

where C is a constant independent of q. On the other hand,

$$\int_\Omega |x|^\alpha u^{2q+2} \mathrm{d}x \le \int_\Omega u^{2q+2} \mathrm{d}x.$$

Then we obtain that

$$\int_\Omega |\nabla u^q|^2 + |u^q|^2 \mathrm{d}x \le \frac{Cq^2}{2q-1}\int_\Omega u^{2q+2}\mathrm{d}x. \tag{4.1}$$

Again by Hölder and Sobolev embedding inequality, we obtain

$$\int_\Omega u^{2q+2}\mathrm{d}x \le \left(\int_\Omega u^{3q}\mathrm{d}x\right)^{2/3}(u^6\mathrm{d}x)^{1/3} \le C\left(\int_\Omega u^{3q}\mathrm{d}x\right)^{2/3}$$

and

$$\left(\int_\Omega u^{4q}\mathrm{d}x\right)^{\frac{1}{4}} \le \left(\int_\Omega |\nabla u^q|^2 + |u^q|^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

Thus, we obtain that

$$\left(\int_\Omega u^{4q}\mathrm{d}x\right)^{\frac{1}{4q}} \le \left[\frac{cq^2}{2q-1}\right]^{\frac{1}{2q}}\left(\int_\Omega u^{3q}\mathrm{d}x\right)^{\frac{1}{3q}}.$$

Now let $q(1)=1$ and $q(n+1)=\frac{4}{3}q(n)$, and one can prove that $q(n)\to+\infty$ as $n\to+\infty$. By the proceeding estimate, we have that

$$\|u\|_{4q(n+1)} \le C\prod_{i=2}^{n+1}\left[\frac{q^2(i)}{2q(i)-1}\right]^{\frac{1}{2q(i)}}\|u\|_{3q(1)}.$$

Notice that $q(n+1)=\left(\frac{4}{3}\right)^n$, and thus, we can prove that

$$\ln\prod_{i=2}^{n+1}\left[\frac{q^2(i)}{2q(i)-1}\right]^{\frac{1}{2q(i)}} = \sum_{i=2}^{n+1}\frac{2\ln q(i)-\ln(2q(i)-1)}{2q(i)} < \infty,$$

and thus, $\prod_{i=2}^{n+1}\left[\frac{q^2(i)}{2q(i)-1}\right]^{\frac{1}{2q(i)}} < \infty$, then we complete the L^∞ estimate.

Let $\lambda>0$ satisfy the previous conditions, and $\{\alpha_n\}$ satisfy $\alpha_n\to 0$ as $n\to+\infty$. We denote that $\{(u_n,v_n)\}$ is the ground state solutions of system (1) and $\{(\tilde u_n,\tilde v_n)\}$ is the radial ground state solutions of system (1).

Following the idea of [8], by contradiction, $u_n\ne\tilde u_n$, $v_n\ne\tilde v_n$, let

$$\omega_n=\frac{u_n-\tilde u_n}{\|u_n-\tilde u_n\|_\infty},\quad h_n=\frac{v_n-\tilde v_n}{\|v_n-\tilde v_n\|_{L^\infty}},$$

then $\omega_n\ne 0$, $h_n\ne 0$ and $\omega_n, h_n\in W^{2,t}(\Omega)$, which is classical Sobolev space, for any $1<t<+\infty$. Let

$$f_1(u,v)=|x|^\alpha u^3+\lambda v \quad\text{and}\quad f_2(u,v)=|x|^\alpha v^3+\lambda u,$$

then

$$\begin{cases}-\Delta\omega_n+\mu_1\omega_n=\dfrac{f_1(u_n,v_n)-f_1(\tilde u_n,\tilde v_n)}{u_n-\tilde u_n}\omega_n,\\ -\Delta h_n+\mu_2 h_n=\dfrac{f_2(u_n,v_n)-f_2(\tilde u_n,\tilde v_n)}{v_n-\tilde v_n}h_n.\end{cases} \tag{4.2}$$

By the L^∞ estimate, we have $\omega_n\to\omega$, $h_n\to h$ in $C^1(\Omega)$, and ω and h satisfy

$$\begin{cases}-\Delta\omega+\mu_1\omega=3u_0^2\omega+\lambda c_1 h,\\ -\Delta h+\mu_2 h=3v_0^2 h+\lambda c_2\omega,\end{cases} \tag{4.3}$$

where $c_1=\lim_{n\to+\infty}\frac{\|v_n-\tilde v_n\|_\infty}{\|u_n-\tilde u_n\|_\infty}$ and $c_2=\frac{1}{c_1}$ are positive constants, and (u_o,v_0) is the uniqueness solution of (1.2).

Now let $\bar{\omega} = \frac{\omega}{c_1}$ and $\bar{h} = h$, we have that

$$\begin{cases} -\Delta\bar{\omega} + \mu_1\bar{\omega} = 3u_0^2\bar{\omega} + \lambda\bar{h}, \\ -\Delta\bar{h} + \mu_2\bar{h} = 3\lambda_2 v_0^2\bar{h} + \lambda\bar{\omega}. \end{cases} \tag{4.4}$$

It is a contradiction, then we complete the proof.

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