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Research Article

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On generalized extragradient implicit method for systems of variational inequalities with constraints of variational inclusion and fixed point problems

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Abstract: In a real Banach space, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of countably many nonexpansive mappings. In this article, we introduce a generalized extragradient implicit method for solving a general system of variational inequalities (GSVI) with the VI and CFPP constraints. Strong convergence of the suggested method to a solution of the GSVI with the VI and CFPP constraints under some suitable assumptions is established.

Keywords: generalized extragradient implicit method, general system of variational inequalities, variational inclusion, common fixed point problem

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1 Introduction

Let H be a real Hilbert space, in which the inner product and induced norm are denoted by the notations $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Given a nonempty, closed, and convex subset $C \subset H$. Let P_C be the metric projection of H onto C. Given a mapping $A : C \to H$. Consider the classical variational inequality problem (VIP) of finding a point $u^* \in C$ s.t. $\langle \mathrm{Au}^*, v - u^*\rangle \ge 0, \quad \forall v \in C$. The solution set of the VIP is denoted by $\mathrm{VI}(C, A)$. In 1976, Korpelevich [1] first designed an extragradient method, i.e., for any initial $u_0 \in C$, the sequence $\{u_m\}$ is generated by

$$\begin{cases} v_m = P_C(u_m - \ell A u_m), \\ u_{m+1} = P_C(u_m - \ell A v_m), \quad \forall m \ge 0, \end{cases} \tag{1.1}$$

with $\ell \in \left(0, \frac{1}{L}\right)$, which has been one of the most popular approaches for solving the VIP till now. In the case of $\mathrm{VI}(C, A) \ne \varnothing$, the sequence $\{u_m\}$ has only weak convergence. Indeed, the convergence of $\{u_m\}$ only requires that the mapping A is monotone and Lipschitz continuous. To the best of our knowledge, Korpelevich's extragradient method has received great attention from many authors, who have improved and modified it in various ways, see e.g., [2–12] and references therein.

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Recently, to solve the variational inclusion (VI) of finding $v^* \in H$ s.t. $0 \in (A+B)v^*$, Takahashi et al. [13] suggested a Halpern-type iterative method, i.e., for any given $v_0, u \in H$, $\{v_m\}$ is the sequence generated by

$$v_{m+1} = \beta_m v_m + (1-\beta_m)\Big(\alpha_m u + (1-\alpha_m)J^B_{\lambda_m}(v_m - \lambda_m A v_m)\Big), \quad \forall m \ge 0, \tag{1.2}$$

where A is an α-inverse-strongly monotone operator on H and B is a maximal monotone operator on H. They proved the strong convergence of $\{v_m\}$ to a solution $v^* \in (A+B)^{-1}0$ of the VI. Subsequently, Pholasa et al. [14] extended the result in [13] to the setting of Banach spaces, and proved the strong convergence of $\{v_m\}$ to a point of $(A+B)^{-1}0$.

In 2010, Takahashi et al. [15] invented a Mann-type Halpern iterative scheme for solving the fixed point problem (FPP) of a nonexpansive mapping $S: C \to C$ and the VI for an α-inverse-strongly monotone mapping $A: C \to H$ and a maximal monotone operator $B: D(B) \subset C \to H$, i.e., for any given $y_1 = y \in C$, $\{y_m\}$ is the sequence generated by

$$y_{m+1} = \beta_m y_m + (1-\beta_m)S\Big(\alpha_m y + (1-\alpha_m)J^B_{\lambda_m}(y_m - \lambda_m A y_m)\Big), \quad \forall m \ge 1, \tag{1.3}$$

where $\{\lambda_m\} \subset (0, 2\alpha)$ and $\{\alpha_m\}, \{\beta_m\} \subset (0,1)$ are such that (i) $\lim_{m\to\infty}\alpha_m = 0$ and $\sum_{m=1}^{\infty}\alpha_m = \infty$; (ii) $0 < a \le \lambda_m \le b < 2\alpha$ and $\lim_{m\to\infty}(\lambda_m - \lambda_{m+1}) = 0$ and (iii) $0 < c \le \beta_m \le d < 1$. They proved the strong convergence of $\{y_m\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Since the VI is important and interesting, many researchers have presented and developed a great number of iterative methods for solving the VI in several approaches, see e.g., [7,13–25] and references therein. Meanwhile, we consider the FPP of finding a point $u^* \in C$ such that $u^* = Su^*$, where $S: C \to C$ is a nonlinear mapping. The solution set of the FPP is denoted by $\mathrm{Fix}(S)$. In the practical life, many mathematical models have been formulated to solve this problem. At present, many mathematicians are interested in finding a common solution to the VI and FPP, i.e., find a point u^* s.t. $u^* \in \mathrm{Fix}(S) \cap (A+B)^{-1}0$.

Assume that $A: C \to H$ is an inverse-strongly monotone mapping, $B: D(B) \subset C \to 2^H$ is a maximal monotone operator, and $S: C \to C$ is a nonexpansive mapping. In 2011, Manaka and Takahashi [22] suggested an iterative process, i.e., for any given $u_0 \in C$, $\{u_m\}$ is the sequence generated by

$$u_{m+1} = \alpha_m u_m + (1-\alpha_m)SJ^B_{\lambda_m}(u_m - \lambda_m A u_m), \quad \forall m \ge 0, \tag{1.4}$$

where $\{\alpha_m\} \subset (0,1)$ and $\{\lambda_m\} \subset (0,\infty)$. They proved the weak convergence of $\{u_m\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$ under some appropriate conditions.

Furthermore, suppose that $q \in (1,2]$ and E is a real Banach space. Let $f: E \to E$ be a ρ-contraction and $S: E \to E$ be a nonexpansive mapping. Let $A: E \to E$ be an α-inverse-strongly accretive mapping of order q and $B: E \to 2^E$ be an m-accretive operator. Recently, to solve the FPP of S and the VI of finding $u^* \in E$ s.t. $0 \in (A+B)u^*$, Sunthrayuth and Cholamjiak [7] suggested a modified viscosity-type extragradient method in the setting of uniformly convex and q-uniformly smooth Banach space E with q-uniform smoothness coefficient κ_q, i.e., for any given $u_0 \in E$, $\{u_m\}$ is the sequence generated by

$$\begin{cases} y_m = J^B_{\lambda_m}(u_m - \lambda_m A u_m), \\ z_m = J^B_{\lambda_m}(u_m - \lambda_m A y_m + r_m(y_m - u_m)), \\ u_{m+1} = \alpha_m f(u_m) + \beta_m u_m + \gamma_m S z_m, \quad \forall m \ge 0, \end{cases} \tag{1.5}$$

where $J^B_{\lambda_m} = (I + \lambda_m B)^{-1}$, $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\}, \{r_m\} \subset (0,1)$, and $\{\lambda_m\} \subset (0,\infty)$ are such that: (i) $\alpha_m + \beta_m + \gamma_m = 1$; (ii) $\lim_{m\to\infty}\alpha_m = 0$ and $\sum_{m=1}^{\infty}\alpha_m = \infty$; (iii) $\{\beta_m\} \subset [a,b] \subset (0,1)$; and (iv) $0 < \lambda \le \lambda_m < \lambda_m/r_m \le \mu < (\alpha q/\kappa_q)^{1/(q-1)}$ and $0 < r \le r_m < 1$. They proved strong convergence of $\{u_m\}$ to a point of $\mathrm{Fix}(S) \cap (A+B)^{-1}0$, which solves a certain hierarchical variational inequality (HVI).

On the other hand, let $J: E \to 2^{E^*}$ be the normalized duality mapping from E into 2^{E^*} defined by $J(x) = \{\phi \in E^*: \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}$, $\forall x \in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. Recall that if E is smooth, then J is single-valued. Let $B_1, B_2: C \to E$ be two nonlinear mappings in a smooth Banach space E. The general system of variational inequalities (GSVI) is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases}\langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C,\\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C,\end{cases} \tag{1.6}$$

where μ_i is a positive constant for $i = 1, 2$. In particular, if $E = H$ a real Hilbert space, it is easy to see that the GSVI (1.6) reduces to the GSVI considered in [3] as follows:

$$\begin{cases}\langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C,\\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C.\end{cases} \tag{1.7}$$

In [3], problem (1.7) is transformed into a fixed point problem in the following way.

Lemma 1.1. [3] *For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.7) *if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSVI}(C, B_1, B_2)$ is the fixed point set of the mapping $G := P_C(I - \mu_1 B_1)P_C(I - \mu_2 B_2)$, and $y^* = P_C(I - \mu_2 B_2)x^*$.*

Recently, using Lemma 1.1, Cai et al. [2] proposed a viscosity implicit rule for solving the GSVI (1.7) with the FPP constraint of an asymptotically nonexpansive mapping T with a sequence $\{\theta_n\}$, i.e., for any given $x_0 \in C$, the sequence $\{x_n\}$ is generated as follows:

$$\begin{cases}u_n = s_n x_n + (1 - s_n)y_n,\\ z_n = P_C(u_n - \mu_2 B_2 u_n),\\ y_n = P_C(z_n - \mu_1 B_1 z_n),\\ x_{n+1} = P_C[\alpha_n f(x_n) + (I - \alpha_n \rho F)T^n y_n], & \forall n \geq 0,\end{cases} \tag{1.8}$$

where $\{\alpha_n\}, \{s_n\} \subset (0, 1]$ are such that (i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^{\infty}\alpha_n = \infty$, and $\sum_{n=0}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$; (ii) $\lim_{n\to\infty} \theta_n/\alpha_n = 0$; (iii) $0 < \varepsilon \leq s_n \leq 1$, $\sum_{n=0}^{\infty}|s_{n+1} - s_n| < \infty$; and (iv) $\sum_{n=0}^{\infty}\|T^{n+1}y_n - T^n y_n\| < \infty$. They proved that the sequence constructed by (1.8) converges strongly to a point of $\mathrm{GSVI}(C, A_1, A_2) \cap \mathrm{Fix}(T)$, which solves a certain HVI.

In a real Banach space E, let the VI indicate a variational inclusion for two accretive operators and let the CFPP denote a common fixed point problem of countably many nonexpansive mappings. In this article, we introduce a generalized extragradient implicit method for solving the GSVI (1.6) with the VI and CFPP constraints. We then prove the strong convergence of the suggested method to a solution of the GSVI (1.6) with the VI and CFPP constraints under some suitable assumptions. Our results improve and extend the corresponding results in Manaka and Takahashi [22], Sunthrayuth and Cholamjiak [7], and Cai et al. [2] to a certain extent.

2 Preliminaries

Let C be a nonempty, closed, and convex subset of a real Banach space E with the dual E^*. For simplicity, we shall use the following notations: $x_n \to x$ indicates the strong convergence of the sequence $\{x_n\}$ to x and $x_n \rightharpoonup x$ denotes the weak convergence of the sequence $\{x_n\}$ to x. Given a self-mapping T on C. We use the notations $\mathbf{R}$ and $\mathrm{Fix}(T)$ to stand for the set of all real numbers and the fixed point set of T, respectively. Recall that T is said to be nonexpansive if $\|Tu - Tv\| \leq \|u - v\|$, $\forall u, v \in C$. A mapping $f : C \to C$ is called a contraction if $\exists \delta \in [0, 1)$ s.t. $\|f(u) - f(v)\| \leq \delta\|u - v\|$, $\forall u, v \in C$. In addition, recall that the normalized duality mapping J defined by

$$J(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^2 = \|\phi\|^2\}, \quad \forall x \in E, \tag{2.1}$$

is the one from E into the family of nonempty (by Hahn-Banach's theorem), weak*, and compact subsets of E^*, satisfying $J(\tau u) = \tau J(u)$ and $J(-u) = -J(u)$ for all $\tau > 0$ and $u \in E$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined as follows:

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|u+v\|}{2} : u, v \in E, \|u\| = \|v\| = 1, \|u - v\| \geq \varepsilon\right\}.$$

The modulus of smoothness of E is the function $\rho_E : \mathbf{R}_+ := [0, \infty) \to \mathbf{R}_+$ defined as follows:

$$\rho_E(\tau) = \sup\left\{\frac{\|u + \tau v\| + \|u - \tau v\|}{2} - 1 : u, v \in E, \|u\| = \|v\| = 1\right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0,\ \forall \varepsilon \in (0, 2]$. It is said to be uniformly smooth if $\lim_{\tau\to 0^+}\rho_E(\tau)/\tau = 0$. In addition, it is said to be q-uniformly smooth with $q > 1$ if $\exists c > 0$ s.t. $\rho_E(t) \leq ct^q, \forall t > 0$. If E is q-uniformly smooth, then $q \leq 2$ and E is also uniformly smooth, and if E is uniformly convex, then E is also reflexive and strictly convex. It is known that Hilbert space H is 2-uniformly smooth. Furthermore, sequence space ℓ_p and Lebesgue space L_p are $\min\{p, 2\}$-uniformly smooth for every $p > 1$ [26].

Let $q > 1$. The generalized duality mapping $J_q : E \to 2^{E^*}$ is defined as follows:

$$J_q(x) = \{\phi \in E^* : \langle x, \phi\rangle = \|x\|^q, \|\phi\| = \|x\|^{q-1}\}, \tag{2.2}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. In particular, if $q = 2$, then $J_2 = J$ is the normalized duality mapping of E. It is known that $J_q(x) = \|x\|^{q-2}J(x),\ \forall x \neq 0$, and that J_q is the subdifferential of the functional $\frac{1}{q}\|\cdot\|^q$. If E is uniformly smooth, the generalized duality mapping J_q is one-to-one and single-valued. Furthermore, J_q satisfies $J_q = J_p^{-1}$, where J_p is the generalized duality mapping of E^* with $\frac{1}{p} + \frac{1}{q} = 1$. Note that no Banach space is q-uniformly smooth for $q > 2$, see [8] for more details.

The following lemma is an immediate consequence of the subdifferential inequality of the functional $\frac{1}{q}\|\cdot\|^q$.

Lemma 2.1. *Let $q > 1$ and E be a real normed space with the generalized duality mapping J_q. Then,*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q(x + y)\rangle, \quad \forall x, y \in E, j_q(x + y) \in J_q(x + y). \tag{2.3}$$

The following lemma can be obtained from the result in [26].

Lemma 2.2. *Let $q > 1$ and $r > 0$ be two fixed real numbers, and let E be uniformly convex. Then, there exist strictly increasing, continuous, and convex functions $g, h : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ and $h(0) = 0$ such that*
(a) $\|\mu u + (1 - \mu)v\|^q \leq \mu\|u\|^q + (1 - \mu)\|v\|^q - \mu(1 - \mu)g(\|u - v\|)$ with $\mu \in [0, 1]$;
(b) $h(\|u - v\|) \leq \|u\|^q - q\langle u, j_q(v)\rangle + (q - 1)\|v\|^q$

for all $u, v \in B_r$ and $j_q(v) \in J_q(v)$, where $B_r := \{y \in E : \|y\| \leq r\}$.

The following lemma is an analogue of Lemma 2.2(a).

Lemma 2.3. *Let $q > 1$ and $r > 0$ be two fixed real numbers, and let E be uniformly convex. Then, there exists a strictly increasing, continuous, and convex function $g : \mathbf{R}_+ \to \mathbf{R}_+$ with $g(0) = 0$ such that*

$$\|\lambda u + \mu v + \nu w\|^q \leq \lambda\|u\|^q + \mu\|v\|^q + \nu\|w\|^q - \lambda\mu g(\|u - v\|)$$

for all $u, v, w \in B_r$ and $\lambda, \mu, \nu \in [0, 1]$ with $\lambda + \mu + \nu = 1$.

Proposition 2.1. [27] *Let $\{S_n\}_{n=0}^{\infty}$ be a sequence of self-mappings on C such that $\sum_{n=1}^{\infty}\sup_{x\in C}\|S_nx - S_{n-1}x\| < \infty$. Then, for each $y \in C$, $\{S_ny\}$ converges strongly to some point of C. Moreover, let S be a self-mapping on C defined by $Sy = \lim_{n\to\infty}S_ny$ for all $y \in C$. Then, $\lim_{n\to\infty}\sup_{x\in C}\|S_nx - Sx\| = 0$.*

Proposition 2.2. [26] *Let $q \in (1, 2]$ be a fixed real number and let E be q-uniformly smooth. Then, $\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + \kappa_q\|y\|^q\ \forall x, y \in E$, where κ_q is the q-uniform smoothness coefficient of E.*

Let D be a subset of C and let Π be a mapping of C into D. Then Π is said to be sunny if $\Pi[\Pi(x) + t(x - \Pi(x))] = \Pi(x)$, whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for $x \in C$ and $t \geq 0$. A mapping Π of C into itself is called a retraction if $\Pi^2 = \Pi$. If a mapping Π of C into itself is a retraction, then $\Pi(z) = z$ for each $z \in R(\Pi)$, where $R(\Pi)$ is the range of Π. A subset D of C is called a sunny nonexpansive retract of C if there exists a sunny nonexpansive retraction from C onto D. In terms of [28], we know that if E is smooth and Π is a retraction of C onto D, then the following statements are equivalent:

(i) Π is sunny and nonexpansive;
(ii) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle, \ \forall x, y \in C$;
(iii) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0, \forall x \in C, y \in D$.

Let $B : C \to 2^E$ be a set-valued operator with $Bx \neq \emptyset, \ \forall x \in C$. Let $q > 1$. An operator B is said to be accretive if for each $x, y \in C, \exists\, j_q(x - y) \in J_q(x - y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq 0, \forall u \in Bx, v \in By$. An accretive operator B is said to be α-inverse-strongly accretive of order q if for each $x, y \in C, \exists j_q(x - y) \in J_q(x - y)$ s.t. $\langle u - v, j_q(x - y)\rangle \geq \alpha\|u - v\|^q, \ \forall u \in Bx, v \in By$ for some $\alpha > 0$. If $E = H$ a Hilbert space, then B is called α-inverse-strongly monotone. An accretive operator B is said to be m-accretive if $(I + \lambda B)C = E$ for all $\lambda > 0$. For an accretive operator B, we define the mapping $J_\lambda^B : (I + \lambda B)C \to C$ by $J_\lambda^B = (I + \lambda B)^{-1}$ for each $\lambda > 0$. Such J_λ^B is called the resolvent of B for $\lambda > 0$.

Lemma 2.4. [21] *Let $B : C \to 2^E$ be an m-accretive operator. Then, the following statements hold:*
(*i*) *the resolvent identity*: $J_\lambda^B x = J_\mu^B\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^B x\right), \quad \forall \lambda, \mu > 0, x \in E$;
(*ii*) *if J_λ^B is a resolvent of B for $\lambda > 0$, then J_λ^B is a firmly nonexpansive mapping with* $\mathrm{Fix}(J_\lambda^B) = B^{-1}0$, *where $B^{-1}0 = \{x \in C : 0 \in Bx\}$*;
(*iii*) *if $E = H$ a Hilbert space, B is a maximal monotone.*

Let $A : C \to E$ be an α-inverse-strongly accretive mapping of order q and $B : C \to 2^E$ be an m-accretive operator. In the sequel, we will use the notation $T_\lambda := J_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \ \forall \lambda > 0$.

Proposition 2.3. [21] *The following statements hold*:
(*i*) $\mathrm{Fix}(T_\lambda) = (A + B)^{-1}0, \ \forall \lambda > 0$;
(*ii*) $\|y - T_\lambda y\| \leq 2\|y - T_r y\|$ *for* $0 < \lambda \leq r$ *and* $y \in C$.

Proposition 2.4. [29] *Let E be uniformly smooth, $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *and $f : C \to C$ be a fixed contraction. For each $t \in (0, 1)$, let $z_t \in C$ be the unique fixed point of the contraction $C \ni z \mapsto tf(z) + (1 - t)Tz$ on C, i.e., $z_t = tf(z_t) + (1 - t)Tz_t$. Then, $\{z_t\}$ converges strongly to a fixed point $x^* \in \mathrm{Fix}(T)$, which solves HVI: $\langle (I - f)x^*, J(x^* - x)\rangle \leq 0, \ \forall x \in \mathrm{Fix}(T)$.*

Proposition 2.5. [21] *Let E be q-uniformly smooth with $q \in (1, 2]$. Suppose that $A : C \to E$ is an α-inverse-strongly accretive mapping of order q. Then, for any given $\lambda \geq 0$,*

$$\|(I - \lambda A)u - (I - \lambda A)v\|^q \leq \|u - v\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Au - Av\|^q, \quad \forall u, v \in C,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 \leq \lambda \leq \left(\frac{q\alpha}{\kappa_q}\right)^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Proposition 2.6. [30] *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that B_1 and $B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Let $G : C \to C$ be a mapping defined by*

$G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ and $\mathrm{GSVI}(C, B_1, B_2)$ denote the fixed point set of G. If $0 \le \mu_1 \le \left(\frac{q\alpha}{\kappa_q}\right)^{\frac{1}{q-1}}$ and $0 \le \mu_2 \le \left(\frac{q\beta}{\kappa_q}\right)^{\frac{1}{q-1}}$, then G is nonexpansive.

Lemma 2.5. [30] *Let E be q-uniformly smooth with $q \in (1, 2]$. Let Π_C be a sunny nonexpansive retraction from E onto C. Suppose that $B_1, B_2 : C \to E$ are two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem* (1.6) *if and only if $x^* \in \mathrm{GSVI}(C, B_1, B_2)$, where $\mathrm{GSV\,I}(C, B_1, B_2)$ is the fixed point set of the mapping $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$, and $y^* = \Pi_C(I - \mu_2 B_2)x^*$.*

Lemma 2.6. [31] *Let E be smooth, $A : C \to E$ be accretive, and Π_C be a sunny nonexpansive retraction from E onto C. Then,* $\mathrm{VI}(C, A) = \mathrm{Fix}(\Pi_C(I - \lambda A))$, $\forall \lambda > 0$, *where* $\mathrm{VI}(C, A)$ *is the solution set of the VIP of finding $z \in C$ s.t. $\langle Az, J(z - y)\rangle \le 0$, $\forall y \in C$.*

Recall that if $E = H$ a Hilbert space, then the sunny nonexpansive retraction Π_C from E onto C coincides with the metric projection P_C from H onto C. Moreover, if E is uniformly smooth and T is a nonexpansive self-mapping on C with $\mathrm{Fix}(T) \ne \emptyset$, then $\mathrm{Fix}(T)$ is a sunny nonexpansive retract from E onto C [32]. By Lemma 2.6, we know that $x^* \in \mathrm{Fix}(T)$ solves the HVI in Proposition 2.4 if and only if x^* solves the fixed point equation $x^* = \Pi_{\mathrm{Fix}(T)} f(x^*)$.

Lemma 2.7. [33] *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ that satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each integer $i \ge 1$. Define the sequence $\{\tau(n)\}_{n \ge n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where integer $n_0 \ge 1$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. Then, the following statements hold:
(*i*) $\tau(n_0) \le \tau(n_0 + 1) \le \cdots$ *and* $\tau(n) \to \infty$;
(*ii*) $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \le \Gamma_{\tau(n)+1}$ $\forall n \ge n_0$.

Lemma 2.8. [34] *Let E be strictly convex, and $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings on C. Suppose that $\bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty}\lambda_n = 1$. Then, a mapping S on C defined by $Sx = \sum_{n=0}^{\infty}\lambda_n T_n x$, $\forall x \in C$, is defined well, nonexpansive operator and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(T_n)$ holds.*

Lemma 2.9. [29] *Let $\{a_n\}$ be a sequence in $[0, \infty)$ such that $a_{n+1} \le (1 - s_n)a_n + s_n \nu_n$, $\forall n \ge 0$, where $\{s_n\}$ and $\{\nu_n\}$ satisfy the conditions:* (*i*) $\{s_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} s_n = \infty$; *and* (*ii*) $\limsup_{n\to\infty}\nu_n \le 0$ *or* $\sum_{n=0}^{\infty}|s_n \nu_n| < \infty$. *Then,* $\lim_{n\to\infty} a_n = 0$.

3 Main results

Throughout this article, we assume that E is a q-uniformly smooth and uniformly convex Banach space with $q \in (1, 2]$. Let C be a nonempty, closed, and convex subset of E and Π_C be a sunny nonexpansive retraction from E onto C. Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0, 1)$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C. Let $A : C \to E$ and $B : C \to 2^E$ be a σ-inverse-strongly accretive mapping of order q and an m-accretive operator, respectively. Suppose that B_1 and $B_2 : C \to E$ are α-inverse-strongly accretive mapping of order q and β-inverse-strongly accretive mapping of order q, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0 \ne \emptyset$.

Algorithm 3.1. Generalized extragradient implicit method for the GSVI (1.6) with the VI and CFPP constraints. Given $x_0 \in C$ arbitrarily. Given the current iterate x_n, compute x_{n+1} as follows:
Step 1. Calculate

$$\begin{cases} w_n = s_n x_n + (1 - s_n)u_n, \\ v_n = \Pi_C(w_n - \mu_2 B_2 w_n), \\ u_n = \Pi_C(v_n - \mu_1 B_1 v_n). \end{cases}$$

Step 2. Calculate $y_n = J_{\lambda_n}^B(u_n - \lambda_n A u_n)$.
Step 3. Calculate $z_n = J_{\lambda_n}^B(u_n - \lambda_n A y_n + r_n(y_n - u_n))$.
Step 4. Calculate $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n z_n$, where $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$.
Set $n := n + 1$ and go to Step 1.

Lemma 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega := \bigcap_{n=0}^{\infty} \mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0$. Then, we observe that

$$p = Gp = S_n p = J_{\lambda_n}^B(p - \lambda_n A p) = J_{\lambda_n}^B\left((1 - r_n)p + r_n\left(p - \frac{\lambda_n}{r_n}Ap\right)\right).$$

By Propositions 2.5 and 2.6, we know that $I - \mu_1 B_1$, $I - \mu_2 B_2$, and $G := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$ are nonexpansive mappings. Then, we obtain that

$$\|u_n - p\| = \|G(s_n x_n + (1 - s_n)u_n) - p\| \le \|s_n(x_n - p) + (1 - s_n)(u_n - p)\| \le s_n\|x_n - p\| + (1 - s_n)\|u_n - p\|,$$

which hence yields

$$\|u_n - p\| \le \|x_n - p\|.$$

Using Lemma 2.4 (ii) and Proposition 2.5, we have

$$\begin{aligned} \|y_n - p\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_n A u_n) - J_{\lambda_n}^B(p - \lambda_n A p)\|^q \le \|(I - \lambda_n A)u_n - (I - \lambda_n A)p\|^q \le \|u_n - p\|^q \\ &\quad - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q, \end{aligned} \tag{3.1}$$

which hence leads to

$$\|y_n - p\| \le \|u_n - p\|.$$

By the convexity of $\|\cdot\|^q$ for all $q \in (1, 2]$ and (3.1), we deduce that

$$\begin{aligned} \|z_n - p\|^q &= \left\|J_{\lambda_n}^B\left((1 - r_n)u_n + r_n\left(y_n - \frac{\lambda_n}{r_n}Ay_n\right)\right) - J_{\lambda_n}^B\left((1 - r_n)p + r_n\left(p - \frac{\lambda_n}{r_n}Ap\right)\right)\right\|^q \\ &\le (1 - r_n)\|u_n - p\|^q + r_n\left\|\left(I - \frac{\lambda_n}{r_n}A\right)y_n - \left(I - \frac{\lambda_n}{r_n}A\right)p\right\|^q \\ &\le (1 - r_n)\|u_n - p\|^q + r_n\left[\|y_n - p\|^q - \frac{\lambda_n}{r_n}\left(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ap\|^q\right] \\ &\le (1 - r_n)\|u_n - p\|^q + r_n\left[\|u_n - p\|^q - \lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q - \frac{\lambda_n}{r_n}\left(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ap\|^q\right] \\ &= \|u_n - p\|^q - r_n\lambda_n(\sigma q - \kappa_q \lambda_n^{q-1})\|Au_n - Ap\|^q - \lambda_n\left(\sigma q - \frac{\kappa_q \lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ap\|^q. \end{aligned} \tag{3.2}$$

This ensures that

$$\|z_n - p\| \le \|u_n - p\|.$$

Hence, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n(f(x_n)-p)+\beta_n(x_n-p)+\gamma_n(S_nz_n-p)\|\\ &\le \alpha_n\|f(x_n)-p\|+\beta_n\|x_n-p\|+\gamma_n\|S_nz_n-p\|\\ &\le \alpha_n(\|f(x_n)-f(p)\|+\|p-f(p)\|)+\beta_n\|x_n-p\|+\gamma_n\|S_nz_n-p\|\\ &\le \alpha_n(\delta\|x_n-p\|+\|p-f(p)\|)+\beta_n\|x_n-p\|+\gamma_n\|x_n-p\|\\ &= (1-\alpha_n(1-\delta))\|x_n-p\|+\alpha_n\|p-f(p)\|\\ &\le \max\left\{\|x_n-p\|, \frac{\|p-f(p)\|}{1-\delta}\right\}.\end{aligned}$$

By induction, we obtain $\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|p-f(p)\|}{1-\delta}\right\}$, $\forall n \ge 0$. Consequently, $\{x_n\}$ is bounded, and so are $\{u_n\}\{v_n\}$, $\{y_n\}$, $\{z_n\}$, $\{S_nz_n\}$, $\{Au_n\}$, $\{Ay_n\}$. This completes the proof. □

We now state and prove the main result of this article.

Theorem 3.1. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.1. Suppose that the following conditions hold*:
$(C1)$: $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;
$(C2)$: $0 < a \le \beta_n \le b < 1$, $0 < c \le s_n < 1$;
$(C3)$: $0 < r \le r_n < 1$ *and* $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < \left(\frac{\sigma q}{\kappa_q}\right)^{\frac{1}{q-1}}$;
$(C4)$: $0 < \mu_1 < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$ *and* $0 < \mu_2 < \left(\frac{\beta q}{\kappa_q}\right)^{\frac{1}{q-1}}$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_nx\| < \infty$ *for any bounded subset D of C. Let $S : C \to C$ be a mapping defined by* $Sx = \lim_{n\to\infty}S_nx$, $\forall x \in C$, *and suppose that* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then,* $x_n \to x^* \in \Omega$, *which is the unique solution to the HVI*: $\langle (I-f)x^*, J(x^*-p)\rangle \le 0$, $\forall p \in \Omega$, *i.e., the fixed point equation* $x^* = \Pi_\Omega f(x^*)$.

Proof. First of all, let $x^* \in \Omega$ and $y^* = \Pi_C(x^* - \mu_2B_2x^*)$. Using Proposition 2.5, we obtain

$$\|v_n-y^*\|^q = \|\Pi_C(w_n-\mu_2B_2w_n)-\Pi_C(x^*-\mu_2B_2x^*)\|^q \le \|w_n-x^*\|^q - \mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q$$

and

$$\|u_n-x^*\|^q = \|\Pi_C(v_n-\mu_1B_1v_n)-\Pi_C(y^*-\mu_1B_1y^*)\|^q \le \|v_n-y^*\|^q - \mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q.$$

Combining the last two inequalities, we have

$$\|u_n-x^*\|^q \le \|w_n-x^*\|^q - \mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q - \mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q.$$

Using Lemmas 2.1, 2.3, and (3.2), we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^q &\le \|\alpha_n(f(x_n)-f(x^*))+\beta_n(x_n-x^*)+\gamma_n(S_nz_n-x^*)\|^q + q\alpha_n\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle\\ &\le \alpha_n\|f(x_n)-f(x^*)\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n\|S_nz_n-x^*\|^q-\beta_n\gamma_ng(\|x_n-S_nz_n\|)\\ &\quad + q\alpha_n\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle\\ &\le \alpha_n\delta\|x_n-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n\Big[\|u_n-x^*\|^q - r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\ &\quad -\lambda_n\left(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n-Ax^*\|^q\Big] - \beta_n\gamma_ng(\|x_n-S_nz_n\|)+q\alpha_n\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle\\ &\le \alpha_n\delta\|x_n-x^*\|^q+\beta_n\|x_n-x^*\|^q+\gamma_n\Big[\|x_n-x^*\|^q-\mu_2(\beta q-\kappa_q\mu_2^{q-1})\|B_2w_n-B_2x^*\|^q\\ &\quad -\mu_1(\alpha q-\kappa_q\mu_1^{q-1})\|B_1v_n-B_1y^*\|^q - r_n\lambda_n(\sigma q-\kappa_q\lambda_n^{q-1})\|Au_n-Ax^*\|^q\\ &\quad -\lambda_n\left(\sigma q-\frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n-Ax^*\|^q\Big] - \beta_n\gamma_ng(\|x_n-S_nz_n\|)\\ &\quad + q\alpha_n\langle f(x^*)-x^*, J_q(x_{n+1}-x^*)\rangle\end{aligned} \tag{3.3}$$

$$= (1 - \alpha_n(1-\delta))\|x_n - x^*\|^q - \gamma_n[\mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2w_n - B_2x^*\|^q + \mu_1(\alpha q - \kappa_q\mu_1^{q-1})\|B_1v_n - B_1y^*\|^q$$
$$+ r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q + \lambda_n\left(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\Bigg] - \beta_n\gamma_n g(\|x_n - S_nz_n\|)$$
$$+ q\alpha_n\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle.$$

For each $n \geq 0$, we set $\Gamma_n = \|x_n - x^*\|^q$, $\varepsilon_n = \alpha_n(1-\delta)$, $\delta_n = q\alpha_n\langle (f - I)x^*, J_q(x_{n+1} - x^*)\rangle$, and

$$\eta_n = \gamma_n[\mu_2(\beta q - \kappa_q\mu_2^{q-1})\|B_2w_n - B_2x^*\|^q + \mu_1(\alpha q - \kappa_q\mu_1^{q-1})\|B_1v_n - B_1y^*\|^q + r_n\lambda_n(\sigma q - \kappa_q\lambda_n^{q-1})\|Au_n - Ax^*\|^q$$
$$+ \lambda_n\left(\sigma q - \frac{\kappa_q\lambda_n^{q-1}}{r_n^{q-1}}\right)\|Ay_n - Ax^*\|^q\Bigg] + \beta_n\gamma_n g(\|x_n - S_nz_n\|).$$

Then, (3.3) can be rewritten as follows:

$$\Gamma_{n+1} \leq (1 - \varepsilon_n)\Gamma_n - \eta_n + \delta_n, \quad \forall n \geq 0, \tag{3.4}$$

and hence,

$$\Gamma_{n+1} \leq (1 - \varepsilon_n)\Gamma_n + \delta_n, \quad \forall n \geq 0. \tag{3.5}$$

We next show the strong convergence of $\{\Gamma_n\}$ by the following two cases:

Case 1. Suppose that there exists an integer $n_0 \geq 1$ such that $\{\Gamma_n\}$ is nonincreasing. Then,

$$\Gamma_n - \Gamma_{n+1} \to 0.$$

From (3.4), we obtain

$$0 \leq \eta_n \leq \Gamma_n - \Gamma_{n+1} + \delta_n - \varepsilon_n\Gamma_n.$$

Since $\varepsilon_n \to 0$ and $\delta_n \to 0$, we have $\eta_n \to 0$. This ensures that $\lim_{n\to\infty} g(\|x_n - S_nz_n\|) = 0$,

$$\lim_{n\to\infty}\|B_2w_n - B_2x^*\| = \lim_{n\to\infty}\|B_1v_n - B_1y^*\| = 0 \tag{3.6}$$

and

$$\lim_{n\to\infty}\|Au_n - Ax^*\| = \lim_{n\to\infty}\|Ay_n - Ax^*\| = 0. \tag{3.7}$$

Note that g is a strictly increasing, continuous, and convex function with $g(0) = 0$. So, it follows that

$$\lim_{n\to\infty}\|x_n - S_nz_n\| = 0. \tag{3.8}$$

On the other hand, using Lemma 2.2(b) and the firm nonexpansivity of Π_C, we have

$$\begin{aligned}\|v_n - y^*\|^q &= \|\Pi_C(w_n - \mu_2B_2w_n) - \Pi_C(x^* - \mu_2B_2x^*)\|^q\\ &\leq \langle w_n - \mu_2B_2w_n - (x^* - \mu_2B_2x^*), J_q(v_n - y^*)\rangle\\ &= \langle w_n - x^*, J_q(v_n - y^*)\rangle + \mu_2\langle B_2x^* - B_2w_n, J_q(v_n - y^*)\rangle\\ &\leq \frac{1}{q}[\|w_n - x^*\|^q + (q-1)\|v_n - y^*\|^q - \tilde{h}_1(\|w_n - x^* - v_n + y^*\|)] + \mu_2\langle B_2x^* - B_2w_n, J_q(v_n - y^*)\rangle,\end{aligned}$$

which hence attains

$$\|v_n - y^*\|^q \leq \|w_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1}.$$

In a similar way, we obtain

$$\begin{aligned}\|u_n - x^*\|^q &= \|\Pi_C(v_n - \mu_1B_1v_n) - \Pi_C(y^* - \mu_1B_1y^*)\|^q\\ &\leq \langle v_n - \mu_1B_1v_n - (y^* - \mu_1B_1y^*), J_q(u_n - x^*)\rangle\\ &= \langle v_n - y^*, J_q(u_n - x^*)\rangle + \mu_1\langle B_1y^* - B_1v_n, J_q(u_n - x^*)\rangle\\ &\leq \frac{1}{q}[\|v_n - y^*\|^q + (q-1)\|u_n - x^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|)] + \mu_1\langle B_1y^* - B_1v_n, J_q(u_n - x^*)\rangle,\end{aligned}$$

which hence attains

$$\begin{aligned}\|u_n - x^*\|^q &\le \|v_n - y^*\|^q - \tilde{h}_2(\|v_n - y^* - u_n + x^*\|) + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1} \\ &\le \|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1} \\ &\quad - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1}.\end{aligned} \tag{3.9}$$

Since $J_{\lambda_n}^B$ is firmly nonexpansive (due to Lemma 2.4 (ii)), by Lemma 2.2(b), we obtain

$$\begin{aligned}\|y_n - x^*\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_nAu_n) - J_{\lambda_n}^B(x^* - \lambda_nAx^*)\|^q \\ &\le \langle (u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*), J_q(y_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*)\|^q + (q-1)\|y_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)],\end{aligned}$$

which, together with (3.1), implies that

$$\begin{aligned}\|y_n - x^*\|^q &\le \|(u_n - \lambda_nAu_n) - (x^* - \lambda_nAx^*)\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|) \\ &\le \|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|).\end{aligned}$$

This, together with (3.2) and (3.9), implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|S_nz_n - x^*\|^q \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[(1 - r_n)\|u_n - x^*\|^q + r_n\|y_n - x^*\|^q] \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{(1 - r_n)\|u_n - x^*\|^q \\ &\quad + r_n[\|u_n - x^*\|^q - h_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)]\} \\ &= \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{\|u_n - x^*\|^q - r_nh_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\{\|x_n - x^*\|^q - \tilde{h}_1(\|w_n - v_n - x^* + y^*\|) - \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1} + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1} \\ &\quad - r_nh_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) \\ &\quad + r_nh_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\quad + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1} + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1},\end{aligned}$$

which immediately yields

$$\begin{aligned}&\gamma_n\{\tilde{h}_1(\|w_n - v_n - x^* + y^*\|) + \tilde{h}_2(\|v_n - u_n + x^* - y^*\|) + r_nh_1(\|u_n - \lambda_n(Au_n - Ax^*) - y_n\|)\} \\ &\quad \le \alpha_n\|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1} + q\mu_1\|B_1y^* - B_1v_n\|\|u_n - x^*\|^{q-1} + q\mu_2\|B_2x^* - B_2w_n\|\|v_n - y^*\|^{q-1}.\end{aligned}$$

Since $\tilde{h}_1$, $\tilde{h}_2$, and h_1 are strictly increasing, continuous, and convex functions with $\tilde{h}_1(0) = \tilde{h}_2(0) = h_1(0) = 0$, from (3.6) and (3.7), we conclude that $\|w_n - v_n - x^* + y^*\| \to 0$, $\|v_n - u_n + x^* - y^*\| \to 0$, and $\|u_n - y_n\| \to 0$ as $n \to \infty$. This together with $w_n = s_nx_n + (1 - s_n)u_n$ ensures that

$$\lim_{n\to\infty} \|x_n - u_n\| = \lim_{n\to\infty} \|w_n - u_n\| = \lim_{n\to\infty} \|u_n - y_n\| = 0. \tag{3.10}$$

In a similar way, we obtain

$$\begin{aligned}\|z_n - x^*\|^q &= \|J_{\lambda_n}^B(u_n - \lambda_nAy_n + r_n(y_n - u_n)) - J_{\lambda_n}^B(x^* - \lambda_nAx^*)\|^q \\ &\le \langle (u_n - \lambda_nAy_n + r_n(y_n - u_n)) - (x^* - \lambda_nAx^*), J_q(z_n - x^*)\rangle \\ &\le \frac{1}{q}[\|(u_n - \lambda_nAy_n + r_n(y_n - u_n)) - (x^* - \lambda_nAx^*)\|^q + (q-1)\|z_n - x^*\|^q \\ &\quad - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)],\end{aligned}$$

which, together with (3.2), implies that

$$\begin{aligned}\|z_n - x^*\|^q &\le \|(u_n - \lambda_n A y_n + r_n(y_n - u_n)) - (x^* - \lambda_n A x^*)\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)\\ &\le \|u_n - x^*\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|).\end{aligned}$$

So, it follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^q &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n\|S_n z_n - x^*\|^q\\ &\le \alpha_n\|f(x_n) - x^*\|^q + \beta_n\|x_n - x^*\|^q + \gamma_n[\|u_n - x^*\|^q - h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|)]\\ &\le \alpha_n\|f(x_n) - x^*\|^q + \|x_n - x^*\|^q - \gamma_n h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|),\end{aligned}$$

which immediately leads to

$$\gamma_n h_2(\|u_n + r_n(y_n - u_n) - \lambda_n(Ay_n - Ax^*) - z_n\|) \le \alpha_n\|f(x_n) - x^*\|^q + \Gamma_n - \Gamma_{n+1}.$$

Note that h_2 is a strictly increasing, continuous, and convex function with $h_2(0) = 0$. Using (3.7) and (3.10), we obtain

$$\lim_{n\to\infty}\|u_n - z_n\| = 0,$$

which, together with (3.10), implies that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.11}$$

Combining (3.8) and (3.11), we obtain

$$\|x_n - S_n x_n\| \le \|x_n - S_n z_n\| + \|S_n z_n - S_n x_n\| \le \|x_n - S_n z_n\| + \|z_n - x_n\| \to 0 \quad (n \to \infty).$$

Moreover, using Proposition 2.1, we obtain

$$\lim_{n\to\infty}\|S_n x_n - S x_n\| = 0.$$

So, it follows that

$$\|Sx_n - x_n\| \le \|Sx_n - S_n x_n\| + \|S_n x_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.12}$$

For each $n \ge 0$, we put $T_{\lambda_n} := J^B_{\lambda_n}(I - \lambda_n A)$. Then, from (3.10), we obtain

$$\lim_{n\to\infty}\|u_n - T_{\lambda_n}u_n\| = 0.$$

Note that $0 < \lambda \le \lambda_n$ for all $n \ge 0$, and using Proposition 2.3 (ii), we obtain from (3.10) that

$$\begin{aligned}\|T_\lambda x_n - x_n\| &\le \|T_\lambda x_n - T_\lambda u_n\| + \|T_\lambda u_n - u_n\| + \|u_n - x_n\|\\ &\le 2\|x_n - u_n\| + \|T_\lambda u_n - u_n\|\\ &\le 2\|x_n - u_n\| + 2\|T_{\lambda_n}u_n - u_n\| \to 0 \quad (n \to \infty).\end{aligned} \tag{3.13}$$

In addition, again from (3.10), we obtain

$$\|Gx_n - x_n\| \le \|Gx_n - Gw_n\| + \|Gw_n - x_n\| \le \|x_n - w_n\| + \|u_n - x_n\| \to 0(n \to \infty). \tag{3.14}$$

We define the mapping $\Phi : C \to C$ by $\Phi x := \theta_1 Sx + \theta_2 Gx + (1 - \theta_1 - \theta_2)T_\lambda x$, $\forall x \in C$, with $\theta_1 + \theta_2 < 1$ for constants $\theta_1, \theta_2 \in (0, 1)$. Then, by Lemma 2.8 and Proposition 2.3 (i), we know that Φ is nonexpansive and

$$\mathrm{Fix}(\Phi) = \mathrm{Fix}(S) \cap \mathrm{Fix}(G) \cap \mathrm{Fix}(T_\lambda) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSV\,I}(C, B_1, B_2) \cap (A + B)^{-1}0(=:\Omega).$$

Taking into account that

$$\|\Phi x_n - x_n\| \le \theta_1\|Sx_n - x_n\| + \theta_2\|Gx_n - x_n\| + (1 - \theta_1 - \theta_2)\|T_\lambda x_n - x_n\|,$$

we deduce from (3.12) to (3.14) that

$$\lim_{n\to\infty}\|\Phi x_n - x_n\| = 0. \tag{3.15}$$

Let $z_s = sf(z_s) + (1 - s)\Phi z_s$, $\forall s \in (0, 1)$. Then, it follows from Proposition 2.4 that $\{z_s\}$ converges strongly to a point $x^* \in \mathrm{Fix}(\Phi) = \Omega$, which solves the HVI as follows:

$$\langle (I - f)x^*, J(x^* - p)\rangle \le 0, \quad \forall p \in \Omega.$$

In addition, from Lemma 2.1, we obtain

$$\begin{aligned}\|z_s - x_n\|^q &= \|s(f(z_s) - x_n) + (1-s)(\Phi z_s - x_n)\|^q\\ &\le (1-s)^q\|\Phi z_s - x_n\|^q + qs\langle f(z_s) - x_n, J_q(z_s - x_n)\rangle\\ &= (1-s)^q\|\Phi z_s - x_n\|^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\langle z_s - x_n, J_q(z_s - x_n)\rangle\\ &\le (1-s)^q(\|\Phi z_s - \Phi x_n\| + \|\Phi x_n - x_n\|)^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q\\ &\le (1-s)^q(\|z_s - x_n\| + \|\Phi x_n - x_n\|)^q + qs\langle f(z_s) - z_s, J_q(z_s - x_n)\rangle + qs\|z_s - x_n\|^q,\end{aligned}$$

which immediately attains

$$\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle \le \frac{(1-s)^q}{qs}(\|z_s - x_n\| + \|\Phi x_n - x_n\|)^q + \frac{qs-1}{qs}\|z_s - x_n\|^q.$$

From (3.15), we have

$$\limsup_{n\to\infty}\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle \le \frac{(1-s)^q}{qs}M + \frac{qs-1}{qs}M = \left(\frac{(1-s)^q + qs - 1}{qs}\right)M, \tag{3.16}$$

where M is a constant such that $\|z_s - x_n\|^q \le M$ for all $n \ge 0$ and $s \in (0,1)$. It is easy to see that $((1-s)^q + qs - 1)/qs \to 0$ as $s \to 0$. Since J_q is norm-to-norm uniformly continuous on bounded subsets of E and $z_s \to x^*$, we obtain

$$\|J_q(x_n - z_s) - J_q(x_n - x^*)\| \to 0 \quad (s \to 0).$$

So, we obtain

$$\begin{aligned}&|\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle|\\ &= |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle + \langle f(x^*) - x^*, J_q(x_n - z_s)\rangle + \langle x^* - z_s, J_q(x_n - z_s)\rangle - \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle|\\ &\le |\langle f(x^*) - x^*, J_q(x_n - z_s) - J_q(x_n - x^*)\rangle| + |\langle f(z_s) - f(x^*), J_q(x_n - z_s)\rangle| + |\langle x^* - z_s, J_q(x_n - z_s)\rangle|\\ &\le \|f(x^*) - x^*\|\|J_q(x_n - z_s) - J_q(x_n - x^*)\| + (1+\delta)\|z_s - x^*\|\|x_n - z_s\|^{q-1}.\end{aligned}$$

Hence, for each $n \ge 0$, we obtain

$$\lim_{s\to 0}\langle f(z_s) - z_s, J_q(x_n - z_s)\rangle = \langle f(x^*) - x^*, J_q(x_n - x^*)\rangle.$$

From (3.16), as $s \to 0$, it follows that

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_n - x^*)\rangle \le 0. \tag{3.17}$$

By (C1) and (3.8), we obtain

$$\|x_{n+1} - x_n\| = \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n z_n - x_n\| \le \alpha_n\|f(x_n) - x_n\| + \gamma_n\|S_n z_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.18}$$

Using (3.17) and (3.18), we have

$$\limsup_{n\to\infty}\langle f(x^*) - x^*, J_q(x_{n+1} - x^*)\rangle \le 0. \tag{3.19}$$

Using Lemma 2.9 and (3.19), we can infer that $\Gamma_n \to 0$ as $n \to \infty$. Thus, $x_n \to x^*$ as $n \to \infty$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{k_i}\}$ of $\{\Gamma_k\}$ s.t. $\Gamma_{k_i} < \Gamma_{k_i+1}$, $\forall i \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. Define the mapping $\tau : \mathbf{N} \to \mathbf{N}$ by

$$\tau(k) := \max\{i \le k : \Gamma_i < \Gamma_{i+1}\}.$$

Using Lemma 2.7, we have

$$\Gamma_{\tau(k)} \le \Gamma_{\tau(k)+1} \quad \text{and} \quad \Gamma_k \le \Gamma_{\tau(k)+1}.$$

Putting $\Gamma_k = \|x_k - x^*\|^q$, $\forall k \in \mathbf{N}$, and using the same inference as in Case 1, we can obtain

$$\lim_{k\to\infty}\|x_{\tau(k)+1} - x_{\tau(k)}\| = 0 \tag{3.20}$$

and

$$\limsup_{k\to\infty}\langle f(x^*) - x^*, J_q(x_{\tau(k)+1} - x^*)\rangle \le 0. \tag{3.21}$$

Because of $\Gamma_{\tau(k)} \le \Gamma_{\tau(k)+1}$ and $\alpha_{\tau(k)} > 0$, we conclude from (3.5) that

$$\|x_{\tau(k)} - x^*\|^q \le \frac{q}{1-\delta}\langle f(x^*) - x^*, J_q(x_{\tau(k)+1} - x^*)\rangle,$$

and hence,

$$\limsup_{k\to\infty}\|x_{\tau(k)} - x^*\|^q \le 0.$$

Thus, we have

$$\lim_{k\to\infty}\|x_{\tau(k)} - x^*\|^q = 0.$$

Using Proposition 2.2 and (3.20), we obtain

$$\begin{aligned}\|x_{\tau(k)+1} - x^*\|^q - \|x_{\tau(k)} - x^*\|^q &\le q\langle x_{\tau(k)+1} - x_{\tau(k)}, J_q(x_{\tau(k)} - x^*)\rangle + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q \\ &\le q\|x_{\tau(k)+1} - x_{\tau(k)}\|\|x_{\tau(k)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q \to 0 \quad (k\to\infty).\end{aligned}$$

Taking into account $\Gamma_k \le \Gamma_{\tau(k)+1}$, we have

$$\|x_k - x^*\|^q \le \|x_{\tau(k)+1} - x^*\|^q \le \|x_{\tau(k)} - x^*\|^q + q\|x_{\tau(k)+1} - x_{\tau(k)}\|\|x_{\tau(k)} - x^*\|^{q-1} + \kappa_q\|x_{\tau(k)+1} - x_{\tau(k)}\|^q.$$

It is easy to see from (3.20) that $x_k \to x^*$ as $k \to \infty$. This completes the proof. □

We also obtain the strong convergence result for the generalized extragradient implicit method in a real Hilbert space H. It is well known that $\kappa_2 = 1$ [26]. Thus, by Theorem 3.1, we derive the following conclusion.

Corollary 3.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $f : C \to C$ be a δ-contraction with constant $\delta \in [0, 1)$ and $\{S_n\}_{n=0}^{\infty}$ be a countable family of nonexpansive self-mappings on C. Let $A : C \to H$ and $B : C \to 2^H$ be a σ-inverse-strongly monotone mapping and a maximal monotone operator, respectively. Suppose that B_1 and $B_2 : C \to H$ are α-inverse-strongly monotone mapping and β-inverse-strongly monotone mapping, respectively. Assume that $\Omega := \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n) \cap \mathrm{GSVI}(C, B_1, B_2) \cap (A + B)^{-1}0 \neq \emptyset$. For any given $x_0 \in C$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by*

$$\begin{cases} w_n = s_n x_n + (1 - s_n)u_n, \\ v_n = P_C(w_n - \mu_2 B_2 w_n), \\ u_n = P_C(v_n - \mu_1 B_1 v_n), \\ y_n = J_{\lambda_n}^B(u_n - \lambda_n A u_n), \\ z_n = J_{\lambda_n}^B(u_n - \lambda_n A y_n + r_n(y_n - u_n)), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n S_n z_n, \quad \forall n \ge 0, \end{cases} \tag{3.22}$$

where $J_{\lambda_n}^B = (I + \lambda_n B)^{-1}$, $\{r_n\}, \{s_n\}, \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset (0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ and $\{\lambda_n\} \subset (0, \infty)$. Suppose that the following conditions hold:

(C1): $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(C2): $0 < a \le \beta_n \le b < 1$ and $0 < c \le s_n < 1$;

(C3): $0 < r \le r_n < 1$ and $0 < \lambda \le \lambda_n < \frac{\lambda_n}{r_n} \le \mu < 2\sigma$;

(C4): $0 < \mu_1 < 2\alpha$ and $0 < \mu_2 < 2\beta$.

Assume that $\sum_{n=0}^{\infty}\sup_{x\in D}\|S_{n+1}x - S_n x\| < \infty$ for any bounded subset D of C. Let $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty}S_n x$, $\forall x \in C$, and suppose that $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. Then, $x_n \to x^* \in \Omega$, which is the unique solution to the HVI: $\langle (I - f)x^*, p - x^*\rangle \ge 0$, $\forall p \in \Omega$, i.e., the fixed point equation $x^* = P_\Omega f(x^*)$.

4 Conclusion

Now, it is well known that the Korpelevich's extragradient method is an important tool for solving a class of variational inequalities. In this article, we extend this method to solve a GSVI in which a VI problem and a fixed point problem are involved. More specifically, we propose a generalized extragradient implicit algorithm [Algorithm 3.1] for solving GSVI (1.6), where the related operators A, B, B_1, and B_2 are all inverse-strongly accretive mappings. At the same time, this extragradient algorithm can be used to solve a fixed point problem of a countable family of nonexpansive self-mappings $\{S_n\}_{n=0}^{\infty}$ and a VI problem. Under some mild conditions, we show that the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to a common point in Ω, which also solves the variational inequality $\langle (I-f)x^*, J(x^*-p)\rangle \le 0,\ \forall p \in \Omega$.

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