

Research Article

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Study on r -truncated degenerate Stirling numbers of the second kind

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Abstract: The degenerate Stirling numbers of the second kind and of the first kind, which are, respectively, degenerate versions of the Stirling numbers of the second kind and of the first kind, appear frequently when we study various degenerate versions of some special numbers and polynomials. The aim of this article is to consider the r -truncated degenerate Stirling numbers of the second kind, which reduce to the degenerate Stirling numbers of the second for $r = 1$, and to investigate their explicit expressions, some properties and related identities, in connection with several other degenerate special numbers and polynomials.

Keywords: r -truncated degenerate Stirling numbers of the second kind, r -truncated degenerate Bernoulli polynomials of order alpha

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1 Introduction

Carlitz [1,2] initiated to study of the degenerate Bernoulli and Euler polynomials and numbers, which are degenerate versions of the Bernoulli and Euler polynomials and numbers. In recent years, studying degenerate versions of some special numbers and polynomials has regained interests of some mathematicians and yielded quite a few interesting results (see [3–11] and references therein). It is noteworthy that studying degenerate versions is not only limited to polynomials but also extended to transcendental functions, like gamma functions (see [12,13]). It is also remarkable that the degenerate umbral calculus is introduced as a degenerate version of the classical umbral calculus (see [14]). Degenerate versions of special numbers and polynomials have been explored by various methods, including combinatorial methods, generating functions, umbral calculus techniques, p -adic analysis, differential equations, special functions, probability theory and analytic number theory.

The Stirling number of the second $S_2(n, k)$ is the number of ways to partition a set of n objects into k nonempty subsets. The (signed) Stirling number of the first kind $S_1(n, k)$ is defined so that the number of permutations of n elements having exactly k cycles is the nonnegative integer $(-1)^{n-k} S_1(n, k) = |S_1(n, k)|$. The degenerate Stirling numbers of the second kind $S_{2,\lambda}(n, k)$ (see (5), (8)) and of the first kind $S_{1,\lambda}(n, k)$ (see (4), (7)) were introduced as degenerate versions of the Stirling numbers of the second and of the first kind, respectively. These degenerate Stirling numbers of both kinds appear frequently when we study degenerate versions of some special numbers and polynomials.

Here we consider the r -truncated degenerate Stirling numbers of the second which reduce to the degenerate Stirling numbers of the second for $r = 1$. We note that the generating function of these numbers

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is obtained by truncating the first r terms of the degenerate exponential function (see (8), (10), (35)). In the same way, the r -truncated degenerate Bernoulli polynomials of order α are defined by truncating the first r terms of the degenerate exponential function in their generating function (see (17)). In some works related to truncated degenerate special numbers and polynomials, we let the reader refer to [4,6,7,15] and references therein.

The aim of this article is by using generating functions to study their explicit expressions, some properties and related identities on the r -truncated degenerate Stirling numbers of the second kind, in connection with the r -truncated degenerate Bernoulli polynomials of order α , the degenerate Bernoulli numbers, the degenerate Stirling numbers of the second kind and the degenerate Bernoulli numbers of order α .

The outline of this article is as follows. In Section 1, we recall the degenerate exponentials and logarithms. We remind the reader of the degenerate Stirling numbers of both kinds. Furthermore, we recall the degenerate Bernoulli polynomials of order α . Section 2 gives the main results of this article. We consider the r -truncated degenerate Stirling numbers of the second kind and find three explicit expressions for those numbers in Theorems 1–3. Then we introduce the r -truncated degenerate Bernoulli polynomials of order α and obtain some results in connection with the r -truncated degenerate Stirling numbers of the second kind. In Theorem 4, the degenerate Bernoulli numbers are expressed in terms of the Stirling numbers of the second kind. In Theorems 5 and 8, some identities involving the r -truncated degenerate Stirling numbers of the second kind (for $r = 2$ and $r = 3$, respectively), the degenerate Bernoulli numbers and the degenerate Stirling numbers of the second are obtained. In Theorem 6, we find an identity connecting the 2-truncated degenerate Stirling numbers of the second kind and the degenerate Bernoulli numbers. In Theorem 7, we obtain an identity relating the 2-truncated degenerate Stirling numbers of the second kind and the degenerate Bernoulli numbers of order α . In the rest of this section, we recall the facts that are needed throughout this article.

For any nonzero $\lambda \in \mathbb{R}$, the degenerate exponentials are defined by

$$e_{\lambda}^x(t) = \sum_{k=0}^{\infty} \frac{(x)_{k,\lambda}}{k!} t^k = (1 + \lambda t)^{\frac{x}{\lambda}} \quad (\text{see [3–7]}), \quad (1)$$

$$(x)_{0,\lambda} = 1, \quad (x)_{n,\lambda} = x(x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda), \quad (n \geq 1). \quad (2)$$

When $x = 1$, $e_{\lambda}(t) = e_{\lambda}^1(t) = (1 + \lambda t)^{\frac{1}{\lambda}}$ (see [8,15]).

In [1,2], Carlitz considered the degenerate Bernoulli polynomials of order α given by

$$\left(\frac{t}{e_{\lambda}(t) - 1} \right)^{\alpha} e_{\lambda}^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(\alpha)}(x) \frac{t^n}{n!}. \quad (3)$$

When $x = 0$, $\beta_{n,\lambda}^{(\alpha)} = \beta_{n,\lambda}^{(\alpha)}(0)$ are called the degenerate Bernoulli numbers of order α .

In particular, for $\alpha = 1$, $\beta_{n,\lambda}(x) = \beta_{n,\lambda}^{(1)}(x)$ are called the degenerate Bernoulli polynomials.

For $n \geq 0$, the degenerate Stirling numbers of the first kind are defined by

$$(x)_n = \sum_{k=0}^n S_{1,\lambda}(n, k)(x)_{k,\lambda} \quad (\text{see [3]}), \quad (4)$$

where

$$(x)_0 = 1, \quad (x)_n = x(x - 1) \cdots (x - n + 1), \quad (n \geq 1) \quad (\text{see [1–11, 14–22]}).$$

As the inversion formula of (4), the degenerate Stirling numbers of the second kind are defined by

$$(x)_{n,\lambda} = \sum_{k=0}^n S_{2,\lambda}(n, k)(x)_k, \quad (n \geq 0) \quad (\text{see [3]}). \quad (5)$$

The degenerate logarithm $\log_{\lambda}(t)$ is the compositional inverse of $e_{\lambda}(t)$ satisfying $e_{\lambda}(\log_{\lambda}(t)) = \log_{\lambda}(e_{\lambda}(t))$. Then we have

$$\log_{\lambda}(1+t) = \frac{1}{\lambda}((1+t)^{\lambda} - 1) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n,\frac{1}{\lambda}}}{n!} t^n \quad (\text{see [3]}). \quad (6)$$

From (4) and (5), we note that

$$\frac{1}{k!}(\log_{\lambda}(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!}, \quad (k \geq 0) \quad (7)$$

and

$$\frac{1}{k!}(e_{\lambda}(t) - 1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k) \frac{t^n}{n!} \quad (\text{see [3]}). \quad (8)$$

Let $f(t) = \sum_{k=0}^{\infty} a_k t^k \in \mathbb{C}[t]$. For $n \geq 0$, the operator $[t^n]$ is defined by

$$[t^n]f(t) = a_n, \quad (n \geq 0) \quad (\text{see [22]}). \quad (9)$$

2 r -Truncated degenerate Stirling numbers of the second kind

For $r \in \mathbb{N}$, we consider the r -truncated degenerate Stirling numbers of the second kind given by

$$\frac{1}{k!} \left(e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l \right)^k = \sum_{n=kr}^{\infty} S_{2,\lambda}^{[r]}(n, kr) \frac{t^n}{n!}, \quad (k \geq 0). \quad (10)$$

We agree that $S_{2,\lambda}^{[r]}(n, kr) = 0$, for $0 \leq n < kr$. Note that $S_{2,\lambda}^{[1]}(n, k) = S_{2,\lambda}(n, k)$, $(n, k \geq 0)$.

From (10), we note that

$$\frac{1}{k!} \left(e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l \right)^k = \frac{1}{k!} \left(\sum_{l=r}^{\infty} \frac{(1)_{l,\lambda}}{l!} t^l \right)^k = \frac{1}{k!} \sum_{n=kr}^{\infty} \left(\sum_{l_1+l_2+\dots+l_k=n} \frac{n! (1)_{l_1,\lambda} (1)_{l_2,\lambda} \dots (1)_{l_k,\lambda}}{l_1! l_2! \dots l_k!} \right) \frac{t^n}{n!}. \quad (11)$$

Thus, by (9) and (11), we obtain the following theorem.

Theorem 1. For $n, k \geq 0$ with $n \geq kr$, we have

$$S_{2,\lambda}^{[r]}(n, kr) = \frac{1}{k!} \sum_{l_1+l_2+\dots+l_k=n} \frac{n! (1)_{l_1,\lambda} (1)_{l_2,\lambda} \dots (1)_{l_k,\lambda}}{l_1! l_2! \dots l_k!}.$$

From the binomial expansion, we note that

$$\begin{aligned} & \frac{1}{k!} \left(e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l \right)^k \\ &= \frac{1}{k!} \sum_{m=0}^k \binom{k}{m} e_{\lambda}^{k-m}(t) (-1)^m \left(\sum_{l=0}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l \right)^m \\ &= \frac{1}{k!} \sum_{m=0}^k \binom{k}{m} \sum_{j=0}^{\infty} \binom{k-m}{j} \frac{t^j}{j!} (-1)^m \sum_{l_1, l_2, \dots, l_m=0}^{r-1} \frac{(1)_{l_1,\lambda} (1)_{l_2,\lambda} \dots (1)_{l_m,\lambda}}{l_1! l_2! \dots l_m!} t^{l_1+\dots+l_m} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{k!} \sum_{m=0}^k \binom{k}{m} (-1)^m \sum_{l_1, l_2, \dots, l_m=0}^{r-1} \frac{n! (1)_{l_1,\lambda} (1)_{l_2,\lambda} \dots (1)_{l_m,\lambda} (k-m)_{n-l_1-l_2-\dots-l_m,\lambda}}{l_1! l_2! \dots l_m! (n-l_1-l_2-\dots-l_m)!} \right) \frac{t^n}{n!}. \end{aligned} \quad (12)$$

Therefore, by comparing the coefficients on both sides of (12), we obtain the following theorem.

Theorem 2. For $n, k \geq 0$, we have

$$\frac{1}{k!} \sum_{m=0}^k \binom{k}{m} (-1)^m \sum_{l_1, l_2, \dots, l_m=0}^{r-1} \frac{n! (1)_{l_1, \lambda} (1)_{l_2, \lambda} \cdots (1)_{l_m, \lambda} (k-m)_{n-l_1-l_2-\dots-l_m, \lambda}}{l_1! l_2! \cdots l_m! (n-l_1-\dots-l_m)!} = \begin{cases} S_{2, \lambda}^{[r]}(n, kr), & \text{if } n \geq kr, \\ 0, & \text{if } 0 \leq n < kr. \end{cases}$$

When $k = 1$ in (10), we have

$$\begin{aligned} \sum_{l=r}^{\infty} \frac{(1)_{l, \lambda}}{l!} t^l &= \left(e_{\lambda}(t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} t^l \right) = \sum_{n=r}^{\infty} S_{2, \lambda}^{[r]}(n, r) \frac{t^n}{n!} = \sum_{n=0}^{\infty} S_{2, \lambda}^{[r]}(n+r, r) \frac{t^{n+r}}{(n+r)!} \\ &= t^r \sum_{n=0}^{\infty} S_{2, \lambda}^{[r]}(n+r, r) \frac{n!}{(n+r)!} \frac{t^n}{n!}. \end{aligned} \quad (13)$$

For $k = 2$, we have

$$\begin{aligned} \left(\sum_{l=r}^{\infty} \frac{(1)_{l, \lambda}}{l!} t^l \right)^2 &= t^{2r} \left(\sum_{j=0}^{\infty} S_{2, \lambda}^{[r]}(j+r, r) \frac{t^j}{(j+r)!} \right) \left(\sum_{l=0}^{\infty} S_{2, \lambda}^{[r]}(l+r, r) \frac{t^l}{(l+r)!} \right) \\ &= t^{2r} \sum_{n=0}^{\infty} \left(\sum_{j=0}^n S_{2, \lambda}^{[r]}(j+r, r) S_{2, \lambda}^{[r]}(n-j+r, r) \frac{n!}{(n-j+r)!(j+r)!} \right) \frac{t^n}{n!}. \end{aligned} \quad (14)$$

Continuing this process, we have

$$\left(\sum_{l=r}^{\infty} \frac{(1)_{l, \lambda}}{l!} t^l \right)^k = t^{kr} \sum_{n=0}^{\infty} \left(\sum_{j_1+j_2+\dots+j_k=n} \frac{n! S_{2, \lambda}^{[r]}(j_1+r, r) S_{2, \lambda}^{[r]}(j_2+r, r) \cdots S_{2, \lambda}^{[r]}(j_k+r, r)}{(j_1+r)!(j_2+r)! \cdots (j_k+r)!} \right) \frac{t^n}{n!}. \quad (15)$$

On the other hand, by (10), we obtain

$$\begin{aligned} \left(\sum_{l=r}^{\infty} \frac{t^l}{l!} (1)_{l, \lambda} \right)^k &= \left(e_{\lambda}(t) - \sum_{l=1}^{r-1} \frac{(1)_{l, \lambda}}{l!} t^l \right)^k \\ &= k! \sum_{n=kr}^{\infty} S_{2, \lambda}^{[r]}(n, kr) \frac{t^n}{n!} \\ &= k! \sum_{n=0}^{\infty} S_{2, \lambda}^{[r]}(n+kr, kr) \frac{t^{n+kr}}{(n+kr)!} \\ &= t^{kr} \sum_{n=0}^{\infty} \frac{k! S_{2, \lambda}^{[r]}(n+kr, kr)}{(n+kr)!} t^n. \end{aligned} \quad (16)$$

Therefore, by (15) and (16), we obtain the following theorem.

Theorem 3. For $n, k \geq 0$, we have

$$\frac{k!}{(n+kr)!} S_{2, \lambda}^{[r]}(n+kr, kr) = \sum_{j_1+j_2+\dots+j_k=n} \frac{S_{2, \lambda}^{[r]}(j_1+r, r) S_{2, \lambda}^{[r]}(j_2+r, r) \cdots S_{2, \lambda}^{[r]}(j_k+r, r)}{(j_1+r)!(j_2+r)! \cdots (j_k+r)!}.$$

Let us consider the r -truncated degenerate Bernoulli polynomials of order α given by

$$\frac{t^{\alpha}}{(e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{t^l}{l!} (1)_{l, \lambda})^{\alpha}} e_{\lambda}^{\alpha}(t) = \sum_{n=0}^{\infty} \beta_{n, \lambda}^{[r-1, \alpha]}(x) \frac{t^n}{n!}. \quad (17)$$

When $x = 0$, $\beta_{n, \lambda}^{[r-1, \alpha]} = \beta_{n, \lambda}^{[r-1, \alpha]}(0)$ are called the r -truncated degenerate Bernoulli numbers of order α .

Note that

$$\begin{aligned}
 t^{\alpha r} &= \left(e_{\lambda}(t) - \sum_{l=0}^{r-1} \frac{t^l}{l!} (1)_{l,\lambda} \right)^{\alpha} \sum_{l=0}^{\infty} \beta_{l,\lambda}^{[r-1,\alpha]} \frac{t^l}{l!} \\
 &= \alpha! \sum_{j=\alpha r}^{\infty} S_{2,\lambda}^{[r]}(j, \alpha r) \frac{t^j}{j!} \sum_{l=0}^{\infty} \beta_{l,\lambda}^{[r-1,\alpha]} \frac{t^l}{l!} \\
 &= \alpha! \sum_{n=\alpha r}^{\infty} \left(\sum_{l=0}^{n-\alpha r} \frac{S_{2,\lambda}^{[r]}(n-l, \alpha r) n!}{(n-l)! l!} \beta_{l,\lambda}^{[r-1,\alpha]} \right) \frac{t^n}{n!} \\
 &= \sum_{n=\alpha r}^{\infty} \left(\alpha! \sum_{l=0}^{n-\alpha r} \binom{n}{l} S_{2,\lambda}^{[r]}(n-l, \alpha r) \beta_{l,\lambda}^{[r-1,\alpha]} \right) \frac{t^n}{n!},
 \end{aligned} \tag{18}$$

where α is a positive integer.

Thus, by (18), we obtain

$$\sum_{l=0}^{n-\alpha r} \binom{n}{l} S_{2,\lambda}^{[r]}(n-l, \alpha r) \beta_{l,\lambda}^{[r-1,\alpha]} = \begin{cases} \frac{(\alpha r)!}{\alpha!}, & \text{if } n = \alpha r, \\ 0, & \text{if } n > \alpha r. \end{cases}$$

It is well known that the partial Bell polynomials are defined by

$$\frac{1}{k!} \left(\sum_{l=1}^{\infty} x_l \frac{t^l}{l!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!},$$

where k is a nonnegative integer.

Thus, we note that

$$\begin{aligned}
 B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) &= \sum_{\substack{l_1+l_2+\dots+l_{n-k+1}=k \\ l_1+2l_2+\dots+(n-k+1)l_{n-k+1}=n}} \frac{n!}{l_1! l_2! \dots l_{n-k+1}!} \left(\frac{x_1}{1!} \right)^{l_1} \left(\frac{x_2}{2!} \right)^{l_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!} \right)^{l_{n-k+1}} \\
 &= n! \sum_{\Lambda_n^k} \prod_{j=1}^{n-k+1} \frac{1}{k_j!} \left(\frac{x_j}{j!} \right)^{k_j},
 \end{aligned}$$

where

$$\Lambda_n^k = \{(k_1, k_2, \dots, k_{n-k+1}) | k_1 + k_2 + \dots + k_{n-k+1} = k, \quad k_1 + 2k_2 + \dots + (n-k+1)k_{n-k+1} = n\}.$$

We observe that

$$\begin{aligned}
 &\frac{1}{1 + \frac{x_1}{1!}(1)_{1,\lambda}t + \frac{x_2}{2!}(1)_{2,\lambda}t^2 + \dots + \frac{x_k}{k!}(1)_{k,\lambda}t^k + \dots} \\
 &= \sum_{k=0}^{\infty} (-1)^k k! \frac{1}{k!} \left(\sum_{l=1}^{\infty} \frac{x_l(1)_{l,\lambda}}{l!} t^l \right)^k \\
 &= 1 + \sum_{k=1}^{\infty} (-1)^k k! \sum_{n=k}^{\infty} B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda}, \dots, x_{n-k+1}(1)_{n-k+1,\lambda}) \frac{t^n}{n!} \\
 &= 1 + \sum_{n=1}^{\infty} \left(\sum_{k=1}^n (-1)^k k! B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda}, \dots, x_{n-k+1}(1)_{n-k+1,\lambda}) \right) \frac{t^n}{n!}.
 \end{aligned} \tag{19}$$

We denote (19) by

$$\frac{1}{\sum_{n=0}^{\infty} \frac{(1)_{n,\lambda}}{n!} x_n t^n} = \sum_{n=0}^{\infty} K_{n,\lambda}(x_1, x_2, \dots) \frac{t^n}{n!}. \tag{20}$$

Then, by (19) and (20), we obtain

$$K_{n,\lambda}(x_1, x_2, \dots) = \sum_{k=1}^n (-1)^k k! B_{n,k}(x_1(1)_{1,\lambda}, x_2(1)_{2,\lambda}, \dots, x_{n-k+1}(1)_{n-k+1,\lambda}), \quad (n \geq 1),$$

$$K_{0,\lambda}(x_1, x_2, \dots) = 1.$$

Note that, from (19) and (20), we have

$$K_{n,\lambda}(1, 1, \dots) = \sum_{k=0}^n (-1)^k k! B_{n,k}((1)_{1,\lambda}, (1)_{2,\lambda}, \dots, (1)_{n-k+1,\lambda}) = \sum_{k=0}^n (-1)^k k! S_{2,\lambda}(n, k).$$

Taking $\alpha = 1$ in (17), we have

$$\frac{t^r}{e_\lambda(t) - \sum_{l=0}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l} e_\lambda^x(t) = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{[r-1,1]}(x) \frac{t^n}{n!}. \quad (21)$$

When $r = 1$, $\beta_{n,\lambda}^{[0,1]}(x) = \beta_{n,\lambda}(x)$ are the degenerate Bernoulli polynomials.

From (21), we note that

$$\begin{aligned} e_\lambda^x(t) &= \frac{1}{t^r} \left(e_\lambda(t) - \sum_{l=1}^{r-1} \frac{(1)_{l,\lambda}}{l!} t^l \right) \sum_{l=0}^{\infty} \beta_{l,\lambda}^{[r-1,1]}(x) \frac{t^l}{l!} \\ &= \frac{1}{t^r} \sum_{j=r}^{\infty} \frac{(1)_{j,\lambda}}{j!} t^j \sum_{l=0}^{\infty} \beta_{l,\lambda}^{[r-1,1]}(x) \frac{t^l}{l!} \\ &= \sum_{j=0}^{\infty} \frac{(1)_{j+r,\lambda}}{(j+r)!} t^j \sum_{l=0}^{\infty} \beta_{l,\lambda}^{[r-1,1]}(x) \frac{t^l}{l!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^n \binom{n}{j} \frac{j! (1)_{j+r,\lambda}}{(j+r)!} \beta_{n-j,\lambda}^{[r-1,1]}(x) \right) \frac{t^n}{n!}. \end{aligned} \quad (22)$$

By (1) and (22), we obtain

$$(x)_{n,\lambda} = \sum_{j=0}^n \binom{n}{j} \frac{j!}{(j+r)!} (1)_{j+r,\lambda} \beta_{n-j,\lambda}^{[r-1,1]}(x), \quad (n \geq 0). \quad (23)$$

In particular, for $n = 0$, $\beta_{0,\lambda}^{[r-1,1]} = \frac{r!}{(1)_{r,\lambda}}$.

By (23), we obtain

$$\begin{aligned} \beta_{1,\lambda}^{[r-1,1]} &= \frac{r!}{(1)_{r,\lambda}} \left(x - \frac{1-r\lambda}{1+r} \right), \\ \beta_{2,\lambda}^{[r-1,1]} &= \frac{r!}{(1)_{r,\lambda}} \left\{ x(x-\lambda) + \frac{2x(1-r\lambda)}{1+r} - \frac{2(1-r\lambda)^2}{(1+r)^2} + \frac{2}{(1+r)(2+r)} (1-r\lambda)(1-(r+1)\lambda) \right\}. \end{aligned}$$

From (6), we note that

$$\frac{1}{t} \log_\lambda(1+t) = \frac{1}{t} \sum_{n=1}^{\infty} \frac{\lambda^{n-1} (1)_{n,\frac{1}{\lambda}}}{n!} t^n = \sum_{n=0}^{\infty} \frac{\lambda^n (1)_{n+1,\frac{1}{\lambda}}}{(n+1)!} t^n. \quad (24)$$

Replacing t by $e_\lambda(t) - 1$ in (24), we obtain

$$\sum_{n=0}^{\infty} \beta_{n,\lambda} \frac{t^n}{n!} = \frac{t}{e_\lambda(t) - 1} = \sum_{k=0}^{\infty} \frac{\lambda^k (1)_{k+1,\frac{1}{\lambda}}}{(k+1)!} \frac{1}{k!} (e_\lambda(t) - 1)^k = \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{\lambda^k (1)_{k+1,\frac{1}{\lambda}}}{k+1} S_{2,\lambda}(n, k) \right) \frac{t^n}{n!}. \quad (25)$$

Comparing the coefficients on both sides of (25), we have

$$\beta_{n,\lambda} = \sum_{k=0}^n \frac{\lambda^k (1)_{k+1,\frac{1}{\lambda}}}{k+1} S_{2,\lambda}(n, k), \quad (n \geq 0). \quad (26)$$

Replacing t by $\log_\lambda(1+t)$ in (3), with $\alpha = 1$, $x = 0$, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\lambda^n(1)_{n+1, \frac{1}{\lambda}}}{(n+1)!} t^n &= \frac{1}{t} \log_\lambda(1+t) = \sum_{k=0}^{\infty} \beta_{k,\lambda} \frac{1}{k!} (\log_\lambda(1+t))^k \\ &= \sum_{k=0}^{\infty} \beta_{k,\lambda} \sum_{n=k}^{\infty} S_{1,\lambda}(n, k) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \beta_{k,\lambda} S_{1,\lambda}(n, k) \right) \frac{t^n}{n!}. \end{aligned} \quad (27)$$

By comparing the coefficients on both sides of (27), we obtain

$$\sum_{k=0}^n \beta_{k,\lambda} S_{1,\lambda}(n, k) = \frac{\lambda^n(1)_{n+1, \frac{1}{\lambda}}}{n+1}, \quad (n \geq 0). \quad (28)$$

Therefore, by (26) and (28), we obtain the following theorem.

Theorem 4. For $n \geq 0$, we have

$$\beta_{n,\lambda} = \sum_{k=0}^n \frac{\lambda^k(1)_{k+1, \frac{1}{\lambda}}}{k+1} S_{2,\lambda}(n, k)$$

and

$$\frac{(1)_{n+1, \frac{1}{\lambda}}}{n+1} \lambda^n = \sum_{k=0}^n \beta_{k,\lambda} S_{1,\lambda}(n, k).$$

From Theorem 4, we note that

$$\begin{aligned} &\sum_{j=0}^n S_{2,\lambda}^{[2]}(n-j+k, 2k) \binom{n+k}{j} \beta_{j,\lambda} \\ &= (n+k)! \sum_{j=0}^n \frac{S_{2,\lambda}^{[2]}(n-j+k, 2k)}{(n-j+k)!} \frac{\beta_{j,\lambda}}{j!} \\ &= (n+k)! \sum_{j=0}^n [t^{n-j+k}] \frac{(e_\lambda(t) - 1 - t)^k}{k!} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k)! \sum_{j=0}^n [t^{n-j}] \frac{(e_\lambda(t) - 1 - t)^k}{k! t^k} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k)! [t^n] \frac{(e_\lambda(t) - 1 - t)^k t}{k! t^k (e_\lambda(t) - 1)} \\ &= (n+k)! [t^n] \sum_{j=0}^k \binom{k}{j} \frac{(-1)^{k-j} (e_\lambda(t) - 1)^j t^{k-j} t}{k! t^k (e_\lambda(t) - 1)} \\ &= \frac{(-1)^k (n+k)!}{k!} [t^n] \frac{t}{e_\lambda(t) - 1} + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j} [t^n] (e_\lambda(t) - 1)^{j-1}}{j! (k-j)! t^{j-1}} \\ &= \frac{(-1)^k (n+k)!}{k!} \frac{\beta_{n,\lambda}}{n!} + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j}}{j! (k-j)!} [t^n] \frac{(e_\lambda(t) - 1)^{j-1}}{t^{j-1}} \\ &= (-1)^k \binom{n+k}{k} \beta_{n,\lambda} + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j} (j-1)!}{j! (k-j)!} [t^n] \sum_{l=0}^{\infty} \frac{S_{2,\lambda}(l+j-1, j-1)}{(l+j-1)!} t^l \\ &= (-1)^k \binom{n+k}{k} \beta_{n,\lambda} + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j}}{j! (k-j)!} \frac{S_{2,\lambda}(n+j-1, j-1)}{(n+j-1)!}. \end{aligned} \quad (29)$$

Therefore, by (29), we obtain the following theorem.

Theorem 5. For $n, k \geq 0$, we have

$$\sum_{j=0}^n S_{2,\lambda}^{[2]}(n-j+k, 2k) \binom{n+j}{k} \beta_{j,\lambda} = (-1)^n \binom{n+k}{k} \beta_{n,\lambda} + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j}}{j(k-j)!(n+j-1)!} S_{2,\lambda}(n+j-1, j-1).$$

Observe that, for any formal power series $f(t) = \sum_{i=0}^{\infty} a_i t^i$, we have

$$[t^n]f'(t) = (n+1)a_{n+1} = (n+1)[t^{n+1}]f(t). \quad (30)$$

By making use of (30), we have

$$\begin{aligned} & \sum_{j=0}^n \binom{n+k-1}{j} S_{2,\lambda}^{[2]}(n-j+k, 2k) \beta_{j,\lambda} \\ &= (n+k-1)! \sum_{j=0}^n \frac{S_{2,\lambda}^{[2]}(n-j+k, 2k) \beta_{j,\lambda}}{(n-j+k-1)! j!} \\ &= (n+k-1)! \sum_{j=0}^n (n-j+k) [t^{n-j+k}] \frac{(e_\lambda(t) - 1 - t)^k}{k!} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! \sum_{j=0}^n [t^{n-j+k-1}] \frac{d}{dt} \left(\frac{(e_\lambda(t) - 1 - t)^k}{k!} \right) [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! \sum_{j=0}^n [t^{n-j+k-1}] \frac{(e_\lambda(t) - 1 - t)^{k-1} (e_\lambda(t) - 1 - \lambda t)}{(k-1)!} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! \sum_{j=0}^n [t^{n-j+k-1}] \frac{(e_\lambda(t) - 1 - t)^{k-1} (e_\lambda(t) - 1 - \lambda t)}{(k-1)!(1+\lambda t)} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! \sum_{j=0}^n [t^{n-j}] \frac{(e_\lambda(t) - 1 - t)^{k-1} (e_\lambda(t) - 1 - \lambda t)}{t^{k-1}(k-1)!(1+\lambda t)} [t^j] \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! [t^n] \frac{(e_\lambda(t) - 1 - t)^{k-1}}{t^{k-2}(k-1)!(1+\lambda t)} - (n+k-1)! \lambda [t^n] \frac{(e_\lambda(t) - 1 - t)^{k-1}}{t^{k-2}(k-1)!(1+\lambda t)} \frac{t}{e_\lambda(t) - 1} \\ &= (n+k-1)! [t^n] \left\{ \sum_{m=k}^{\infty} \sum_{l=k}^m \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2)}{(l+k-2)!} (-\lambda)^{m-l} t^m \right. \\ & \quad \left. - \lambda \sum_{m=k}^{\infty} \sum_{j=k}^m \sum_{l=k}^j \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2)}{(l+k-2)!} (-\lambda)^{j-l} \frac{\beta_{m-j,\lambda}}{(m-j)!} t^m \right\} \\ &= (n+k-1)! \sum_{l=k}^n \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2)}{(l+k-2)!} (-\lambda)^{n-l} \\ & \quad - \lambda (n+k-1)! \sum_{j=k}^n \sum_{l=k}^j \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2) \beta_{n-j,\lambda} (-\lambda)^{j-l}}{(l+k-2)!(n-j)!}. \end{aligned} \quad (31)$$

Therefore, by (31), we obtain the following theorem.

Theorem 6. For $n, k \geq 1$, with $n \geq k$, we have

$$\begin{aligned} & \sum_{j=0}^n \binom{n+k-1}{j} S_{2,\lambda}^{[2]}(n-j+k, 2k) \beta_{j,\lambda} \\ &= (n+k-1)! \left\{ \sum_{l=k}^n \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2)}{(l+k-2)!} (-\lambda)^{n-l} + \sum_{j=k}^n \sum_{l=k}^j \frac{S_{2,\lambda}^{[2]}(l+k-2, 2k-2) \beta_{n-j,\lambda} (-\lambda)^{j-l+1}}{(l+k-2)!(n-j)!} \right\}. \end{aligned}$$

Now, we observe that

$$\begin{aligned}
 & \sum_{j=0}^n S_{2,\lambda}^{[2]}(n-j+k, 2k) \binom{n+k}{j} \beta_{j,\lambda}^{(k)} \\
 &= (n+k)! \sum_{j=0}^n \frac{S_{2,\lambda}^{[2]}(n-j+k, 2k)}{(n-j+k)!} \frac{\beta_{j,\lambda}^{(k)}}{j!} \\
 &= (n+k)! \sum_{j=0}^n [t^{n-j+k}] \frac{(e_\lambda(t) - 1 - t)^k}{k!} [t^j] \left(\frac{t}{e_\lambda(t) - 1} \right)^k \\
 &= \frac{(n+k)!}{k!} \sum_{j=0}^n [t^{n-j}] \frac{(e_\lambda(t) - 1 - t)^k}{t^k} [t^j] \left(\frac{t}{e_\lambda(t) - 1} \right)^k \\
 &= \frac{(n+k)!}{k!} [t^n] \left(1 - \frac{t}{e_\lambda(t) - 1} \right)^k \\
 &= \frac{(n+k)!}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} [t^n] \left(\frac{t}{e_\lambda(t) - 1} \right)^{k-j} \\
 &= \frac{(n+k)!}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \frac{\beta_{n,\lambda}^{(k-j)}}{n!}.
 \end{aligned} \tag{32}$$

Therefore, by (32), we obtain the following theorem.

Theorem 7. For $n, k \geq 0$, we have

$$\sum_{j=0}^n S_{2,\lambda}^{[2]}(n-j+k, 2k) \binom{n+k}{j} \beta_{j,\lambda}^{(k)} = \binom{n+k}{k} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \beta_{n,\lambda}^{(k-j)}.$$

From Theorems 4 and 5, we have

$$\begin{aligned}
 & \sum_{j=0}^n \binom{n+k}{j+k} S_{2,\lambda}^{[3]}(j+k, 3k) \beta_{n-j,\lambda} = (n+k)! \sum_{j=0}^n \frac{S_{2,\lambda}^{[3]}(j+k, 3k)}{(j+k)!} \frac{\beta_{n-j,\lambda}}{(n-j)!} \\
 &= (n+k)! \sum_{j=0}^n [t^{j+k}] \frac{(e_\lambda(t) - 1 - t - (1)_{2,\lambda} t^2/2)^k}{k!} [t^{n-j}] \frac{t}{e_\lambda(t) - 1} \\
 &= (n+k)! \sum_{j=0}^n [t^j] \frac{(e_\lambda(t) - 1 - t - (1)_{2,\lambda} t^2/2)^k}{t^k k!} [t^{n-j}] \left(\frac{t}{e_\lambda(t) - 1} \right) \\
 &= (n+k)! [t^n] \frac{(e_\lambda(t) - 1 - t - (1)_{2,\lambda} t^2/2)^k t}{t^k k! (e_\lambda(t) - 1)} \\
 &= (n+k)! [t^n] \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \frac{(e_\lambda(t) - 1)^j (t + (1-\lambda)\frac{t^2}{2})^{k-j} t}{k! (e_\lambda(t) - 1) t^k} \\
 &= (n+k)! [t^n] \frac{(-1)^k (1 + (1-\lambda)\frac{t}{2})^k t}{k! (e_\lambda(t) - 1)} + \frac{(n+k)!}{k!} [t^n] \\
 &\quad \times \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} (e_\lambda(t) - 1)^{j-1} t^{1-j} \left(1 + (1-\lambda)\frac{t}{2} \right)^{k-j} \\
 &= (n+k)! [t^n] \sum_{m=0}^{\infty} \left((-1)^k \sum_{j=0}^m \binom{k}{j} (1-\lambda)^j \frac{\beta_{m-j,\lambda}}{2^j (m-j)! k!} \right) t^m \\
 &\quad + \frac{(n+k)!}{k!} [t^n] \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} (e_\lambda(t) - 1)^{j-1} \sum_{l=0}^{k-j} \binom{k-j}{l} (1-\lambda)^l \frac{t^{l-j+1}}{2^l}
 \end{aligned} \tag{33}$$

$$\begin{aligned}
&= (-1)^k(n+k)! \sum_{j=0}^n \binom{k}{j} (1-\lambda)^j \frac{1}{2^j(n-j)!k!} \beta_{n-j,\lambda} \\
&\quad + (n+k)! [t^n] \sum_{j=1}^k \frac{(-1)^{k-j}}{j(k-j)!} \sum_{l=j-1}^{\infty} S_{2,\lambda}(l, j-1) \frac{t^l}{l!} \sum_{l=1}^{k-j} \binom{k-j}{l} \frac{(1-\lambda)^l}{2^l} t^{l-j+1} \\
&= (-1)^k(n+k)! \sum_{j=0}^n \binom{k}{j} (1-\lambda)^j \frac{1}{2^j(n-j)!k!} \beta_{n-j,\lambda} \\
&\quad + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j}}{j(k-j)!} \sum_{l=1}^n \binom{k-j}{l} \frac{S_{2,\lambda}(n-l+j-1, j-1)(1-\lambda)^l}{2^l(n-l+j-1)!}.
\end{aligned}$$

Therefore, by (33), we obtain the following theorem.

Theorem 8. For $n, k \geq 0$, we have

$$\begin{aligned}
\sum_{j=0}^n \binom{n+k}{j+k} S_{2,\lambda}^{[3]}(j+k, 3k) \beta_{n-j,\lambda} &= (-1)^k(n+k)! \sum_{j=0}^n \binom{k}{j} (1-\lambda)^j \frac{1}{2^j(n-j)!k!} \beta_{n-j,\lambda} \\
&\quad + (n+k)! \sum_{j=1}^k \frac{(-1)^{k-j}}{j(k-j)!} \sum_{l=1}^n \binom{k-j}{l} \frac{S_{2,\lambda}(n-l+j-1, j-1)(1-\lambda)^l}{2^l(n-l+j-1)!}.
\end{aligned} \tag{34}$$

Remark 9. As the counterpart of (10), we may consider the r -truncated degenerate Stirling numbers of the first kind given by

$$\frac{1}{k!} \left(\log_{\lambda}(1+t) - \sum_{l=1}^{r-1} \frac{(1)_{l,\frac{1}{\lambda}} t^{l-1}}{l!} \right)^k = \sum_{n=kr}^{\infty} S_{1,\lambda}^{[r]}(n, kr) \frac{t^n}{n!}, \tag{35}$$

where r is a positive integer. These numbers will be investigated in a forthcoming article.

3 Conclusion

In recent years, we have witnessed that many degenerate versions of quite a few special numbers and polynomials were investigated and some nice results were obtained by adopting various tools.

In this article, we considered the r -truncated degenerate Stirling numbers of the second, which reduce to the degenerate Stirling numbers of the second for $r = 1$, and studied by using generating functions their explicit expressions, some properties and related identities on those numbers, in connection with several other degenerate special numbers and polynomials.

As one of our future projects, we would like to continue this line of research, namely, to explore various degenerate versions of some special numbers and polynomials, and to find their applications to physics, science and engineering.

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