Research Article

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# Delta waves of the isentropic relativistic Euler system coupled with an advection equation for Chaplygin gas

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**Abstract:** Most previous studies concerning delta waves have been focused on the overcompressible ones. To study the non-overcompressible delta waves, this article is concerned with the isentropic relativistic Euler system coupled with an advection equation for Chaplygin gas. The Riemann problem is completely solved. The solutions exhibit four kinds of wave patterns: the first contains three contact discontinuities; the second includes a single overcompressible delta wave, and the third and fourth involve a contact discontinuity and a non-overcompressible delta wave.

**Keywords:** relativistic Euler system, advection equation, Chaplygin gas, delta waves, Riemann problem

**MSC 2020:** 35L65, 35L67, 35B30

## 1 Introduction

A delta wave is a generalization of an ordinary shock wave. Speaking informally, it is a kind of discontinuity, on which at least one of the state variables may develop an extreme concentration in the form of a weighted Dirac delta function with the discontinuity as its support. From the physical point of view, a delta wave represents the concentration of the mass. If all characteristics are not outcoming from the discontinuity on both sides, the delta wave is said to be overcompressible; otherwise, it is non-overcompressible.

Since Korchinski [1] constructed his Riemann solutions by using generalized delta functions, people started to explore the existence and uniqueness of delta waves. Over the past three decades, such research studies have become very active. There are lots of authors who have obtained a great many excellent achievements; see [2–16] and references cited therein. Regrettably, all the delta waves in these investigations are overcompressible. Excitingly, Cheng [17] has recently found a kind of non-overcompressible delta wave when he solved the Riemann problem of the following isentropic Cargo-LeRoux model [18,19]:

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ (\rho v)_t + (\rho v^2 + p + q)_x = 0, \\ (\rho q)_t + (\rho v q)_x = 0 \end{cases} \tag{1.1}$$

for the Chaplygin gas, which is characterized by the equation of state [20–22]

$$p = -\frac{1}{\rho}. \tag{1.2}$$

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Besides, Cheng [23] has also derived a kind of contact delta wave in the Riemann solutions to the pressureless isentropic Cargo-LeRoux model ((1.1) with $p \equiv 0$).

The aim of this article is to extend the study in [17] to the relativistic situation. To this end, we couple the isentropic Euler system of conservation laws of baryon numbers and momentum for a perfect fluid in special relativity [24–26] with an advection equation as follows:

$$\begin{cases} \left(\dfrac{n}{\sqrt{1-v^2/c^2}}\right)_t + \left(\dfrac{nv}{\sqrt{1-v^2/c^2}}\right)_x = 0, \\ \left(\dfrac{(\rho+p/c^2)v}{1-v^2/c^2}\right)_t + \left(\dfrac{(\rho+p/c^2)v^2}{1-v^2/c^2} + p + q\right)_x = 0, \\ \left(\dfrac{nq}{\sqrt{1-v^2/c^2}}\right)_t + \left(\dfrac{nvq}{\sqrt{1-v^2/c^2}}\right)_x = 0, \end{cases} \tag{1.3}$$

where $\rho$ and $v$ represent the proper energy density and particle speed, respectively; $p = p(\rho)$ is the pressure; $n$ is the proper number density of baryons; $c$ is the speed of light; and $q$ (satisfying $q_t + vq_x = 0$) is the advected quantity. The advection equation is the basis for mathematical models of continuum mechanics. The concept of advection is widely used in the fields of physics and engineering to describe the transport of a substance or quantity by bulk flux. The model (1.3) can describe the relativistic fluids affected by a quantity that is advected along the particle speed, such as the hydrostatic pressure.

The proper number density of baryons is determined by the first law of thermodynamics as follows:

$$\theta\,\mathrm{d}S = \frac{1}{n}\mathrm{d}\rho - \frac{\rho + p/c^2}{n^2}\mathrm{d}n,$$

where $\theta$ is the temperature and $S$ is the entropy per baryon. For isentropic fluids ($S$ = const.), it is reduced to

$$\frac{\mathrm{d}n}{n} = \frac{\mathrm{d}\rho}{\rho + p/c^2},$$

that is,

$$n = n(\rho) = n_0 \exp\left(\int_\alpha^\rho \frac{\mathrm{d}s}{s + p(s)/c^2}\right), \quad (\text{where } n_0 \text{ and } \alpha \text{ are constants}). \tag{1.4}$$

Formally, the Newtonian limit ($c \to \infty$) of (1.3) + (1.4) is just (1.1).

We consider the Riemann problem for (1.3) + (1.4) + (1.2) with initial data

$$(\rho, v, q)(x, t = 0) = \begin{cases} (\rho_-, v_-, q_-), & x < 0, \\ (\rho_+, v_+, q_+), & x > 0. \end{cases} \tag{1.5}$$

The system (1.3) + (1.4) + (1.2) is strictly hyperbolic and has three linearly degenerate characteristics; the classical waves include three kinds of contact discontinuity $J^-$, $J^0$, and $J^+$. It is concluded that when the initial data (1.5) satisfy any one of the conditions (2.19)–(2.21), the wave pattern $J^- + J^0 + J^+$ serves as a solution to the Riemann problem. In order to construct the Riemann solutions for the rest of the initial data, first, we introduce the overcompressible delta wave and clarify its existence conditions, with which it is concluded that when any one of the conditions (4.14)–(4.17) is true, the Riemann problem can be solved by a single overcompressible delta wave besides two constant states; second, we introduce a kind of non-overcompressible delta wave and derive the existence conditions, by which we construct the Riemann solutions by the structure containing a contact discontinuity $J^-$ on the left and a non-overcompressible delta wave on the right for any one of the conditions (4.22)–(4.24) and by the structure including a non-overcompressible delta wave on the left and a contact discontinuity $J^+$ on the right for any one of the conditions (4.31)–(4.33). Thus the Riemann problem is solved for all initial data by four kinds of structures. The jump of $q$ is the cause of non-overcompressible delta waves.

One can observe that the structures for $q_- = q_+$ are coincident with those of the Riemann solutions to the isentropic relativistic Euler system for Chaplygin gas [11]; besides, as $c \to +\infty$, these structures just tend to those of the Riemann solutions to the isentropic Cargo-LeRoux system for Chaplygin gas [17].

The arrangement of the article is as follows. In Section 2, we solve the Riemann problem by the contact discontinuities for some initial data. In Section 3, we introduce delta waves and clarify the generalized Rankine-Hugoniot relation. In Section 4, we solve the Riemann problem by using overcompressible and non-overcompressible delta waves for the rest of the initial data.

## 2 Riemann solutions only containing contact discontinuities

In this section, we present some preliminary knowledge and construct the Riemann solutions through contact discontinuities. The physically relevant region for solutions is

$$\mathscr{V} = \left\{(\rho, v, q) : \rho \geq \frac{1}{c}, \quad |v| < c, \quad |q| < c\right\}, \tag{2.1}$$

which implies that the sonic speed $\sqrt{p'(\rho)}$ should be equal to or less than the light speed $c$. Putting equation (1.2) into equation (1.4), by suitably choosing $n_0$ and $\alpha$, one can easily reach

$$n = n(\rho) = \rho\sqrt{\left(1 + \frac{1}{\rho c}\right)\left(1 - \frac{1}{\rho c}\right)}. \tag{2.2}$$

The characteristics of the system (1.3) + (1.4) + (1.2) and the associated right eigenvectors are

$$\lambda_- = \frac{v - \sqrt{p'(\rho)}}{1 - v\sqrt{p'(\rho)}\,/c^2} = \frac{v - \frac{1}{\rho}}{1 - \frac{v}{\rho c^2}}, \quad r_- = \left(\frac{-1}{1 - v^2/c^2}, \frac{\sqrt{p'(\rho)}}{\rho + p(\rho)/c^2}, 0\right)^T,$$

$$\lambda_0 = v, \quad r_0 = (\rho^2, 0, -1)^T,$$

$$\lambda_+ = \frac{v + \sqrt{p'(\rho)}}{1 + v\sqrt{p'(\rho)}\,/c^2} = \frac{v + \frac{1}{\rho}}{1 + \frac{v}{\rho c^2}}, \quad r_+ = \left(\frac{1}{1 - v^2/c^2}, \frac{\sqrt{p'(\rho)}}{\rho + p(\rho)/c^2}, 0\right)^T.$$

A direct calculation gives

$$\nabla\lambda_i \cdot r_i \equiv 0 \quad (i = -, 0, +), \tag{2.3}$$

where $\nabla = (\partial_\rho, \partial_v, \partial_q)$ is the gradient operation. Therefore, the system is strictly hyperbolic ($-c \leq \lambda_- < \lambda_0 < \lambda_+ \leq c$) and fully linearly degenerate. The linear degeneracy excludes the possibility of shocks and rarefaction waves in solutions.

Since the considered system and initial data remain invariant under a uniform expansion of coordinates $t \to \alpha t$, $x \to \alpha x$, $\alpha > 0$, the solution is only connected with $\xi = x/t$. Thus, we should seek the self-similar solution $(\rho, v, q)(x, t) = (\rho, v, q)(\xi)$. At this time, the system becomes

$$\begin{cases} -\xi\left(\dfrac{n}{\sqrt{1 - v^2/c^2}}\right)_\xi + \left(\dfrac{nv}{\sqrt{1 - v^2/c^2}}\right)_\xi = 0, \\ -\xi\left(\dfrac{(\rho + p/c^2)v}{1 - v^2/c^2}\right)_\xi + \left(\dfrac{(\rho + p/c^2)v^2}{1 - v^2/c^2} + p + q\right)_\xi = 0, \\ -\xi\left(\dfrac{nq}{\sqrt{1 - v^2/c^2}}\right)_\xi + \left(\dfrac{nvq}{\sqrt{1 - v^2/c^2}}\right)_\xi = 0 \end{cases} \tag{2.4}$$

and the initial data change to the boundary condition

$$(\rho, v, q)(\pm\infty) = (\rho_\pm, v_\pm, q_\pm). \tag{2.5}$$

This is a two-point boundary value problem of first-order ordinary differential equations with the boundary value at the infinity.

Any smooth solution satisfies

$$\begin{pmatrix} \dfrac{nv - \xi n}{(\rho + p/c^2)\sqrt{1 - v^2/c^2}} & \dfrac{c^2 n - \xi nv}{c^2(1 - v^2/c^2)^{3/2}} & 0 \\ \dfrac{(p' + v^2) - \xi(1 + p'/c^2)v}{1 - v^2/c^2} & \dfrac{2(\rho + p/c^2)v - \xi(\rho + p/c^2)(1 + v^2/c^2)}{(1 - v^2/c^2)^2}, & 1 \\ 0 & 0 & v - \xi \end{pmatrix} \begin{pmatrix} \mathrm{d}\rho \\ \mathrm{d}v \\ \mathrm{d}q \end{pmatrix} = 0. \tag{2.6}$$

It provides only general solutions (constant states)

$$(\rho, v, q) = \text{constant}, \quad \left(\rho \geq \frac{1}{c}\right). \tag{2.7}$$

For a bounded discontinuity at $\xi = \tau$, the Rankine-Hugoniot relation reads

$$\begin{cases} -\tau\left[\dfrac{n}{\sqrt{1 - v^2/c^2}}\right] + \left[\dfrac{nv}{\sqrt{1 - v^2/c^2}}\right] = 0, \\ -\tau\left[\dfrac{(\rho + p/c^2)v}{1 - v^2/c^2}\right] + \left[\dfrac{(\rho + p/c^2)v^2}{1 - v^2/c^2} + p + q\right] = 0, \\ -\tau\left[\dfrac{nq}{\sqrt{1 - v^2/c^2}}\right] + \left[\dfrac{nvq}{\sqrt{1 - v^2/c^2}}\right] = 0, \end{cases} \tag{2.8}$$

where $[G] = G_l - G_r$ is the jump of $G$ across discontinuity. From equation (2.8), we can solve three kinds of contact discontinuities

$$J^- : \begin{cases} \tau = \dfrac{v_l - \frac{1}{\rho_l}}{1 - \frac{v_l}{\rho_l c^2}} = \dfrac{v_r - \frac{1}{\rho_r}}{1 - \frac{v_r}{\rho_r c^2}}, \\ q_l = q_r, \end{cases} \tag{2.9}$$

$$J^0 : \begin{cases} \tau = v_l = v_r, \\ q_l - \dfrac{1}{\rho_l} = q_r - \dfrac{1}{\rho_r}, \end{cases} \tag{2.10}$$

and

$$J^+ : \begin{cases} \tau = \dfrac{v_l + \frac{1}{\rho_l}}{1 + \frac{v_l}{\rho_l c^2}} = \dfrac{v_r + \frac{1}{\rho_r}}{1 + \frac{v_r}{\rho_r c^2}}, \\ q_l = q_r. \end{cases} \tag{2.11}$$

Now we solve the Riemann problem by the structure consisting of three kinds of contact discontinuities. Let

$$\lambda_{--} = \frac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} < \lambda_{++} = \frac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}}. \tag{2.12}$$

Consider the following system:

$$\begin{cases} \begin{cases} \lambda_{--} = \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} = \dfrac{v_{1*} - \frac{1}{\rho_{1*}}}{1 - \frac{v_{1*}}{\rho_{1*} c^2}}, \\ q_- = q_{1*}, \end{cases} \\ \begin{cases} v_{1*} = v_{2*}, \\ q_{1*} - \dfrac{1}{\rho_{1*}} = q_{2*} - \dfrac{1}{\rho_{2*}}, \end{cases} \\ \begin{cases} \dfrac{v_{2*} + \frac{1}{\rho_{2*}}}{1 + \frac{v_{2*}}{\rho_{2*} c^2}} = \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} = \lambda_{++}, \\ q_{2*} = q_+ \end{cases} \\ 0 < \dfrac{1}{\rho_{1*}} \le c, \quad 0 < \dfrac{1}{\rho_{2*}} \le c, \end{cases} \tag{2.13}$$

which is equivalent to

$$\begin{cases} \begin{cases} (\lambda_{++} - \lambda_{--})\left(\dfrac{1}{\rho_{1*}} \cdot \dfrac{1}{\rho_{2*}}\right) + (\lambda_{--}\lambda_{++} - c^2)\left(\dfrac{1}{\rho_{1*}} + \dfrac{1}{\rho_{2*}}\right) + c^2(\lambda_{++} - \lambda_{--}) = 0, \\ q_- - \dfrac{1}{\rho_{1*}} = q_+ - \dfrac{1}{\rho_{2*}}, \\ 0 < \dfrac{1}{\rho_{1*}} \le c, \quad 0 < \dfrac{1}{\rho_{2*}} \le c, \end{cases} \\ \begin{cases} v_{1*} = v_{2*} = \dfrac{\lambda_{--} + \frac{1}{\rho_{1*}}}{1 + \frac{\lambda_{--}}{\rho_{1*} c^2}} = \dfrac{\lambda_{++} - \frac{1}{\rho_{2*}}}{1 - \frac{\lambda_{++}}{\rho_{2*} c^2}}, \\ q_{1*} = q_-, \\ q_{2*} = q_+. \end{cases} \end{cases} \tag{2.14}$$

From the first two equations in (2.14), it can be solved that

$$\begin{cases} \dfrac{1}{\rho_{1*}} = \dfrac{2(c^2 - \lambda_{--}\lambda_{++}) - (\lambda_{++} - \lambda_{--})(q_+ - q_-) \pm \sqrt{\Delta}}{2(\lambda_{++} - \lambda_{--})}, \\ \dfrac{1}{\rho_{2*}} = \dfrac{2(c^2 - \lambda_{--}\lambda_{++}) + (\lambda_{++} - \lambda_{--})(q_+ - q_-) \pm \sqrt{\Delta}}{2(\lambda_{++} - \lambda_{--})} \end{cases} \tag{2.15}$$

with

$$\Delta = 4(c^2 - {\lambda_{--}}^2)(c^2 - {\lambda_{++}}^2) + (\lambda_{++} - \lambda_{--})^2(q_+ - q_-)^2 > 0.$$

To guarantee $\frac{1}{\rho_{1*}} \le c$ and $\frac{1}{\rho_{2*}} \le c$, we should choose

$$\begin{cases} \dfrac{1}{\rho_{1*}} = \dfrac{2(c^2 - \lambda_{--}\lambda_{++}) - (\lambda_{++} - \lambda_{--})(q_+ - q_-) - \sqrt{\Delta}}{2(\lambda_{++} - \lambda_{--})}, \\ \dfrac{1}{\rho_{2*}} = \dfrac{2(c^2 - \lambda_{--}\lambda_{++}) + (\lambda_{++} - \lambda_{--})(q_+ - q_-) - \sqrt{\Delta}}{2(\lambda_{++} - \lambda_{--})}, \end{cases} \tag{2.16}$$

that is,

$$\begin{cases} \dfrac{1}{\rho_{1*}} = \dfrac{\frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_+ - q_-)}{2}, \\ \dfrac{1}{\rho_{2*}} = \dfrac{\frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_- - q_+)}{2}. \end{cases} \tag{2.17}$$

Observing that

$$2(c^2 - \lambda_{--}\lambda_{++}) > \sqrt{\Delta}, \quad \lambda_{++} > \lambda_{--},$$

one can find that $\frac{1}{\rho_{1*}} > 0$ and $\frac{1}{\rho_{2*}} > 0$ are satisfied if and only if either

$$q_- = q_+$$

or

$$q_- > q_+, \quad \frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} > q_- - q_+ \Leftrightarrow \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_- - q_+$$

or

$$q_- < q_+, \quad \frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} > q_+ - q_- \Leftrightarrow \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_+ - q_-.$$

At this time, returning to (2.14), we have

$$\begin{cases} v_{1*} = v_{2*} = \dfrac{2\lambda_{--} + \frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_+ - q_-)}{2 + \frac{\lambda_{--}}{c^2}\left(\frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_+ - q_-)\right)} \\ \qquad = \dfrac{2\lambda_{++} - \frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_- - q_+)}{2 - \frac{\lambda_{++}}{c^2}\left(\frac{2(c^2 - \lambda_{--}\lambda_{++}) - \sqrt{\Delta}}{\lambda_{++} - \lambda_{--}} - (q_- - q_+)\right)}, \\ q_{1*} = q_-, \\ q_{2*} = q_+. \end{cases} \tag{2.18}$$

Thus, we can conclude that when the initial data (1.5) satisfy

$$\lambda_{--} < \lambda_{++}, \quad q_- = q_+ \tag{2.19}$$

or

$$\lambda_{--} < \lambda_{++}, \quad q_- > q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_- - q_+ \tag{2.20}$$

or

$$\lambda_{--} < \lambda_{++}, \quad q_- < q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_+ - q_-, \tag{2.21}$$

the Riemann problem (1.3) + (1.4) + (1.2) and (1.5) is solvable with three contact discontinuities:

$$(\rho, v, q)(\xi) = \begin{cases} (\rho_-, v_-, q_-), & -\infty < \xi < \tau_-, \\ (\rho_{1*}, v_{1*}, q_{1*}), & \tau_- < \xi < \tau_0, \\ (\rho_{2*}, v_{2*}, q_{2*}), & \tau_0 < \xi < \tau_+, \\ (\rho_+, v_+, q_+), & \tau_+ < \xi < +\infty, \end{cases} \tag{2.22}$$

where $(\rho_{1*}, v_{1*}, q_{1*})$ and $(\rho_{2*}, v_{2*}, q_{2*})$ are shown by equations (2.17) and (2.18); $(\rho_-, v_-, q_-)$ and $(\rho_{1*}, v_{1*}, q_{1*})$ are connected by a contact discontinuity $J^-$ with the speed $\tau_- = \lambda_{--}$; $(\rho_+, v_+, q_+)$ and $(\rho_{2*}, v_{2*}, q_{2*})$ are connected by a contact discontinuity $J^+$ with the speed $\tau_+ = \lambda_{++}$; and two intermediate states $(\rho_{1*}, v_{1*}, q_{1*})$ and $(\rho_{2*}, v_{2*}, q_{2*})$ are connected by a contact discontinuity $J^0$ with the speed $\tau_0 = v_{1*} = v_{2*}$. We illustrate the structure, denoted by $J^- + J^0 + J^-$, in Figure 1.

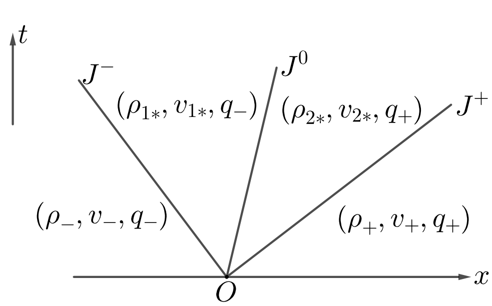


**Figure 1:** Riemann solution: $J^- + J^0 + J^-$.

## 3 Generalized Rankine-Hugoniot relation of delta waves

For the rest of the initial data, it can be found that the Riemann problem cannot be solved only by using three kinds of contact discontinuities. Motivated by [11,17], we introduce delta waves to solve the Riemann problem.

We define a two-dimensional weighted Dirac delta function $w(s)\delta(S)$ supported on a smooth curve $S$ parameterized as $x = x(s)$, $t = t(s)(c \le s \le d)$ by

$$\langle w(s)\delta(S), \varphi(x,t)\rangle = \int_c^d w(s)\varphi(x(s), t(s))\mathrm{d}s \tag{3.1}$$

for all the test functions $\varphi(x,t) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$.

Let $\Omega$ be a region in $(x,t)$-plane and cut into a left part $\Omega_l$ and a right part $\Omega_r$ by a smooth curve $S : x = x(t)$. Seek the delta wave in the form

$$(\rho, v, q)(x,t) = \begin{cases} (\rho_l, v_l, q_l)(x,t), & (x,t) \in \Omega_l, \\ (w(t)\delta_S, v_\delta(t), q_\delta(t)), & (x,t) \in S, \\ (\rho_r, v_r, q_r)(x,t), & (x,t) \in \Omega_r \end{cases} \tag{3.2}$$

with

$$\frac{1}{\rho} = \begin{cases} \dfrac{1}{\rho_l}, & (x,t) \in \Omega_l, \\ 0, & (x,t) \in S, \\ \dfrac{1}{\rho_r}, & (x,t) \in \Omega_r, \end{cases} \tag{3.3}$$

where $(\rho_l, v_l, q_l)$ and $(\rho_r, v_r, q_r)(t,x)$ are piecewise smooth solutions of (1.3) + (1.4) + (1.2) and $w(t)$, $v_\delta(t)$, and $q_\delta(t)$ are smooth functions.

We assert that if the following ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = v_\delta(t), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{w(t)}{\sqrt{1 - v_\delta^2(t)/c^2}}\right) = -v_\delta(t)\left[\dfrac{n}{\sqrt{1 - v^2/c^2}}\right] + \left[\dfrac{nv}{\sqrt{1 - v^2/c^2}}\right], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{w(t)v_\delta(t)}{1 - v_\delta^2(t)/c^2}\right) = -v_\delta(t)\left[\dfrac{(\rho + p/c^2)v}{1 - v^2/c^2}\right] + \left[\dfrac{(\rho + p/c^2)v^2}{1 - v^2/c^2} + p + q\right], \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{w(t)q_\delta(t)}{\sqrt{1 - v_\delta^2(t)/c^2}}\right) = -v_\delta(t)\left[\dfrac{nq}{\sqrt{1 - v^2/c^2}}\right] + \left[\dfrac{nvq}{\sqrt{1 - v^2/c^2}}\right] \end{cases} \tag{3.4}$$

are satisfied, where $[G] = \bar{G}_l - \bar{G}_r$, $\bar{G}_l$, and $\bar{G}_r$ are the limits of $G_l(x,t)$ and $G_r(x,t)$ on the discontinuity $S$, respectively, then the distribution (3.2) is a solution of the system (1.3) + (1.4) + (1.2) in the distributional sense, that is, it satisfies

$$\begin{aligned} &\iint_{\Omega_l}\left(\left(\frac{n(\rho_l)}{\sqrt{1 - v_l^2/c^2}}\right)\varphi_t + \left(\frac{n(\rho_l)v_l}{\sqrt{1 - v_l^2/c^2}}\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t + \iint_{\Omega_r}\left(\left(\frac{n(\rho_r)}{\sqrt{1 - v_r^2/c^2}}\right)\varphi_t + \left(\frac{n(\rho_r)v_r}{\sqrt{1 - v_r^2/c^2}}\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t \\ &+ \left\langle \frac{w(t)}{\sqrt{1 - (v_\delta(t))^2/c^2}}\delta(S), \varphi_t \right\rangle + \left\langle \frac{w(t)v_\delta(t)}{\sqrt{1 - (v_\delta(t))^2/c^2}}\delta(S), \varphi_x \right\rangle = 0, \end{aligned}$$

$$\iint_{\Omega_l}\left(\left(\frac{(\rho_l+p(\rho_l)/c^2)v_l}{1-v_l^2/c^2}\right)\varphi_t+\left(\frac{(\rho_l+p(\rho_l)/c^2)v_l^2}{1-v_l^2/c^2}+p(\rho_l)+q_l\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t+\iint_{\Omega_r}\left(\left(\frac{(\rho_r+p(\rho_r)/c^2)v_r}{1-v_r^2/c^2}\right)\varphi_t\right.$$
$$\left.+\left(\frac{(\rho_r+p(\rho_r)/c^2)v_r^2}{1-v_r^2/c^2}+p(\rho_r)+q_r\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t+\left\langle\frac{w(t)}{1-(v_\delta(t))^2/c^2}\delta(S),\varphi_t\right\rangle+\left\langle\frac{w(t)v_\delta(t)}{1-(v_\delta(t))^2/c^2}\delta(S),\varphi_x\right\rangle=0$$

and

$$\iint_{\Omega_l}\left(\left(\frac{n(\rho_l)q_l}{\sqrt{1-v_l^2/c^2}}\right)\varphi_t+\left(\frac{n(\rho_l)v_lq_l}{\sqrt{1-v_l^2/c^2}}\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t+\iint_{\Omega_r}\left(\left(\frac{n(\rho_r)q_r}{\sqrt{1-v_r^2/c^2}}\right)\varphi_t+\left(\frac{n(\rho_r)v_rq_r}{\sqrt{1-v_r^2/c^2}}\right)\varphi_x\right)\mathrm{d}x\mathrm{d}t$$
$$+\left\langle\frac{w(t)q_\delta(t)}{\sqrt{1-(v_\delta(t))^2/c^2}}\delta(S),\varphi_t\right\rangle+\left\langle\frac{w(t)v_\delta(t)q_\delta(t)}{\sqrt{1-(v_\delta(t))^2/c^2}}\delta(S),\varphi_x\right\rangle=0$$

for all the test functions $\varphi(x,t)\in C_0^\infty(\mathbb{R}\times\mathbb{R}^+)$. The proof is similar to those in the articles [5,6], and we omit it.

Relations in equation (3.4) are called the generalized Rankine-Hugoniot relation of the delta wave, which reflects the exact relationship among the location, propagation speed, weight, and assignment of $v$ and $q$ on the discontinuity.

Let us consider the delta wave in the form (3.2) with the assumption that $(\bar\rho_l,\bar v_l,\bar q_l)=(\rho_1,v_1,q_1)$ and $(\bar\rho_r,\bar v_r,\bar q_r)=(\rho_2,v_2,q_2)$ are constant, $x(0)=0$, and $w(0)=0$. In view of knowledge concerning delta waves [1–16], it is found that $v_\delta(t)=:v_\delta$ and $q_\delta(t)=:q_\delta$ are constants, and $w(t)=:w_0t$ is a linear function of $t$. Then, the generalized Rankine-Hugoniot relation becomes

$$\begin{cases}x(t)=v_\delta t,\\ \dfrac{w_0}{\sqrt{1-v_\delta^2/c^2}}=-v_\delta\left[\dfrac{n}{\sqrt{1-v^2/c^2}}\right]+\left[\dfrac{nv}{\sqrt{1-v^2/c^2}}\right],\\ \dfrac{w_0v_\delta}{1-v_\delta^2/c^2}=-v_\delta\left[\dfrac{(\rho+p/c^2)v}{1-v^2/c^2}\right]+\left[\dfrac{(\rho+p/c^2)v^2}{1-v^2/c^2}+p+q\right],\\ \dfrac{w_0q_\delta}{\sqrt{1-v_\delta^2/c^2}}=-v_\delta\left[\dfrac{nq}{\sqrt{1-v^2/c^2}}\right]+\left[\dfrac{nvq}{\sqrt{1-v^2/c^2}}\right].\end{cases}\tag{3.5}$$

One can observe that if $v_\delta$ is determined, then the remaining is determined accordingly.

Denote

$$\begin{aligned}E&=\left[\frac{n}{\sqrt{1-v^2/c^2}}\right]=\left[\frac{\rho\sqrt{1-1/(\rho^2c^2)}}{\sqrt{1-v^2/c^2}}\right]=[\rho m],\\ F&=\left[\frac{nv}{\sqrt{1-v^2/c^2}}\right]=\left[\frac{\rho v\sqrt{1-1/(\rho^2c^2)}}{\sqrt{1-v^2/c^2}}\right]=[\rho vm],\\ G&=\left[\frac{(\rho+p/c^2)v}{1-v^2/c^2}\right]=\left[\frac{\rho v(1-1/(\rho^2c^2))}{1-v^2/c^2}\right]=[\rho vm^2],\\ H&=\left[\frac{(\rho+p/c^2)v^2}{1-v^2/c^2}+p+q\right]=\left[\frac{\rho v^2(1-1/(\rho^2c^2))}{1-v^2/c^2}+p+q\right]=[\rho v^2m^2+p+q].\end{aligned}\tag{3.6}$$

From the second and third equations in (3.5), we obtain

$$L(v_\delta):=(Ev_\delta-F)f(v_\delta)-Gv_\delta+H=0\tag{3.7}$$

with

$$f(v_\delta)=\frac{v_\delta}{\sqrt{1-v_\delta^2/c^2}},\quad f'(v_\delta)=\frac{1}{(1-v_\delta^2/c^2)^{3/2}}>0.$$

It is indeed a quartic equation, so it is difficult to obtain an explicit expression for $v_\delta$.

One can calculate that

$$\begin{aligned}L(v_\delta) &= (Ev_\delta - F)f(v_\delta) - Gv_\delta + H\\ &= \left\{\rho_1 m_1(v_\delta - v_1)(f(v_\delta) - v_1 m_1) - \frac{1}{\rho_1}\right\} + \left\{\rho_2 m_2(v_2 - v_\delta)(f(v_\delta) - v_2 m_2) + \frac{1}{\rho_2}\right\} + q_1 - q_2\\ &=: L_1(v_\delta) + L_2(v_\delta) + q_1 - q_2\end{aligned}$$

and

$$L'(v_\delta) = \rho_1 m_1\{(f(v_\delta) - v_1 m_1) + (v_\delta - v_1)f'(v_\delta)\} + \rho_2 m_2\{(v_2 m_2 - f(v_\delta)) + (v_2 - v_\delta)f'(v_\delta)\} =: L_1'(v_\delta) + L_2'(v_\delta),$$

where

$$m_i = \frac{\sqrt{1 - 1/(\rho_i^2 c^2)}}{\sqrt{1 - v_i^2/c^2}}, \quad i = 1, 2.$$

# 4 Riemann solutions involving delta waves

In this section, we will construct the Riemann solutions by using overcompressible delta wave and non-overcompressible delta waves respectively.

## 4.1 Riemann solution containing overcompressible delta wave

A delta wave in the form (3.2) is overcompressible, denoted by $\delta$, if it satisfies the condition

$$\frac{\bar{v}_l - \frac{1}{\bar{\rho}_l}}{1 - \frac{\bar{v}_l}{\bar{\rho}_l c^2}} \geq v_\delta(t) \geq \frac{\bar{v}_r + \frac{1}{\bar{\rho}_r}}{1 + \frac{\bar{v}_r}{\bar{\rho}_r c^2}}, \tag{4.1}$$

which means that all the characteristics on both sides of the delta wave are not outcoming from the discontinuity.

Let us study the conditions of existence for an overcompressilbe delta wave. Consider the above constant limit delta wave with $(\bar{\rho}_l, \bar{v}_l, \bar{q}_l) = (\rho_1, v_1, q_1)$ and $(\bar{\rho}_r, \bar{v}_r, \bar{q}_r) = (\rho_2, v_2, q_2)$.

Let

$$\frac{v_1 - \frac{1}{\rho_1}}{1 - \frac{v_1}{\rho_1 c^2}} > \frac{v_2 + \frac{1}{\rho_2}}{1 + \frac{v_2}{\rho_2 c^2}}. \tag{4.2}$$

First, with

$$f\left(\frac{v_1 - \frac{1}{\rho_1}}{1 - \frac{v_1}{\rho_1 c^2}}\right) = \frac{\frac{v_1 - \frac{1}{\rho_1}}{1 - \frac{v_1}{\rho_1 c^2}}}{\sqrt{1 - \left(\frac{v_1 - \frac{1}{\rho_1}}{1 - \frac{v_1}{\rho_1 c^2}}\right)^2 \Big/ c^2}} = \frac{\left(v_1 - \frac{1}{\rho_1}\right)\cdot c}{\sqrt{(c - \frac{v_1}{\rho_1 c})^2 - \left(v_1 - \frac{1}{\rho_1}\right)^2}} = \frac{v_1 - \frac{1}{\rho_1}}{\sqrt{\left(1 - \frac{1}{\rho_1^2 c^2}\right)\left(1 - \frac{v_1^2}{c^2}\right)}} \tag{4.3}$$

and

$$f\left(\frac{v_2 + \frac{1}{\rho_2}}{1 + \frac{v_2}{\rho_2 c^2}}\right) = \frac{\frac{v_2 + \frac{1}{\rho_2}}{1 + \frac{v_2}{\rho_2 c^2}}}{\sqrt{1 - \left(\frac{v_2 + \frac{1}{\rho_2}}{1 + \frac{v_2}{\rho_2 c^2}}\right)^2 \Big/ c^2}} = \frac{v_2 + \frac{1}{\rho_2}}{\sqrt{\left(1 - \frac{1}{\rho_2^2 c^2}\right)\left(1 - \frac{v_2^2}{c^2}\right)}}, \tag{4.4}$$

we can calculate that

$$
\begin{aligned}
L_1\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right) &= \rho_1\frac{\sqrt{1-\frac{1}{\rho_1^2c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}}\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}-v_1\right)\left(f\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)-v_1\frac{\sqrt{1-\frac{1}{\rho_1^2c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}}\right)-\frac{1}{\rho_1}\\
&= \rho_1\frac{\sqrt{1-\frac{1}{\rho_1^2c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}}\frac{1}{\rho_1}\frac{\frac{v_1^2}{c^2}-1}{1-\frac{v_1}{\rho_1c^2}}\left(\frac{v_1-\frac{1}{\rho_1}}{\sqrt{\left(1-\frac{1}{\rho_1^2c^2}\right)\left(1-\frac{v_1^2}{c^2}\right)}}-v_1\frac{\sqrt{1-\frac{1}{\rho_1^2c^2}}}{\sqrt{1-\frac{v_1^2}{c^2}}}\right)-\frac{1}{\rho_1}\\
&= -\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}+v_1\frac{1-\frac{1}{\rho_1^2c^2}}{1-\frac{v_1}{\rho_1c^2}}-\frac{1}{\rho_1}\\
&= 0
\end{aligned}
\tag{4.5}
$$

and

$$
\begin{aligned}
L_2\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right) &= \rho_2\frac{\sqrt{1-\frac{1}{\rho_2^2c^2}}}{\sqrt{1-\frac{v_2^2}{c^2}}}\left(v_2-\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)\left(f\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)-v_2\frac{\sqrt{1-\frac{1}{\rho_2^2c^2}}}{\sqrt{1-\frac{v_2^2}{c^2}}}\right)+\frac{1}{\rho_2}\\
&= \rho_2\frac{\sqrt{1-\frac{1}{\rho_2^2c^2}}}{\sqrt{1-\frac{v_2^2}{c^2}}}\frac{1}{\rho_2}\frac{\frac{v_2^2}{c^2}-1}{1+\frac{v_2}{\rho_2c^2}}\left(\frac{v_2+\frac{1}{\rho_2}}{\sqrt{\left(1-\frac{1}{\rho_2^2c^2}\right)\left(1-\frac{v_2^2}{c^2}\right)}}-v_2\frac{\sqrt{1-\frac{1}{\rho_2^2c^2}}}{\sqrt{1-\frac{v_2^2}{c^2}}}\right)+\frac{1}{\rho_2}\\
&= 0.
\end{aligned}
\tag{4.6}
$$

In consideration of (2.1), it is also easy to obtain from (4.3) and (4.4) that

$$
\begin{aligned}
f\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right) &< v_1m_1,\\
f\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right) &> v_2m_2.
\end{aligned}
$$

Since $f(v_\delta)$ is monotonely increasing, it follows that

$$
\begin{aligned}
L_1'(v_\delta) &= \rho_1m_1\{(f(v_\delta)-v_1m_1)+(v_\delta-v_1)f'(v_\delta)\}<0 \quad \text{for } v_\delta<\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}<v_1,\\
L_2'(v_\delta) &= \rho_2m_2\{(v_2m_2-f(v_\delta))+(v_2-v_\delta)f'(v_\delta)\}<0 \quad \text{for } v_\delta>\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}>v_2,
\end{aligned}
\tag{4.7}
$$

and furthermore,

$$
L'(v_\delta)=L_1'(v_\delta)+L_2'(v_\delta)<0 \quad \text{for } v_\delta\in\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}},\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right). \tag{4.8}
$$

Second, it is obvious that

$$L\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)=L_1\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)+q_1-q_2=\left\{\rho_1m_1\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}-v_1\right)\left(f\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)-v_1m_1\right)-\frac{1}{\rho_1}\right\}+q_1-q_2 \tag{4.9}$$

and

$$L\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)=L_2\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)+q_1-q_2=\left\{\rho_2m_2\left(v_2-\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)\left(f\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)-v_2m_2\right)+\frac{1}{\rho_2}\right\}+q_1-q_2. \tag{4.10}$$

Note that due to (4.2) and (4.7), it holds that

$$L_1\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)>L_1\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)=0$$

and

$$L_2\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)<L_2\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)=0.$$

Thus, by using the zero point theorem in mathematical analysis and taking the monotonicity of $L(v_\delta)$ in account, we conclude that if and only if either

$$q_1=q_2 \tag{4.11}$$

or

$$q_1>q_2,\quad \left\{\rho_2m_2\left(v_2-\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)\left(v_2m_2-f\left(\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right)\right)-\frac{1}{\rho_2}\right\}\geq q_1-q_2 \tag{4.12}$$

or

$$q_1<q_2,\quad \left\{\rho_1m_1\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}-v_1\right)\left(f\left(\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}\right)-v_1m_1\right)-\frac{1}{\rho_1}\right\}\geq q_2-q_1, \tag{4.13}$$

(3.7) has a unique solution

$$v_\delta\in\left[\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}},\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}\right],$$

that is, an overcompressilbe delta wave exists. Besides, when

$$\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}=\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}},$$

there exists an overcompressilbe delta wave with

$$v_\delta=\frac{v_2+\frac{1}{\rho_2}}{1+\frac{v_2}{\rho_2c^2}}=\frac{v_1-\frac{1}{\rho_1}}{1-\frac{v_1}{\rho_1c^2}}$$

if and only if

$$q_1=q_2.$$

Thus, we can conclude that when the initial data (1.5) satisfy either

$$\lambda_{--} > \lambda_{+-}, \quad q_- = q_+ \tag{4.14}$$

or

$$\begin{cases} \lambda_{--} > \lambda_{+-}, \quad q_- > q_+, \\ \rho_+ m_+ \left( v_+ - \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} \right)\left( v_+ m_+ - f\left( \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} \right)\right) - \dfrac{1}{\rho_+} \geq q_- - q_+ \end{cases} \tag{4.15}$$

or

$$\begin{cases} \lambda_{--} > \lambda_{+-}, \quad q_- < q_+, \\ \rho_- m_- \left( \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} - v_- \right)\left( f\left( \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} \right) - v_- m_- \right) - \dfrac{1}{\rho_-} \geq q_+ - q_- \end{cases} \tag{4.16}$$

or

$$\lambda_{--} = \lambda_{+-}, \quad q_- = q_+, \tag{4.17}$$

the Riemann problem (1.3) + (1.4) + (1.2) and (1.5) is solvable with a single overcompressible delta wave:

$$(\rho, v, q)(x, t) = \begin{cases} (\rho_-, v_-, q_-), & x < x(t), \\ (w_0 t \delta(x - x(t)), v_\delta, q_\delta), & x = x(t), \\ (\rho_+, v_+, q_+), & x > x(t) \end{cases} \tag{4.18}$$

satisfying the relations (3.5) and (4.1). The structure, denoted by $\delta$, is illustrated in Figure 2.

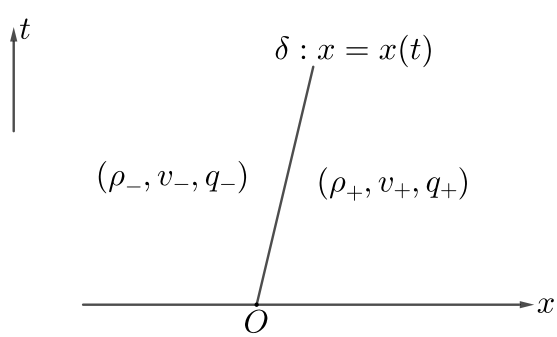


**Figure 2:** Riemann solution: $\delta$.

## 4.2 Riemann solution involving a left-contact non-overcompressible delta wave

To look at the Riemann solution for the critical initial data

$$\lambda_{--} < \lambda_{++}, \quad q_- > q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} = q_- - q_+,$$

we fix $(\rho_\pm, v_\pm)$ and consider the limit of the solution (2.22) with initial data (2.20) as

$$q_- - q_+ \to \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2}.$$

One can easily calculate that

$$\begin{cases} \rho_{1*} \to \dfrac{1 - \lambda_{--}\lambda_{++}/c^2}{\lambda_{++} - \lambda_{--}} := \rho_*, \\ \rho_{2*} \to +\infty, \\ \tau_0 = v_{1*} = v_{2*} \to \lambda_{++} := v_*, \\ \int\limits_{\tau_0}^{\tau_+} \rho \mathrm{d}\xi \to 1 - \dfrac{\lambda_{++}^2}{c^2} \neq 0, \end{cases}$$

which show that $J^0$ and $J^+$ will coincide to form a new nonlinear hyperbolic wave at $\xi = v_*$, where $\rho(\xi)$ has the same singularity as a weighted Dirac delta function. As a result, the limit is the structure: $(\rho_-, v_-, q_-)$ is connected to $(\rho_*, v_*, q_-)$ by a contact discontinuity $J^-$ with $\tau = \lambda_{--}$, and $(\rho_*, v_*, q_-)$ is connected to $(\rho_+, v_+, q_+)$ by a delta wave with $x/t = u_*$ (Figure 3).

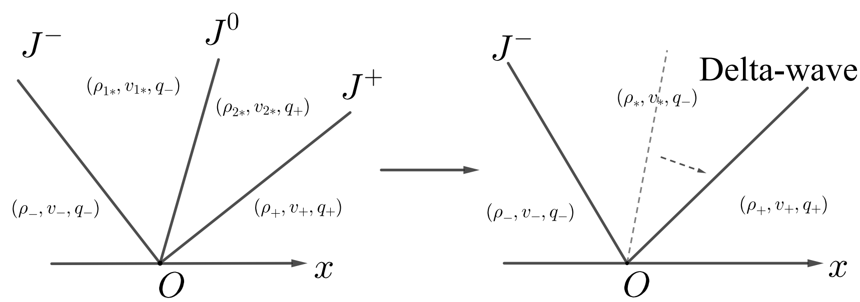


**Figure 3:** Limit of the Riemann solution as $q_- - q_+ \to \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2}$.

Observing that for the limited delta wave, the characteristics $\lambda_-$ and $\lambda_0$ on the left are outcoming from and contact to the discontinuity, respectively, we introduce the following delta wave. If a delta wave satisfies

$$v_\delta(t) = \bar{v}_l \geq \frac{\bar{v}_r + \frac{1}{\bar{\rho}_r}}{1 + \frac{\bar{v}_r}{\bar{\rho}_r c^2}}, \tag{4.19}$$

we call it a left-contact non-overcompressible delta wave, denoted by $\delta_0^l$, for which the characteristics $\lambda_-$, $\lambda_0$, and $\lambda_+$ on the left are outcoming from, contact to, and coming into the discontinuity, respectively, and the characteristics on the right are not outcoming from the discontinuity.

Consider the above constant limit delta wave with $(\bar{\rho}_l, \bar{v}_l, \bar{q}_l) = (\rho_1, v_1, q_1)$ and $(\bar{\rho}_r, \bar{v}_r, \bar{q}_r) = (\rho_2, v_2, q_2)$. With (4.19), we have

$$L(v_1) = -\frac{1}{\rho_1} + \left\{\rho_2 m_2(v_2 - v_1)(f(v_1) - v_2 m_2) + \frac{1}{\rho_2}\right\} + q_1 - q_2 = 0,$$

which gives the condition of existence for the left-contact non-overcompressible delta wave

$$q_1 - q_2 = \frac{1}{\rho_1} - \left\{\rho_2 m_2(v_2 - v_1)(f(v_1) - v_2 m_2) + \frac{1}{\rho_2}\right\} = \frac{1}{\rho_1} - L_2(v_1) > 0. \tag{4.20}$$

Besides, from the generalized Rankine-Hugoniot relation (3.5), we can obtain

$$\begin{cases} x(t) = v_1 t, \\ w_0 = \dfrac{\sqrt{1 - v_1^2/c^2}}{\sqrt{1 - v_2^2/c^2}}(v_1 - v_2)n_2, \\ v_\delta = v_1, \\ q_\delta = q_2. \end{cases} \tag{4.21}$$

When the initial data (1.5) satisfy either

$$\lambda_{--} < \lambda_{++}, \quad q_- > q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} \le q_- - q_+ \tag{4.22}$$

or

$$\begin{cases} \lambda_{--} > \lambda_{++}, \quad q_- > q_+, \\ \rho_+ m_+ \left( v_+ - \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} \right)\left( v_+ m_+ - f\left( \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} \right)\right) - \dfrac{1}{\rho_+} < q_- - q_+ \end{cases} \tag{4.23}$$

or

$$\lambda_{--} = \lambda_{++}, \quad q_- > q_+, \tag{4.24}$$

we suggest the following solution to the Riemann problem (1.3) + (1.4) + (1.2) and (1.5):

$$(\rho, v, q)(x,t) = \begin{cases} (\rho_-, v_-, q_-), & x < \tau t, \\ (\rho_*, v_*, q_-), & \tau t < x < v_* t, \\ (w_0 t\delta(x - v_* t), v_\delta, q_\delta), & x = v_* t, \\ (\rho_+, v_+, q_+), & x > v_* t, \end{cases} \tag{4.25}$$

that is, $(\rho_-, v_-, q_-)$ is connected to $(\rho_*, v_*, q_-)$ by a contact discontinuity $J^-$ with $\tau = \lambda_{--}$, and $(\rho_*, v_*, q_-)$ is connected to $(\rho_+, v_+, q_+)$ by a left-contact non-overcompressible delta wave, where it holds that

$$\begin{cases} \lambda_{--} = \dfrac{v_- - \frac{1}{\rho_-}}{1 - \frac{v_-}{\rho_- c^2}} = \dfrac{v_* - \frac{1}{\rho_*}}{1 - \frac{v_*}{\rho_* c^2}}, \\ q_- - q_+ = \dfrac{1}{\rho_*} - \left\{ \rho_+ m_+ (v_+ - v_*)(f(v_*) - v_+ m_+) + \dfrac{1}{\rho_+} \right\} \end{cases} \tag{4.26}$$

and

$$\begin{cases} w_0 = \dfrac{\sqrt{1 - v_*^2/c^2}}{\sqrt{1 - v_+^2/c^2}}(v_* - v_+)n_+, \\ v_\delta = v_*, \\ q_\delta = q_+. \end{cases} \tag{4.27}$$

This structure, denoted by $J^- + \delta_0^l$, is shown in Figure 4.

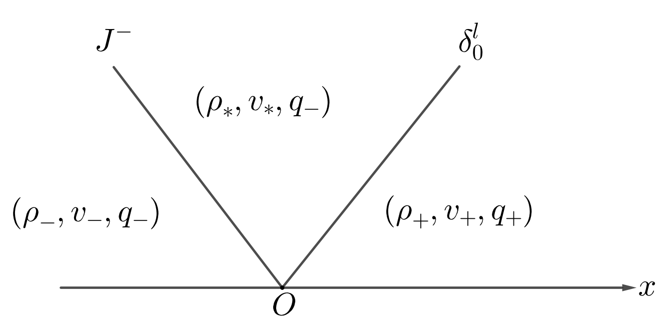


**Figure 4:** Riemann solution: $J^- + \delta_0^l$.

Let us show the existence and uniqueness of $v_*$ satisfying $v_* \geq \lambda_{++}$ and $v_* > \lambda_{--}$. From (4.26), we have

$$M(v_*) = q_- - q_+ - \frac{v_* - \lambda_{--}}{1 - v_*\lambda_{--}/c^2} + \left\{\rho_+ m_+(v_+ - v_*)(f(v_*) - v_+ m_+) + \frac{1}{\rho_+}\right\} = 0,$$

in which, similarly to (4.6),

$$\left\{\rho_+ m_+(v_+ - v_*)(f(v_*) - v_+ m_+) + \frac{1}{\rho_+}\right\}\Bigg|_{v_*=\lambda_{++}} = 0.$$

On the one side, it holds that

$$M'(v_*) = -\frac{1 - \lambda_{--}^2/c^2}{(1 - v_*\lambda_{--}/c^2)^2} + \rho_+ m_+\{-(f(v_*) - v_+ m_+) + (v_+ - v_*)f'(v_*)\} < 0$$

for $v_* \in [\lambda_{++}, c)$ because of $f(v_*) \geq f(\lambda_{++}) > v_+ m_+$. On the other hand, it is true that

$$M(\lambda_{++}) = q_- - q_+ - \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{++}\lambda_{--}/c^2} \geq 0, \quad M(c) \to -\infty$$

for (4.22),

$$M(\lambda_{--}) = q_- - q_+ + \left\{\rho_+ m_+(v_+ - \lambda_{--})(f(\lambda_{--}) - v_+ m_+) + \frac{1}{\rho_+}\right\} > 0, \quad M(c) \to -\infty$$

for (4.23), and

$$M(\lambda_{--} = \lambda_{++}) = q_- - q_+ > 0, \quad M(c) \to -\infty$$

for (4.24). In virtue of the monotonicity of $M(v_*)$, by zero point theorem in mathematical analysis, it is known that there exists a unique $v_*$ satisfying $v_* \geq \lambda_{++}$ and $v_* > \lambda_{--}$.

## 4.3 Riemann solution involving a right-contact non-overcompressible delta wave

In order to see the Riemann solution for the critical initial data

$$\lambda_{--} < \lambda_{++}, \quad q_- < q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} = q_+ - q_-,$$

let us fix $(\rho_\pm, v_\pm)$ and study the limit of the solution (2.22) with initial data (2.21) as follows:

$$q_+ - q_- \to \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2}.$$

It can be obtained that

$$\begin{cases} \rho_{1*} \to +\infty, \\ \rho_{2*} \to \dfrac{1 - \lambda_{--}\lambda_{++}/c^2}{\lambda_{++} - \lambda_{--}} := \rho_*, \\ \tau_0 = v_{1*} = v_{2*} \to \lambda_{--} := v_*, \\ \displaystyle\int_{\tau_-}^{\tau_0} \rho \mathrm{d}\xi \to 1 - \frac{\lambda_{--}^2}{c^2} \neq 0, \end{cases}$$

that is, $J^0$ and $J^-$ will coincide to form a new nonlinear hyperbolic wave at $\xi = \lambda_{--}$, where $\rho(\xi)$ has the same singularity as a weighted Dirac delta function. As a result, the limit is the structure: $(\rho_-, v_-, q_-)$ is connected to $(\rho_*, v_*, q_+)$ by a delta wave with $x/t = u_*$, and $(\rho_*, v_*, q_+)$ is connected to $(\rho_+, v_+, q_+)$ by a contact discontinuity $J^+$ with $\tau = \lambda_{++}$ (Figure 5).

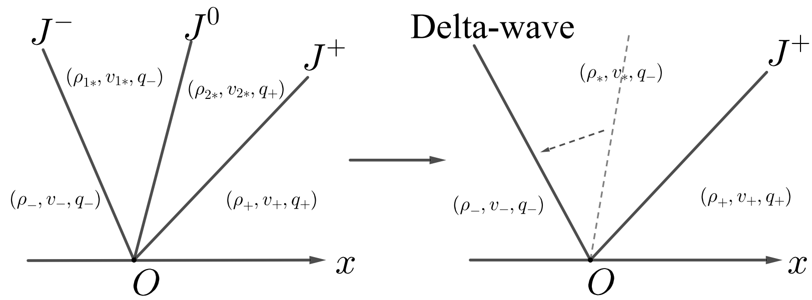


**Figure 5:** Limit of the Riemann solution as $q_+ - q_- \to \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2}$.

Observing that for the limit delta wave, the characteristics $\lambda_+$ and $\lambda_0$ on the right are outcoming from and contact to the discontinuity respectively, we introduce the right-contact non-overcompressible delta wave, denoted by $\delta_0^r$, by which we mean that it satisfies

$$\frac{\bar{v}_l - \frac{1}{\bar{\rho}_l}}{1 - \frac{\bar{v}_l}{\bar{\rho}_l c^2}} \geq \bar{v}_r = v_\delta(t), \tag{4.28}$$

that is, the characteristics $\lambda_-$, $\lambda_0$, and $\lambda_+$ on the right are incoming into, contact to, and outcoming from the discontinuity, respectively, and the characteristics on the left are not outcoming from the discontinuity.

Consider the above constant limit delta wave with $(\bar{\rho}_l, \bar{v}_l, \bar{q}_l) = (\rho_1, v_1, q_1)$ and $(\bar{\rho}_r, \bar{v}_r, \bar{q}_r) = (\rho_2, v_2, q_2)$. With (4.28), we have

$$L(v_2) = \left\{\rho_1 m_1 (v_2 - v_1)(f(v_2) - v_1 m_1) - \frac{1}{\rho_1}\right\} + \frac{1}{\rho_2} + q_1 - q_2 = 0,$$

which gives the condition of existence for the left-contact non-overcompressible delta wave

$$q_1 - q_2 = -\frac{1}{\rho_2} - \left\{\rho_1 m_1 (v_2 - v_1)(f(v_2) - v_1 m_1) - \frac{1}{\rho_1}\right\} = -\frac{1}{\rho_2} - L_1(v_2) < 0. \tag{4.29}$$

Then, from the generalized Rankine-Hugoniot relation (3.5), it follows that

$$\begin{cases} x(t) = v_2 t, \\ w_0 = \dfrac{\sqrt{1 - v_2^2/c^2}}{\sqrt{1 - v_1^2/c^2}}(v_1 - v_2) n_1, \\ v_\delta = v_2, \\ q_\delta = q_1. \end{cases} \tag{4.30}$$

It is suggested that when the initial data (1.5) satisfy either

$$\lambda_{--} < \lambda_{++}, \quad q_- < q_+, \quad \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} \leq q_+ - q_- \tag{4.31}$$

or

$$\begin{cases} \lambda_{--} > \lambda_{++}, \quad q_- < q_+, \\ \rho_- m_- \left( \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} - v_- \right) \left( f\left( \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} \right) - v_- m_- \right) - \dfrac{1}{\rho_-} < q_+ - q_- \end{cases} \tag{4.32}$$

or

$$\lambda_{--} = \lambda_{++}, \quad q_- < q_+, \tag{4.33}$$

the solution of the Riemann problem (1.3) + (1.4) + (1.2) and (1.5) is

$$
(\rho, v, q)(x, t) = \begin{cases} (\rho_-, v_-, q_-), & x < v_* t, \\ (w_0 t\delta(x - v_* t), v_\delta, q_\delta), & x = v_* t, \\ (\rho_*, v_*, q_+), & v_* t < x < \tau t, \\ (\rho_+, v_+, q_+), & x > \tau t, \end{cases} \tag{4.34}
$$

that is, $(\rho_-, v_-, q_-)$ is connected to $(\rho_*, v_*, q_+)$ by a right-contact non-overcompressible delta wave, and $(\rho_*, v_*, q_+)$ is connected to $(\rho_+, v_+, q_+)$ by a contact discontinuity $J^+$ with $\tau = \lambda_{++}$, where it holds that

$$
\begin{cases} q_+ - q_- = \dfrac{1}{\rho_*} + \left\{\rho_- m_-(v_* - v_-)(f(v_*) - v_- m_-) - \dfrac{1}{\rho_-}\right\}, \\ \lambda_{++} = \dfrac{v_+ + \frac{1}{\rho_+}}{1 + \frac{v_+}{\rho_+ c^2}} = \dfrac{v_* + \frac{1}{\rho_*}}{1 + \frac{v_*}{\rho_* c^2}} \end{cases} \tag{4.35}
$$

and

$$
\begin{cases} w_0 = \dfrac{\sqrt{1 - v_*^2/c^2}}{\sqrt{1 - v_-^2/c^2}} n_-(v_- - v_*), \\ v_\delta = v_*, \\ q_\delta = q_-. \end{cases} \tag{4.36}
$$

We show this structure, denoted by $\delta_0^r + J^+$, in Figure 6.

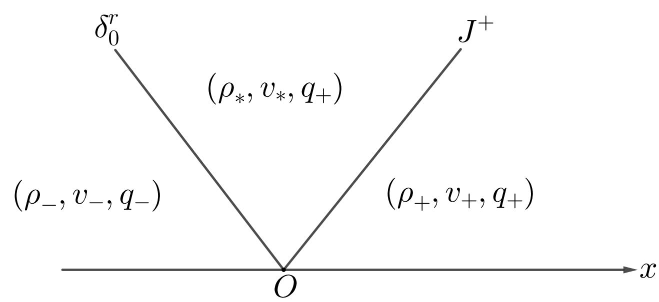


**Figure 6:** Riemann solution: $\delta_0^r + J^+$.

We also show the existence and uniqueness of $v_*$ satisfying $v_* < \lambda_{++}$ and $v_* \le \lambda_{--}$. From (4.35), we have

$$
N(v_*) = q_+ - q_- - \frac{v_* - \lambda_{++}}{v_* \lambda_{++}/c^2 - 1} - \left\{\rho_- m_-(v_* - v_-)(f(v_*) - v_- m_-) - \frac{1}{\rho_-}\right\} = 0,
$$

where

$$
\left\{\rho_- m_-(v_* - v_-)(f(v_*) - v_- m_-) - \frac{1}{\rho_-}\right\}\Bigg|_{v_* = \lambda_{--}} = 0.
$$

First, it holds that

$$
N'(v_*) = \frac{1 - \lambda_{++}^2/c^2}{(1 - v_* \lambda_{--}/c^2)^2} - \rho_- m_-\{(f(v_*) - v_- m_-) + (v_* - v_-)f'(v_*)\} > 0
$$

for $v_* \in (-c, \lambda_{--}]$. Second, we have

$$
N(-c) \to -\infty, \quad N(\lambda_{--}) = q_+ - q_- - \frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{++}\lambda_{--}/c^2} \ge 0
$$

for (4.31),

$$N(-c) \to -\infty, \quad N(\lambda_{++}) = q_+ - q_- - \left\{\rho_- m_-(\lambda_{++} - v_-)(f(\lambda_{++}) - v_- m_-) - \frac{1}{\rho_-}\right\} > 0$$

for (4.32), and

$$N(-c) \to -\infty, \quad N(\lambda_{--} = \lambda_{++}) = q_+ - q_- > 0$$

for (4.33). By zero point theorem of monotone functions in mathematical analysis, it is known that there exists a unique $v_*$ satisfying $v_* < \lambda_{++}$ and $v_* \le \lambda_{--}$.

So far, we have solved the Riemann problem (1.3) + (1.4) + (1.2) and (1.5) for all initial data by four kinds of wave patterns. The solutions and corresponding initial conditions can be summarized as follows:

(1) $q_- = q_+$

(1.1) when $\lambda_{--} < \lambda_{++}$, the solution is $J^- + J^0 + J^+$,

(1.2) when $\lambda_{--} \ge \lambda_{++}$, the solution is $\delta$;

(2) $q_- > q_+$

(2.1) $\lambda_{--} < \lambda_{++}$

(2.1.1) when $\frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_- - q_+$, the solution is $J^- + J^0 + J^+$,

(2.1.2) when $\frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} \le q_- - q_+$, the solution is $J^- + \delta_0^l$,

(2.2) $\lambda_{--} > \lambda_{++}$

(2.2.1) when $\rho_+ m_+(v_+ - \lambda_{--})(v_+ m_+ - f(\lambda_{--})) - \frac{1}{\rho_+} \ge q_- - q_+$, the solution is $\delta$,

(2.2.2) when $\rho_+ m_+(v_+ - \lambda_{--})(v_+ m_+ - f(\lambda_{--})) - \frac{1}{\rho_+} < q_- - q_+$, the solution is $J^- + \delta_0^l$,

(2.3) $\lambda_{--} = \lambda_{++}$, the solution is $J^- + \delta_0^l$;

(3) $q_- < q_+ \lambda_{--} < \lambda_{++}$

(3.1.1) when $\frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} > q_+ - q_-$, the solution is $J^- + J^0 + J^+$,

(3.1.2) when $\frac{\lambda_{++} - \lambda_{--}}{1 - \lambda_{--}\lambda_{++}/c^2} \le q_+ - q_-$, the solution is $\delta_0^r + J^+$,

(3.2) $\lambda_{--} > \lambda_{++}$

(3.2.1) when $\rho_- m_-(\lambda_{++} - v_-)(f(\lambda_{++}) - v_- m_-) - \frac{1}{\rho_-} \ge q_+ - q_-$, the solution is $\delta$,

(3.2.2) when $\rho_- m_-(\lambda_{++} - v_-)(f(\lambda_{++}) - v_- m_-) - \frac{1}{\rho_-} < q_+ - q_-$, the solution is $\delta_0^r + J^+$,

(3.3) $\lambda_{--} = \lambda_{++}$, the solution is $\delta_0^r + J^+$.

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