Research Article

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A monotone iteration for a nonlinear Euler-Bernoulli beam equation with indefinite weight and Neumann boundary conditions

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Abstract: In this article, we focus on the existence of positive solutions and establish a corresponding iterative scheme for a nonlinear fourth-order equation with indefinite weight and Neumann boundary conditions

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda h(x)f(y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases}$$

where k_1 and k_2 are constants, $\lambda > 0$ is a parameter, $h(x) \in L^1(0, 1)$ may change sign, and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$, $\mathbb{R}^+ := [0, \infty)$. We first discuss the sign properties of Green's function for the elastic beam boundary value problem, and then we establish some new results of the existence of positive solutions to this problem if the nonlinearity f is monotone on $\mathbb{R}^+$. The technique for dealing with this article relies on a monotone iteration technique and Schauder's fixed point theorem. Finally, an example is presented to illustrate the application of our main results.

Keywords: Euler-Bernoulli beam equations, positive solutions, indefinite weight, monotone iteration technique

MSC 2020: 34B15, 34B18, 34B27, 34C23, 34G20

1 Introduction

In this article, we aim to investigate the existence of positive solutions and establish a corresponding iterative scheme for a nonlinear Euler-Bernoulli beam equation with Neumann boundary conditions (for short NBVP)

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda h(x)f(y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases} \tag{1.1}$$

where k_1 and k_2 are constants, $\lambda > 0$ is a parameter, $h(x) \in L^1(0, 1)$ may change sign and $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R})$, $\mathbb{R}^+ := [0, \infty)$. This problem is always used to describe the sliding braces at both ends of an elastic beam.

Boundary value problems of ordinary differential equations are of great importance in both theory and application, many of which come from classical mechanics and electricity. For example, the equation of the

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elastic beam studied in this article is derived from the description of the deformation of the elastic beam in material mechanics. In material mechanics, the boundary value problem of the fourth-order differential equation is used to describe the deformation of the elastic beam in the equilibrium state. In particular, the elastic beam equation is also called the Euler-Bernoulli beam equation. In the last few decades, the equation in (1.1) with clamped beam boundary condition

$$y(0) = y(1) = y'(0) = y'(1) = 0 \tag{1.2}$$

has attracted the attention of many scholars, which describes the deformations of elastic beams with both fixed end-point, see [1–7] as well as references therein. In addition, equation (1.1) with the Lidstone boundary condition

$$y(0) = y(1) = y''(0) = y''(1) = 0 \tag{1.3}$$

also has received a lot of attention, since it models the stationary states of the deflection of an elastic beam with both hinged ends, see [8–11] and references therein.

Some nonlinear analysis tools have been used to investigate the existence of solutions for the fourth-order elastic beam equation with boundary conditions (1.2) and (1.3), such as lower and upper solutions [1,12–14], a monotone iterative technique [8,15–17], Krasnosel'skii fixed point theorem [5,11,18,19], fixed-point index [8], Leray-Schauder degree [20], and bifurcation theory [10,21–23]. In particular, by using the bifurcation techniques, Ma [10] considered the existence and multiplicity results of differential equation

$$y^{(4)}(x) + \eta y''(x) - \zeta y(x) = \lambda h(x) f(y(x)), \quad x \in (0, 1) \tag{1.4}$$

with boundary condition (1.3), where $f \in C(\mathbb{R}, \mathbb{R})$ satisfies $yf(y) > 0$ for all $y \neq 0$, and $\eta \in (-\infty, +\infty)$ and $\zeta \in [0, +\infty)$ are constants and satisfies the key condition $\frac{\eta}{\pi^2} + \frac{\zeta}{\pi^4} < 1$. Now, the interesting question is whether we could give a more general condition on η and ζ?

In 2014, by using lower and upper solution methods, Vrabel [14] discussed the existence of the solution following beam equation:

$$y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = f(x, y(x)), \quad x \in (0, 1) \tag{1.5}$$

with hinged end condition (1.3). Here, constants k_1 and k_2 satisfy the following condition:

$$k_1 < k_2 < 0. \tag{1.6}$$

Now, if we take $\eta = k_1 + k_2$ and $\zeta = -k_1k_2$, then the left sides of equations (1.4) and (1.5) are the same. However, it is easy to see that the condition (1.6) is more general than the condition $\frac{\eta}{\pi^2} + \frac{\zeta}{\pi^4} < 1$. Later, Ma et al. [12,13] discussed the same problem (1.5) with Lidstone boundary condition (1.3) under the restrictive conditions

$$0 < k_1 < k_2 < \pi^2 \quad \text{and} \quad k_1 < 0 < k_2 < \pi^2 \tag{1.7}$$

and obtained the existence of a solution by using lower and upper methods. It is noted that Vrabel [14], Ma et al. [12,13] only obtained the positivity of Green's function under the conditions (1.6) and (1.7). Naturally, the question is: could we obtain sign properties of Green's function when k_1 and k_2 change and the positive solution to these kinds of problems under similar conditions?

Roughly speaking, these tools which we talked about earlier cannot be applied directly if the weight function is allowed to change sign. The likely reason is the lack of any *a priori* estimate over the set of possible solutions. As far as we know, boundary value problems with sign-changing weight functions arise from population modeling. In this model, a weight function changes sign corresponding to the fact that the intrinsic population growth rate is positive at some points and negative at others, for details, readers can see Cantrell and Cosner [24]. It is precise because of this fact that many scholars have become more and more interested in boundary value problems with sign-changing weight functions.

In particular, for the case of fourth-order boundary value problem with sign-changing weight functions, Ma et al. [23] considered nonlinear boundary value problem with a sign-changing weight function

$$y^{(4)}(x) = rh(x)f(y(x)), \quad x \in (0,1) \tag{1.8}$$

with simply supported beam conditions (1.3), where r is a parameter, $f \in C(\mathbb{R},\mathbb{R})$ and $h : [0,1] \to \mathbb{R}$ is a continuous function which attains both positive and negative values. They determine the principal eigenvalues of the linear problem corresponding to (1.8). With this result in hands, they establish the existence of positive solutions for problem (1.8) via the global bifurcation theory.

It is worth pointing out that studies [21] and [23] only discussed the existence of nodal or positive solutions for fourth-order equations with a sign-changing weight function under the simply supported beam conditions (1.3), respectively. However, to the best of our knowledge, a fourth-order equation with a sign-changing weight function under Neumann boundary conditions despite its simple-looking structure, is considered as a difficult problem in the literature. Therefore, relatively little is known about problem (1.1). The possible reason is that fewer techniques are available for problem (1.1), and the popular comparison principle appears to be completely inapplicable (see Yan [25]). To overcome these difficulties, we use a method to overcome the change sign of the weight function, which is the monotone iterative technique. We also refer readers to Pei and Chang [17] and Yao [5] for this approach under the fourth-order case. Nevertheless, the difference between these works and our work consists of the fact that our weight function may change signs. Motivated by the aforementioned studies, the main purpose of this article is to establish a monotone convergent iterative scheme for nonlinear NBVP (1.1), which avoids the aforementioned inconvenient, and then obtain some new existence results on the positive solutions for the problem (1.1) with indefinite weight.

To sum up, all the ideas mentioned in the Introduction, we try to discuss the existence of positive solutions for the nonlinear NBVP (1.1) under more general conditions like (1.6) and (1.7). Our method is also suitable for problem (1.5), (1.6) and problem (1.5), and (1.7). To obtain it, we first discuss the properties of Green's function $G(x,s)$ for NBVP (1.1). We will investigate the sign properties of Green's function as the constants k_1 and k_2 change, see Section 2. Based on the properties of Green's function, we try to discuss the existence of positive solutions for nonlinear NBVP (1.1) by using a monotone iterative technique.

To make our statements precise, we have to recall two common notations. Namely,

$$h^+(x) := \max\{h(x),0\} \quad \text{and} \quad h^-(x) := \max\{-h(x),0\} \quad \text{for } x \in [0,1].$$

It is not difficult to see that $h^\pm \geq 0$ and $h(x) = h^+(x) - h^-(x)$, $x \in [0,1]$. Throughout this article, we use the following assumptions:

(A1) $h(x) \in L^1(0,1)$ may change sign;
(A2) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing on $\mathbb{R}^+$;
(A3) $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nonincreasing on $\mathbb{R}^+$.

This article is divided into six sections. From the previous discussion, we find that the appearance of k_1 and k_2 leads to the absence of the positivity of Green's function in NBVP (1.1), which greatly increases the complexity of the calculation of Green's function. On this basis, in the second part of this article, we discuss the properties of Green's function in detail according to the different classifications of k_1 and k_2. Including the case of $k_1 \leq k_2 < 0$, $k_1 < 0 < k_2 \leq \pi^2/4$ and $0 < k_1 < k_2 \leq \pi^2/4$, respectively. Some preliminary results which we shall require are introduced. Sections 3 and 4 are devoted to establishing the existence results of positive solutions of NBVP (1.1) by constructing a monotone iteration scheme. In Section 5, we give a supplementary theorem and an example to illustrate our main results and iteration schemes. Finally, in the last section, we review the main problems and the main results and make a summary statement for the problems with other kinds of boundary conditions in this article.

2 Preliminaries

Let $X = C[0,1]$ be a Banach space, with its usual norm $\|y\| = \max\{|y(x)|, x \in [0,1]\}$ for all $y \in X$.

Define linear operator $L : D(L) \to X$ as follows:

$$Ly := y^{(4)} + (k_1 + k_2)y'' + k_1k_2y, \quad y \in D(L),$$

where $D(L) := \{y \in C^4[0,1] : y'(0) = y'(1) = y'''(0) = y'''(1) = 0\}$. To obtain Green's function $G(x,s)$ of $Ly = 0$, we define another linear operator

$$L_1y := y'' + k_1y, \quad D(L_1) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\};$$
$$L_2y := y'' + k_2y, \quad D(L_2) := \{y \in C^2[0,1] : y'(0) = y'(1) = 0\}.$$

It is easy to verify that Green's function of $Ly = L_2 \circ (L_1y)$ and $Ly = 0$ is

$$G(x,s) := \int_0^1 G_2(x,t)G_1(t,s)\mathrm{d}t, \quad (x,s) \in [0,1] \times [0,1], \tag{2.1}$$

where $G_i(t,s)$ ([26]) are (symmetric) Green's functions for $L_iy = 0$ $(i = 1, 2)$ and

$$G_i(t,s) = \begin{cases} \begin{cases} -\dfrac{\cosh[\sqrt{-k_i}(1-t)]\cosh(\sqrt{-k_i}s)}{\sqrt{-k_i}\sinh\sqrt{-k_i}}, & s \le t, \\ -\dfrac{\cosh[\sqrt{-k_i}(1-s)]\cosh(\sqrt{-k_i}t)}{\sqrt{-k_i}\sinh\sqrt{-k_i}}, & t \le s, \end{cases} & k_i \in (-\infty, 0), \\ \begin{cases} \dfrac{\cos[\sqrt{k_i}(1-t)]\cos(\sqrt{k_i}s)}{\sqrt{k_i}\sin\sqrt{k_i}}, & s \le t, \\ \dfrac{\cos[\sqrt{k_i}(1-s)]\cos(\sqrt{k_i}t)}{\sqrt{k_i}\sin\sqrt{k_i}}, & t \le s, \end{cases} & k_i \in (0, \pi^2/4). \end{cases}$$

But for $t > s$, we have

$$G_1(t,s) = \frac{\cosh(\sqrt{-k_1}(1-t))\cdot\cosh(\sqrt{-k_1}s)}{\sqrt{-k_1}\sinh\sqrt{-k_1}}.$$

This function is obviously positive for $k_1 < 0$ and negative for $k_1 \in (0, \pi^2/4)$. Consequently, $G : [0,1] \times [0,1] \to \mathbb{R}$. In particular, we refer to the study by Wang et al. [4] for details of the calculation of equality (2.1), and some of the results are discussed in the following sections.

2.1 Green's function and its sign properties in case $k_1 \le k_2 < 0$

From $k_1 \le k_2 < 0$, let $k_1 = -\alpha^2$, $k_2 = -\beta^2$, where α and β are constants greater than zero that satisfy $\alpha \ge \beta$. Therefore, we divide two cases as follows:

Case 1. $\alpha = \beta > 0$

From the theory of Green's function, we can obtain the explicit expression of Green's function of (1.1) as follows:

$$\overline{G}(x,s) = \begin{cases} \dfrac{\sinh\alpha\cosh[\alpha(1-s)][\cosh(\alpha x) - \alpha x\sinh(\alpha x)]}{2\alpha^3\sinh^2\alpha} \\ \quad + \dfrac{\alpha\cosh(\alpha x)[\cosh(\alpha s + s\sinh\alpha\sinh[\alpha(1-s)]]}{2\alpha^3\sinh^2\alpha}, & s \le x, \\ \dfrac{\sinh\alpha\cosh[\alpha(1-x)][\cosh(\alpha s) - \alpha s\sinh(\alpha s)]}{2\alpha^3\sinh^2\alpha} \\ \quad + \dfrac{\alpha\cosh(\alpha s)[\cosh(\alpha x + x\sinh\alpha\sinh[\alpha(1-x)]]}{2\alpha^3\sinh^2\alpha}, & x \le s. \end{cases}$$

Case 2. $\alpha > \beta > 0$

Thus, the concrete expression of Green's function of problem (1.1) is

$$\overline{G}(x,s)=\begin{cases}\dfrac{1}{\alpha^2-\beta^2}\left[\dfrac{\cosh[\beta(1-s)]\cosh(\beta x)}{\beta\sinh\beta}-\dfrac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\right], & 0\le x\le s\le 1,\\ \dfrac{1}{\alpha^2-\beta^2}\left[\dfrac{\cosh[\beta(1-x)]\cosh(\beta s)}{\beta\sinh\beta}-\dfrac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\right], & 0\le s\le x\le 1.\end{cases}$$

Lemma 2.1. [4, Theorem 2.1] *If $\alpha,\beta\in(0,+\infty)$ with $\alpha\ge\beta$, then Green's function of problem* (1.1) *satisfies*

$$\overline{G}(x,s)>0,\quad (x,s)\in[0,1]\times[0,1].$$

2.2 Green's function and its sign properties in case $k_1<0<k_2\le\pi^2/4$

From $k_1<0<k_2\le\pi^2/4$, let $k_1=-\alpha^2$, $k_2=\beta^2$ and $\alpha\in(0,+\infty)$, $\beta\in(0,\pi/2]$. Thus, the concrete expression of Green's function of problem (1.1) is

$$-\widetilde{G}(x,s)=\begin{cases}\dfrac{1}{\alpha^2+\beta^2}\left[\dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta}+\dfrac{\cosh[\alpha(1-s)]\cosh(\alpha x)}{\alpha\sinh\alpha}\right], & 0\le x\le s\le 1,\\ \dfrac{1}{\alpha^2+\beta^2}\left[\dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}+\dfrac{\cosh[\alpha(1-x)]\cosh(\alpha s)}{\alpha\sinh\alpha}\right], & 0\le s\le x\le 1.\end{cases}$$

The properties of Green's function $\widetilde{G}(x,s)$ are given as follows:

Lemma 2.2. ([4], Theorem 2.1) *If $\alpha\in(0,+\infty)$, $\beta\in(0,\pi/2]$, then*

$$\widetilde{G}(x,s)<0,\quad (x,s)\in[0,1]\times[0,1].$$

Remark 2.3. It is worth noting that we obtain $\widetilde{G}(x,s)<0$ with the case of $k_1<0<k_2\le\pi^2/4$. At this point, if the problem we are studying (1.1) is transformed into the following form:

$$\begin{cases}y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)+\lambda h(x)f(y(x))=0, & x\in[0,1],\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0,\end{cases}\tag{2.2}$$

then the results obtained in this article still held true for the above problems.

2.3 Green's function and its sign properties in case $0<k_1<k_2\le\pi^2/4$

From $0<k_1<k_2\le\pi^2/4$, let $k_1=\alpha^2$, $k_2=\beta^2$, and $0<\alpha<\beta\le\pi/2$. Therefore, we divide two cases as follows:

Case 3. $\alpha=\beta<\pi/2$

Then, we can obtain the concrete expression of Green's function of problem (1.1) as follows:

$$G(x,s)=\begin{cases}\dfrac{\sin\alpha\cos[\alpha(1-s)][\cos(\alpha x)+\alpha x\sin(\alpha x)]}{2\alpha^3\sin^2\alpha}+\dfrac{\alpha\cos(\alpha x)[\cos(\alpha s-s\sin\alpha\sin[\alpha(1-s)]]}{2\alpha^3\sin^2\alpha}, & s\le x,\\ \dfrac{\sin\alpha\cos[\alpha(1-x)][\cos(\alpha s)+\alpha s\sin(\alpha s)]}{2\alpha^3\sin^2\alpha}+\dfrac{\alpha\cos(\alpha s)[\cos(\alpha x-x\sin\alpha\sin[\alpha(1-x)]]}{2\alpha^3\sin^2\alpha}, & x\le s.\end{cases}$$

In particular, if $\alpha=\beta=\pi/2$, then $t=s=0$ or $t=s=1$, $G(x,s)$ contains zero.

Case 4. $0<\alpha<\beta<\pi/2$

In this case, the concrete expression of Green's function of problem (1.1) is

$$G(x,s)=\begin{cases}\dfrac{1}{\beta^2-\alpha^2}\left[\dfrac{\cos[\alpha(1-s)]\cos(\alpha x)}{\alpha\sin\alpha}-\dfrac{\cos[\beta(1-s)]\cos(\beta x)}{\beta\sin\beta}\right], & 0\le x\le s\le 1,\\ \dfrac{1}{\beta^2-\alpha^2}\left[\dfrac{\cos[\alpha(1-x)]\cos(\alpha s)}{\alpha\sin\alpha}-\dfrac{\cos[\beta(1-x)]\cos(\beta s)}{\beta\sin\beta}\right], & 0\le s\le x\le 1.\end{cases}$$

The properties of Green's function $G(x,s)$ are given as follows:

Lemma 2.4. ([4], Theorem 2.1) *If* $0<\alpha<\beta\le\pi/2$, *then*

$$G(x,s)>0,\quad (x,s)\in[0,1]\times[0,1].$$

Remark 2.5. It should be noted that in the three cases discussed in this article, if the parameter $k_1=0$ or $k_2=0$, the operator Ly has eigenvalue $\bar{\lambda}=0$ and $Ly=0$ has nontrivial solution $y\equiv C$ $(C\ne 0)$. Therefore, according to the Fredholm alternative theorem, there is no solution to the problem (1.1), so the parameters in this article meet the requirement that $k_1k_2\ne 0$ are always valid.

Based on the sign of Green's function of NBVP (1.1), without loss of generality, we discuss the case of $0<k_1<k_2\le\pi^2/4$.

First, let us briefly recall the concept of cone. That is:

Definition 2.6. [27] Let E be a real Banach space. A nonempty closed convex set $\mathcal{K}\subset E$ is said to be a cone provided that

(a) $\mu y\in\mathcal{K}$ for all $y\in\mathcal{K}$ and $\mu\ge 0$;
(b) If $y\in\mathcal{K}$ and $-y\in\mathcal{K}$, then $y=0$.

In the sequel, we recall the Schauder's fixed-point theorem, which will be used along the article.

Lemma 2.7. [28] *Let P be a compact convex set in a Banach space X and let $\mathcal{L}$ be a continuous mapping of P into itself. Then, $\mathcal{L}$ has a fixed point, that is,*

$$\mathcal{L}y=y,$$

for some $y\in P$.

It is well known that NBVP (1.1) has a solution $y=y(x)$ if and only if y solves the operator equation

$$y(x)=\lambda\int_0^1 G(x,s)h(s)f(y(s))\mathrm{d}s=:\mathcal{L}y(x),$$

whose domain will be $C^{+}[0,1]$, the cone of continuous nonnegative functions. The operator $\mathcal{L}$ does not leave this cone invariant since we allow $h(x)$ to change its sign. Here, $G(x,s)$ denote the Green's function for the linear boundary value problem

$$\begin{cases}y^{(4)}(x)+(k_1+k_2)y''(x)+k_1k_2y(x)=u(x), & x\in(0,1),\\ y'(0)=y'(1)=y'''(0)=y'''(1)=0\end{cases}$$

with $u\in C[0,1]$, for details, see Section 2 in this article.

Remark 2.8. It is not difficult to check, from Section 2, that $G(x,s)$ is continuous and nonnegative on $(x,s)\in[0,1]\times[0,1]$. Moreover, we need to give it a proper estimate, that is,

$$0<G(x,s)\le\mathcal{M}=\max_{0\le x,s\le 1}G(x,s)\quad\text{for } x,s\in[0,1],\tag{2.3}$$

which plays a significant role in giving an example corresponding our main result.

For the sake of brevity, we denote

$$\mathcal{H}^+ := \{x \in [0,1] : h(x) \geq 0\} \quad \text{and} \quad \mathcal{H}^- := \{x \in [0,1] : h(x) \leq 0\}.$$

3 The case of $f(y(x))$ nondecreasing

Lemma 3.1. *Suppose* (A1) *and* (A2) *hold. If there are two bounded measurable functions* $\phi_0(x)$ *and* $\psi_0(x)$ *on* $[0,1]$, *such that they satisfy*

(h_1) $0 \leq \phi_0(x) \leq \psi_0(x)$ *for* $x \in \mathcal{H}^+$, $0 \leq \psi_0(x) \leq \phi_0(x)$ *for* $x \in \mathcal{H}^-$;

(h_2) $\mathcal{L}\psi_0(x) \leq \psi_0(x)$ *for* $x \in \mathcal{H}^+$, $\mathcal{L}\psi_0(x) \leq \phi_0(x)$ *for* $x \in \mathcal{H}^-$;

(h_3) $\mathcal{L}\phi_0(x) \geq \phi_0(x)$ *for* $x \in \mathcal{H}^+$, $\mathcal{L}\phi_0(x) \geq \phi_0(x)$ *for* $x \in \mathcal{H}^-$, *then* $\phi_1(x)$ *and* $\psi_1(x)$ *also satisfy the corresponding inequalities, where*

$$\phi_1(x) = \begin{cases} \mathcal{L}\phi_0(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\psi_0(x), & x \in \mathcal{H}^- \end{cases}$$

and

$$\psi_1(x) = \begin{cases} \mathcal{L}\psi_0(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\phi_0(x), & x \in \mathcal{H}^-. \end{cases}$$

Proof. Due to the fact that

$$h(x) = h^+(x) - h^-(x) \quad \text{and} \quad x \in [0,1],$$

the operator $\mathcal{L}$ can be rewritten as

$$\mathcal{L}\phi(x) = \lambda \int_{\mathcal{H}^+} G(x,s)h^+(s)f(\phi(s))\mathrm{d}s - \lambda \int_{\mathcal{H}^-} G(x,s)h^-(s)f(\phi(s))\mathrm{d}s.$$

For simplicity, we write

$$\mathcal{L}\phi(x) = \mathcal{L}_1\phi(x) - \mathcal{L}_2\phi(x) \quad \text{and} \quad x \in [0,1],$$

where $\mathcal{L}_i\phi(x)$, $(i = 1, 2)$ are both monotones in the sense that $\phi(x) \leq \psi(x)$ implies $\mathcal{L}_i\phi(x) \leq \mathcal{L}_i\psi(x)$. $\mathcal{L}_1$ acts on $C^+([0,1] \cap \mathcal{H}^+)$ and $\mathcal{L}_2$ on $C^+([0,1] \cap \mathcal{H}^-)$. Note that ($h_1$) implies that

$$\mathcal{L}\phi_0(x) = \mathcal{L}_1\phi_0(x) - \mathcal{L}_2\phi_0(x) \leq \mathcal{L}_1\psi_0(x) - \mathcal{L}_2\psi_0(x) = \mathcal{L}\psi_0(x). \tag{3.1}$$

This immediately implies that $\phi_1(x)$ and $\psi_1(x)$ satisfy condition (h_1). More precisely, they hold

$$0 \leq \phi_1(x) = \mathcal{L}\phi_0(x) \leq \mathcal{L}\psi_0(x) = \phi_1(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$0 \leq \psi_1(x) = \mathcal{L}\phi_0(x) \leq \mathcal{L}\psi_0(x) = \psi_1(x) \quad \text{for } x \in \mathcal{H}^-.$$

The condition (h_2) for $\psi_1(x)$ is that

$$\mathcal{L}\psi_1(x) \leq \psi_1(x) = \mathcal{L}\psi_0(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$\mathcal{L}\psi_1(x) \leq \phi_1(x) = \mathcal{L}\psi_0(x) \quad \text{for } x \in \mathcal{H}^-.$$

More precisely, we have

$$\mathcal{L}\psi_1(x) \le \mathcal{L}\psi_0(x) \quad \text{for } x \in [0,1].$$

According to conditions (h_2) and (h_3), we have

$$\mathcal{L}\psi_1(x) = \mathcal{L}_1\psi_1(x) - \mathcal{L}_2\psi_1(x) = \mathcal{L}_1(\mathcal{L}\psi_0(x)) - \mathcal{L}_2(\mathcal{L}\phi_0(x)) \le \mathcal{L}_1\psi_0(x) - \mathcal{L}_2\psi_0(x) = \mathcal{L}\psi_0(x). \tag{3.2}$$

Therefore, condition (h_2) is satisfied. Similarly, condition (h_3) for $\phi_1(x)$ is that

$$\mathcal{L}\phi_1(x) \ge \phi_1(x) = \mathcal{L}\phi_0(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$\mathcal{L}\phi_1(x) \ge \psi_1(x) = \mathcal{L}\psi_0(x) \quad \text{for } x \in \mathcal{H}^-.$$

Now,

$$\mathcal{L}\phi_1(x) = \mathcal{L}_1\phi_1(x) - \mathcal{L}_2\phi_1(x) = \mathcal{L}_1(\mathcal{L}\phi_0(x)) - \mathcal{L}_2(\mathcal{L}\psi_0(x)) \ge \mathcal{L}_1\phi_0(x) - \mathcal{L}_2\phi_0(x) = \mathcal{L}\phi_0(x) \tag{3.3}$$

gives conditions (h_3) for $\phi_1(x)$. Consequently, the proof is completed. □

Theorem 3.2. *Assume that* (A1) *and* (A2) *hold. If there exist two bounded measurable functions* $\phi_0(x)$ *and* $\psi_0(x)$ *on* [0,1] *that satisfy conditions* (h_1), (h_2), *and* (h_3) *of* Lemma 3.1, *then, NBVP* (1.1) *has at least one positive solution.*

Proof. Define

$$\phi_{n+1}(x) = \begin{cases} \mathcal{L}\phi_n(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\psi_n(x), & x \in \mathcal{H}^- \end{cases} \tag{3.4}$$

and

$$\psi_{n+1}(x) = \begin{cases} \mathcal{L}\psi_n(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\phi_n(x), & x \in \mathcal{H}^-. \end{cases} \tag{3.5}$$

Note that, in general, $\phi_n(x)$ and $\psi_n(x)$ are not continuous. We know, by Lemma 3.1 and Section 1, that $(\phi_n(x), \psi_n(x))$ satisfies conditions (h_1), (h_2), and (h_3) of Lemma 3.1. Then, it follows from (3.1), (3.2), and (3.3) that

$$0 \le \mathcal{L}\phi_n(x) \le \mathcal{L}\phi_{n+1}(x) \le \cdots \le \mathcal{L}\psi_{n+1}(x) \le \mathcal{L}\psi_n(x) \le \mathcal{L}\psi_0(x).$$

Therefore, $\mathcal{L}\phi_n(x)$ is monotonically increasing and satisfies

$$\mathcal{L}\phi_n(x) \to \widetilde{\phi}(x) \quad \text{for } n \to \infty;$$

meanwhile, $\mathcal{L}\psi_n(x)$ is monotonically decreasing and satisfies

$$\mathcal{L}\psi_n(x) \to \widetilde{\psi}(x) \quad \text{for } n \to \infty$$

and

$$\widetilde{\phi}(x) \le \widetilde{\psi}(x).$$

Since

$$\begin{aligned} \mathcal{L}\phi_{n+1}(x) &= \mathcal{L}_1\phi_{n+1}(x) - \mathcal{L}_2\phi_{n+1}(x) = \mathcal{L}_1(\mathcal{L}\phi_n(x)) - \mathcal{L}_2(\mathcal{L}\psi_n(x)); \\ \mathcal{L}\psi_{n+1}(x) &= \mathcal{L}_1\psi_{n+1}(x) - \mathcal{L}_2\psi_{n+1}(x) = \mathcal{L}_1(\mathcal{L}\psi_n(x)) - \mathcal{L}_2(\mathcal{L}\phi_n(x)), \end{aligned}$$

we may apply Lebesgue's dominated convergence theorem to have

$$\widetilde{\phi}(x) = \mathcal{L}_1\widetilde{\phi}(x) - \mathcal{L}_2\widetilde{\psi}(x); \tag{3.6}$$

$$\widetilde{\psi}(x) = \mathcal{L}_1\widetilde{\psi}(x) - \mathcal{L}_2\widetilde{\phi}(x). \tag{3.7}$$

Thus, both $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$ are continuous on [0,1].

In view of (3.4) and (3.5), we have $\phi_{n+1}(x) \to \widetilde{\phi}(x)$ on $\mathcal{H}^+$ by definition of $\widetilde{\phi}(x)$. On the other hand, $\phi_{n+1}(x) = \mathcal{L}\phi_n(x)$ on $\mathcal{H}^+$, so $\phi_{n+1}(x) \to \mathcal{L}\widetilde{\phi}(x)$. Thus, $\widetilde{\phi}(x) = \mathcal{L}\widetilde{\phi}(x)$ on $\mathcal{H}^+$. On $\mathcal{H}^-$, we have $\varphi_{n+1} = \mathcal{L}\phi_n(x) \to \widetilde{\phi}(x)$ by definition of $\widetilde{\phi}(x)$. However, $\phi_{n+1}(x) = \mathcal{L}\psi_n(x)$ on $\mathcal{H}^-$, so $\phi_{n+1}(x) \to \widetilde{\psi}(x)$ on $\mathcal{H}^-$, $\mathcal{L}\widetilde{\psi}(x) = \widetilde{\phi}(x)$.

In a similar method, we have

$$\widetilde{\phi}(x) = \begin{cases} \mathcal{L}\widetilde{\phi}(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\widetilde{\psi}(x), & x \in \mathcal{H}^-, \end{cases}$$

and

$$\widetilde{\psi}(x) = \begin{cases} \mathcal{L}\widetilde{\psi}(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\widetilde{\phi}(x), & x \in \mathcal{H}^- \end{cases}$$

and $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$ are fixed points of $\mathcal{L}^2$.

Now, we consider the convex region in $C[0, 1]$ defined by

$$C = \{g(x)|\widetilde{\phi}(x) \le g(x) \le \widetilde{\psi}(x), \quad x \in [0, 1]\}.$$

This is invariant under $\mathcal{L}$ since

$$\mathcal{L}g(x) = \mathcal{L}_1 g(x) - \mathcal{L}_2 g(x) \le \mathcal{L}_1\widetilde{\psi}(x) - \mathcal{L}_2\widetilde{\phi}(x) = \widetilde{\psi}(x)$$

by (3.7). Similarly,

$$\mathcal{L}g(x) = \mathcal{L}_1 g(x) - \mathcal{L}_2 g(x) \ge \mathcal{L}_1\widetilde{\phi}(x) - \mathcal{L}_2\widetilde{\psi}(x) = \widetilde{\phi}(x).$$

Since $G(x, s)$ is continuous on $(x, s) \in [0, 1] \times [0, 1]$, $h(x) \in L^1(0, 1)$ implies that $\{(\mathcal{L}g(x)) : g(x) \in C\}$ is uniformly equicontinuous, it follows from the Arzelá-Ascoli theorem that $\mathcal{L}$ restricted to C is a compact operator.

We can know, by Lemma 2.7, that $\mathcal{L}$ has a fixed point. Hence, NBVP (1.1) has at least one positive solution. Consequently, the proof is complete. □

Remark 3.3. The order interval

$$\{g(x)|\mathcal{L}\phi_0(x) \le g(x) \le \mathcal{L}\psi_0(x), \quad x \in [0, 1]\}$$

already is invariant and one could use Schauder's theorem directly. But the iteration improves the estimates. In general, one might expect the two functions $\widetilde{\phi}(x)$ and $\widetilde{\psi}(x)$ to be the same. Therefore, one could in effect construct the solution numerically, see Section 5 for more precise statements.

4 The case of $f(y(x))$ nonincreasing

Lemma 4.1. *Suppose* (*A*1) *and* (*A*3) *hold. If there are two bounded measurable functions* $\phi_0^*(x)$ *and* $\psi_0^*(x)$ *on* [0, 1] *such that they satisfy*

(h_1') $0 \le \phi_0^*(x) \le \psi_0^*(x)$ *for* $x \in \mathcal{H}^+$, $0 \le \psi_0^*(x) \le \phi_0^*(x)$ *for* $x \in \mathcal{H}^-$;

(h_2') $\mathcal{L}\phi_0^*(x) \le \psi_0^*(x)$ *for* $x \in \mathcal{H}^+$, $\mathcal{L}\phi_0^*(x) \le \phi_0^*(x)$ *for* $x \in \mathcal{H}^-$;

(h_3') $\mathcal{L}\psi_0^*(x) \ge \phi_0^*(x)$ *for* $x \in \mathcal{H}^+$, $\mathcal{L}\psi_0^*(x) \ge \psi_0^*(x)$ *for* $x \in \mathcal{H}^-$.

Then, $\phi_1^*(x)$ *and* $\psi_1^*(x)$ *also satisfy the corresponding inequalities, where*

$$\phi_1^*(x) = \begin{cases} \mathcal{L}\psi_0^*(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\phi_0^*(x), & x \in \mathcal{H}^- \end{cases}$$

and

$$\psi_1^*(x) = \begin{cases} \mathcal{L}\phi_0^*(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\psi_0^*(x), & x \in \mathcal{H}^-. \end{cases}$$

Note that (h_1') *here is the same as* (h_1) *in* Lemma 3.1.

Proof. Due to the fact that

$$h(x) = h^+(x) - h^-(x) \quad \text{and} \quad x \in [0, 1],$$

the operator $\mathcal{L}$ can be rewritten as

$$\mathcal{L}\phi(x) = \lambda \int_{\mathcal{H}^+} G(x, s)h^+(s)f(\phi(s))\mathrm{d}s - \lambda \int_{\mathcal{H}^-} G(x, s)h^-(s)f(\phi(s))\mathrm{d}s.$$

For convenience, we write

$$\mathcal{L}\phi(x) = \mathcal{L}_1\phi(x) - \mathcal{L}_2\phi(x) \quad \text{for } x \in [0, 1],$$

where $\mathcal{L}_i\phi(x) \quad (i = 1, 2)$ are both monotone in the sense that $\phi(x) \le \psi(x)$ implies $\mathcal{L}_i\phi(x) \ge \mathcal{L}_i\psi(x)$. $\mathcal{L}_1$ acts on $C^+([0, 1] \cap \mathcal{H}^+)$ and $\mathcal{L}_2$ on $C^+([0, 1] \cap \mathcal{H}^-)$. Note that (h_1') implies that

$$\mathcal{L}\phi_0^*(x) = \mathcal{L}_1\phi_0^*(x) - \mathcal{L}_2\phi_0^*(x) \ge \mathcal{L}_1\psi_0^*(x) - \mathcal{L}_2\psi_0^*(x) = \mathcal{L}\psi_0^*(x). \tag{4.1}$$

This immediately implies that $\phi_1^*(x)$ and $\psi_1^*(x)$ satisfy condition (h_1'). To be specific, it holds

$$0 \le \phi_1^*(x) = \mathcal{L}\psi_0^*(x) \le \mathcal{L}\phi_0^*(x) = \psi_1^*(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$0 \le \psi_1^*(x) = \mathcal{L}\psi_0^*(x) \le \mathcal{L}\phi_0^*(x) = \phi_1^*(x) \quad \text{for } x \in \mathcal{H}^-.$$

The condition (h_2') for $\phi_1^*(x)$ is that

$$\mathcal{L}\phi_1^*(x) \le \psi_1^*(x) = \mathcal{L}\phi_0^*(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$\mathcal{L}\phi_1^*(x) \le \phi_1^*(x) = \mathcal{L}\phi_0^*(x) \quad \text{for } x \in \mathcal{H}^-.$$

More precisely, we have

$$\mathcal{L}\phi_1^*(x) \le \mathcal{L}\phi_0^*(x) \quad \text{for } x \in [0, 1].$$

According to conditions (h_2') and (h_3'), we have

$$\mathcal{L}\phi_1^*(x) = \mathcal{L}_1\phi_1^*(x) - \mathcal{L}_2\phi_1^*(x) = \mathcal{L}_1(\mathcal{L}\psi_0^*(x)) - \mathcal{L}_2(\mathcal{L}\phi_0^*(x)) \le \mathcal{L}_1\phi_0^*(x) - \mathcal{L}_2\phi_0^*(x) = \mathcal{L}\phi_0^*(x). \tag{4.2}$$

Therefore, condition (h_2') is satisfied. Similarly, condition (h_3') for $\psi_1^*(x)$ is that

$$\mathcal{L}\psi_1^*(x) \ge \phi_1^*(x) = \mathcal{L}\psi_0^*(x) \quad \text{for } x \in \mathcal{H}^+$$

and

$$\mathcal{L}\psi_1^*(x) \ge \psi_1^*(x) = \mathcal{L}\psi_0^*(x) \quad \text{for } x \in \mathcal{H}^-.$$

Now,

$$\mathcal{L}\psi_1^*(x) = \mathcal{L}_1\psi_1^*(x) - \mathcal{L}_2\psi_1^*(x) = \mathcal{L}_1(\mathcal{L}\phi_0^*(x)) - \mathcal{L}_2(\mathcal{L}\psi_0^*(x)) \ge \mathcal{L}_1\psi_0^*(x) - \mathcal{L}_2\psi_0^*(x) = \mathcal{L}\psi_0^*(x) \tag{4.3}$$

gives conditions (h_3') for $\psi_1^*(x)$. Consequently, the proof is completed. □

Theorem 4.2. *Assume that (A1) and (A3) hold. If there exist two bounded measurable functions $\phi_0^*(x)$ and $\psi_0^*(x)$ on $[0, 1]$ that satisfy conditions (h_1'), (h_2'), and (h_3') of* Lemma 4.1. *Then, NBVP* (1.1) *has at least one positive solution.*

Proof. Define

$$\phi_{n+1}^*(x) = \begin{cases} \mathcal{L}\phi_n^*(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\psi_n^*(x), & x \in \mathcal{H}^- \end{cases} \tag{4.4}$$

and

$$\psi_{n+1}^*(x) = \begin{cases} \mathcal{L}\psi_n^*(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\phi_n^*(x), & x \in \mathcal{H}^-. \end{cases} \tag{4.5}$$

Note that, in general, $\phi_n^*(x)$ and $\psi_n^*(x)$ are not continuous. We know, by Lemma 4.1 and Section 1, that $(\phi_n^*(x), \psi_n^*(x))$ satisfies conditions (h_1'), (h_2'), and (h_3') of Lemma 4.1. Then, it follows from (4.1), (4.2), and (4.3) that

$$0 \le \mathcal{L}\psi_n^*(x) \le \mathcal{L}\psi_{n+1}^*(x) \le \cdots \le \mathcal{L}\phi_{n+1}^*(x) \le \mathcal{L}\phi_n^*(x) \le \mathcal{L}\phi_0^*(x).$$

Therefore, $\mathcal{L}\phi_n^*(x)$ is monotonically increasing and satisfies

$$\mathcal{L}\phi_n^*(x) \to \overline{\phi}(x) \quad \text{for } n \to \infty;$$

meanwhile, $\mathcal{L}\psi_n(x)$ is monotonically decreasing and satisfies

$$\mathcal{L}\psi_n^*(x) \to \overline{\psi}(x) \quad \text{for } n \to \infty$$

and

$$\overline{\psi}(x) \le \overline{\phi}(x).$$

Since

$$\mathcal{L}\psi_{n+1}^*(x) = \mathcal{L}_1\psi_{n+1}^*(x) - \mathcal{L}_2\psi_{n+1}^*(x) = \mathcal{L}_1(\mathcal{L}\psi_n^*(x)) - \mathcal{L}_2(\mathcal{L}\phi_n^*(x));$$
$$\mathcal{L}\phi_{n+1}^*(x) = \mathcal{L}_1\phi_{n+1}^*(x) - \mathcal{L}_2\phi_{n+1}^*(x) = \mathcal{L}_1(\mathcal{L}\phi_n^*(x)) - \mathcal{L}_2(\mathcal{L}\psi_n^*(x)),$$

we may apply Lebesgue's dominated convergence theorem to have

$$\overline{\psi}(x) = \mathcal{L}_1\overline{\psi}(x) - \mathcal{L}_2\overline{\phi}(x); \tag{4.6}$$

$$\overline{\phi}(x) = \mathcal{L}_1\overline{\phi}(x) - \mathcal{L}_2\overline{\psi}(x). \tag{4.7}$$

Thus, both $\overline{\psi}(x)$ and $\overline{\phi}(x)$ are continuous on [0,1].

In view of (4.4) and (4.5), we have $\phi_{n+1}^*(x) \to \overline{\phi}(x)$ on $\mathcal{H}^+$ by definition of $\overline{\psi}(x)$. On the other hand, $\phi_{n+1}^*(x) = \mathcal{L}\phi_n^*(x)$ on $\mathcal{H}^+$, so $\phi_{n+1}^*(x) \to \mathcal{L}\overline{\phi}(x)$. Thus, $\overline{\phi}(x) = \mathcal{L}\overline{\phi}(x)$ on $\mathcal{H}^+$. On $\mathcal{H}^-$, we have $\psi_{n+1}^* = \mathcal{L}\phi_n^*(x) \to \overline{\phi}(x)$ by definition of $\overline{\phi}(x)$. But, $\phi_{n+1}^*(x) = \mathcal{L}\psi_n^*(x)$ on $\mathcal{H}^-$, so $\phi_{n+1}^*(x) \to \overline{\psi}(x)$ on $\mathcal{H}^-$, $\mathcal{L}\overline{\psi}(x) = \overline{\phi}(x)$.

In a similar way, we have

$$\overline{\phi}(x) = \begin{cases} \mathcal{L}\overline{\phi}(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\overline{\psi}(x), & x \in \mathcal{H}^-, \end{cases}$$

and

$$\overline{\psi}(x) = \begin{cases} \mathcal{L}\overline{\psi}(x), & x \in \mathcal{H}^+, \\ \mathcal{L}\overline{\phi}(x), & x \in \mathcal{H}^- \end{cases}$$

and $\overline{\phi}(x)$ and $\overline{\psi}(x)$ are fixed points of $\mathcal{L}^2$.

Now, we consider the convex region in $C[0,1]$ defined by

$$\mathcal{D} = \{\eta(x) | \overline{\psi}(x) \le \eta(x) \le \overline{\phi}(x), \quad x \in [0,1]\}.$$

This is invariant under $\mathcal{L}$ since

$$\mathcal{L}\eta(x) = \mathcal{L}_1\eta(x) - \mathcal{L}_2\eta(x) \le \mathcal{L}_1\overline{\psi}(x) - \mathcal{L}_2\overline{\phi}(x) = \overline{\psi}(x)$$

by (4.7). Similarly,

$$\mathcal{L}\eta(x) = \mathcal{L}_1\eta(x) - \mathcal{L}_2\eta(x) \ge \mathcal{L}_1\overline{\phi}(x) - \mathcal{L}_2\overline{\psi}(x) = \widetilde{\phi}(x).$$

Since $G(x,s)$ is continuous on $(x,s) \in [0,1] \times [0,1]$, $h(x) \in L^1(0,1)$ implies that $\{(\mathcal{L}\eta(x)) : \eta(x) \in \mathcal{D}\}$ is uniformly equicontinuous, which follows from the Arzelà-Ascoli theorem that $\mathcal{L}$ restricted to $\mathcal{D}$ is a compact operator.

We can know, by Lemma 2.7, that $\mathcal{L}$ has a fixed point. Hence, NBVP (1.1) has at least one positive solution. Consequently, the proof is complete. □

5 Supplementary theorem

In this section, we shall construct examples where the previous main results. To this end, we make the following assumption:

(A4) There exists a positive real number $\mathcal{X}$ such that

$$\int_0^x h^+(s)\mathrm{d}s \ge (1+\mathcal{X})\int_0^x h^-(s)\mathrm{d}s \quad \text{for } x \in [0,1].$$

Clearly, we can build, from condition (A4), that weight function $h(x)$ is sufficiently positive near 0. In the following, we only make an example corresponding to Theorem 3.2, and the examples corresponding to Theorem 4.2 are similar.

Theorem 5.1. *Assume that* (A1), (A2), *and* (A4) *hold. If* $f(0) > 0$. *Then, there is* $m_0 \in (0,+\infty) \cup \{+\infty\}$ *satisfying*

$$f(m) \le f(0)(1+\mathcal{X}), \quad m \in [0, m_0).$$

Moreover, there exists $\lambda_0 > 0$ *such that NBVP* (1.1) *has at least one positive solution for* $0 < \lambda \le \lambda_0$, *where*

$$\lambda_0 := \frac{m}{\mathcal{M}f(0)\left[\int_0^1 h(s)\mathrm{d}s + \mathcal{X}\int_0^1 h^+(s)\mathrm{d}s\right]}.$$

Proof. We shall construct $\phi_0(x)$ and $\psi_0(x)$ such that the conditions (h_1)–(h_3) in Lemma 3.1 are satisfied.

Assume

$$\phi_0(x) = \begin{cases} 0, & x \in \mathcal{H}^+, \\ m, & x \in \mathcal{H}^- \end{cases}$$

and

$$\psi_0(x) = \begin{cases} m, & x \in \mathcal{H}^+, \\ 0, & x \in \mathcal{H}^-, \end{cases}$$

where $m \in [0, m_0)$ is a constant. Then, condition (h_1) is satisfied. Now, the condition (h_2) is

$$\mathcal{L}\psi_0 = \mathcal{L}_1\psi_0 - \mathcal{L}_2\psi_0 = \mathcal{L}_1(m) - \mathcal{L}_2(0) \le m \quad \text{for } x \in [0,1]; \tag{5.1}$$

in addition, condition (h_3) is

$$\mathcal{L}\phi_0 = \mathcal{L}_1\phi_0 - \mathcal{L}_2\phi_0 = \mathcal{L}_1(0) - \mathcal{L}_2(m) \geq 0 \quad \text{for } x \in [0, 1]. \tag{5.2}$$

Let

$$\xi^{\pm}(x) \equiv \int_1^1 G(x, s)h^{\pm}(s)\mathrm{d}s, \tag{5.3}$$

then conditions (5.1) and (5.2), respectively, become

$$\lambda[\xi^+(x)f(m) - \xi^-(x)f(0)] \leq m \tag{5.4}$$

and

$$\lambda[\xi^+(x)f(0) - \xi^-(x)f(m)] \geq 0 \tag{5.5}$$

with $f(0) > 0$. We first consider (5.5). Define

$$\xi(x) = \xi^+(x) - (1 + \mathcal{X})\xi^-(x). \tag{5.6}$$

In view of (5.3) and (5.6) that $\xi(x)$ is a solution of the following boundary value problem having the form

$$\begin{cases} \xi^{(4)}(x) + (k_1 + k_2)\xi''(x) + k_1k_2\xi(x) - [h^+(x) - (1 + \mathcal{X})h^-(x)] = 0, & x \in [0, 1], \\ \xi'(0) = \xi'(1) = \xi'''(0) = \xi'''(1) = 0. \end{cases}$$

It follows that

$$\xi^{(4)}(x) + (k_1 + k_2)\xi''(x) + k_1k_2\xi(x) = h^+(x) - (1 + \mathcal{X})h^-(x)$$

and

$$\xi(x) = \int_0^1 G(x, s)[h^+(s) - (1 + \mathcal{X})h^-(s)]\mathrm{d}s.$$

Clearly, we can infer, from condition (A4), that $\xi(x)$ is non-negative, to be specific, we have

$$\xi^+(x) \geq (1 + \mathcal{X})\xi^-(x) \quad \text{for } x \in [0, 1].$$

Therefore, (5.5) is satisfied if

$$f(m) \leq (1 + \mathcal{X})f(0).$$

We select such an m and argue that (5.4) can now be satisfied for small λ. To prove this, we will give a proper estimate. According to (5.3), we have

$$\xi^+(x) - \xi^-(x) = \int_0^1 G(x, s)h(s)\mathrm{d}s.$$

Moreover, it follows from (2.3) that

$$\xi^+(x) \leq \xi^-(x) + \mathcal{M}\int_0^1 h(s)\mathrm{d}s;$$

consequently, we have

$$\begin{aligned}
f(m)\xi^+(x) - f(0)\xi^-(x) &\le [f(m) - f(0)]\xi^-(x) + \mathcal{M}f(m)\int_0^1 h(s)ds \\
&\le \mathcal{M}[f(m) - f(0)]\int_0^1 h^-(s)ds + \mathcal{M}f(m)\int_0^1 h(s)ds \\
&\le \mathcal{M}\mathcal{X}f(0)\int_0^1 h^-(s)ds + \mathcal{M}(1+\mathcal{X})f(0)\int_0^1 h(s)ds \\
&= \mathcal{M}f(0)\left[\mathcal{X}\int_0^1 h^-(s)ds + (1+\mathcal{X})\int_0^1 h(s)ds\right] \\
&= \mathcal{M}f(0)\left[(1+\mathcal{X})\int_0^1 h^+(s)ds - \int_0^1 h^-(s)ds\right] \\
&= \mathcal{M}f(0)\left[\int_0^1 h(s)ds + \mathcal{X}\int_0^1 h^+(s)ds\right].
\end{aligned}$$

Therefore, (5.4) is satisfied if

$$\lambda \le \frac{m}{\mathcal{M}f(0)\left[\int_0^1 h(s)ds + \mathcal{X}\int_0^1 h^+(s)ds\right]}.$$

It follows from Lemma 3.1 and Theorem 3.2 that NBVP (1.1) has at least one positive solution for

$$0 < \lambda \le \lambda_0 := \frac{m}{\mathcal{M}f(0)[\int_0^1 h(s)ds + \mathcal{X}\int_0^1 h^+(s)ds]}.$$ □

Example. Consider the following fourth-order Neumann boundary value problem

$$\begin{cases} y^{(4)}(x) + \left(\dfrac{\pi^2}{9} + 1\right)y''(x) + \dfrac{\pi^2}{9}y(x) = \lambda h(x)f(y(x)), & x \in [0,1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0, \end{cases}$$

in this example, we can take $f(m) = f(0)(1+\mathcal{X})$, where $f(y(x)) = e^y$, $m = \ln(1+\mathcal{X})$, and $f(y(x)) = 1 + y^p$, then $m = \mathcal{X}^{\frac{1}{p}}$, $p > 0$. Weight function $h(x)$ is as follows:

$$h(x) = \begin{cases} \alpha, & x \in [0, 1/3]; \\ \text{linear}, & x \in [1/3, 2/3]; \\ -\beta, & x \in [1/3, 1] \end{cases}$$

with $0 < \beta < \alpha$. We can easily verify that $1 + \mathcal{X} = \alpha/\beta$ will work for condition (A4). In particular, if $h(x) = -x + 9/10$, $x \in [0,1]$, then, we can also easily verify that for any $\mathcal{X} \in (0, 80]$ will work for (A4).

Remark 5.2. We have given a sufficient condition for small parameter λ of the NBVP (1.1), which involves $h^+(x)$ being sufficiently positive, that is, the condition (A4). Some such hypothesis is necessary, but we believe that λ small is a correct condition.

6 Conclusion

At the end of the article, we summarize the main work of this article. In this article, we focus on the existence of positive solutions for nonlinear fourth-order equation with Neumann boundary conditions

$$\begin{cases} y^{(4)}(x) + (k_1 + k_2)y''(x) + k_1k_2y(x) = \lambda h(x)f(y(x)), & x \in [0, 1], \\ y'(0) = y'(1) = y'''(0) = y'''(1) = 0. \end{cases}$$

We first discuss the sign properties of Green's function for the elastic beam boundary value problem, and then we establish some new results of existence of positive solutions to this problem if the nonlinearity f is monotone on $\mathbb{R}^+$. The technique for dealing with this article relies on a monotone iteration technique and Schauder's fixed point theorem. Finally, an example is presented to illustrate the application of our main results.

Note that the results of Theorems 3.2, 4.2, and 5.1 are satisfied for the cases of $0 < k_1 < k_2 \le \pi^2/4$. In particular, the conclusion of Theorems 3.2, 4.2, and 5.1 for $k_1 < 0 < k_2 \le \pi^2/4$ also apply when NBVP (1.1) is converted into the problem (2.2) as Remark 2.3. On account of the proof is similar to Theorems 3.2, 4.2, and 5.1, so we omit it here.

In particular, the Euler-Bernoulli beam equation (1.1) with boundary condition $y(0) = y(1) = y'(0) = y'(1) = 0$, Lidstone boundary condition $y(0) = y(1) = y''(0) = y''(1) = 0$, and boundary condition $y(0) = y'(1) = y''(0) = y'''(1) = 0$, respectively. The conclusions drawn in this article still apply to these three types of boundary value problems, and we will not dwell on them here, but leave the rest of the details to the reader.

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