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Research Article

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Multiplicity and minimality of periodic solutions to fourth-order super-quadratic difference systems

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Abstract: In this article, we use the Nehari manifold method to study a class of fourth-order even difference systems. First, we show that there exist multiple periodic solutions to the non-autonomous system. Second, we obtain sufficient conditions to guarantee the existence of solutions with prescribed minimal periods to the autonomous system. Our results generalize the results in reference.

Keywords: difference equations, periodic solutions, critical point theory, multiplicity

MSC 2020: 39A23, 49J35

1 Introduction

On one hand, many differential equations cannot be solved directly. To simulate the solutions of differential equations numerically, we need to discretize differential equations, which are difference equations. On the other hand, difference equations have been widely used as mathematical models to describe real-life situations. Therefore, it is worthwhile to explore this topic. As we all know, periodic phenomena often appear in nature. Thus, the existence of periodic solutions is an important subject in discrete dynamical systems. Many mathematicians used all kinds of fixing point theorems to study the existence of periodic solutions to discrete systems. In 2003, Guo and Yu established the variational structure of the difference equations [1–3]. Since then, discrete dynamical systems have attracted the attention of many researchers, for example, the boundary value problems [4], periodic solutions and subharmonic solutions [1–3,5], positive solutions [6], solutions with a minimal period [7], homoclinic orbits [8], and heteroclinic orbits [9]. Focusing on the fourth-order difference equation, we refer to [10,11] on periodic solutions and [12,13] on boundary value problems.

In this article, we are interested in the existence of periodic solutions with prescribed minimal periods to difference equations by variational methods. To introduce the results, we give some notations. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ be the set of all natural numbers, integers, and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $Z[a] = \{a, a+1, \ldots\}$ and $Z[a, b] = \{a, a+1, \ldots, b\}$ when $a \le b$.

Focusing on the periodic solutions with a minimal period, Yu et al. [7] studied the existence of a solution to the following equation:

$$\triangle^2 x_{n-1} + A\sin x_n = f(n),\ n \in \mathbb{Z},\ x_n \in \mathbb{R}. \tag{1.1}$$

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Using the critical point theory, the authors obtained sufficient conditions to ensure a subharmonic solution with a minimum period.

In 2006, Guo and Yu [5] studied the second-order difference equation

$$\triangle^2 x_{n-1} + f(x_n) = 0, x_n \in \mathbb{R}. \tag{1.2}$$

When f is super linear, using the Z_P geometrical index theory, the authors proved that (1.2) possessed at least $p - 1$ distinct Z_p-orbits with the minimal period p. They also gave an example to illustrate that $p - 1$ is the best lower bound.

In 2008, using the dual least action principle and perturbation method, Long [14] studied the existence of a subharmonic solution with a minimal period to the second-order sub-quadratic discrete Hamiltonian system. In 2010, using the minimax theory and pseudo Z_p index theory, Long [15] obtained sufficient conditions to guarantee the existence and multiplicity of periodic solutions with minimal periods to non-convex super-quadratic discrete Hamiltonian systems. For more results from this direction, we refer to [16,17].

In 2014, using the Nehari manifold method, Xiao [18] studied the existence of periodic solutions with prescribed minimal periods to second-order difference equations when the nonlinear term is both sub-quadratic and super-quadratic.

In 2018, Liu et al. [19] considered a fourth-order nonlinear difference equation

$$\triangle^4 u_{n-2} = f(n, u_n), n \in \mathbb{Z}, u_n \in \mathbb{R}. \tag{1.3}$$

Using the linking theorem and a variational technique, the authors obtained some criteria for the existence of periodic solutions with minimal periods to (1.3). For more results in this direction, we refer to [20,21].

In 2020, Yang [22] used the critical point theory to study the existence of at least a periodic solution with a minimal period for fourth-order nonlinear difference equations.

Note that most of the results are on low-dimensional difference equations. In this article, we focus on the existence of periodic solutions to difference systems. The rest of the article is divided into two parts. In Section 2, we study the multiplicity of periodic solutions to the fourth-order non-autonomous difference system. In Section 3, we study the existence of periodic solutions with prescribed minimal periods to the fourth-order difference system.

2 Non-autonomous system

2.1 Variational structure

In this section, we consider the non-autonomous fourth-order difference system

$$\Delta^4 x_{n-2} + f(n, x_n) = 0, \quad n \in \mathbb{Z}, x_n \in \mathbb{R}^m, m \in \mathbb{N}. \tag{2.1}$$

Assume that:

(F1) there exists an even function $F \in C^2(\mathbb{Z} \times \mathbb{R}^m, \mathbb{R})$ such that $f(n, z)$ is the gradient of $F(n, z)$ with respect to z, i.e.,

$$F(-n, -z) = F(n, z), \quad f(n, z) = \nabla_z F(n, z), \quad \forall (t, z) \in \mathbb{Z} \times \mathbb{R}^m;$$

(F2) there exists a positive integer $T > 4$ such that

$$F(n + T, z) = F(n, z), \quad \forall (n, z) \in \mathbb{Z} \times \mathbb{R}^m;$$

(F3) there exists an $\alpha > 1$ such that

$$0 < \alpha(f(t, z), z) \le (f'(t, z)z, z), \quad \forall z \in \mathbb{R}^m \backslash \{0\},$$

where $f'(t, \cdot)$ denotes the Hermite matrix of $F(t, \cdot)$.

Let H be the space of vector sequences $x = (\cdots, x_{-n}, \ldots, x_0, x_1, \ldots, x_n, \ldots) = \{x_n\}_{n=-\infty}^{+\infty}$, i.e.,

$$H = \{x | x = \{x_n\}_{n=-\infty}^{\infty}, \quad x_n = (x_n^1, x_n^2, \ldots, x_n^m)^\tau \in \mathbb{R}^m, n \in \mathbb{Z}\}.$$

For any $x, y \in H$, $a, b \in \mathbb{R}$, define

$$ax + by := \{ax_n + by_n\}_{n=-\infty}^{\infty}.$$

Then, H is a vector space. Define the subspace H_T of H as follows:

$$H_T = \{x \in H | x_{n+T} = x_n, \quad \forall n \in \mathbb{Z}\}.$$

Define the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ on H_T by

$$\|x\|^2 = \sum_{n=1}^{T} |x_n|^2, \quad \langle x, y\rangle = \sum_{n=1}^{T} (x_n, y_n), \quad \forall x, y \in H_T,$$

where $|\cdot|$ and $(\cdot,\cdot)$ denote the usual norm and inner product in $\mathbb{R}^m$. Define L^s norm on H_T by

$$\|x\|_s = \left(\sum_{n=1}^{T} |x|^s\right)^{\frac{1}{s}}.$$

Then, L^s norm is equivalent to $\|\cdot\|$, and there exists $C_s > 0$ such that

$$\frac{1}{C_s}\|x\|_s \le \|x\| \le C_s\|x\|_s, \quad \forall x \in H_T. \tag{2.2}$$

Define a linear map $\phi : H_T \to \mathbb{R}^{mT}$ by

$$\psi x = (x_1^\tau, x_2^\tau, \ldots, x_T^\tau)^\tau.$$

Then, ϕ is a linear homeomorphism with $\|x\| = \|\psi x\|$ and $(H_T, \langle\cdot,\cdot\rangle)$ is a Hilbert space, which can be identified with $\mathbb{R}^{mT}$.

The functional J, defined on the Hilbert space H_T, is

$$J(x) = \frac{1}{2}\sum_{n=1}^{T} |\Delta^2 x_{n-1}|^2 - \sum_{n=1}^{T} F(n, x_n).$$

Arguing similarly as reference [2], we can prove the following result.

Lemma 2.1. *Assume that F satisfies* (F1)–(F3). *The functional J is twice continuously differentiable on H_T. Critical points of J correspond to T periodic solutions of* (2.1).

Due to the identification of H_T with $\mathbb{R}^{mT}$, we write $x \in H_T$ as $x = (x_1^\tau, x_2^\tau, \ldots, x_T^\tau)^\tau$. For convenience, $J(x)$ is written as

$$J(x) = \frac{1}{2}x^\tau D x - \sum_{n=1}^{T} F(n, x_n), \tag{2.3}$$

where

$$D = \begin{pmatrix} 6I & -4I & I & 0 & \ldots & I & -4I \\ -4I & 6I & -4I & I & \ldots & 0 & I \\ I & -4I & 6I & -4I & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ I & 0 & 0 & 0 & \ldots & 6I & -4I \\ -4I & I & 0 & 0 & \ldots & -4I & 6I \end{pmatrix}_{mT\times mT},$$

and I is the $m \times m$ identity matrix.

By direct computation, we have

$$\lambda_k = 16\sin^4\frac{k\pi}{T}, \quad k = 0, 1, \ldots, T-1.$$

0 is an eigenvalue of A of multiplicity m. It has m eigenvectors given by

$$\eta_{0,i} = \frac{1}{\sqrt{T}}(e_1, e_2, \ldots, e_T),\ e_j = (0, \ldots, 0, 1(i\text{-th component}), 0, \ldots, 0),\ j \in Z[1, T],\ i \in Z[1, m].$$

Other eigenvalues of A are positive. When T is odd, for any $k = 1, 2, \ldots, (T-1)/2$, λ_k is the eigenvalue of A with multiplicity $2m$ and $\lambda_k = \lambda_{T-k}$ and the maximal eigenvalue is $16\cos^4(\pi/2T)$. When T is even, for any $k = 1, 2, \ldots, T/2-1$, λ_k is the eigenvalue of A with multiplicity $2m$. Denote by $\lfloor\cdot\rfloor$ the upper integral function. For any k, $1 \le k \le \lfloor(T-1)/2\rfloor$, λ_k has eigenvalues as follows:

$$\eta_{k,i}^{(1)} = \frac{1}{\sqrt{T}}(\eta_{k1}^{(l)}, \eta_{k2}^{(l)}, \ldots, \eta_{kT}^{(l)});\quad \eta_{kj}^{(l)} = (0, \ldots, 0, \cos\frac{jk\pi}{T}(i\,\text{th component}), 0, \ldots, 0), \quad j \in Z[1, T], i \in Z[1, m],$$

and

$$\eta_{k,i}^{(2)} = \frac{2}{\sqrt{T}}(\eta_{k1}^{(2)}, \eta_{k2}^{(2)}, \ldots, \eta_{kT}^{(2)});\quad \eta_{kj}^{(2)} = \left(0, \ldots, 0, \sin\frac{jk\pi}{T}(i\,\text{th component}), 0, \ldots, 0\right), \quad j \in Z[1, T],\ i \in Z[1, m].$$

Furthermore, if T is even, $\lambda_{T/2} = 16$ is the maximal eigenvalue of A of multiplicity of m. Its eigenvectors are

$$\eta_{e,i} = \frac{1}{\sqrt{T}}(e_1', e_2', \ldots, e_T'),\ e_j' = (0, \ldots, 0, (-1)^{j-1}(i\,\text{th component}), 0, \ldots, 0), \quad j \in Z[1, T],\ i \in Z[1, m].$$

Define the spaces W_l $(l = 1, 2, 3, 4)$ as follows:

$$W_1 = \operatorname{span}\{\eta_{0,i} | i \in Z[1, m]\},\ W_2 = \operatorname{span}\left\{\eta_{k,i}^{(1)} | k = 1, 2, \ldots, \left\lfloor\frac{T-1}{2}\right\rfloor, \quad i \in Z[1, m]\right\},$$
$$W_3 = \operatorname{span}\left\{\eta_{k,i}^{(2)} | k = 1, 2, \ldots, \left\lfloor\frac{T-1}{2}\right\rfloor, i \in Z[1, m]\right\}, \quad W_4 = \operatorname{span}\{\eta_{e,i} | i \in Z[1, m]\}.$$

Then, $\dim(W_1) = m$, $\dim(W_2) = \dim(W_3) = m\lfloor(T-1)/2\rfloor$ and $\dim(W_4) = m$ if T is even. Also, we have

$$H_T = W_1 \oplus W_2 \oplus W_3 \oplus W_4, \quad \text{when } T \text{ is even};$$
$$H_T = W_1 \oplus W_2 \oplus W_3, \quad \text{when } T \text{ is odd}.$$

Thus, for any $x = (x_1, x_2, \ldots, x_T) \in H_T$, x_n has the form of

$$x_n = a + \sum_{k=1}^{\frac{T}{2}-1}\left(a_k\cos\frac{kn\pi}{T} + b_k\sin\frac{kn\pi}{T}\right) + (-1)^{n-1}b, \quad \text{when } T \text{ is even},$$
$$x_n = a + \sum_{k=1}^{\frac{T-1}{2}}\left(a_k\cos\frac{kn\pi}{T} + b_k\sin\frac{kn\pi}{T}\right), \quad \text{when } T \text{ is odd},$$

where $a, b, a_k, b_k (k = 1, 2, \ldots, \lfloor(T-1)/2\rfloor)$ are all constant vectors of $\mathbb{R}^m$.

Now, we define a subspace of H_T as follows:

$$E_T = \{x \in H_T \quad | x_{-n} = -x_n\}.$$

Obviously, $E_T = W_3$. Thus, $\dim(E_T) = m\lfloor(T-1)/2\rfloor$. For any $x \in E_T$, no matter whether T is even or odd, x_n has the Fourier expansion as follows:

$$x_n = \sum_{k=1}^{\left\lfloor\frac{T-1}{2}\right\rfloor} b_k\sin\frac{kn\pi}{T}.$$

Lemma 2.2. [18] *The critical points of J restricted to E_T are also critical points of J on the whole space, which corresponds to periodic solutions of system* (2.1).

Next, we will study the functional J restricted to E_T. For simplicity, we denote $J|_{E_T}$ by J again. Then, the matrix A restricted to E_T has only $m\lfloor(T-1)/2\rfloor$ eigenvalues. Denote the minimal eigenvalue and the maximum eigenvalue by $\lambda_{\min}$ and $\lambda_{\max}$. Then,

$$\lambda_{\min} = 16\sin^4\frac{\pi}{T}, \quad \lambda_{\max} = \lambda_{\lfloor(T-1)/2\rfloor}. \tag{2.4}$$

A sequence $\{x^k\}$ is called a (PS) sequence for J if $\{J(x^k)\}$ is bounded and $J'(x^k) \to 0$ as $k \to +\infty$. J is said to satisfy the (PS) condition if every (PS) sequence contains a convergent subsequence.

Suppose that $\mathcal{H}$ is a real Banach space and M is a closed symmetric C^1–submanifold of $\mathcal{H}$ with $0 \notin M$. Suppose that $\phi \in C^1(M, \mathbb{R})$.

Lemma 2.3. [23] *Suppose that ϕ is even and bounded below. Define*

$$c_j := \inf_{A\in\Gamma_j}\sup_{x\in A}\phi(x), \quad j = 1, 2, \ldots,$$

where $\Gamma_j := \{A \subset M : A = -A, A \subset \mathcal{H}\backslash\{0\}, A$ is compact, and $\gamma(A) \geq j\}$. Here, $\gamma(A)$ is the Lusternik-Schnirelmann category of A. If $\Gamma_k \neq \varnothing$ for some $k \geq 1$ and if ϕ satisfies the (PS)$_c$ *for all $c = c_j$, $j = 1, 2, \ldots, k$, then ϕ has at least k distinct pairs of critical points.*

Lemma 2.4. [24] *If ϕ is bounded below and the* (*PS*) *condition is satisfied, then $c := \inf_M\phi$ is attained and is a critical value of ϕ.*

2.2 Main results and proofs

In this subsection, we consider the existence of multiple periodic solutions of system (2.1), where F satisfies (F1), (F2), and (F3).

Lemma 2.5. [25] *If F satisfies* (*F1*), (*F2*), *and* (*F3*), *then $0 < (1+\alpha)F(t,x) \leq (f(t,x), x)$ for all $x \in \mathbb{R}^m\backslash\{0\}$. Also,*

$$F(t,x) \leq \overline{M}|x|^{\alpha+1} \quad \text{when } |x| \leq 1 \text{ and } F(t,x) \geq \underline{M}|x|^{\alpha+1} \text{ when } |x| \geq 1,$$

where $\overline{M} = \max_{\{t|0\leq t\leq T\}}\max_{|x|=1}F(t,x)$ and $\underline{M} = \min_{\{t|0\leq t\leq T\}}\min_{|x|=1}F(t,x)$.

Let $h(x) = \langle J'(x), x\rangle$. Define a Nehari manifold $\mathcal{M}$ on E_T as follows:

$$\mathcal{M} = \{x \in E_T\backslash\{0\}|h(x) = 0\}.$$

Obviously, $0 \notin \mathcal{M}$.

Arguing similarly as [26], one can prove the following lemma.

Lemma 2.6. *$\mathcal{M}$ is C^1-manifold with dimension $m\lfloor(T-1)/2\rfloor - 1$. If x_0 is a critical point of J restricted on $\mathcal{M}$, then x_0 is also a critical point of J restricted on E_T.*

By Lemmas 2.2 and 2.6, the critical points of J restricted on $\mathcal{M}$ correspond to periodic solutions of (2.1).

Lemma 2.7. *Given $x \in E_T\backslash\{0\}$, there exists a unique $t_x > 0$ such that $t_x x \in \mathcal{M}$.*

Proof. Given $x \in E_T\backslash\{0\}$, define $\varphi_x(t) := J(tx)$ for $t \in [0, +\infty)$. Obviously, $\varphi_x \in C^2$. It is easy to check that $\varphi_x'(t) = 0$ if and only if $tx \in \mathcal{M}$.

If $0 < t < 1/\|x\|$, then $|tx_n| < 1$. It follows from Lemma 2.5 that $|F(n, tx_n)| \le \overline{M}|tx_n|^{\alpha+1}$. Thus,

$$\varphi_x(t) = \frac{1}{2}\langle tx, A(tx)\rangle - \sum_{n=1}^{T} F(n, tx_n) \ge \frac{t^2}{2}\lambda_{\min}\|x\|^2 - t^{1+\alpha}\overline{M}C_{\alpha+1}^{\alpha+1}\|x\|^{\alpha+1},$$

where $C_{\alpha+1}$ is the constant defined by (2.2) by setting $s = \alpha + 1$. Since $\alpha > 1$, there exists a $t_1 > 0$ depending only on $\|x\|$ such that

$$\varphi_x(t) \ge \frac{t^2}{4}\lambda_{\min}\|x\|^2, \quad \forall t \in (0, t_1].$$

Denote by $B_0(x) = \{n \in Z[1, T] | x_n \ne 0\}$. Obviously, $B_0(x) \ne \varnothing$. If $t > 1/\min\{|x_n| | n \in B_0(x)\}$, then $t|x_n| > 1$ for all $n \in B_0(x)$. It follows from Lemma 2.5 again that

$$\begin{aligned}\varphi_x(t) &= \frac{1}{2} < tx, A(tx) > - \sum_{n=1}^{T} F(n, tx_n)\\ &\le \frac{t^2}{2}\lambda_{\max}\|x\|^2 - t^{1+\alpha}\underline{M}\sum_{n\in B_0(x)} |x_n|^{\alpha+1}\\ &\le \frac{t^2}{2}\lambda_{\max}\|x\|^2 - t^{1+\alpha}\underline{M}C_{\alpha+1}^{-(\alpha+1)}\|x\|^{\alpha+1}.\end{aligned}$$

Since $\alpha > 1$, $\varphi_x(t) < 0$ when $t > 1$ is large enough. Thus, Rolle's Mean Value Theorem implies that there exists a $t_x > 0$ such that

$$\varphi_x'(t_x) = 0. \tag{2.5}$$

Claim: There exists a unique $t_x > 0$ satisfying (2.5).

Suppose, to the opposite, that there exist $0 < t_1 < t_2$ satisfying (2.5). By a direct computation, we have

$$\varphi_x'(t) = t\langle x, Ax\rangle - \sum_{n=1}^{T}(f(n, tx_n), x_n), \quad \varphi_x''(t) = \langle x, Ax\rangle - \sum_{n=1}^{T}(f'(n, tx_n)x_n, x_n).$$

For $i = 1, 2$, since $\varphi_x'(t_i) = 0$, then $t_i\langle x, Ax\rangle = \sum_{n=1}^{T}(f(n, t_ix_n), x_n)$. It follows from (F3) that

$$\varphi_x''(t_i) = \frac{1}{t_i^2}\sum_{n=1}^{T}[(f(n, t_ix_n), t_ix_n) - (f'(n, t_ix_n)t_ix_n, t_ix_n)] < 0, \quad i = 1, 2. \tag{2.6}$$

Thus, there exists $t_3 \in (t_1, t_2)$ satisfying $\varphi_x(t_3) = \min_{t_1<t_3<t_2}\varphi_x(t)$. Consequently, $\varphi_x'(t_3) = 0$ and $\varphi_x''(t_3) \ge 0$. However, by a similar argument as (2.6), $\varphi_x''(t_3) < 0$, which is a contradiction. Thus, t_x is unique. □

Lemma 2.8. *t_x is continuous in x.*

Proof. Suppose $x_n \to x_0 \ne 0$. Denote t_{x_n} and t_{x_0} by constants such that

$$J\left(t_{x_n}x_n\right) = \sup_{t\in\mathbb{R}} J(tx_n), J\left(t_{x_0}x_0\right) = \sup_{t\in\mathbb{R}} J(tx_0).$$

It suffices to show that $t_{x_n} \to t_{x_0}$ after passing to a subsequence.

Suppose to the opposite that $t_{x_n} \to t_0 \ne t_{x_0}$. Choose $\varepsilon \in (0, [J(t_{x_0}x_0) - J(t_0x_0)]/3)$. The continuity of J implies that, for sufficient large n,

$$|J\left(t_{x_n}x_n\right) - J(t_0x_0)| < \varepsilon, \quad |J\left(t_{x_0}x_0\right) - J\left(t_{x_0}x_n\right)| < \varepsilon.$$

Thus,

$$J\left(t_{x_n}x_n\right) - J\left(t_{x_0}x_n\right) = J\left(t_{x_n}x_n\right) - J(t_0x_0) + J(t_0x_0) - J\left(t_{x_0}x_0\right) + J\left(t_{x_0}x_0\right) - J\left(t_{x_0}x_n\right) < 0.$$

It follows that $\sup_{t\in\mathbb{R}}J(tx_n) = J(t_{x_n}x_n) < J(t_{x_0}x_n)$, which is a contradiction. Thus, t_x is continuous in x. □

Since $\varphi_x'(t_x) = 0$ and $\varphi_x''(t_x) < 0$, then $\varphi_x(t_x) = \max_{t\in(0,\infty)}\varphi_x(t)$. Hence, $J(tx)$ restricted on $(0, \infty)$ attains its maximum at t_x.

Lemma 2.9. *J satisfies the (PS) condition on $\mathcal{M}$.*

Proof. Assume that $\{x^k\} \subset \mathcal{M}$ is a (PS) sequence for J. Then, there exists $M_1 \geq 0$ such that $|J(x^k)| \leq M_1$ for all $m \in \mathbb{N}$ and $J'(x^k) \to 0$ as $k \to \infty$. Set

$$B_1(x^k) = \{n \in Z[1, T]||x_n^k|\leq 1\}, \quad B_2(x^k) = \{n \in Z[1, T]||x_n^k| > 1\}.$$

Since F is continuous, there exists $M_2 > 0$ such that

$$|\underline{M}|x|^{1+\alpha} - F(n, x)| \leq M_2, \quad \forall n \in Z[1, T], x \in \mathbb{R}^m \text{ with } |x| \leq 1.$$

Then,

$$\begin{aligned}
-M_1 \leq J(x^k) &= \frac{1}{2}\langle x^k, Ax^k\rangle - \sum_{n=1}^{T} F(n, x_n^k) \\
&\leq \frac{1}{2}\lambda_{\max}\|x^k\|^2 - \sum_{n\in B_1(x^k)} F(n, x_n^k) - \underline{M} \sum_{n\in B_2(x^k)} |x_n^k|^{1+\alpha} \\
&\leq \frac{1}{2}\lambda_{\max}\|x^k\|^2 + \sum_{n\in B_1(x^k)} [\underline{M}|x_n^k|^{1+\alpha} - F(n, x_n^k)] - \underline{M}C_{1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha} \\
&\leq \frac{1}{2}\lambda_{\max}\|x^k\|^2 + TM_2 - \underline{M}C_{1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha}.
\end{aligned}$$

Thus, $\underline{M}C_{1,1+\alpha}^{-(1+\alpha)}\|x^k\|^{1+\alpha} - 1/2\lambda_{\max}\|x^k\|^2 \leq M_1 + TM_2$. Since $\alpha > 1$, $\{\|x^k\|\}$ is bounded. Since E_T is a finite dimensional space, there exists a convergent subsequence of $\{x^k\}$. □

Denote S^1 the unit sphere of E_T. Define a new map

$$g : S^1 \to \mathcal{M} \quad x \mapsto t_x x.$$

It follows from Lemma 2.7 that g is a bijection whose inverse g^{-1} is given by $g^{-1}(x) = x/\|x\|$. According to Lemma 2.8, g is continuous. Thus, g is the homomorphism between S^1 and $\mathcal{M}$.

Lemma 2.10. $M_1 = \inf_{x\in\mathcal{M}}J(x) > 0$.

Proof. For any $x \in \mathcal{M}$, since $J(x) = \sup_{t\in(0,+\infty)}J(tx) = \sup_{t\in(0,+\infty)}J(tx/\|x\|)$, it follows that

$$\inf_{x\in\mathcal{M}} J(x) = \inf_{x\in\mathcal{M}}\sup_{t\in(0,+\infty)} J(tx) = \inf_{x\in S^1}\sup_{t\in(0,+\infty)} J(tx).$$

To prove that $M_1 > 0$, one only needs to show that $\inf_{x\in S^1}\sup_{t\in(0,+\infty)}J(tx) > 0$.

Arguing similarly as Lemma 2.7, there exists $t_4 > 0$, which is independent of x, such that

$$\varphi_x(t) \geq \lambda_{\min}t^2/4, \quad \forall 0 < t \leq t_4, \quad \forall x \in S^1.$$

Setting $t = t_4/2$, one obtains

$$J\left(\frac{t_4}{2}x\right) = \varphi_x\left(\frac{t_4}{2}\right) \geq \frac{\lambda_{\min}t_4^2}{16} > 0, \quad \forall x \in S^1.$$

Thus,

$$M_1 = \inf_{x\in S^1}\sup_{t\in(0,\infty)} J(tx) \geq \frac{\lambda_{\min}t_4^2}{16} > 0. \qquad \square$$

For the non-autonomous system, we have the following two results.

Theorem 2.11. *Suppose that F satisfies* (F1), (F2), *and* (F3). *Then system* (2.1) *has at least* $m[(T-1)/2]-1$ *distinct pairs of different T-periodic solutions.*

Proof. Because of (F2), $\mathcal{M}$ is a closed symmetric manifold and $0 \notin \mathcal{M}$. It follows from Lemma 2.6 that $\mathcal{M}$ is a C^1 manifold with dimension $m\lfloor(T-1)/2\rfloor-1$. By Lemmas 2.9 and 2.10, J is bounded from below and satisfies the (PS) condition. It is easy to check that J is even. Then, Lemma 2.3 implies that J has at least $m\lfloor(T-1)/2\rfloor-1$ distinct pairs of critical points. Thus, (2.1) possesses at least $m\lfloor(T-1)/2\rfloor-1$ distinct pairs of periodic solutions. □

3 Autonomous difference system

In this section, consider the autonomous difference system:

$$\triangle^2 x_{n-1} + f(x_n) = 0, \quad n \in \mathbb{Z}. \tag{3.1}$$

Assume that

(F4) there exists an even function $F \in C^1(\mathbb{R}^m, \mathbb{R})$ such that $f(z)$ is the gradient of $F(z)$, i.e.,

$$F(-z) = F(z), f(z) = \nabla_z F(z), \quad \forall z \in \mathbb{R}^m.$$

Given $T > 4$, the variational functional corresponding to (3.1) defined on H_T is

$$I(x) = \sum_{n=1}^{T}\left[\frac{1}{2}|\triangle x_n|^2 - F(x_n)\right] = \frac{1}{2}\langle Dx, x\rangle - \sum_{n=1}^{T} F(x_n). \tag{3.2}$$

If F satisfies (F4), then I is continuously differentiable. The critical points of I restricted on E_T are also critical points of I on the whole space H_T, which correspond to T periodic solutions to (3.1). Now we restrict the functional I to E_T and find its critical points.

3.1 Global (AR) conditions

Assume that F satisfies

(F5) and that there exists an even function $F \in C^2(\mathbb{R}^m, \mathbb{R})$ such that $f(z)$ is the gradient of $F(z)$, i.e.,

$$F(-z) = F(z), f(z) = \nabla_z F(z), \quad \forall z \in \mathbb{R}^m.$$

(F6) (Global (AR) conditions) and that there exists $\alpha' > 1$ such that

$$0 < \alpha'(f(z), z) \le (f'(z)z, z), \quad \forall z \in \mathbb{R}^m \backslash \{0\}.$$

If F satisfies (F5) and (F6), then I is twice continuously differentiable. We define the Nehari manifold $\overline{\mathcal{M}}$ on E_T as follows:

$$\overline{\mathcal{M}} = \{x \in E_T \backslash \{0\} | \langle I'(x), x\rangle = 0\}.$$

With a similar argument as reference [26], one can prove the following lemma.

Lemma 3.1. $\overline{\mathcal{M}}$ *is C^1-manifold with dimension $m\lfloor(T-1)/2\rfloor-1$. If x_0 is a critical point of I restricted on $\overline{\mathcal{M}}$, then x_0 is also a critical point of I restricted on E_T.*

Arguing similarly as Subsection 2.2, one can check the following facts:

(i) $0 < (1+\alpha)F(x) \le (f(x), x)$ for all $x \in \mathbb{R}^m \backslash \{0\}$. Also,

$$F(x) \le \overline{M}'|x|^{\alpha+1} \quad \text{when } |x| \le 1 \textit{ and } F(x) \ge \underline{M}'|x|^{\alpha+1} \quad \text{when } |x| \ge 1,$$

where $\overline{M}' = \max_{|x|=1} F(x)$ and $\underline{M}' = \min_{|x|=1} F(x)$;

(ii) $\overline{\mathcal{M}}$ is a C^1 manifold;

(iii) critical points of I restricted on $\overline{\mathcal{M}}$ are also critical points of I restricted on E_T;

(iv) for any $x \in E_T \backslash \{0\}$, there exists a unique t_x such that $t_x x \in \overline{\mathcal{M}}$ and $I(t_x x) = \max_{t \in (0,\infty)} I(tx)$;

(v) I restricted on $\overline{\mathcal{M}}$ satisfies the (PS) condition;

(vi) I restricted on $\overline{\mathcal{M}}$ is bounded from below and $M_2 = \inf_{x \in \overline{\mathcal{M}}} I(x) > 0$.

Let us state the main result of this subsection.

Theorem 3.2. *Suppose that F satisfies* (F5) *and* (F6). *Then, for any integer $T > 4$, system* (3.1) *possesses at least a periodic solution with the minimal period T.*

Proof. It follows from (iii) and (vi) that M_2 is a critical value. Denote by $\tilde{x}$ the critical point of I corresponding to M_2. Then, $\tilde{x}$ is a T periodic solution of (3.1). It is easy to check that $\tilde{x}$ is a nonconstant periodic solution.

Claim: $\tilde{x}$ has T as its minimal period.

Suppose, to the opposite, that there exists a positive integer $k \ge 2$ such that $\tilde{x}$ has T/k as its minimal period. Define $\tilde{y} = \{\tilde{y}_l | l \in Z[1, T]\}$ as follows:

$$\tilde{y}_l = \tilde{x}_{\left\lfloor \frac{l-1}{k} \right\rfloor + 1}.$$

Since $\tilde{x} \in \overline{\mathcal{M}}$, then $\tilde{y} \in E_T \backslash \{0\}$. Thus, there exists a $r_{\tilde{y}} > 0$ such that $r_{\tilde{y}}\tilde{y} \in \overline{\mathcal{M}}$. Then,

$$\begin{aligned} I\big(r_{\tilde{y}}\tilde{y}\big) &= \sum_{n=1}^{T} \left[\frac{1}{2} |\triangle r_{\tilde{y}}\tilde{y}_n|^2 - F\big(r_{\tilde{y}}\tilde{y}_n\big) \right] \\ &= \frac{1}{2} \sum_{n=1}^{T/k} |\triangle r_{\tilde{y}}\tilde{x}_n|^2 - \sum_{n=1}^{T} F\big(r_{\tilde{y}}\tilde{x}_n\big) \\ &< \sum_{n=1}^{T} \left[\frac{1}{2} |\triangle r_{\tilde{y}}\tilde{x}_n|^2 - F\big(r_{\tilde{y}}\tilde{x}_n\big) \right] \\ &= I\big(r_{\tilde{y}}\tilde{x}\big) \le I(\tilde{x}) = \inf_{x \in \overline{\mathcal{M}}} I(x). \end{aligned}$$

This contradicts with $r_{\tilde{y}}\tilde{y} \in \overline{\mathcal{M}}$. Hence, $\tilde{x}$ has T as its minimal period. □

3.2 Strictly monotonic case

In this subsection, we study the periodic solutions of (3.1) under the assumption (F4) and the following assumptions:

(F7) $F(x) \ge 0$ for all $x \in \mathbb{R}^m$,

(F8) $f(x) = o(|x|)$ as $x \to 0$ in $\mathbb{R}^m$,

(F9) $F(x)/|x|^2 \to \infty$ as $|x| \to \infty$,

(F10) there exists $\alpha > 2$ and $C > 0$ such that $|f(x)| \le C(1 + |x|^{\alpha-1})$,

(F11) for any $x \in \mathbb{R}^m$ with $|x| = 1$, the map $s \mapsto (f(sx_n), x_n)/s$ is strictly increasing on $(-\infty, 0)$ and on $(0, \infty)$.

According to (2.4),

$$\lambda_{\min}\|x\|^2 \leq \|x\|_0 := \langle Dx, x\rangle \leq \lambda_{\max}\|x\|^2.$$

Thus, $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms. For $s \geq 1$, there exists $\overline{C}_s > 0$ such that

$$\frac{1}{\overline{C}_s}\|x\|_s \leq \|x\|_0 \leq \overline{C}_s\|x\|_s, \quad \forall x \in E_T, \tag{3.3}$$

where $\overline{C}_2 := \max\{1/\lambda_{\min}, \lambda_{\max}\}$. Then, the functional I can be rewritten as follows:

$$I(x) = \frac{1}{2}\|x\|_0^2 - \sum_{n=1}^{P} F(x_n), \tag{3.4}$$

Given $x \in E_T\backslash\{0\}$, define $g_x : \mathbb{R}^* \to \mathbb{R}$ by setting $g_x(s) = I(sx)$. One can easily verify that g_x is continuously differentiable. In particular, if $x \in S^1$, where S^1 denotes the unit sphere of E_T, then $g_x \in C^1(\mathbb{R}^*, \mathbb{R})$.

Lemma 3.3. *Assume that F satisfies* (*F*4) *and* (*F*7)–(*F*11). *Given* $x \in S^1$, *there exists a unique positive constant* s_x *depending on* x *such that*

$$g_x(s_x) = \max_{s\in\mathbb{R}^*} g_x(s), \quad g_x'(s) > 0, \ \forall 0 < s < s_x, \quad g_x'(s) < 0, \ \forall s > s_x. \tag{3.5}$$

Proof. First, we will show that $g_x(s) > 0$ in a small interval. Thanks to (F8), we have

$$|f(sx)| = o(s|x|) \quad \text{for } s \text{ small enough}.$$

Consequently, we have $F(x) = o(s^2|x|^2)$ for s small enough. Substituting the preview inequality into (3.4), there exists $s_1 > 0$ small enough such that

$$g_x(s) \geq \frac{1}{4}s^2, \quad \forall s \leq s_1.$$

Set

$$B_3(x) = \{n \in Z[1, T]||x_n|{\leq}1\}, \quad B_4(x) = \{n \in Z[1, T]||x_n| > 1\}.$$

Thanks to the condition (F9), for any $M_3 > 0$, there exists $R_{M_3} > 0$ such that $F(x) \geq M_3|x|^2$ for all $|x| \geq R_{M_3}$. Consequently, for large enough s, we have

$$g_x(s) \leq \left[\frac{1}{2} - M_3\overline{C}_2^{-2}\right]s^2 - \sum_{n\in B_3(x)}[F(sx_n) - M_3 s^2|x_n|^2]. \tag{3.6}$$

Obviously, $\frac{1}{2} - M_3\overline{C}_2^{-2} < 0$ if $M_3 > \overline{C}^2/2$. For s large enough, one has $|sx_n| \geq R_{M_3}$ for all $x_n \in B_4(x)$ and the right side of the (3.6) is negative. Hence, there exists $s_x \in [0, s]$ such that $g_x(s_x) = \max_{s\in\mathbb{R}^*} g_x(s) > 0$ and $g_x'(s_x) = 0$.

Finally, we will show that there exists a unique s_x such that $g_x(s_x) = \max_{s\in\mathbb{R}^*} g_x(s)$. By calculating the derivative of g_x, one obtains

$$g_x'(s) = s - \sum_{n=1}^{T}(f(sx_n), x_n) = s\left[1 - \sum_{n=1}^{T}\frac{(f(sx_n), x_n)}{s}\right]. \tag{3.7}$$

Thanks to $(F11)$, g_x' has the unique zero, which is s_x, and $g_x'(s) > 0$ for all $s \in (0, s_x)$, $g_x'(s) < 0$ for all $s > s_x$. Then, the proof of Lemma 3.3 is completed. □

Remark 3.4. Take $x \in E_T\backslash\{0\}$. Then, Lemma 3.3 states that there exists a unique $s_x > 0$ such that $g_x(s_x) = \sup_{s\in\mathbb{R}^*} g_x(s) = \sup_{s\in\mathbb{R}^*} I(s\|x\|\cdot x/\|x\|)$ and $g_x'(s) > 0$ for all $s \in (0, s_x)$, $g_x'(s) < 0$ for all $s > s_x$.

Define the Nehari manifold by setting

$$\mathcal{N} = \left\{s_x x | x \in E_T \backslash \{0\}, \quad g_x(s_x) = \max_{s \in \mathbb{R}^*} g_x(s)\right\},$$

or equivalently,

$$\mathcal{N} = \left\{s_x x | x \in S^1, g_x(s_x) = \max_{s \in \mathbb{R}^*} g_x(s)\right\}.$$

The main result in this subsection reads as follows.

Theorem 3.5. *Assume that f satisfies* (F4) *and* (F7)–(F11). *Then, for any given positive constant $T > 4$, system* (3.1) *admits a non-constant T periodic solution with minimal period T.*

To prove our main results, we state a useful result that has been proved in [24] (see Theorem 3.5 for more details).

Lemma 3.6. *Let $\mathbb{F}$ be a Hilbert space and suppose that $\Phi(x) = \frac{1}{2}\|x\|^2 - \phi(x)$, where*

(i) *$\phi'(x) = o(\|x\|)$ as $x \to 0$ in $\mathbb{F}$,*
(ii) *$s \mapsto \phi'(sx)x/s$ is strictly increasing for all $x \neq 0$ and $s > 0$,*
(iii) *$\phi(sx)/s^2 \to \infty$ uniformly for x on weakly compact subsets of* $\mathbb{F}\backslash\{0\}$ as $s \to \infty$,
(iv) *ϕ' is completely continuous.*

Then, equation $\Phi'(x) = 0$ has a ground state solution.

To prove Theorem 3.5, put $\mathbb{F} = E_T$, $\Phi(x) = I(x)$ and $\phi(x) = \sum_{n=1}^{T} F(x_n)$. Let us check that all conditions of Lemma 3.6 hold.

Lemma 3.7. *If F satisfies* (F6) *and* (F8), *then $\phi'(x) = o(\|x\|_0)$ as $x \to 0$ in E_T.*

Proof. Since V satisfies (F8), for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x)| < \varepsilon|x|, \quad \forall |x| < \delta.$$

If $\|x\|_0 < \delta/\overline{C}$, then $|x_n| \le \|x\| \le \overline{C}_2\|x\|_0 < \delta$. Subsequently,

$$\langle\phi'(x), y\rangle_0 = \sum_{n=1}^{T}(f(x_n), y_n) \le \varepsilon \sum_{n=1}^{T}[|x_n|\cdot|y_n|] \le \frac{\varepsilon}{2}\|x\|_2 \cdot \|y\|_2 \le \frac{\varepsilon}{2}\overline{C}_2^2\|x\|_0 \cdot \|y\|_0, \quad \forall y \in E_T. \tag{3.8}$$

This implies that $\|\phi'(x)\| \le \frac{\varepsilon}{2}\overline{C}_2^2\|x\|_0$. Since ε is arbitrary, one has $\phi'(x)| = o(\|x\|_0)$. Thus, (i) of Lemma 3.6 holds. □

Lemma 3.8. *Assume that* (F10) *and* (F11) *are satisfied. Then, $s \mapsto \langle\phi'(sx), x\rangle/s$ is strictly increasing for all $|x| = 1$ and $s > 0$.*

Proof. For all $x \in E_T\backslash\{0\}$ such that $|x| = 1$, set

$$B_3(x) = \{n \in Z[1, T]||x_n| \neq 0\}.$$

For any $s > 0$, one has

$$\frac{\langle\phi'(sx), x\rangle_0}{s} = \sum_{n=1}^{T}\frac{(f(sx_n), x_n)}{s} = \sum_{n \in B_3(x)}\frac{\left(f\left(s|x_n|\frac{x_n}{|x_n|}\right), \frac{x_n}{|x_n|}\right)}{s|x_n|}|x_n|^2.$$

Thanks to assumption (F11), the map $s \mapsto \langle\phi'(sx), x\rangle_0/s$ is strictly increasing on $(0, \infty)$. Hence, (ii) of Lemma 3.6 holds. □

Lemma 3.9. *If F satisfies (F9), then $\phi(sx)/s^2 \to \infty$ uniformly for x on weakly compact subsets of $E_T\backslash\{0\}$ as $s \to \infty$.*

Proof. Let $\mathcal{X} \subset E_T\backslash\{0\}$ be a weakly compact set and let $\{y^k\} \subset \mathcal{X}$. It suffices to show that if $s_k \to \infty$ as $k \to \infty$, then so does a subsequence of $\phi(s_k y^k)/s_k^2$. Passing to a subsequence, $y^k \to y^0$ in E_T. Obviously, $y^0 \neq 0$. Set $B_4(y^0) = \{n \in Z[1, T] \,||y_n^0| \neq 0\}$. Assumption (F9) yields

$$\frac{\phi(s_k y^k)}{s_k^2} = \sum_{n \in B_4(y^0)} \frac{F(s_k y^k)}{|s_k y^k|^2} |y^k|^2 \to \infty \,\text{as}\, k \to \infty.$$

Hence (iii) of Lemma 3.6 holds. □

Now we are in a position to prove Theorem 3.5.

Proof of Theorem 3.5. Since E_T is finite-dimensional, then ϕ' is completely continuous. Thus, (iv) of Lemma 3.6 holds. According to Lemmas 3.7, 3.8, and 3.9, all conditions of Lemma 3.6 are satisfied. Applying Lemma 3.6, one can obtain that I restricted to $\mathcal{N}$ has a ground state solution x^0. As we can see in the proof of Theorem 12 in [24], $I(x^0) > 0$. Consequently, x^0 is not a trivial solution.

Suppose, to the opposite, that there exists a positive integer $k \geq 2$ such that x^0 has T/k as its minimal period. Define $\widetilde{x} = \{\widetilde{x}_l | l \in Z[1, T]\}$ as follows:

$$\widetilde{x}_l = x^0_{\left\lfloor \frac{l-1}{k} \right\rfloor + 1}.$$

Since $x^0 \in \mathcal{N}$, then $\widetilde{x} \in E_T\backslash\{0\}$. Thus, there exists $r_{\widetilde{x}} > 0$ such that $r_{\widetilde{x}}\widetilde{x} \in \mathcal{N}$. Then,

$$\begin{aligned} I\left(r_{\widetilde{x}}\widetilde{x}\right) &= \sum_{n=1}^{T} \left[\frac{1}{2} |\triangle r_{\widetilde{x}}\widetilde{x}_n|^2 - F\left(r_{\widetilde{x}}\widetilde{x}_n\right)\right] \\ &= \frac{1}{2} \sum_{n=1}^{T/k} |\triangle r_{\widetilde{x}} x_n^0|^2 - \sum_{n=1}^{T} F\left(r_{\widetilde{x}} x_n^0\right) \\ &< \sum_{n=1}^{T} \left[\frac{1}{2} |\triangle r_{\widetilde{x}} x_n^0|^2 - F\left(r_{\widetilde{x}} x_n^0\right)\right] \\ &= I\left(r_{\widetilde{x}} x^0\right) \leq I(x^0) = \inf_{x \in \mathcal{N}} I(x). \end{aligned}$$

This contradicts with $r_{\widetilde{x}}\widetilde{x} \in \mathcal{N}$. Hence, x^0 has T as its minimal period. □

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