Research Article

Yuanfei Li* and Xuejiao Chen

Phragmén-Lindelöf alternative results and structural stability for Brinkman fluid in porous media in a semi-infinite cylinder

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Abstract: This article investigates the spatial behavior of the solutions of the Brinkman equations in a semi-infinite cylinder. We no longer require the solutions to satisfy any *a priori* assumptions at infinity. Using the energy estimation method and the differential inequality technology, the differential inequality about the solutions is derived. By solving this differential inequality, it is proved that the solutions grow polynomially or decay exponentially with spatial variables. In the case of decay, the structural stability of Brinkman fluid is also proved.

Keywords: Phragmén-Lindelöf alternative results, Brinkman equations, energy estimation, structural stability

MSC 2020: 35B40, 35K50, 35Q35

1 Introduction

The Brinkman equations are often used to describe flow in a porous medium, which have been discussed in the books of Nield and Bejan [1], Straughan [2], and Hewitt [3]. Many scholars in the literature have paid attention to the spatial attenuation of Brinkman equations on a semi-infinite cylinder. Payne and Song [4] considered the fluid in porous media controlled by Brinkman equations. The Saint-Venant-type decay of the solutions on a semi-infinite cylinder is obtained. For more works on fluid equations, one can see [5–11]. These articles need to assume that the solutions satisfy certain *a priori* assumptions at the infinity of the cylinder.

However, the classical Phragmén-Lindelöf alternative theorem does not need such *a priori* assumption, but proves that the solutions either decay exponentially or increase exponentially with the distance from the finite end of the cylinder. In the past few decades, the Phragmén-Lindelöf alternative research has received a lot of attention (see [12–17]). The above articles mainly focus on linear problems.

In this article, we suppose that a porous medium occupies the interior of a semi-infinite cylindrical pipe of arbitrary cross-section and generators parallel to the x_3 axis. We let R denote the semi-infinite cylinder, i.e.,

$$R = \{\boldsymbol{x} = (x_1, x_2, x_3)|(x_1, x_2) \in D, x_3 \geq 0\},$$

where D is the cross section of the cylinder (Figure 1).

The Brinkman equations we consider in this article can be written as follows:

*** Corresponding author: Yuanfei Li,** School of Data Science, Guangzhou Huashang College, Guangzhou, Guangdong 511300, PR China, e-mail: liyuanfei@gdhsc.edu.cn
Xuejiao Chen: School of Data Science, Guangzhou Huashang College, Guangzhou, Guangdong 511300, PR China

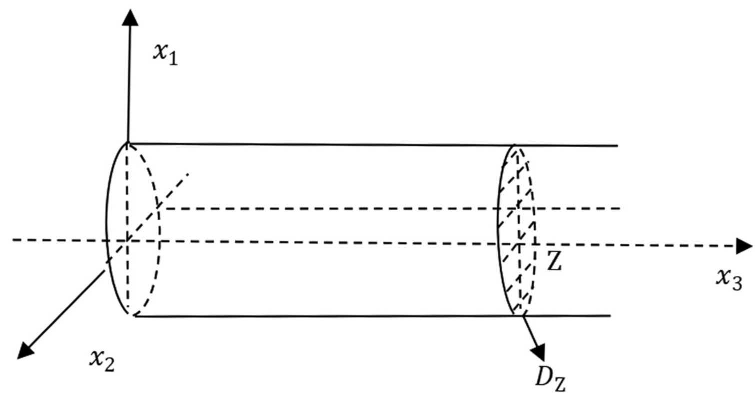

Figure 1: Cylindrical pipe.

$$-\nu\Delta u_i + u_i = -p_i + g_i T, \quad \text{in } R \times (0, t), \tag{1.1}$$

$$u_{i,i} = 0, \quad \text{in } R \times (0, t), \tag{1.2}$$

$$\partial_t T + u_i T_i = \Delta T, \quad \text{in } R \times (0, t), \tag{1.3}$$

where $i = 1, 2, 3$; u_i is the velocity field; T is the temperature of the fluid; p is the pressure; $\nu > 0$ represents the Brinkman coefficient; and $\boldsymbol{g} = (g_1, g_2, g_3)$ is the gravity field, which are prescribed functions. For simplicity, we assume that $\boldsymbol{g}$ satisfies

$$|\boldsymbol{g}| \le 1, \quad |\nabla \boldsymbol{g}| \le 1. \tag{1.4}$$

We use commas for derivation, repeated English subscripts for summation from 1 to 3, and repeated Greek subscripts for summation from 1 to 2, e.g., $u_{i,j}u_{i,j} = \sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}\right)^2$ and $u_{\alpha,\beta}u_{\alpha,\beta} = \sum_{\alpha,\beta=1}^{2}\left(\frac{\partial u_\alpha}{\partial x_\beta}\right)^2$.

Equations (1.1)–(1.3) also satisfy the following initial-boundary conditions:

$$u_i(x_1, x_2, x_3, t) = 0,\ T(x_1, x_2, x_3, t) = 0, \quad \text{on } \partial D \times \{x_3 > 0\} \times (0, t), \tag{1.5}$$

$$u_i(x_1, x_2, 0, t) = f_i(x_1, x_2, t), \quad T(x_1, x_2, 0, t) = h(x_1, x_2, t), \quad \text{on } D \times (0, t), \tag{1.6}$$

$$u_i(x_1, x_2, x_3, 0) = 0, \quad T(x_1, x_2, x_3, 0) = 0, \quad \text{in } R, \tag{1.7}$$

where f_i and h are continuously differentiable and ∂D is the boundary of D.

We also introduce the following notations:

$$R_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, x_3 > z \ge 0\},$$

$$D_z = \{(x_1, x_2, x_3) | (x_1, x_2) \in D, x_3 = z \ge 0\},$$

where z is a running variable along the x_3 axis.

This article studies the Phragmén-Lindelöf-type alternative theorem of equations (1.1)–(1.7) on R. Because our model contains nonlinear terms and pressure terms that are difficult to deal with, how to set the energy function is the key. As far as we know, there are few relevant results on the Phragmén-Lindelöf-type alternative results of such nonlinear equations. The second work of this article is to study the structural stability of Brinkman equations in the case of decay. This type of structural stability is proposed by Hirsch and Smale in their book [18], which mainly studies whether small changes in the coefficients of the equations can cause great changes in the solution. A large number of results have been achieved in the past few decades, see [19–29]. However, these results only considered the bounded region and have not paid enough attention to the structural stability of the solutions of partial differential equations in a semi-infinite cylinder. Therefore, the research of this article is very meaningful and can provide a reference for the alternative research of other types of nonlinear equations.

2 Preliminary

Here are some lemmas that will be often used.

Lemma 2.1. [4] *If $w\,|_{\partial D} = 0$, then*

$$\lambda_1 \int_D w^2 \mathrm{d}A \le \int_D w_\alpha w_\alpha \mathrm{d}A,$$

where λ_1 is the smallest positive eigenvalue of

$$\Delta_2 \varphi + \lambda\varphi = 0, \quad \text{in } D,\ \varphi = 0, \ \text{on } \partial D,$$

where Δ_2 is a two-dimensional Laplace operator.

Lemma 2.2. [30] *There exists a positive constant k_1 depending on region D such that*

$$\left(\int_D w^4 \mathrm{d}A\right)^{\frac{1}{2}} \le k_1\left[\int_D w^2 \mathrm{d}A + \int_D w_\alpha w_\alpha \mathrm{d}A\right].$$

Lemma 2.3. [31] *If g is a continuously differentiable function on D and $\int_D g\mathrm{d}A = 0$, then there exists a vector function $\mathbf{w} = (w_1, w_2)$ such that*

$$w_{\alpha,\alpha} = g, \ \text{in } D,\ w_\alpha = 0, \ \text{on } \partial D,$$

and a positive constant C depending only on the geometry of D such that

$$\int_D w_{\alpha,\beta} w_{\alpha,\beta} \mathrm{d}A \le C \int_D w_{\alpha,\alpha} w_{\alpha,\alpha}\, \mathrm{d}A.$$

To obtain the Phragmén-Lindelöf-type alternative result of the solutions to (1.1)–(1.3), we define

$$F(z,t) = \delta_1 F_1(z,t) + F_2(z,t) + F_3(z,t), \tag{2.1}$$

where δ_1 is a positive constant and $F_i(z,t)$ $(i = 1, 2, 3)$ are defined as follows:

$$F_1(z,t) = -\int_0^t \int_{D_z} e^{-\omega\eta} p u_3 \mathrm{d}A\mathrm{d}\eta + \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,3} u_i \mathrm{d}A\mathrm{d}\eta, \tag{2.2}$$

$$F_2(z,t) = \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,33} u_{i,3} \mathrm{d}A\mathrm{d}\eta + \int_0^t \int_{D_z} e^{-\omega\eta} p_3 u_{3,3} \mathrm{d}A\mathrm{d}\eta - \int_0^t \int_{D_z} e^{-\omega\eta} g_i T u_{i,3} \mathrm{d}A\mathrm{d}\eta, \tag{2.3}$$

$$F_3(z,t) = -\frac{1}{2}\int_0^t \int_{D_z} e^{-\omega\eta} u_3 T^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t \int_{D_z} e^{-\omega\eta} T_3 T \mathrm{d}A\mathrm{d}\eta. \tag{2.4}$$

In (2.2)–(2.4), ω is a positive constant. Let z_0 be a positive constant that satisfies $z > z_0 \ge 0$. Using the divergence theorem, equations (1.1)–(1.3), and the initial-boundary conditions (1.5)–(1.7), we have

$$\begin{aligned} F_1(z,t) - F_1(z_0,t) &= -\int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta} u_i p_i \mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \nu \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}[u_i \Delta u_i + u_{i,j} u_{i,j}] \mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta \\ &= \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}[\nu u_{i,j} u_{i,j} + u_i u_i] \mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta - \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta} g_i u_i T \mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{2.5}$$

Second, differentiating equation (1.1) with respect to x_3, we have

$$-\nu\Delta u_{i,3} + u_{i,3} + p_{i3} - [g_i T]_3 = 0.$$

Therefore, we have

$$\int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}[-\nu\Delta u_{i,3} + u_{i,3} + p_{i3} - [g_i T]_3]u_{i,3}\mathrm{d}A\mathrm{d}\eta = 0. \tag{2.6}$$

By the divergence theorem, we have from (2.6)

$$\begin{aligned}
&\nu \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}u_{i,j3}u_{i,j3}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta \\
&= \nu \int_0^t \int_{D_z} e^{-\omega\eta}u_{i,33}u_{i,3}\mathrm{d}A\mathrm{d}\eta - \nu \int_0^t \int_{D_{z_0}} e^{-\omega\eta}u_{i,33}u_{i,3}\mathrm{d}A\mathrm{d}\eta + \int_0^t \int_{D_z} e^{-\omega\eta}p_3 u_{3,3}\mathrm{d}A\mathrm{d}\eta \\
&\quad - \int_0^t \int_{D_{z_0}} e^{-\omega\eta}p_3 u_{3,3}\mathrm{d}A\mathrm{d}\eta - \int_0^t \int_{D_z} e^{-\omega\eta}g_i T u_{i,3}\mathrm{d}A\mathrm{d}\eta + \int_0^t \int_{D_{z_0}} e^{-\omega\eta}g_i T u_{i,3}\mathrm{d}A\mathrm{d}\eta \\
&\quad + \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}g_i T u_{i,33}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned} \tag{2.7}$$

We have, from (2.7) and (2.3),

$$\begin{aligned}
F_2(z,t) - F_2(z_0,t) &= \nu \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}u_{i,j3}u_{i,j3}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta \\
&\quad - \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}g_i T u_{i,33}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta.
\end{aligned} \tag{2.8}$$

Similar to (2.5), we have

$$\begin{aligned}
F_3(z,t) - F_3(z_0,t) &= -\int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}Tu_i T_i\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}(T_i T)_i\mathrm{d}A\mathrm{d}\eta \\
&= \int_0^t \int_{z_0}^z \int_{D_\xi} e^{-\omega\eta}\left[T_i T_i + \frac{1}{2}\omega T^2\right]\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \frac{1}{2}e^{-\omega t}\int_{z_0}^z \int_{D_\xi} T^2\mathrm{d}A\mathrm{d}\xi.
\end{aligned} \tag{2.9}$$

Combining (2.2), (2.3), and (2.4), we have

$$\begin{aligned}
F(z,t) - F(z_0,t) &= -\delta_1 \int_0^t \int_{D_z} e^{-\omega\eta}pu_3\mathrm{d}A\mathrm{d}\eta + \int_0^t \int_{D_z} e^{-\omega\eta}p_3 u_{3,3}\mathrm{d}A\mathrm{d}\eta + \nu \int_0^t \int_{D_z} e^{-\omega\eta}u_{i,33}u_{i,3}\mathrm{d}A\mathrm{d}\eta \\
&\quad + \delta_1\nu \int_0^t \int_{D_z} e^{-\omega\eta}u_{i,3}u_i\mathrm{d}A\mathrm{d}\eta - \int_0^t \int_{D_z} e^{-\omega\eta}g_i T u_{i,3}\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_0^t \int_{D_z} e^{-\omega\eta}u_3 T^2\mathrm{d}A\mathrm{d}\eta \\
&\quad + \int_0^t \int_{D_z} e^{-\omega\eta}T_3 T\mathrm{d}A\mathrm{d}\eta \\
&\doteq \sum_{i=1}^7 I_i.
\end{aligned} \tag{2.10}$$

Combining (2.5), (2.8), and (2.9), we have

$$\begin{aligned} F(z,t)-F(z_0,t) &= \int_0^t\int_{z_0}^z\int_{D_\xi} e^{-\omega\eta}\left[\delta_1 \nu u_{i,j}u_{i,j}+\delta_1 u_iu_i+\nu u_{i,j3}u_{i,j3}+u_{i,3}u_{i,3}+T_iT_i+\frac{1}{2}\omega T^2\right]\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{z_0}^z\int_{D_\xi}T^2\mathrm{d}A\mathrm{d}\xi-\delta_1\int_0^t\int_{z_0}^z\int_{D_\xi}e^{-\omega\eta}g_iu_iT\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta-\int_0^t\int_{z_0}^z\int_{D_\xi}e^{-\omega\eta}g_iTu_{i,33}\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta. \end{aligned} \tag{2.11}$$

From (2.11), we have

$$\begin{aligned} \frac{\partial}{\partial z}F(z,t) &= \int_0^t\int_{D_z} e^{-\omega\eta}\left[\delta_1 \nu u_{i,j}u_{i,j}+\delta_1 u_iu_i+\nu u_{i,j3}u_{i,j3}+u_{i,3}u_{i,3}+T_iT_i+\frac{1}{2}\omega T^2\right]\mathrm{d}A\mathrm{d}\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{D_z}T^2\mathrm{d}A-\delta_1\int_0^t\int_{D_z}e^{-\omega\eta}g_iu_iT\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{D_z}e^{-\omega\eta}g_iTu_{i,33}\mathrm{d}A\mathrm{d}\eta. \end{aligned} \tag{2.12}$$

We have the following lemma.

Lemma 2.4. *For the function $F(z,t)$ defined in* (2.1) *and $\int_D f_3\mathrm{d}A=0$, the following differential inequality is satisfied*:

$$|F(z,t)|\le b_1\left[\frac{\partial F}{\partial z}(z,t)\right]+b_2\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}}, \tag{2.13}$$

where $b_1=\max\left\{2\sqrt{\frac{C}{\lambda_1}},\frac{1}{\sqrt{\delta_1}}+\frac{\sqrt{\nu}}{\sqrt{2\delta_1}}+\frac{\sqrt{2}}{\sqrt{\omega}},\frac{\sqrt{\nu}}{\sqrt{2}}+\sqrt{\nu\delta_1}+\frac{1}{\sqrt{\omega}}\right\}$ *and* $b_2=\frac{1}{2}\sqrt{k_1}\max\left\{\frac{4}{\omega},1\right\}$.

Proof. Using the Hölder inequality and the Young inequality, we have

$$\left|-\delta_1\int_0^t\int_{D_z}e^{-\omega\eta}g_iu_iT\mathrm{d}A\mathrm{d}\eta\right|\le\frac{1}{2}\delta_1\int_0^t\int_{D_z}e^{-\omega\eta}u_iu_i\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\delta_1\int_0^t\int_{D_z}e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta, \tag{2.14}$$

$$\left|\int_0^t\int_{D_z}e^{-\omega\eta}g_iTu_{i,33}\mathrm{d}A\mathrm{d}\eta\right|\le\frac{1}{2}\nu\int_0^t\int_{D_z}e^{-\omega\eta}u_{i,33}u_{i,33}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2\sqrt{\nu}}\int_0^t\int_{D_z}e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta. \tag{2.15}$$

Inserting (2.14) into (2.15) and choosing $\delta_1+\frac{1}{\sqrt{\nu}}\le\frac{1}{2}\omega$, we have

$$\begin{aligned} \frac{\partial}{\partial z}F(z,t) &\ge \int_0^t\int_{D_z} e^{-\omega\eta}\left[\delta_1 \nu u_{i,j}u_{i,j}+\frac{1}{2}\delta_1 u_iu_i+\nu u_{i,j3}u_{i,j3}+\frac{1}{2}u_{i,3}u_{i,3}+T_iT_i+\frac{1}{4}\omega T^2\right]\mathrm{d}A\mathrm{d}\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{D_z}T^2\mathrm{d}A, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \frac{\partial}{\partial z}F(z,t) &\le \int_0^t\int_{D_z} e^{-\omega\eta}\left[\delta_1 \nu u_{i,j}u_{i,j}+\frac{3}{2}\delta_1 u_iu_i+\nu u_{i,j3}u_{i,j3}+\frac{3}{2}u_{i,3}u_{i,3}+T_iT_i+\frac{3}{4}\omega T^2\right]\mathrm{d}A\mathrm{d}\eta \\ &+\frac{1}{2}e^{-\omega t}\int_{D_z}T^2\mathrm{d}A. \end{aligned} \tag{2.17}$$

Next, we will bound I_i $(i=1,2,\ldots,7)$ by $\frac{\partial}{\partial z}F(z,t)$. To do this, we note that

$$\int_{D_z} u_3 \mathrm{d}A = \int_D u_3 \mathrm{d}A + \int_0^z \int_{D_\xi} u_{3,3} \mathrm{d}A \mathrm{d}\xi = \int_D u_3 \mathrm{d}A - \int_0^z \int_{D_\xi} u_{\alpha,\alpha} \mathrm{d}A \mathrm{d}\xi = \int_D f_3 \mathrm{d}A.$$

This shows that the area mean value of u_3 over each cross section is the same as that over D. Since $\int_D f_3 \mathrm{d}A = 0$, we have $\int_{D_z} u_3 \mathrm{d}A = 0$. According to Lemma 2.3, there exists a vector function $\boldsymbol{v} = (v_1, v_2)$ such that

$$v_{\alpha,\alpha} = u_3, \text{ in } D; \quad v_\alpha = 0, \quad \text{on } \partial D.$$

Therefore, using (1.1), we have

$$\begin{aligned} I_1 &= -\delta_1 \int_0^t \int_{D_z} e^{-\omega\eta} p v_{\alpha,\alpha} \mathrm{d}A \mathrm{d}\eta \\ &= \delta_1 \int_0^t \int_{D_z} e^{-\omega\eta} p_\alpha v_\alpha \mathrm{d}A \mathrm{d}\eta \\ &= \delta_1 \int_0^t \int_{D_z} e^{-\omega\eta} [\nu \Delta u_\alpha + g_\alpha T - u_\alpha] v_\alpha \mathrm{d}A \mathrm{d}\eta \\ &= -\delta_1 \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,\beta} v_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta + \delta_1 \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,33} v_\alpha \mathrm{d}A \mathrm{d}\eta \\ &\quad + \delta_1 \int_0^t \int_{D_z} e^{-\omega\eta} g_\alpha v_\alpha T \mathrm{d}A \mathrm{d}\eta - \delta_1 \int_0^t \int_{D_z} e^{-\omega\eta} u_\alpha v_\alpha \mathrm{d}A \mathrm{d}\eta \\ &\doteq I_{11} + I_{12} + I_{13} + I_{14}. \end{aligned} \tag{2.18}$$

Using the Hölder inequality, the Young inequality, and Lemmas 2.1 and 2.3, we have

$$\begin{aligned} |I_{11}| &\leq \delta_1 \nu \left[\int_0^t \int_{D_z} e^{-\omega\eta} v_{\alpha,\beta} v_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,\beta} u_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta \right]^{\frac{1}{2}} \\ &\leq \delta_1 \nu \sqrt{C} \left[\int_0^t \int_{D_z} e^{-\omega\eta} v_{\alpha,\alpha} v_{\alpha,\alpha} \mathrm{d}A \mathrm{d}\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,\beta} u_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta \right]^{\frac{1}{2}} \\ &\leq \delta_1 \nu \sqrt{C} \left[\int_0^t \int_{D_z} e^{-\omega\eta} u_3^2 \mathrm{d}A \mathrm{d}\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,\beta} u_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta \right]^{\frac{1}{2}} \\ &\leq \delta_1 \nu \sqrt{\frac{C}{\lambda_1}} \left[\int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\alpha} u_{3,\alpha} \mathrm{d}A \mathrm{d}\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,\beta} u_{\alpha,\beta} \mathrm{d}A \mathrm{d}\eta \right]^{\frac{1}{2}} \\ &\leq \frac{1}{2} \delta_1 \nu \sqrt{\frac{C}{\lambda_1}} \int_0^t \int_{D_z} e^{-\omega\eta} u_{i,j} u_{i,j} \mathrm{d}A \mathrm{d}\eta, \end{aligned} \tag{2.19}$$

$$\begin{aligned} |I_{12}| &\leq \delta_1 \nu \left[\int_0^t \int_{D_z} e^{-\omega\eta} v_\alpha v_\alpha \mathrm{d}A \mathrm{d}\eta \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,33} u_{\alpha,33} \mathrm{d}A \mathrm{d}\eta \right]^{\frac{1}{2}} \\ &\leq \frac{1}{2} \frac{\sqrt{\delta_1 C}}{\lambda_1} \left[\delta_1 \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{3,\alpha} u_{3,\alpha} \mathrm{d}A \mathrm{d}\eta + \nu \int_0^t \int_{D_z} e^{-\omega\eta} u_{\alpha,33} u_{\alpha,33} \mathrm{d}A \mathrm{d}\eta \right], \end{aligned} \tag{2.20}$$

$$
\begin{aligned}
|I_{13}| &\leq \delta_1\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}v_\alpha v_\alpha \mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \delta_1\frac{1}{\sqrt{\lambda_1}}\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}v_{\alpha,\beta}v_{\alpha,\beta}\mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \delta_1\sqrt{\frac{C}{\lambda_1}}\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}v_{\alpha,\alpha}v_{\alpha,\alpha}\mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \delta_1\sqrt{\frac{C}{\lambda_1}}\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_3^2\mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \frac{\sqrt{2C\delta_1}}{\sqrt{\lambda_1\omega}}\left[\frac{1}{2}\delta_1\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_3^2\mathrm{d}A\mathrm{d}\eta + \frac{1}{4}\omega\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right],
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
|I_{14}| &\leq \delta_1\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}v_\alpha v_\alpha \mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_\alpha u_\alpha\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \delta_1\sqrt{\frac{C}{\lambda_1}}\left[\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_3^2\mathrm{d}A\mathrm{d}\eta \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_\alpha u_\alpha\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\leq \frac{1}{2}\delta_1\sqrt{\frac{C}{\lambda_1}}\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_i u_i\mathrm{d}A\mathrm{d}\eta.
\end{aligned}
\tag{2.22}
$$

Inserting (2.19)–(2.22) into (2.18) and noting (2.16), we have

$$
|I_1| \leq 2\sqrt{\frac{C}{\lambda_1}}\left[\frac{\partial}{\partial z}F(z,t)\right],
\tag{2.23}
$$

where we have used the condition $\delta_1 + 1 < \frac{1}{2}\omega$.

For I_2, we use (1.1) and (1.2) to have

$$
\begin{aligned}
I_2 &= \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}[\nu\Delta u_3 + g_3T - u_3]u_{3,3}\mathrm{d}A\mathrm{d}\eta\\
&= \nu\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_{3,\alpha\alpha}u_{3,3}\mathrm{d}A\mathrm{d}\eta + \nu\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_{3,33}u_{3,3}\mathrm{d}A\mathrm{d}\eta\\
&\quad + \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}g_3Tu_{3,3}\mathrm{d}A\mathrm{d}\eta - \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_3u_{3,3}\mathrm{d}A\mathrm{d}\eta\\
&= -\nu\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_{3,\alpha}u_{3,\alpha3}\mathrm{d}A\mathrm{d}\eta + \nu\int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_{3,33}u_{3,3}\mathrm{d}A\mathrm{d}\eta\\
&\quad + \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}g_3Tu_{3,3}\mathrm{d}A\mathrm{d}\eta - \int\limits_0^t\int\limits_{D_z} e^{-\omega\eta}u_3u_{3,3}\mathrm{d}A\mathrm{d}\eta\\
&\doteq I_{21} + I_{22} + I_{23} + I_{24}.
\end{aligned}
\tag{2.24}
$$

Using the Hölder inequality and the Young inequality, we have

$$
\begin{aligned}
|I_{21}| &\le \nu\left[\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,\alpha}u_{3,\alpha}\mathrm{d}A\mathrm{d}\eta\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,\alpha3}u_{3,\alpha3}\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\le \frac{1}{2\sqrt{\delta_1}}\left[\delta_1\nu\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,\alpha}u_{3,\alpha}\mathrm{d}A\mathrm{d}\eta+\nu\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,\alpha3}u_{3,\alpha3}\mathrm{d}A\mathrm{d}\eta\right],
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
|I_{22}| &\le \nu\left[\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,33}^2\mathrm{d}A\mathrm{d}\eta\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}}\\
&\le \frac{\sqrt{\nu}}{\sqrt{2\delta_1}}\left[\delta_1\nu\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,33}^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}A\mathrm{d}\eta\right],
\end{aligned}
\tag{2.26}
$$

$$
|I_{23}| \le \frac{\sqrt{2}}{\sqrt{\omega}}\left[\frac{1}{4}\omega\int_0^t\int_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}A\mathrm{d}\eta\right],
\tag{2.27}
$$

$$
|I_{24}| \le \frac{1}{2\sqrt{\delta_1}}\left[\frac{1}{2}\delta_1\int_0^t\int_{D_z} e^{-\omega\eta}u_3^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}A\mathrm{d}\eta\right].
\tag{2.28}
$$

Inserting (2.25)–(2.28) into (2.24), we have

$$
|I_2| \le \left[\frac{1}{\sqrt{\delta_1}}+\frac{\sqrt{\nu}}{\sqrt{2\delta_1}}+\frac{\sqrt{2}}{\sqrt{\omega}}\right]\left[\frac{\partial}{\partial z}F(z,t)\right].
\tag{2.29}
$$

Using the Hölder inequality and the Young inequality, we have

$$
|I_3| \le \frac{\sqrt{\nu}}{\sqrt{2}}\left[\nu\int_0^t\int_{D_z} e^{-\omega\eta}u_{i,33}u_{i,33}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}A\mathrm{d}\eta\right],
\tag{2.30}
$$

$$
|I_4| \le \sqrt{\nu\delta_1}\left[\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\delta_1\int_0^t\int_{D_z} e^{-\omega\eta}u_iu_i\mathrm{d}A\mathrm{d}\eta\right],
\tag{2.31}
$$

$$
|I_5| \le \frac{1}{\sqrt{\omega}}\left[\frac{1}{2}\int_0^t\int_{D_z} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\omega\int_0^t\int_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right],
\tag{2.32}
$$

$$
|I_7| \le \frac{1}{\sqrt{\omega}}\left[\int_0^t\int_{D_z} e^{-\omega\eta}T_3^2\mathrm{d}A\mathrm{d}\eta+\frac{1}{4}\omega\int_0^t\int_{D_z} e^{-\omega\eta}T^2\mathrm{d}A\mathrm{d}\eta\right].
\tag{2.33}
$$

Combining (2.30)–(2.33), we have

$$
|I_3|+|I_4|+|I_5|+|I_7| \le \left[\frac{\sqrt{\nu}}{\sqrt{2}}+\sqrt{\nu\delta_1}+\frac{1}{\sqrt{\omega}}\right]\left[\frac{\partial}{\partial z}F(z,t)\right].
\tag{2.34}
$$

Using the Hölder inequality, the Young inequality, and Lemma 2.2, we have

$$
\begin{aligned}
|I_6| &\le \frac{1}{2}\left[\int_0^t\int_{D_z} e^{-\omega\eta}T^4 \mathrm{d}A\mathrm{d}\eta \int_0^t\int_{D_z} e^{-\omega\eta}u_3u_3 \mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}\sqrt{k_1}\left[\int_0^t\int_{D_z} e^{-\omega\eta}T^2 \mathrm{d}A\mathrm{d}\eta + \int_0^t\int_{D_z} e^{-\omega\eta}T_\alpha T_\alpha \mathrm{d}A\mathrm{d}\eta\right]\left[\int_0^t\int_{D_z} e^{-\omega\eta}u_3u_3 \mathrm{d}A\mathrm{d}\eta\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}\sqrt{k_1}\max\left\{\frac{4}{\omega}, 1\right\}\left[\frac{\partial}{\partial z}F(z,t)\right]^{\frac{3}{2}}.
\end{aligned}
\tag{2.35}
$$

Inserting (2.23), (2.29), (2.34), and (2.35) into (2.10), we can obtain Lemma 2.4. □

3 Main result

Based on Lemma 4, we can obtain the following theorem.

Theorem 3.1. *Let* (u_i, T) *be a solution of equations* (1.1)–(1.3) *with the initial-boundary conditions* (1.5) and (1.6) *in* R, *then for fixed t, either*

$$
\lim_{z\to\infty}\left\{z^{-3}\left[\int_0^t\int_{z_0}^z\int_{D_\xi} e^{-\omega\eta}\left(\delta_1\nu u_{i,j}u_{i,j} + \frac{3}{2}\delta_1 u_iu_i + T_iT_i + \frac{3}{4}\omega T^2\right)\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \frac{1}{2}e^{-\omega t}\int_{z_0}^z\int_{D_\xi} T^2\mathrm{d}A\mathrm{d}\xi\right]\right\} \ge C_1 \tag{3.1}
$$

holds or

$$
\begin{aligned}
&\int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\left[\delta_1\nu u_{i,j}u_{i,j} + \frac{1}{2}\delta_1 u_iu_i + T_iT_i + \frac{1}{4}\omega T^2\right]\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \frac{1}{2}e^{-\omega t}\int_z^\infty\int_{D_\xi} T^2\mathrm{d}A\mathrm{d}\xi \\
&\le\ b_6Q^2(0,t)e^{-\frac{2z}{b_5}} + b_5Q(0,t)e^{-\frac{z}{b_5}}
\end{aligned}
\tag{3.2}
$$

holds, where C_1, b_5, *and* b_6 *are positive constants and* $Q(0, t)$ *will be defined in* (3.16).

Proof. We consider (2.1) for two cases.

Case I. $\exists z_0 \ge 0$ such that $F(z_0, t) \ge 0$.

From (2.11), we know that $\frac{\partial}{\partial z}F(z,t) \ge 0$. So, we have $F(z,t) \ge F(z_0,t) \ge 0$, $z \ge z_0$. Therefore, (2.1) can be written as

$$
F(z,t) \le b_1\left[\frac{\partial F}{\partial z}(z,t)\right] + b_2\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}}, \quad z \ge z_0. \tag{3.3}
$$

Using the Young inequality, we have

$$
\left[\frac{\partial F}{\partial z}(z,t)\right] = \left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{4}\cdot\frac{2}{3}}\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}\cdot\frac{1}{3}} \le \frac{2}{3}\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{4}} + \frac{1}{3}\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}}. \tag{3.4}
$$

Inserting (3.4) into (3.3), we have

$$
F(z,t) \le b_3\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{4}} + b_4\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}}, \quad z \ge z_0,
$$

where $b_3 = \frac{2}{3}b_1$ and $b_4 = \frac{1}{3}b_1 + b_2$. Therefore, we have

$$F(z,t)+\frac{b_3^2}{4b_4}\le b_4\left[\left(\frac{\partial F}{\partial z}(z,t)\right)^{\frac{3}{4}}+\frac{b_3}{2b_4}\right]^2,\quad z\ge z_0. \tag{3.5}$$

From (3.5), it follows that

$$\frac{\partial F}{\partial z}(z,t)\ge\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{\frac{4}{3}},\quad z\ge z_0.$$

So, we have

$$\left\{2b_4\frac{1}{\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{\frac{1}{3}}}+b_3\frac{1}{\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{\frac{4}{3}}}\right\}\cdot d\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right] \tag{3.6}$$
$$\ge 1,\ \ z\ge z_0.$$

Integrating (3.6) from z_0 to z, we have

$$\begin{aligned}&3b_4\left\{\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{\frac{2}{3}}-\left[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{\frac{2}{3}}\right\}\\&-3b_3\left\{\left[\sqrt{\frac{1}{b_4}F(z,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{-\frac{1}{3}}-\left[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{-\frac{1}{3}}\right\}\ge z-z_0,\ \ z\ge z_0.\end{aligned} \tag{3.7}$$

We discard the second and third terms at the left end of (3.7). In the first term of (3.7), we use the following inequality:

$$\sqrt{a+b}\le\sqrt{a}+\sqrt{b},\quad a,b\ge 0,$$

to have

$$3b_4\left[\sqrt{\frac{1}{b_4}F(z,t)}\right]^{\frac{2}{3}}\ge z-z_0-3b_3\left[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{-\frac{1}{3}}.$$

Therefore, we have

$$F(z,t)\ge\left\{\frac{1}{3\sqrt[3]{b_4^2}}(z-z_0)-\frac{b_3}{\sqrt[3]{b_4^2}}\left[\sqrt{\frac{1}{b_4}F(z_0,t)+\frac{b_3^2}{4b_4^2}}-\frac{b_3}{2b_4}\right]^{-\frac{1}{3}}\right\}^3. \tag{3.8}$$

On the other hand, we integrate (2.12) from z_0 to z to obtain

$$F(z,t)-F(z_0,t)\le\int_0^t\int_{z_0}^z\int_{D_\xi}e^{-\omega\eta}\left[\delta_1\nu u_{i,j}u_{i,j}+\frac{3}{2}\delta_1u_iu_i+T_iT_i+\frac{3}{4}\omega T^2\right]\mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta+\frac{1}{2}e^{-\omega t}\int_{z_0}^z\int_{D_\xi}T^2\mathrm{d}A\mathrm{d}\xi. \tag{3.9}$$

Combining (3.8) and (3.9), we can obtain (3.1).

Case II. $\forall z\ge 0$ such that $F(z,t)<0$, then we have from (2.1)

$$-F(z,t)\le b_1\left[\frac{\partial F}{\partial z}(z,t)\right]+b_2\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}},\quad z\ge 0. \tag{3.10}$$

Using the Young inequality again, we have

$$\left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{3}{2}} = \left[\frac{\partial F}{\partial z}(z,t)\right]^{\frac{1}{2}}\left[\frac{\partial F}{\partial z}(z,t)\right]^{2\cdot\frac{1}{2}} \leq \frac{1}{2}\left[\frac{\partial F}{\partial z}(z,t)\right] + \frac{1}{2}\left[\frac{\partial F}{\partial z}(z,t)\right]^2. \tag{3.11}$$

Inserting (3.11) into (3.10), we have

$$-F(z,t) \leq b_5\left[\frac{\partial F}{\partial z}(z,t)\right] + b_6\left[\frac{\partial F}{\partial z}(z,t)\right]^2, \quad z \geq 0, \tag{3.12}$$

where $b_5 = b_1 + \frac{1}{2}b_2$ and $b_6 = \frac{1}{2}b_2$. It follows from (3.12) that

$$\frac{\partial F}{\partial z}(z,t) \geq \sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}.$$

So, we have

$$\left\{-2b_6 - b_5\frac{1}{\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}}\right\} d\left\{\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right\} \geq 1, \quad z \geq 0. \tag{3.13}$$

Integrating (3.13) from 0 to z, we have

$$\begin{aligned} &2b_6\left[\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}}\right] + b_5 ln\left[\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right] \\ &\quad - b_5 ln\left[\sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right] \leq -z. \end{aligned} \tag{3.14}$$

Dropping the first term on the left of (3.14), we have

$$b_5 \ln\left[\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right] \leq -z + 2b_6\sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}} + b_5 \ln\left[\sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right].$$

Therefore, we obtain

$$\sqrt{-\frac{F(z,t)}{b_6} + \frac{b_5^2}{4b_6^2}} \leq Q(0,t)e^{-\frac{z}{b_5}} + \frac{b_5}{2b_6}, \tag{3.15}$$

where

$$Q(0,t) = \left[\sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}} - \frac{b_5}{2b_6}\right] e^{\frac{2b_6}{b_5}\sqrt{-\frac{F(0,t)}{b_6} + \frac{b_5^2}{4b_6^2}}}. \tag{3.16}$$

Squaring (3.15), we have

$$-F(z,t) \leq b_6 Q^2(0,t)e^{-\frac{2z}{b_5}} + b_5 Q(0,t)e^{-\frac{z}{b_5}}. \tag{3.17}$$

So, we have $\lim_{z\to\infty}[-F(z,t)] = 0$. Now, we integrate (2.11) from z to ∞ to obtain

$$-F(z,t) \geq \int_0^t\int_z^\infty\int_{D_\xi} e^{-\omega\eta}\left[\delta_1 \nu u_{i,j}u_{i,j} + \frac{1}{2}\delta_1 u_i u_i + T_i T_i + \frac{1}{4}\omega T^2\right] \mathrm{d}A\mathrm{d}\xi\mathrm{d}\eta + \frac{1}{2}e^{-\omega t}\int_z^\infty\int_{D_\xi} T^2 \mathrm{d}A\mathrm{d}\xi. \tag{3.18}$$

Combining (3.17) and (3.18), we can obtain (3.2). □

Remark 3.2. Theorem 3.1 shows that the solution of equations (1.1)–(1.3) grows polynomially or decays exponentially as $z \to \infty$, and the growth rate is at least as fast as z^{-3}.

Remark 3.3. To make the decay estimate explicit, we have to derive the upper bounds for $-F(0,t)$. To do this, we let

$$\widetilde{\boldsymbol{f}}(\boldsymbol{x}, t) = \boldsymbol{f}(\boldsymbol{x}, t)e^{-\sigma z}, \quad \sigma > 0.$$

Using a similar methods of [4, 5], it is easy to derive that $-F(0, t)$ can be bounded in terms of known data.

4 Continuous dependence result

In this section, we derive the continuous dependence on the Brinkman coefficient ν in the case of decay. To do this, we first give a useful lemma.

Lemma 4.1. (See [32]) *Let R be a bounded simply connected region in $\mathbb{R}^3$ with the Lipschitz boundary ∂R. Then, given any Dirichlet integrable function v satisfying $\int_R v\mathrm{d}x = 0$, there exist a vector field with components $\overline{w}_i$ $(i = 1, 2, 3)$, which is Dirichlet integrable and vanishes on ∂R, and a dimensionless constant k_3 depending only on the geometry of R such that*

$$\overline{w}_{i,i} = v, \ \text{in } R,$$

and

$$\int_R \overline{w}_{i,j}\overline{w}_{i,j}\mathrm{d}x \le k_3 \int_R [\overline{w}_{j,j}]^2\mathrm{d}x.$$

Suppose that u_i, p, T and u_i^*, p^*, T^* be the solutions of equations (1.1)–(1.7) but corresponding to deferent ν and ν^*, respectively. Let

$$v_i = u_i - u_i^*, \quad \Sigma = T - T^*, \quad \pi = p - p^*, \quad \tilde{\nu} = \nu - \nu^*, \tag{4.1}$$

then v_i, Σ, and π satisfy the following system:

$$-\tilde{\nu}\Delta u_i - \nu^*\Delta v_i + v_i = -\pi_i + g_i\Sigma, \quad \text{in } R \times (0, \tau), \tag{4.2}$$

$$v_{i,i} = 0, \quad \text{in } R \times (0, \tau), \tag{4.3}$$

$$\partial_t\Sigma + u_i\Sigma_i + v_iT_i^* = \Delta\Sigma, \quad \text{in } R \times (0, \tau), \tag{4.4}$$

$$v_i(x_1, x_2, x_3, t) = 0, \Sigma(x_1, x_2, x_3, t) = 0, \quad \text{on } \partial D \times \{x_3 > 0\} \times (0, \tau), \tag{4.5}$$

$$v_i(x_1, x_2, 0, t) = 0, \quad \Sigma(x_1, x_2, 0, t) = 0, \quad \text{on } D \times (0, \tau), \tag{4.6}$$

$$v_i(x_1, x_2, x_3, 0) = 0, \quad \Sigma(x_1, x_2, x_3, 0) = 0, \quad \text{in } R. \tag{4.7}$$

Next, we derive the continuous dependence result.

We multiply (4.2) with $(\xi - z)e^{-\omega\eta}v_i$ and integrate in $R_z \times [0, t]$ to obtain

$$\int_0^t \int_{R_z} (\xi - z)e^{-\omega\eta}[-\tilde{\nu}\Delta u_i - \nu^*\Delta v_i + v_i + \pi_i - g_i\Sigma]v_i\mathrm{d}x\mathrm{d}\eta = 0,$$

from which it follows that

$$\begin{aligned}
&\int_0^t \int_{R_z} (\xi - z)e^{-\omega\eta}[\nu^* v_{i,j}v_{i,j} + v_iv_i]\mathrm{d}x\mathrm{d}\eta \\
&= -\tilde{\nu}\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,3}v_i\mathrm{d}x\mathrm{d}\eta - \nu^*\int_0^t \int_{R_z} e^{-\omega\eta}v_{i,3}v_i\mathrm{d}x\mathrm{d}\eta + \int_0^t \int_{R_z} e^{-\omega\eta}\pi v_3\mathrm{d}x\mathrm{d}\eta \\
&\quad - \tilde{\nu}\int_0^t \int_{R_z} (\xi - z)e^{-\omega\eta}u_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta + \int_0^t \int_{R_z} (\xi - z)e^{-\omega\eta}g_i\Sigma v_i\mathrm{d}x\mathrm{d}\eta \doteq \sum_{i=1}^5 J_i.
\end{aligned} \tag{4.8}$$

Using the Schwarz inequality and (3.2), we obtain

$$
\begin{aligned}
J_1 &\leq \left[\int_0^t \int_{R_z} e^{-\omega\eta}\tilde{\nu}^2 u_{i,3}u_{i,3}\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\left[\int_0^t \int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}} \\
&\leq \delta_1\nu\tilde{\nu}^2\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,3}u_{i,3}\mathrm{d}x\mathrm{d}\eta + \frac{1}{4\delta_1\nu}\int_0^t \int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta,
\end{aligned} \tag{4.9}
$$

$$
J_2 \leq \frac{\sqrt{\nu^*}}{2}\int_0^t \int_{R_z} e^{-\omega\eta}\nu^* v_{i,3}v_{i,3}\mathrm{d}x\mathrm{d}\eta + \frac{\sqrt{\nu^*}}{2}\int_0^t \int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta. \tag{4.10}
$$

Obviously, $\int_D v_3\mathrm{d}A = 0$. So, according to Lemma 4.1, there exists a vector function $\overline{w} = (\overline{w}_1, \overline{w}_2, \overline{w}_3)$ such that

$$
\overline{w}_{i,i} = v_3, \ \text{ in } R, \quad \overline{w}_i = 0, \ \text{ on } \partial R.
$$

$$
\begin{aligned}
J_3 &= \int_0^t \int_{R_z} e^{-\omega\eta}\pi\overline{w}_{i,i}\mathrm{d}x\mathrm{d}\eta = -\int_0^t \int_{R_z} e^{-\omega\eta}\pi_i\overline{w}_i\mathrm{d}x\mathrm{d}\eta \\
&= \int_0^t \int_{R_z} e^{-\omega\eta}[-\tilde{\nu}\Delta u_i - \nu^*\Delta v_i + v_i - g_i\Sigma]\overline{w}_i\mathrm{d}x\mathrm{d}\eta \\
&= \tilde{\nu}\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,j}\overline{w}_{i,j}\mathrm{d}x\mathrm{d}\eta + \nu^*\int_0^t \int_{R_z} e^{-\omega\eta}v_{i,j}\overline{w}_{i,j}\mathrm{d}x\mathrm{d}\eta \\
&\quad + \int_0^t \int_{R_z} e^{-\omega\eta}v_i\overline{w}_i\mathrm{d}x\mathrm{d}\eta - \int_0^t \int_{R_z} e^{-\omega\eta}g_i\Sigma\overline{w}_i\mathrm{d}x\mathrm{d}\eta \\
&\doteq J_{31} + J_{32} + J_{33} + J_{34}.
\end{aligned} \tag{4.11}
$$

Using the Schwarz inequality and Lemma 4.1, we obtain

$$
\begin{aligned}
J_{31} &\leq \tilde{\nu}\left[\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\left[\int_0^t \int_{R_z} e^{-\omega\eta}\overline{w}_{i,j}\overline{w}_{i,j}\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}} \\
&\leq \tilde{\nu}\sqrt{k_3}\left[\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\left[\int_0^t \int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}} \\
&\leq \tilde{\nu}^2\delta_1\nu\int_0^t \int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{k_3}{4\delta_1\nu}\int_0^t \int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta,
\end{aligned} \tag{4.12}
$$

$$
J_{32} \leq \frac{\sqrt{k_3\nu^*}}{2}\left[\nu^*\int_0^t \int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta + \int_0^t \int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right], \tag{4.13}
$$

$$
J_{33} \leq \frac{\sqrt{k_3}h}{2\pi}\left[\int_0^t \int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta + \int_0^t \int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right], \tag{4.14}
$$

$$
J_{34} \leq \frac{\sqrt{k_3}h}{2\pi}\left[\int_0^t \int_{R_z} e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta + \int_0^t \int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right]. \tag{4.15}
$$

Inserting (4.12)–(4.15) into (4.11), we have

$$\begin{aligned} J_3 \le \tilde{\nu}^2\delta_1\nu \int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{\sqrt{k_3}\,h}{2\pi}\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta + \left[\frac{k_3}{4\delta_1\nu} + \frac{\sqrt{k_3\nu^*}}{2} + \frac{\sqrt{k_3}\,h}{\pi}\right] \\ \times \int_0^t\int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta + \frac{\sqrt{k_3\nu^*}}{2}\left[\nu^*\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta\right]. \end{aligned} \tag{4.16}$$

Using the Schwarz inequality, we obtain

$$J_4 \le \frac{1}{2\nu^*}\tilde{\nu}^2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\nu^*\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta, \tag{4.17}$$

$$J_5 \le \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta. \tag{4.18}$$

Inserting (4.9), (4.10), and (4.16)–(4.18) into (4.8), we have

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\nu^*v_{i,j}v_{i,j} + v_iv_i]\mathrm{d}x\mathrm{d}\eta \\ &\le \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta + \frac{\sqrt{k_3}\,h}{2\pi}\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta \\ &\quad + \left[\frac{1}{4\delta_1\nu} + \frac{\sqrt{\nu^*}}{2} + \frac{k_3}{4\delta_1\nu} + \frac{\sqrt{k_3\nu^*}}{2} + \frac{\sqrt{k_3}\,h}{\pi}\right]\int_0^t\int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta \\ &\quad + \left[\frac{\sqrt{\nu^*}}{2} + \frac{\sqrt{k_3\nu^*}}{2}\right]\left[\nu^*\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta\right] \\ &\quad + \frac{1}{2\nu^*}\tilde{\nu}^2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta + 2\tilde{\nu}^2\delta_1\nu\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta. \end{aligned} \tag{4.19}$$

Now, we multiply (4.4) with $(\xi - z)e^{-\omega\eta}\Sigma$ and integrate in $R_z \times [0, t]$ to obtain

$$\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\partial_\eta\Sigma + u_i\Sigma_i + v_iT_i^* - \Delta\Sigma]\Sigma\mathrm{d}x\mathrm{d}\eta = 0,$$

from which it follows that

$$\begin{aligned} &\frac{1}{2}\int_{R_z}(\xi - z)e^{-\omega t}\Sigma^2\mathrm{d}x\,|_{\eta=t} + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\left[\frac{1}{2}\omega\Sigma^2 + \Sigma_i\Sigma_i\right]\mathrm{d}x\mathrm{d}\eta \\ &= -\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_3\Sigma\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{R_z} e^{-\omega\eta}u_3\Sigma^2\mathrm{d}x\mathrm{d}\eta \\ &\quad + \int_0^t\int_{R_z} e^{-\omega\eta}T^*\Sigma v_3\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_iT^*\Sigma_i\mathrm{d}x\mathrm{d}\eta \\ &\doteq \sum_{i=1}^4\hat{J}_i. \end{aligned} \tag{4.20}$$

Using the Schwarz inequality, we obtain

$$\begin{aligned}\widehat{J}_1 &\le \sqrt{\frac{2}{\omega}}\left[\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}\omega\Sigma^2\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\left[\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_3^2\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\\ &\le \sqrt{\frac{2}{\omega}}\left[\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}\omega\Sigma^2\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_3^2\mathrm{d}x\mathrm{d}\eta\right].\end{aligned} \tag{4.21}$$

Using Lemma 2.2, we have

$$\begin{aligned}\widehat{J}_2 &\le \int_0^t\int_z^\infty\left[\int_{D_\xi} e^{-\omega\eta}u_3^2\mathrm{d}A\right]^{\frac{1}{2}}\left[\int_{D_\xi} e^{-\omega\eta}\Sigma^4\mathrm{d}A\right]^{\frac{1}{2}}\mathrm{d}\xi\mathrm{d}\eta\\ &\le \sqrt{k_1}\int_0^t\int_z^\infty\left[-2\int_{R_\xi} e^{-\omega\eta}u_3u_{3,3}\mathrm{d}x\right]^{\frac{1}{2}}\left[\int_{D_\xi} e^{-\omega\eta}\Sigma^2\mathrm{d}A + \int_{D_\xi} e^{-\omega\eta}\Sigma_\alpha\Sigma_\alpha\mathrm{d}A\right]\mathrm{d}\xi\mathrm{d}\eta\\ &\le \sqrt{2k_1}\max_{t\in[0,\tau]}\left\{\left[\int_{R_z} e^{-\omega\eta}u_3^2\mathrm{d}x\int_{R_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}x\right]^{\frac{1}{4}}\right\}\times\left[\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_\alpha\Sigma_\alpha\mathrm{d}A\mathrm{d}x\mathrm{d}\eta\right].\end{aligned} \tag{4.22}$$

Let

$$T_M = \max_{R\times[0,\tau]}\{h(x_1, x_2, t)\},$$

we have

$$\begin{aligned}\widehat{J}_3 &\le \sqrt{\frac{2}{\omega}}T_M\left[\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}\omega\Sigma^2\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\left[\int_0^t\int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right]^{\frac{1}{2}}\\ &\le \sqrt{\frac{2}{\omega}}T_M\left[\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}\omega\Sigma^2\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta\right],\end{aligned} \tag{4.23}$$

$$\widehat{J}_4 \le \frac{1}{2}\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\Sigma_i\Sigma_i\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}T_M^2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta. \tag{4.24}$$

Inserting (4.21)–(4.24) into (4.20), we have

$$\begin{aligned}&\frac{1}{2}\int_{R_z}(\xi - z)e^{-\omega t}\Sigma^2\mathrm{d}x\bigg|_{\eta=t} + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}\left[\frac{1}{2}\omega\Sigma^2 + \frac{1}{2}\Sigma_i\Sigma_i\right]\mathrm{d}x\mathrm{d}\eta\\ &\le \left[\sqrt{\frac{2}{\omega}} + \sqrt{2k_1}\frac{2}{\omega}\max_{t\in[0,\tau]}\left\{\left[\int_{R_z} e^{-\omega\eta}u_3^2\mathrm{d}x\int_{R_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}x\right]^{\frac{1}{4}}\right\} + \sqrt{\frac{2}{\omega}}T_M\right]\int_0^t\int_{R_z} e^{-\omega\eta}\frac{1}{2}\omega\Sigma^2\mathrm{d}x\mathrm{d}\eta\\ &\quad + \left[\sqrt{\frac{2}{\omega}} + \sqrt{2k_1}\frac{2}{\omega}\max_{t\in[0,\tau]}\left\{\left[\int_{R_z} e^{-\omega\eta}u_3^2\mathrm{d}x\int_{R_z} e^{-\omega\eta}u_{3,3}^2\mathrm{d}x\right]^{\frac{1}{4}}\right\}\right]\\ &\quad \times\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_i\Sigma_i\mathrm{d}x\mathrm{d}\eta + \sqrt{\frac{2}{\omega}}T_M\int_0^t\int_{R_z} e^{-\omega\eta}v_3^2\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}T_M^2\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta.\end{aligned} \tag{4.25}$$

If we define

$$\Psi(z,t)=\int\limits_{R_z}(\xi-z)e^{-\omega t}\Sigma^2\mathrm{d}x\,|_{\eta=t}+\int\limits_0^t\int\limits_{R_z}(\xi-z)e^{-\omega\eta}[\omega\Sigma^2+\Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta+\frac{1}{2}\omega\int\limits_0^t\int\limits_{R_z}(\xi-z)e^{-\omega\eta}[v^*v_{i,j}v_{i,j}+v_iv_i]\mathrm{d}x\mathrm{d}\eta,\tag{4.26}$$

then we have

$$-\frac{\partial}{\partial z}\Psi(z,t)=\int\limits_{R_z}e^{-\omega t}\Sigma^2\mathrm{d}x\,|_{\eta=t}+\int\limits_0^t\int\limits_{R_z}e^{-\omega\eta}[\omega\Sigma^2+\Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta+\frac{1}{2}\omega\int\limits_0^t\int\limits_{R_z}e^{-\omega\eta}[v^*v_{i,j}v_{i,j}+v_iv_i]\mathrm{d}x\mathrm{d}\eta.\tag{4.27}$$

Combining (4.19) and (4.25) and choosing $\omega\geq 4T_M^2$, we have

$$\Psi(z,t)\leq b_7\left[-\frac{\partial}{\partial z}\Psi(z,t)\right]+\frac{1}{2v^*}\tilde{v}^2\omega\int\limits_0^t\int\limits_{R_z}(\xi-z)e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta+2\tilde{v}^2\delta_1 v\omega\int\limits_0^t\int\limits_{R_z}e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta,\tag{4.28}$$

where

$$\begin{aligned}b_7=\max\Bigg\{&2\sqrt{\frac{2}{\omega}}+\sqrt{2k_1}\frac{4}{\omega}\max_{t\in[0,\tau]}\left\{\left[\int\limits_{R_z}e^{-\omega\eta}u_3^2\mathrm{d}x\int\limits_{R_z}e^{-\omega\eta}u_{3,3}^2\mathrm{d}x\right]^{\frac{1}{4}}\right\}+2\sqrt{\frac{2}{\omega}}T_M\\&+\frac{2\sqrt{k_3}h}{h\omega},\frac{1}{4\delta_1 v}+\frac{\sqrt{v^*}}{2}+\frac{k_3}{4\delta_1 v}+\frac{\sqrt{k_3v^*}}{2}+\frac{\sqrt{k_3}h}{\pi}+2\sqrt{\frac{2}{\omega}}T_M\Bigg\}.\end{aligned}\tag{4.29}$$

In view of (3.2), we have

$$\int\limits_0^t\int\limits_{R_z}e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta\leq\frac{b_6}{v\delta_1}Q^2(0,t)e^{-\frac{2z}{b_5}}+\frac{b_5}{v\delta_1}Q^2(0,t)e^{-\frac{z}{b_5}}.\tag{4.30}$$

Therefore,

$$\int\limits_0^t\int\limits_{R_z}(\xi-z)e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\eta=\int\limits_0^t\int\limits_z^\infty\int\limits_{R_\xi}e^{-\omega\eta}u_{i,j}u_{i,j}\mathrm{d}x\mathrm{d}\xi\mathrm{d}\eta\leq\frac{b_6b_5}{2v\delta_1}Q^2(0,t)e^{-\frac{2z}{b_5}}+\frac{b_5^2}{v\delta_1}Q^2(0,t)e^{-\frac{z}{b_5}}.\tag{4.31}$$

Inserting (4.30) and (4.31) into (4.28), we have

$$\Psi(z,t)\leq b_7\left[-\frac{\partial}{\partial z}\Psi(z,t)\right]+b_8\tilde{v}^2e^{-\frac{2z}{b_5}}+b_9\tilde{v}^2e^{-\frac{z}{b_5}},\tag{4.32}$$

where $b_8=\frac{b_6b_5}{4v^*v\delta_1}Q^2(0,t)\omega+b_6Q^2(0,t)\omega$ and $b_9=\frac{b_5^2}{2v^*v\delta_1}Q^2(0,t)\omega+b_5Q^2(0,t)\omega$. It follows that from (4.32)

$$\frac{\partial}{\partial z}\left\{\Psi(z,t)e^{\frac{z}{b_7}}\right\}\leq b_8\tilde{v}^2e^{-\frac{2z}{b_5}+\frac{z}{b_7}}+b_9\tilde{v}^2e^{-\frac{z}{b_5}+\frac{z}{b_7}}.\tag{4.33}$$

If $b_5\neq 2b_7$ and $b_5\neq b_7$, then we integrate (4.33) from 0 to z to obtain

$$\Psi(z,t)\leq\Psi(0,t)e^{-\frac{z}{b_7}}+\frac{b_8}{-\frac{2}{b_5}+\frac{1}{b_7}}\tilde{v}^2\left[e^{-\frac{2z}{b_5}}-e^{-\frac{z}{b_7}}\right]+\frac{b_9}{-\frac{1}{b_5}+\frac{1}{b_7}}\tilde{v}^2\left[e^{-\frac{z}{b_5}}-e^{-\frac{z}{b_7}}\right].\tag{4.34}$$

If $b_5=2b_7$, then we integrate (4.33) from 0 to z to obtain

$$\Psi(z,t)\leq\Psi(0,t)e^{-\frac{z}{b_7}}+b_8\tilde{v}^2ze^{-\frac{z}{b_7}}+2b_9b_7\tilde{v}^2\left[e^{-\frac{z}{2b_7}}-e^{-\frac{z}{b_7}}\right].\tag{4.35}$$

If $b_5 = b_7$, then we integrate (4.33) from 0 to z to obtain

$$\Psi(z,t) \le \Psi(0,t)e^{-\frac{z}{b_7}} + b_8 b_7 \tilde{\nu}^2\Big[e^{-\frac{z}{b_7}} - e^{-\frac{2z}{b_7}}\Big] + b_9\tilde{\nu}^2 z e^{-\frac{z}{b_7}}. \tag{4.36}$$

To obtain the main result, we have to derive the upper bound for $\Psi(0,t)$. In (4.32), we choose $z = 0$ to have

$$\Psi(0,t) \le b_7\left[-\frac{\partial}{\partial z}\Psi(0,t)\right] + b_8\tilde{\nu}^2 + b_9\tilde{\nu}^2. \tag{4.37}$$

Therefore, we only need to derive the bound for $-\frac{\partial}{\partial z}\Psi(0,t)$. In (4.27), we choose $z = 0$ to have

$$-\frac{\partial}{\partial z}\Psi(0,t) = \int_R e^{-\omega t}\Sigma^2 \mathrm{d}x\Bigg|_{\eta=t} + \int_0^t\int_R e^{-\omega\eta}[\omega\Sigma^2 + \Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_R e^{-\omega\eta}[\nu^* v_{i,j}v_{i,j} + v_i v_i]\mathrm{d}x\mathrm{d}\eta. \tag{4.38}$$

On the other hand, combining (4.8) and (4.20) and using the initial-boundary conditions (4.5)–(4.7), we obtain

$$\begin{aligned}
-\frac{\partial}{\partial z}\Psi(0,t) &\le -\frac{1}{2}\tilde{\nu}\omega\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_{R_z} e^{-\omega\eta}g_i\Sigma v_i\mathrm{d}x\mathrm{d}\eta + \int_0^t\int_{R_z} e^{-\omega\eta}v_i T\Sigma_i\mathrm{d}x\mathrm{d}\eta\\
&\le \frac{1}{4\nu^*}\tilde{\nu}^2\omega\int_0^t\int_{R_z} e^{-\omega\eta}u_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{1}{4}\omega\nu^*\int_0^t\int_{R_z} e^{-\omega\eta}v_{i,j}v_{i,j}\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma^2\mathrm{d}x\mathrm{d}\eta\\
&\quad + \frac{1}{8}\omega\int_0^t\int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta + \frac{2}{\omega}T_M^2\int_0^t\int_{R_z} e^{-\omega\eta}\Sigma_i\Sigma_i\mathrm{d}x\mathrm{d}\eta + \frac{1}{8}\omega\int_0^t\int_{R_z} e^{-\omega\eta}v_iv_i\mathrm{d}x\mathrm{d}\eta.
\end{aligned} \tag{4.39}$$

Since $\omega \ge 4T_M^2$, we combine (4.30), (4.38), and (4.39) to have

$$-\frac{\partial}{\partial z}\Psi(0,t) \le \frac{1}{2}\left[-\frac{\partial}{\partial z}\Psi(0,t)\right] + \tilde{\nu}^2\omega\frac{b_6}{4\nu^*\nu\delta_1}Q^2(0,t)e^{-\frac{2z}{b_5}} + \tilde{\nu}^2\omega\frac{b_5}{4\nu^*\nu\delta_1}Q^2(0,t)e^{-\frac{z}{b_5}}.$$

Therefore, we have

$$-\frac{\partial}{\partial z}\Psi(0,t) \le \tilde{\nu}^2\omega\frac{b_6}{2\nu^*\nu\delta_1}Q^2(0,t) + \tilde{\nu}^2\omega\frac{b_5}{2\nu^*\nu\delta_1}Q^2(0,t). \tag{4.40}$$

Inserting (4.40) into (4.37), we have

$$\Psi(0,t) \le b_{10}\tilde{\nu}^2, \tag{4.41}$$

where $b_{10} = \omega\frac{b_6 b_7}{2\nu^*\nu\delta_1}Q^2(0,t) + \omega\frac{b_5 b_7}{2\nu^*\nu\delta_1}Q^2(0,t) + b_8 + b_9$. Inserting (4.41) into (4.34)–(4.35) and combining with (4.26), we can obtain the following theorem.

Theorem 4.1. *Let* (u, T) *and* (u^*, T^*) *be solutions of* (1.1)–(1.7) *corresponding to the coefficients* ν and ν^*, *respectively. If* $\forall z \ge 0$ *such that* $F(z,t) < 0$ *and* $\int_{D_0} f_3\mathrm{d}A = 0$, *then the solutions of equations* (1.1)–(1.7) *depend continuously on the effective viscosity coefficient* ν. *Specifically, if* $b_5 \ne 2b_7$ *and* $b_5 \ne b_7$, *then we obtain*

$$\begin{aligned}
&\int_{R_z}(\xi - z)e^{-\omega t}\Sigma^2\mathrm{d}x\Bigg|_{\eta=t} + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\omega\Sigma^2 + \Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\nu^* v_{i,j}v_{i,j} + v_iv_i]\mathrm{d}x\mathrm{d}\eta\\
&\le\ b_{10}\tilde{\nu}^2 e^{-\frac{z}{b_7}} + \frac{b_8}{-\frac{2}{b_5} + \frac{1}{b_7}}\tilde{\nu}^2\Big[e^{-\frac{2z}{b_5}} - e^{-\frac{z}{b_7}}\Big] + \frac{b_9}{-\frac{1}{b_5} + \frac{1}{b_7}}\tilde{\nu}^2\Big[e^{-\frac{z}{b_5}} - e^{-\frac{z}{b_7}}\Big].
\end{aligned}$$

If $b_5 = 2b_7$, *then we obtain*

$$\int_{R_z}(\xi - z)e^{-\omega t}\Sigma^2\mathrm{d}x\Bigg|_{\eta=t} + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\omega\Sigma^2 + \Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\nu^* v_{i,j}v_{i,j} + v_iv_i]\mathrm{d}x\mathrm{d}\eta$$
$$\leq b_{10}\tilde{v}^2e^{-\frac{z}{b_7}} + \frac{b_8}{b_7}\tilde{v}^2ze^{-\frac{z}{b_7}} + 2b_9\tilde{v}^2\left[e^{-\frac{z}{2b_7}} - e^{-\frac{z}{b_7}}\right].$$

If $b_5 = b_7$, *then we obtain*

$$\int_{R_z}(\xi - z)e^{-\omega t}\Sigma^2\mathrm{d}x\Bigg|_{\eta=t} + \int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\omega\Sigma^2 + \Sigma_i\Sigma_i]\mathrm{d}x\mathrm{d}\eta + \frac{1}{2}\omega\int_0^t\int_{R_z}(\xi - z)e^{-\omega\eta}[\nu^* v_{i,j}v_{i,j} + v_iv_i]\mathrm{d}x\mathrm{d}\eta$$
$$\leq b_{10}\tilde{v}^2e^{-\frac{z}{b_7}} + b_8\tilde{v}^2\left[e^{-\frac{z}{b_7}} - e^{-\frac{2z}{b_7}}\right] + \frac{b_9}{b_7}\tilde{v}^2ze^{-\frac{z}{b_7}}.$$

5 Conclusion

In this article, we define the Brinkman equations (1.1)–(1.3) in a semi-infinite cylinder and the Phragmén-Lindelöf alternative result and the continuous dependence result on ν are obtained. However, there are still some deeper problems to be studied in this article. We note that Quintanilla [13] considered the spatial selectivity of solutions of several kinds of partial differential equations with a radius defined in the outer region of the sphere for the first time, in which the so-called outer region of the sphere is

$$\Omega = \{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^2 \geq R_0^2, R_0 > 0\}.$$

Li et al. [17,33] studied the selectivity of the wave equations in the region Ω, and obtained the rapid attenuation rate and growth rate. However, this type of research has not received sufficient attention. Therefore, it will be a meaningful topic to study the spatial properties of solutions of Brinkman equations on Ω.

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