Research Article

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Compact mappings and s-mappings at subsets

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Abstract: Almost s-mappings and almost compact mappings have been introduced and studied. In this article, we continue to research some questions related to the almost s-images (resp., almost compact images) of metric spaces. The following results are obtained. (1) A space X is a quotient and almost compact image of a metric space if and only if X is a sequential space having a cs^*-network which is point-regular at nonisolated points, which gives an affirmative answer to Question 4.9 in the article "S. Lin, X. W. Ling, and Y. Ge, *Point-regular covers and sequence-covering compact mappings*, Topology Appl. **271** (2020), 106987." (2) There exists a bi-quotient and almost compact image of a metric space satisfying no base, which is point-countable at nonisolated points, which gives negative answers to Question 3.1 in the article "X. W. Ling and S. Lin, *On open almost s-images of metric spaces*, Adv. Math. (China) **48** (2019), no. 4, 489–496" and Question 3.7 in the article "X. W. Ling, S. Lin, and W. He, *Point-countable covers and sequence-covering s-mappings at subsets*, Topology Appl. **290** (2021), 107572." (3) Some characterizations of countably bi-quotient and almost s-images (resp., pseudo-open and almost compact images) of metric spaces.

Keywords: almost s-mappings, almost compact mappings, point-countable family, point-regular family, cs^*-networks

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1 Introduction

The metrization problem is one of the central topics in the study of general topology, and numerous metrization theorems provide a broad stage for the discussion of generalized metric spaces [1]. The theory of generalized metric spaces has injected some new vitalities into the development of general topology [2]. Metrizability can be characterized in terms of sequences of open coverings [3]. Nowadays, it is widely recognized that the method of systems of coverings is one of the main tools for classifying spaces [1]. It was discovered that systems of coverings can be used very effectively to construct some natural mappings of metrizable spaces onto spaces admitting such systems of coverings [4]. This method led to a mutual classification of spaces and mappings based on the interaction of systems of coverings and mappings [2,5].

In 1960, Ponomarev [6] proved every space with a point-countable base can be characterized as an open and s-image of a metric space. In 1962, Arhangel'skiĭ [7, Theorem 1] proved that every space with a point-regular base can be characterized as an open and compact image of a metric space. The aforementioned two results have become extremely important theorems in the theory of spaces and mappings and laid

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a foundation for its development [1]. After that, spaces with some types of point-countable (resp., point-regular) covers were described as various continuous images of metric spaces [1,2]. Recently, the study of the relationships between certain s-images (resp., compact images) of metric spaces and spaces with point-countable (resp., point-regular) covers becomes one of the central research topics in general topology [1,4,8–17].

Arhangel'skiĭ [10, p. 218] and Lin et al. [13, Definition 4.1(2)] introduced the notion of almost s-mappings and almost compact mappings, respectively. However, there is an open and almost s-image (resp., almost compact image) of a metric space, which is not an open and s-image (resp., compact image) of a metric space [10,15]. A characterization of open and almost s-images (resp., almost compact images) of metric spaces was given as follows.

Theorem 1.1. [14, Theorem 2.1] *The followings are equivalent for a space X.*
(1) *X is an open and almost s-image of a metric space.*
(2) *X has a base which is point-countable at nonisolated points.*

Theorem 1.2. [13, Theorem 3.4] *The followings are equivalent for a space X.*
(1) *X is an open and almost compact image of a metric space.*
(2) *X has a base which is point-regular at nonisolated points.*

Since the set of nonisolated points is a special subset in a topological space, we can further discuss point-countable (resp., point-regular) covers at arbitrary subsets in topological spaces. In [13], Lin et al. gave some characterizations about point-regular covers at arbitrary subsets. In [16], Ling et al. gave some characterizations about point-countable covers at arbitrary subsets. But these studies are not complete, inspired by [13,16], we continue to discuss the point-countable (resp., point-regular) covers at arbitrary subsets in topological spaces and solve some related questions [13–15].

Inspired by Theorem 1.1, Ling et al. characterized quotient and almost s-images of metric spaces as follows.

Theorem 1.3. [16, Corollary 3.5] *The followings are equivalent for a space X.*
(1) *X is a quotient and almost s-image of a metric space.*
(2) *X is a sequential space with a point-countable cs^*-network at nonisolated points.*

It is interesting to investigate the following question.

Question 1.4. [13, Question 4.9] Are the following equivalent for a space X?
(1) X is a quotient and almost compact image of a metric space.
(2) X is a sequential space with a point-regular cs^*-network at nonisolated points.

It is well known that a space X is a countably bi-quotient and s-image of a metric space if and only if X is an open and s-image of a metric space (i.e., X has a point-countable base) [18, Theorem 1.1], and a space X is a pseudo-open and compact image of a metric space if and only if X is an open and compact image of a metric space (i.e., X has a point-regular base) [19]. The following questions were formed by Theorems 1.1 and 1.2.

Question 1.5. [14, Question 3.1] Does a countably bi-quotient and almost s-image of a metric space have a base which is point-countable at nonisolated points?

Question 1.6. [15, Question 3.7] Does a pseudo-open and almost compact image of a metric space have a base which is point-regular at nonisolated points?

In this article, we will give an affirmative answer to Question 1.4 (see Corollary 3.6), present an example to give negative answers to Questions 1.5 and 1.6 (see Example 4.1), and further obtain some characterizations of countably bi-quotient and almost s-images (resp., pseudo-open and almost compact images) of metric spaces (see Corollaries 4.4 and 3.8).

2 Mappings or networks at subsets

In this article, all spaces are T_2, and all mappings are continuous and onto. Recall some related concepts and notations. Let τ_X denote the topology for a space X. For a family $\mathscr{P}$ of subsets of a space X, $x \in X$ and $A \subset X$, put

$$(\mathscr{P})_x = \{P \in \mathscr{P} : x \in P\}, \quad \mathrm{st}(x, \mathscr{P}) = \bigcup(\mathscr{P})_x \quad \text{and} \quad \mathrm{st}(A, \mathscr{P}) = \bigcup_{x \in A} \mathrm{st}(x, \mathscr{P}).$$

The family $\mathscr{P}$ is said to be *point-countable* (resp., *point-finite*) at A if the family $(\mathscr{P})_x$ is countable (resp., finite) for each $x \in A$.

Let X be a topological space. A subset P of X is called a *sequential neighborhood* of a point x in X if, for each sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converging to the point x, there exists $m \in \mathbb{N}$ such that $\{x\} \cup \{x_n : n \geqslant m\} \subset P$, i.e., the sequence $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. A subset P of X is called a *sequentially open set* if P is a sequential neighborhood of each point in P. The space X is called a *sequential space* [3] if every sequentially open set of X is open in X. Let

$$I(X) = \{x : x \text{ is an isolated point of } X\}, \quad NI(X) = X \backslash I(X),$$
$$S(X) = \{x : \{x\} \text{ is a sequentially open set in } X\}, \quad NS(X) = X \backslash S(X).$$

The purpose of this section is to introduce some mappings or certain networks at subsets and to discuss some relationships between them.

Definition 2.1. Let $f : X \to Y$ be a mapping and $A \subset Y$.

(1) f is called an *s-mapping* (resp., a *boundary s-mapping*) at A [16, Definition 2.1] if $f^{-1}(y)$ (resp., the boundary $\partial f^{-1}(y)$) is a separable set in X for each $y \in A$; f is called an *almost s-mapping* [10, p. 218] if f is an s-mapping at $NI(Y)$.

(2) f is called a *compact mapping* (resp., *boundary-compact mapping*) at A [16, Definition 2.1] if $f^{-1}(y)$ (resp., the boundary $\partial f^{-1}(y)$) is a compact set in X for each $y \in A$; f is called an *almost compact mapping* [13, Definition 4.1(2)] if f is a compact mapping at $NI(Y)$.

(3) f is called a (*countably*) *bi-quotient mapping* [2, Definition 2.1.1(3) and p. 113] at A if, for each $y \in A$ and each (countable) family $\mathscr{U}$ of open subsets in X, which covers $f^{-1}(y)$, there is a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $y \in [f(\bigcup\mathscr{U}')]^\circ$ in Y.

(4) f is called a *strictly countably bi-quotient mapping* [20, Definition 2.2] at A if, for each $y \in A$ and each countable family $\mathscr{U}$ of open subsets in X which covers $f^{-1}(y)$, there is an element U of $\mathscr{U}$ such that $y \in [f(U)]^\circ$ in Y.

(5) f is called an *open mapping* at A if, for each $y \in A$ and each $x \in f^{-1}(y)$, then $y \in [f(U)]^\circ$ in Y whenever U is a neighborhood of x in X.

(6) f is called an *almost-open mapping* at A if, for each $y \in A$, there exists a point $x \in f^{-1}(y)$ such that $y \in [f(U)]^\circ$ in Y whenever U is a neighborhood of x in X.

(7) f is called a *pseudo-open mapping* at A if, for each $y \in A$ and $f^{-1}(y) \subset U$ with U open in X, then $y \in [f(U)]^\circ$ in Y.

(8) f is called a *sequentially quotient mapping* [2, Definition 2.1.4(3)] at A if, whenever $\{y_n\}_{n\in\mathbb{N}}$ is a sequence converging to a point $y \in A$ in Y, there are a convergent sequence $\{x_i\}_{i\in\mathbb{N}}$ in X and a subsequence $\{y_{n_i}\}_{i\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ with each $x_i \in f^{-1}(y_{n_i})$.

The mapping f is called a P-mapping if f is the P-mapping at Y, where P is defined in (1)–(8).

It is known that open mappings ⇒ almost-open mappings ⇒ bi-quotient mappings (resp., strictly countably bi-quotient mappings) ⇒ countably bi-quotient mappings ⇒ pseudo-open mappings ⇒ quotient mappings [20].

Lemma 2.2. *Let $f : X \to Y$ be a mapping and $A \subset Y$. If f is a pseudo-open and boundary-compact mapping at A, then f is bi-quotient at A.*

Proof. Let $y \in A$ and $\mathscr{U}$ be a family of open subsets in X, which covers $f^{-1}(y)$. Since f is boundary-compact at A, there exists a finite subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $\partial f^{-1}(y) \subset \bigcup \mathscr{U}'$. We can assume that there exists $U \in \mathscr{U}'$ such that $U \cap f^{-1}(y) \neq \varnothing$, whence $y \in f(U)$. Let $V = [f^{-1}(y)]^\circ \cup \bigcup \mathscr{U}'$. Then $f^{-1}(y) \subset V$. Since f is pseudo-open at A, we have that $y \in [f(V)]^\circ \subset f(f^{-1}(y) \cup \bigcup \mathscr{U}') = \{y\} \cup f(\bigcup \mathscr{U}') = f(\bigcup \mathscr{U}')$. So $y \in [f(\bigcup \mathscr{U}')]^\circ$. Therefore, f is bi-quotient at A. □

Lemma 2.3. *Let $f : X \to Y$ be a mapping and $A \subset Y$. Suppose that $\partial f^{-1}(y)$ is Lindelöf in X for each $y \in A$.*
(1) *If f is countably bi-quotient at A, then f is bi-quotient at A.*
(2) *If f is strictly countably bi-quotient at A, then f is almost-open at A.*

Proof. (1) Let $y \in A$ and $\mathscr{U}$ be a family of open subsets in X which covers $f^{-1}(y)$. Since the set $\partial f^{-1}(y)$ is Lindelöf, there exists a countable subfamily $\mathscr{U}'$ of $\mathscr{U}$ such that $\partial f^{-1}(y) \subset \bigcup \mathscr{U}'$ and $y \in f(\bigcup \mathscr{U}')$, whence $f^{-1}(y) \subset [f^{-1}(y)]^\circ \cup \bigcup \mathscr{U}'$. Since f is countably bi-quotient at A, there exists a finite subfamily $\mathscr{U}''$ of $\mathscr{U}'$ such that $y \in [f(\bigcup \mathscr{U}'')]^\circ$. Hence, f is bi-quotient at A.

(2) If f is not almost-open at A, then there exists a point $y \in A$ such that for every $x \in f^{-1}(y)$ there is an open neighborhood U_x at x in X satisfying $y \notin [f(U_x)]^\circ$. Then y is a nonisolated point in Y. Since $\partial f^{-1}(y)$ is Lindelöf, there exists a subset $\{x_i : i \in \mathbb{N}\} \subset f^{-1}(y)$ such that $\partial f^{-1}(y) \subset \bigcup \{U_{x_i} : i \in \mathbb{N}\}$, whence $f^{-1}(y) \subset [f^{-1}(y)]^\circ \cup \bigcup \{U_{x_i} : i \in \mathbb{N}\}$. Since f is strictly countably bi-quotient at A, $y \in [f(U_{x_i})]^\circ$ for some $i \in \mathbb{N}$, which is a contradiction. Hence, f is almost-open at A. □

Let X be a space and $A \subset X$. The space X is called a *first-countable space at A* if each point of A has a countable neighborhood base in X; the space X is called a *Fréchet space at A* if for any subset $B \subset X$ and $x \in A \cap \overline{B}$, there is a sequence in B converging to x in X.

Remark 2.4. If X is Fréchet at a point $x \in X$ and U is a sequential neighborhood of x in X, then U is a neighborhood of x.

Lemma 2.5. *Let $f : X \to Y$ be a mapping and $A \subset Y$.*
(1) *If Y is a first-countable space at A and f is a sequentially quotient mapping at A, then f is countably bi-quotient at A.*
(2) *If Y is a Fréchet space at A and f is a sequentially quotient mapping at A, then f is pseudo-open at A.*
(3) *If X is a Fréchet space at $f^{-1}(A)$ and f is a pseudo-open mapping at A, then Y is a Fréchet space at A and f is sequentially quotient at A.*

Proof. (1) If f is not countably bi-quotient at A, then there exist $y \in A$ and a countable family $\{U_i : i \in \mathbb{N}\}$ of open subsets in X covering $f^{-1}(y)$ such that for every $n \in \mathbb{N}$,

$$y \in Y \setminus \left[f\left(\bigcup_{i \le n} U_i \right) \right]^\circ = \overline{Y \setminus f\left(\bigcup_{i \le n} U_i \right)}.$$

Since Y is first-countable at A, there is $y_n \in Y \setminus f(\bigcup_{i \le n} U_i)$ for each $n \in \mathbb{N}$ such that $y_n \to y$. Since f is sequentially quotient at A, there are a convergent sequence $\{x_j\}_{j \in \mathbb{N}}$ in X and a subsequence $\{y_{n_j}\}_{j \in \mathbb{N}}$ of $\{y_n\}_{n \in \mathbb{N}}$ with each $x_j \in f^{-1}(y_{n_j})$. Let $x = \lim_{j \to \infty} x_j$. Then $x \in f^{-1}(y)$. It follows that there exists $m \in \mathbb{N}$ such that $x \in U_m$.

Hence, there is $k \in \mathbb{N}$ such that $x_j \in U_m$ and $m \leqslant n_j$ for each $j \geqslant k$. So $y_{n_j} \in f(U_m)$ for each $j \geqslant k$, which is a contradiction.

(2) Statement (2) holds by a similar proof in (1).

(3) Let $B \subset Y$ and $y \in A \cap \overline{B}$. If $f^{-1}(y) \cap \overline{f^{-1}(B)} = \varnothing$, then $f^{-1}(y) \subset X \backslash \overline{f^{-1}(B)}$. Since f is pseudo-open at A, we have that

$$y \in [f(X \backslash \overline{f^{-1}(B)})]^{\circ} \subset [f(X \backslash f^{-1}(B))]^{\circ} = (Y \backslash B)^{\circ} = Y \backslash \overline{B},$$

which is a contradiction. So there exists $x \in f^{-1}(y) \cap \overline{f^{-1}(B)}$. It follows from Fréchet property at x that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ in $f^{-1}(B)$ such that $x_n \to x$. Then $\{f(x_n)\}_{n \in \mathbb{N}}$ is the sequence in B converging to y. Hence, Y is a Fréchet space at A.

Let $y_n \to y \in A$ with each $y_n \neq y$. Put $C = \bigcup_{n \in \mathbb{N}} C_n$, where each $C_n = f^{-1}(y_n)$. We claim that $f^{-1}(y) \cap \overline{C} \neq \varnothing$. Assume that $f^{-1}(y) \cap \overline{C} = \varnothing$, then $f^{-1}(y) \subset (X \backslash C)^{\circ}$. Since f is pseudo-open at A, we have that $y \in [f(X \backslash C)]^{\circ}$. It follows that there is $m \in \mathbb{N}$ such that $y_m \in f(X \backslash C)$, which is a contradiction. Hence, $f^{-1}(y) \cap \overline{C} \neq \varnothing$, and put $x \in f^{-1}(y) \cap \overline{C}$. There exists a sequence $\{x_i\}_{i \in \mathbb{N}}$ in C such that $x_i \to x \notin C$ in X. We may assume that there is a subsequence $\{y_{n_i}\}_{i \in \mathbb{N}}$ of $\{y_n\}_{n \in \mathbb{N}}$ such that each $x_i \in f^{-1}(y_{n_i})$. Therefore, f is sequentially quotient at A. □

Lemma 2.6. *The followings are equivalent for a space X and $A \subset X$.*

(1) *X is a first-countable space at A.*
(2) *X is the image of a metric space under an almost-open and boundary-compact mapping at A.*
(3) *X is the image of a metric space under a countably bi-quotient and boundary s-mapping at A.*
(4) *X is the image of a metric space under a pseudo-open and boundary-compact mapping at A.*

Proof. By a similar proof of [15, Lemma 3.8], we have that (1) $\Rightarrow$ (2). Obviously, (2) $\Rightarrow$ (4). By Lemma 2.2, (4) $\Rightarrow$ (3). Next, we will show that (3) $\Rightarrow$ (1).

Let $f : M \to X$ be a countably bi-quotient and boundary s-mapping at A, where M is metrizable. Suppose that $\mathscr{B}$ is a point-countable base of M and $x \in A$.

If $\partial f^{-1}(x) = \varnothing$, then the set $f^{-1}(x)$ is open in M. It follows from the fact that f is countably bi-quotient at A that $x \in [f(f^{-1}(x))]^{\circ} \subset \{x\}$, i.e., x is an isolated point of X, whence x has a countable neighborhood base in X.

Suppose that $\partial f^{-1}(x) \neq \varnothing$. Let $\mathscr{B}' = \{B \in \mathscr{B} : B \cap \partial f^{-1}(x) \neq \varnothing\}$. It is well known that every point-countable family of open subsets in a separable space is countable. Thus, the separable set $\partial f^{-1}(x)$ meets at most countably many elements of $\mathscr{B}$. Then $f(\mathscr{B}')$ is nonempty and countable, and it can be denoted by $\{P_i\}_{i \in \mathbb{N}}$. Put

$$\mathscr{P} = \left\{ \left(\bigcup_{\alpha \in \Lambda} P_\alpha \right)^{\circ} : \Lambda \text{ is a finite subset of } \mathbb{N} \right\}.$$

Then $\mathscr{P}$ is countable. If U is an arbitrary neighborhood of x in X, then $\partial f^{-1}(x) \subset f^{-1}(x) \subset f^{-1}(U)$, thus there exists $\mathscr{B}_1 \subset \mathscr{B}'$ such that $\partial f^{-1}(x) \subset \bigcup \mathscr{B}_1 \subset f^{-1}(U)$. Hence, $f^{-1}(x) = [f^{-1}(x)]^{\circ} \cup \partial f^{-1}(x) \subset [f^{-1}(x)]^{\circ} \cup \bigcup \mathscr{B}_1 \subset f^{-1}(U)$. Since f is countably bi-quotient at A, there is a finite subfamily $\mathscr{B}_2$ of $\mathscr{B}_1$ such that $x \in [f([f^{-1}(x)]^{\circ} \cup \bigcup \mathscr{B}_2)]^{\circ} = [f(\bigcup \mathscr{B}_2)]^{\circ}$. It follows from $[f(\bigcup \mathscr{B}_2)]^{\circ} \in \mathscr{P}$ and $[f(\bigcup \mathscr{B}_2)]^{\circ} \subset U$ that x has a countable neighborhood base in X.

In summary, X is a first-countable space at A. □

Definition 2.7. Let $\mathscr{P}$ be a family of subsets of a space X and $A \subset X$. The family $\mathscr{P}$ is called a *cs*-network* at A for X [16, Definition 2.6(2)] if, for each $x \in A$, any sequence $\{x_n\}_{n \in \mathbb{N}}$ in X converging to x and $x \in U \in \tau_X$, there exist a subsequence $\{x_{n_i}\}_{i \in \mathbb{N}}$ of $\{x_n\}_{n \in \mathbb{N}}$ and $P \in \mathscr{P}$ such that $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$; the family $\mathscr{P}$ is called a *cs*-network* for X if it is a *cs*-network* at X [21, Definition 3].

A family $\mathscr{P}$ of subsets of a space X is called a *network* at a point $x \in X$ [3] if $x \in \bigcap \mathscr{P}$, and for each neighborhood U of x in X, there is $P \in \mathscr{P}$ such that $P \subset U$.

Definition 2.8. Let X be a space and $A \subset X$. A sequence $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ of families of subsets in X is called a *point-star network* at A for X [13, Definition 2.4] if $\{\mathrm{st}(x, \mathscr{P}_i)\}_{i\in\mathbb{N}}$ is a network at x in X for each $x \in A$; $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ is called a *point-star network* for X [22, Definition 5(2)] if $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ is a point-star network at X.

Definition 2.9. Let $\mathscr{P}$ be a family of subsets of a space X and $A \subset X$.
(1) The family $\mathscr{P}$ is called a *uniform cover* at A for X [13, Definition 2.3(1)] if, for $x \in A$, each countably infinite subset $\mathscr{P}'$ of $(\mathscr{P})_x$ is a network at x in X; $\mathscr{P}$ is called a *uniform cover* for X [23] if $\mathscr{P}$ is a uniform cover at X.
(2) The family $\mathscr{P}$ is called a *point-regular cover* at A for X [13, Definition 2.3(2)] if, for each $x \in A$ and $x \in U \in \tau_X$, the family $\{P \in (\mathscr{P})_x : P \not\subset U\}$ is finite; $\mathscr{P}$ is called a *point-regular cover* [23] for X if $\mathscr{P}$ is a point-regular cover at X.
(3) The family $\mathscr{P}$ is called a *cs*-cover* at A for X if $\{x_n\}_{n\in\mathbb{N}}$ is a sequence converging to a point $x \in A$ in X, then there exists $P \in \mathscr{P}$ such that some subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P; $\mathscr{P}$ is called a *cs*-cover* for X [24] if $\mathscr{P}$ is a cs^*-cover at X.

Definition 2.10. Suppose that $\mathscr{P}$ is a family of subsets of a space X such that, for each $x \in X$, there is a countable subfamily of $\mathscr{P}$, which is a network at x in X. Let $\mathscr{P} = \{P_\alpha : \alpha \in \Lambda\}$, which is no repetition by indexes in the enumeration, and Λ be endowed with the discrete topology. Put

$$M = \left\{\alpha = (\alpha_i) \in \Lambda^\omega : \left\{P_{\alpha_i}\right\}_{i\in\mathbb{N}} \text{ forms a network at some point } x_\alpha \text{ in } X\right\}.$$

Define a function $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. Then $(f, M, X, \mathscr{P})$ is called *Ponomarev's system* [25, p. 296].

Definition 2.11. Let $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ be a sequence of subset families in a space X, satisfying for each $x \in X$ and each $i \in \mathbb{N}$, there is $P_{x,i} \in \mathscr{P}_i$ such that the family $\{P_{x,i}\}_{i\in\mathbb{N}}$ is a network at x. For each $i \in \mathbb{N}$, let $\mathscr{P}_i = \{P_\alpha : \alpha \in \Lambda_i\}$, and Λ_i be endowed with the discrete topology. Put

$$M = \{\alpha = (\alpha_i) \in \prod_{i\in\mathbb{N}} \Lambda_i : \left\{P_{\alpha_i}\right\}_{i\in\mathbb{N}} \text{ forms a network at some point } x_\alpha \text{ in } X\}.$$

Define a function $f : M \to X$ by $f(\alpha) = x_\alpha$ for each $\alpha \in M$. Then, $(f, M, X, \{\mathscr{P}_i\})$ is called *Ponomarev's system* [25, p. 296].

Ponomarev's system is one of the important methods to construct metric spaces, and it is also a basic tool to discuss the images of metric spaces under certain mappings [1,2].

Lemma 2.12. *Let $(f, M, X, \mathscr{P})$ be Ponomarev's system and $x \in X$. Then*
(1) *M is a metric space, and $f : M \to X$ is a mapping* [25, Lemma 1(1)].
(2) *$f^{-1}(x)$ is separable in M if and only if $(\mathscr{P})_x$ is countable* [14, Lemma 1.3].

Lemma 2.13. *Let $(f, M, X, \{\mathscr{P}_i\})$ be Ponomarev's system and $x \in X$. Then*
(1) *M is a metric space, and $f : M \to X$ is a mapping* [15, p. 4].
(2) *If $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ is a point-star network at x in X, then $f^{-1}(x)$ is compact in M if and only if each $\mathscr{P}_i$ is point-finite at x* [15, Lemma 2.6].
(3) *f is a sequentially quotient mapping at x if and only if $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ is a sequence of cs^*-cover at x for X* [26, Theorem 2.7(2)].

The result (3) in Lemma 2.13 was proved for each point x in the space X, but it is easy to see that it is also correct for each fixed point x in X.

3 On sequentially quotient and compact mappings

The purpose of this section is to give some characterizations of the images of metric spaces under sequentially quotient and compact mappings at subsets. Furthermore, we give an affirmative answer to Question 1.4, and improve and perfect the characterizations of the images of metric spaces under compact mappings at subsets.

Lemma 3.1. [13, Lemma 3.1] *Let $\mathscr{P}$ be a family of subsets of a space X and $A \subset X$. Then the followings are equivalent.*

(1) *$\mathscr{P}$ is a point-regular cover at A for X.*
(2) *$\mathscr{P}$ is a uniform cover at A for X.*
(3) *For each $x \in A$, if $\{P_n : n \in \mathbb{N}\}$ is an infinite set of $(\mathscr{P})_x$ and U is a sequential neighborhood of x in X, then there is $m \in \mathbb{N}$ such that $P_n \subset U$ for all $n \geqslant m$.*

Proposition 3.2. *Suppose that A is a subset of a space X. If X has a point-regular cs^*-network at A for X, then X has a sequence of point-countable cs^*-covers at A, which is a point-star network at A for X.*

Proof. Let $\mathscr{P}$ be a point-regular cs^*-network at A for X. We can assume that $\mathscr{P}$ is closed under finite intersections (see [13, Lemma 3.1]) and $\{\{x\} : x \in S(X) \cap A\} \subset \mathscr{P}$.

Claim a. $\mathscr{P}$ is point-countable at A (see [13, Claim 1 in the proof of Lemma 3.2]).

Put

$$\mathscr{P}^m = \{H \in \mathscr{P} : \quad \text{If } H \subset P \in \mathscr{P}, \quad \text{then } P = H\},$$
$$\mathscr{P}' = (\mathscr{P} \backslash \mathscr{P}^m) \cup \{\{x\} : x \in S(X) \cap A\}.$$

Claim b. If $x \in A$ and $P \in (\mathscr{P})_x$, then there exists $H \in \mathscr{P}^m$ such that $P \subset H$.

To the contrary, assume that there exists an infinite subset $\{P_n : n \in \mathbb{N}\}$ of $\mathscr{P}$ such that $P \subsetneq P_1 \subsetneq P_2 \subsetneq \ldots P_n \subsetneq P_{n+1} \subsetneq \cdots$. Then there exists a point $y \neq x$ such that $\{x, y\} \subset P_n$ for each $n \in \mathbb{N}$. Hence, $\{P_n : n \in \mathbb{N}\} \subset \{P \in (\mathscr{P})_x : P \not\subset X \backslash \{y\}\}$, which is a contradiction.

Claim c. $\mathscr{P}'$ is a point-regular cs^*-network at A for X.

It suffices to prove that $\mathscr{P}'$ is a cs^*-network at $A \cap NS(X)$ for X. Let $x \in A \cap NS(X)$, $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to x and $x \in U \in \tau_X$. We may assume that every $x_n \neq x$. Since $\mathscr{P}$ is a cs^*-network at A for X, there exist $P_1 \in \mathscr{P}$ and some subsequence S_1 of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x\} \cup S_1 \subset P_1 \subset U$. Pick $y \in S_1$, then there exist $P_2 \in \mathscr{P}$ and some subsequence S_2 of S_1 such that $\{x\} \cup S_2 \subset P_2 \subset U \backslash \{y\}$. Put $P = P_1 \cap P_2$, then $x \in P \subsetneq P_1 \subset U$ and P contains a subsequence of $\{x_n\}_{n\in\mathbb{N}}$. Hence, $P \in \mathscr{P}'$. It implies that $\mathscr{P}'$ is a cs^*-network at A for X. So the proof of (c) is completed.

Let

$$\mathscr{P}_1 = \mathscr{P}^m, \quad \mathscr{P}_{n+1} = \left[\left(\mathscr{P} \backslash \bigcup_{i \leqslant n} \mathscr{P}_i\right) \cup \{\{x\} : x \in S(X) \cap A\}\right]^m, \quad n \in \mathbb{N}.$$

It follows from Claims (a)–(c) that $\mathscr{P} = \bigcup_{n\in\mathbb{N}} \mathscr{P}_n$ and every $\mathscr{P}_n$ is a point-countable cs^*-cover at A. We claim that $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ is a point-star network at A for X. Let $x \in A$ and $P_n \in (\mathscr{P}_n)_x$, $\forall n \in \mathbb{N}$. If $x \in S(X)$, there exists $m \in \mathbb{N}$ such that $P_m = \{x\}$, whence $\{P_n : n \in \mathbb{N}\}$ is a network at x in X. If $x \in NS(X)$, since $P_1, P_2, \ldots P_n, \ldots$ are distinct, it follows from Lemma 3.1 that $\{P_n : n \in \mathbb{N}\}$ is a network at x in X. Therefore, $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ is a point-star network at A for X. □

Lemma 3.3. [8, Theorem 2.3] *The followings are equivalent for a space X.*

(1) *X has a point-regular cs^*-network.*
(2) *X has a sequence of point-finite cs^*-covers, which is a point-star network for X.*
(3) *X is a sequentially quotient and compact image of a metric space.*

The following is the main result in this section, which generalizes the aforementioned lemma.

Theorem 3.4. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X has a point-regular cs^*-network at A for X.*
(2) *X has a uniform cs^*-network at A for X.*
(3) *X has a sequence of point-finite cs^*-covers of A, which is a point-star network at A for X.*
(4) *X is the image of a metric space under a sequentially quotient and compact mapping at A.*

Proof. By Lemma 3.1, we have that (1) $\Leftrightarrow$ (2). Next, we will prove that (1) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Let $\mathscr{P}$ be a point-regular cs^*-network at A for X. We can assume that $\mathscr{P}$ is closed under finite intersections and $\{\{x\} : x \in S(X) \cap A\} \subset \mathscr{P}$. It follows from Proposition 3.2 that the family $\mathscr{P}$ can be expressed as $\bigcup_{n\in\mathbb{N}}\mathscr{P}_n$, where $\{\mathscr{P}_n\}_{n\in\mathbb{N}}$ is a sequence of cs^*-covers of A, which is a point-star network at A for X and each $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$. Put $\mathscr{P}|_A = \{P \cap A : P \in \mathscr{P}\}$. Because $\mathscr{P}|_A$ is a point-regular cs^*-network for the subspace A, it follows from Lemma 3.3 that the family $\mathscr{P}|_A$ can be expressed as $\bigcup_{n\in\mathbb{N}}\mathscr{Q}_n$, where $\{\mathscr{Q}_n\}_{n\in\mathbb{N}}$ is a point-star network consisting of point-finite cs^*-covers in the space A and each $\mathscr{Q}_{n+1}$ refines $\mathscr{Q}_n$. For each $n \in \mathbb{N}$, put

$$\mathscr{R}_n = \mathscr{Q}_n \cup \{\{x\} \cup (\mathrm{st}(x, \mathscr{P}_n)\backslash A) : x \in A\}.$$

It is obvious that $\mathscr{R}_n$ is a point-finite family at A for X. Let $L = \{x_i\}_{i\in\mathbb{N}}$ be a sequence converging to some point $x \in A$ in X. If $|\{i \in \mathbb{N} : x_i \in A\}| = \omega$, since $\mathscr{Q}_n$ is a cs^*-cover for the space A, there exists $Q \in \mathscr{Q}_n$ such that some subsequence L' of L is eventually in Q; if $|\{i \in \mathbb{N} : x_i \in A\}| < \omega$, since $\mathscr{P}_n$ is a cs^*-cover at A for X, there exists $P \in \mathscr{P}_n$ such that some subsequence L'' of L is eventually in P, it follows that the sequence L'' is eventually in the set $\{x\} \cup (\mathrm{st}(x, \mathscr{P}_n)\backslash A)$. It implies that $\mathscr{R}_n$ is a cs^*-cover at A for X. We claim that $\{\mathscr{R}_n\}_{n\in\mathbb{N}}$ is a point-star network at A for X.

Let $x \in A$ and U be a neighborhood of x in X. There exist $i, j \in \mathbb{N}$ such that $\mathrm{st}(x, \mathscr{Q}_i) \subset U \cap A$ and $\mathrm{st}(x, \mathscr{P}_j) \subset U$. Put $m = \max\{i, j\}$. Then

$$\mathrm{st}(x, \mathscr{R}_m) = \mathrm{st}(x, \mathscr{Q}_m) \cup (\mathrm{st}(x, \mathscr{P}_m)\backslash A) \subset U.$$

Hence, $\{\mathscr{R}_n\}_{n\in\mathbb{N}}$ is a point-star network at A for X.

(3) $\Rightarrow$ (4). Suppose that X has a sequence $\{\mathscr{P}_i\}_{i\in\mathbb{N}}$ of point-finite cs^*-covers of A which is a point-star network at A for X. For each $x \in X\backslash A$ and $i \in \mathbb{N}$, we may assume that $\{x\} \in \mathscr{P}_i$. Let $(f, M, X, \{\mathscr{P}_i\})$ be Ponomarev's system. It follows from Lemma 2.13 that M is a metric space and $f : M \to X$ is a sequentially quotient and compact mapping at A.

(4) $\Rightarrow$ (1). Suppose that $f : M \to X$ is a sequentially quotient and compact mapping at A, where M is a metric space. Let $\{\mathscr{B}_i\}_{i\in\mathbb{N}}$ be a sequence of locally finite open covers of M such that each $\mathscr{B}_{i+1}$ refines $\mathscr{B}_i$; and for every compact subset K of M, $\{\mathrm{st}(K, \mathscr{B}_i) : i \in \mathbb{N}\}$ is a neighborhood base of K in M [3, Exercises, 5.4.E(a)]. For each $i \in \mathbb{N}$, put $\mathscr{P}_i = \{f(B) : B \in \mathscr{B}_i\}$, then $\mathscr{P}_i$ is a cover of X. Since f is compact at A, it follows that $\mathscr{P}_i$ is point-finite at A. Put $\mathscr{P} = \bigcup_{i\in\mathbb{N}}\mathscr{P}_i$. We claim that $\mathscr{P}$ is a point-regular cs^*-network at A for X.

Since f is sequentially quotient at A and $\bigcup_{i\in\mathbb{N}}\mathscr{B}_i$ is a base for M, it is easy to verify that the family $\mathscr{P}$ is a cs^*-network at A for X. For each $x \in A$, let V be an open neighborhood of x in X. Since the compact subset $f^{-1}(x)$ of M satisfies $f^{-1}(x) \subset f^{-1}(V)$, it follows that there exists $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathscr{B}_n) \subset f^{-1}(V)$. Then for each $i \geqslant n$, $\mathrm{st}(x, \mathscr{P}_i) \subset \mathrm{st}(x, \mathscr{P}_n) \subset V$. Thus, $\{P \in (\mathscr{P})_x : P \not\subset V\} \subset (\bigcup_{i<n}\mathscr{P}_i)_x$ is finite. Hence, $\mathscr{P}$ is a point-regular cs^*-network at A for X. □

Corollary 3.5. *The followings are equivalent for a space X.*
(1) *X has a cs^*-network which is point-regular at $NI(X)$ for X.*
(2) *X has a point-regular cs^*-network at $NI(X)$ for X.*
(3) *X is a sequentially quotient and almost compact image of a metric space.*

It is known that suppose that M is a sequential space then a mapping $f : M \to X$ is quotient if and only if X is a sequential space and f is sequentially quotient [2, Propositions 2.1.12(2)(5) and 2.3.1(1)]. By Corollary 3.5, we have the following corollary, which gives an affirmative answer to Question 1.4.

Corollary 3.6. *The followings are equivalent for a space X.*
(1) *X is a quotient and almost compact image of a metric space.*
(2) *X is a sequential space with a point-regular cs^*-network at nonisolated points.*

Corollary 3.7. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under a pseudo-open and compact mapping at A.*
(2) *X is the image of a metric space under a bi-quotient and compact mapping at A.*
(3) *X is a first-countable space at A and has a point-regular cs^*-network at A for X.*

Proof. It follows from Lemmas 2.6, 2.5(3), and Theorem 3.4 that (1) ⇒ (3). By Theorem 3.4 and Lemma 2.5(1), we have that (3) ⇒ (2). Obviously, (2) ⇒ (1). □

The following corollary is related to Question 1.6, which gives a characterization of the image of a metric space under a pseudo-open and almost compact mapping.

Corollary 3.8. *The followings are equivalent for a space X.*
(1) *X is a pseudo-open and almost compact image of a metric space.*
(2) *X is a bi-quotient and almost compact image of a metric space.*
(3) *X is a first-countable space having a cs^*-network which is point-regular at nonisolated points.*

At the end of this section, we discuss a version at subsets of Theorem 1.2. For ease of reading and proof, we quote the following results.

Lemma 3.9. [13, Theorems 3.3 and 3.4] *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under an open and compact mapping at A.*
(2) *There exists a compact mapping at A, $f : M \to X$ from a metric space M satisfying the following condition: for each $x \in A$, there is a point $z \in f^{-1}(x)$ such that $f(U)$ is a sequential neighborhood of x in X whenever U is a neighborhood of z in M.*
(3) *X has a point-regular base at A for X.*

Theorem 3.10. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under an almost-open and compact mapping at A.*
(2) *X is the image of a metric space under a strictly countably bi-quotient and compact mapping at A.*
(3) *X has a point-regular base at A for X.*

Proof. By Lemma 2.3(2), (1) ⇔ (2). By Lemma 3.9, (3) ⇒ (1) and (1) ⇒ (3). □

The following result is a supplement to Theorem 1.2.

Corollary 3.11. *The followings are equivalent for a space X*
(1) *X is an almost-open (or a strictly countably bi-quotient) and almost compact image of a metric space.*
(2) *X has a base which is point-regular at $NI(X)$.*

4 On countably bi-quotient s-mappings

In this section, we present an example to give negative answers to Questions 1.5 and 1.6, see Example 4.1. Furthermore, we consider what conditions need to be given so that the answer to Question 1.5 is affirmative, see Corollary 4.12, and improve and perfect the characterizations of the images of metric spaces under s-mappings at subsets.

Example 4.1. Consider two subsets in the plane $\mathbb{R}^2$:

$$C_i = \{(a, b) \in \mathbb{R}^2 : b = i \text{ and } 0 \leq a \leq 1\}, \quad \text{where } i = 1, 2,$$

and let $X = C_1 \cup C_2$. Specify the neighborhood base $\mathscr{B}(x)$ of each point $x \in X$ as follows: let $\mathscr{B}(x) = \{\{x\}\}$ for $x \in C_2$; and let $\mathscr{B}(x) = \{B_k(x) : k \in \mathbb{N}\}$ for $x = (a, 1) \in C_1$, where

$$B_k(x) = \{x\} \cup \{(a', b') \in X : 0 < |a - a'| < 1/k\}.$$

The space X is called *the Alexandroff double lines space* [3, Example 3.1.26]. It is not difficult to verify that X is a Hausdorff, compact, and first-countable space.

(1) X has no base that is point-countable at $NI(X)$. Suppose not, let $\mathcal{B}$ be a base which is point-countable at $NI(X)$. C_1 is separable metrizable, let $D \subset C_1$ be a countable dense subset, and let $\mathcal{B}_1 = \{B \in \mathcal{B} : B \cap D \neq \varnothing\}$, then $\mathcal{B}_1$ is countable. It implies that $\{\{x\} : x \in C_2\} \cup \mathcal{B}_1$ is a σ-discrete base of X. This is a contradiction since X is not metrizable. So then X has no base which is point-regular at $NI(X)$ by Claim in the proof of Proposition 3.2, and X is not a strictly countably bi-quotient and almost s-image of a metric space (see Corollary 4.7).

Obviously, $NI(X) = C_1$ and C_1 is compact in X. However, C_1 is not a G_δ-set in X. So X has no base which is point-countable at $NI(X)$.

(2) X has a point-regular cs^*-network at $NI(X)$ for X, so then X is a bi-quotient and almost compact image of a metric space by Corollary 3.8.

Let $Z = \mathbb{R} \times \{1\}$. It follows from the metrizability of $\mathbb{R}$ that Z has a point-regular base $\mathscr{U}$. Let $\mathscr{V} = \{U \cap C_1 : U \in \mathscr{U}\}$. For each $x \in C_1$, put

$$\mathscr{W}_x = \bigcup\{\{x\} \cup [B_k(x) \cap C_2] : k \in \mathbb{N}\} \quad \text{and} \quad \mathscr{W} = \mathscr{V} \cup \bigcup\{\mathscr{W}_x : x \in C_1\}.$$

It is easy to see that $\mathscr{W}$ is a point-regular cs^*-network at $NI(X)$ for X.

Although the answer to Question 1.5 is negative, it is interesting to obtain a characterization of the image of a metric space under a countably bi-quotient and almost s-mapping. A basic result in this direction is the following lemma.

Lemma 4.2. [16, Theorem 3.2] *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under a sequentially quotient and s-mapping at A.*
(2) *X has a point-countable cs^*-network at A for X.*

Theorem 4.3. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under a (countably) bi-quotient and s-mapping at A.*
(2) *X is first-countable at A and has a point-countable cs^*-network at A for X.*

Proof. By Lemma 2.3(1), it is known that the image of a metric space under a bi-quotient and s-mapping at A coincides with the image of a metric space under a countably bi-quotient and s-mapping at A.

By Lemmas 2.6, 2.5, and 4.2, we have that (1) $\Leftrightarrow$ (2). □

Corollary 4.4. *The followings are equivalent for a space X.*
(1) *X is a (countably) bi-quotient and almost s-image of a metric space.*
(2) *X is a first-countable space having a cs^*-network, which is point-countable at $NI(X)$.*

For the sake of completeness and analogy Theorem 3.10 and Corollary 3.7, we will further give some characterizations of the images of metric spaces under open (resp., strictly countably bi-quotient, or pseudo-open) and s-mappings at A.

Corollary 4.5. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under a pseudo-open and s-mapping at A.*
(2) *X is a Fréchet space at A and has a point-countable cs^*-network at A for X.*

Proof. By Lemmas 2.5(3) and 4.2, we have that (1) ⇒ (2). By Lemmas 4.2 and 2.5(2), we have that (2) ⇒ (1). □

Theorem 4.6. *The followings are equivalent for a space X and $A \subset X$.*
(1) *X is the image of a metric space under an open and s-mapping at A.*
(2) *X is the image of a metric space under a strictly countably bi-quotient and s-mapping at A.*
(3) *X has a point-countable base at A for X.*

Proof. Clearly, (1) ⇒ (2). Next, we will prove that (2) ⇒ (3) ⇒ (1).

(2) ⇒ (3). Let $f: M \to X$ be a strictly countably bi-quotient and s-mapping at A and let $\mathscr{B}$ be a point-countable base of M. For each $x \in A$, let $\mathscr{B}_x = \{B \in \mathscr{B} : B \cap f^{-1}(x) \neq \varnothing\}$ and $\mathscr{P}_x = f(\mathscr{B}_x)$. Since the set $f^{-1}(x)$ is separable, families $\mathscr{B}_x$ and $\mathscr{P}_x$ are non-empty and countable. Put $\mathscr{P} = \bigcup_{x\in A}\mathscr{P}_x$ and $\mathscr{Q} = \{P^\circ : P \in \mathscr{P}\}$. Then $\mathscr{Q}$ is point-countable at A. Next, we will show that $\mathscr{Q}$ is a base at A for X. If $x \in A$ and U is an arbitrary neighborhood of x in X, then $f^{-1}(x) \subset f^{-1}(U)$; thus, there exists $\mathscr{B}'_x \subset \mathscr{B}_x$ such that $f^{-1}(x) \subset \bigcup\mathscr{B}'_x \subset f^{-1}(U)$. Since f is a strictly countably bi-quotient mapping at A, there is $B \in \mathscr{B}'_x$ such that $x \in [f(B)]^\circ$. It follows from $[f(B)]^\circ \in \mathscr{Q}$ and $[f(B)]^\circ \subset U$ that X has the point-countable base at A for X.

(3) ⇒ (1). Let $\mathscr{P}$ be a point-countable base at A for X. We may assume that $\{\{x\} : x \in X\backslash A\} \subset \mathscr{P}$ and put $\mathscr{P} = \{P_\alpha : \alpha \in \Lambda\}$. Then $\mathscr{P}$ is a network for X. Let $(f, M, X, \mathscr{P})$ be Ponomarev's system. By Lemma 2.12, M is a metric space and f is an s-mapping at A.

For each $\alpha = (\alpha_i) \in M$ and $k \in \mathbb{N}$, put

$$B(\alpha_1, \alpha_2, \ldots, \alpha_k) = \{(\gamma_i) \in M : \alpha_i = \gamma_i \text{ if } i \leqslant k\}.$$

Then $f(B(\alpha_1, \alpha_2, \ldots, \alpha_k)) = \bigcap_{i\leqslant k}P_{\alpha_i}$ (see [2, part (4.2) of Proposition 2.4.4]) and $\{B(\alpha_1, \alpha_2, \ldots, \alpha_k) : k \in \mathbb{N}\}$ is a local base at α in M.

Let $x \in A$ and $z = (\alpha_i) \in f^{-1}(x)$. If V is a neighborhood of z in M, there exists $k \in \mathbb{N}$ such that $B(\alpha_1, \alpha_2, \ldots, \alpha_k) \subset V$, so then $\bigcap_{i\leqslant k}P_{\alpha_i} = f(B(\alpha_1, \alpha_2, \ldots, \alpha_k)) \subset f(V)$, thus $f(V)$ is a neighborhood of x in X. Then f is an open mapping at A. □

The following result is a supplement to Theorem 1.1.

Corollary 4.7. *The followings are equivalent for a space X.*
(1) *X is an almost-open (or a strictly countably bi-quotient) and almost s-image of a metric space.*
(2) *X has a base which is point-countable at $NI(X)$.*

At the end of this section, we discuss the conditions under which Question 1.5 has a positive answer.

Recall two related concepts [27]. Let $\mathscr{P}$ be a family of subsets of a space X and $A \subset X$. Put $\mathscr{P}^{<\omega} = \{\mathscr{F} \subset \mathscr{P} : \mathscr{F} \text{ is finite}\}$. The family $\mathscr{P}$ is said to satisfy (BM) at A if, whenever $x \in A$ and U is a neighborhood of x in X, there exists $\mathscr{F} \in \mathscr{P}^{<\omega}$ such that

$$x \in \bigcap\mathscr{F}, \quad x \in (\bigcup\mathscr{F})^\circ \quad \text{and} \quad \bigcup\mathscr{F} \subset U.$$

$\mathscr{P}$ is called a *minimal interior cover* of A if $A \subset (\bigcup\mathscr{P})^\circ$ and $A \not\subset (\bigcup\mathscr{H})^\circ$ for any proper subset $\mathscr{H}$ of $\mathscr{P}$.

Lemma 4.8. *Let X be a space and $A \subset X$. Suppose that X is a first-countable space at A and $\mathscr{P}$ is a point-countable cs^*-network at A for X, then $\mathscr{P}$ satisfies (BM) at A.*

Proof. For each $x \in A$, put

$$\mathscr{F}_x = \{\bigcup\mathscr{F} : \mathscr{F} \in (\mathscr{P})_x^{<\omega} \text{ and } x \in (\bigcup\mathscr{F})^\circ\}.$$

Then $\mathscr{F}_x$ is countable. To complete the proof, we only need to show that the family $\mathscr{F}_x$ is a network at x in X.

To the contrary, assume that there exists a neighborhood G of x in X with $F \not\subset G$ for each $F \in \mathscr{F}_x$. Put

$$\{P \in (\mathscr{P})_x : P \subset G\} = \{P_i : i \in \mathbb{N}\}; \quad F_n = \bigcup\{P_i : i \leq n\}, n \in \mathbb{N}.$$

So each F_n is not a neighborhood of x in X. Since X is first-countable at x and $(\mathscr{P})_x$ is a cs^*-network at x in X, for each $i \in \mathbb{N}$, there are a sequence T_i converging to the point x and $n_i \in \mathbb{N}$ such that $T_i \subset P_{n_{i+1}} \backslash F_{n_i}$ and $n_{i+1} > n_i$. Put $T = \{x\} \cup \bigcup \{T_i : i \in \mathbb{N}\}$. It follows that there is a sequence $\{x_k\}_{k \in \mathbb{N}}$ in T converging to x, which meets infinitely many sequences T_i. Then there exists $i \in \mathbb{N}$ such that P_i contains a subsequence $\{x_{k_m}\}_{m \in \mathbb{N}}$ of $\{x_k\}_{k \in \mathbb{N}}$. So there are $m, j \in \mathbb{N}$ such that $j \geqslant i$ and $x_{k_m} \in T_j$, whence $x_{k_m} \in P_i \cap (X \backslash F_{n_j}) = \varnothing$, which is a contradiction. Thus, $\mathscr{F}_x$ is a network at x in X. □

Lemma 4.9. *Let X be a space and $A \subset X$. Suppose that X is a first-countable space at A and has a point-countable cs^*-network at A for X. If $|\partial A| \leqslant \omega$, then X has a point-countable base at A for X.*

Proof. Let $\mathscr{P}$ be a point-countable cs^*-network at A for X. Let $\mathscr{Q} = \bigcup \{\mathscr{Q}_x : x \in A\}$, where each

$$\mathscr{Q}_x = \{P \cap A : P \in (\mathscr{P})_x\} \bigcup \{\{x\} \cup (P \backslash A) : P \in (\mathscr{P})_x \text{ and } x \in \overline{P \backslash A}\}.$$

It is easy to check that $\mathscr{Q}$ is a point-countable cs^*-network at A for X. It follows from Lemma 4.8 that the family $\mathscr{Q}$ satisfies (BM) at A. Let

$$\Phi = \{\mathscr{F} \in \mathscr{Q}^{<\omega} : A \cap \bigcap \mathscr{F} \neq \varnothing\}.$$

For each $\mathscr{F} \in \Phi$, put $V(\mathscr{F}) = [\bigcup \mathscr{H}(\mathscr{F}) \cap \mathscr{Q}]^\circ$, where

$$\mathscr{H}(\mathscr{F}) = \{H \subset X : \mathscr{F} \text{ is a minimal interior cover of } H \cap A\}.$$

Put

$$\mathscr{V} = \{V(\mathscr{F}) : \mathscr{F} \in \Phi\}.$$

Claim 1. $\mathscr{V}$ is a base at A for X.

Let $x \in A$ and U be a neighborhood of x in X. By condition (BM), there exist $\mathscr{F}, \mathscr{B} \in \Phi$ such that $x \in (\bigcap \mathscr{F}) \cap (\bigcap \mathscr{B})$ and $x \in (\bigcup \mathscr{B})^\circ \subset \bigcup \mathscr{B} \subset (\bigcup \mathscr{F})^\circ \subset U$. We may assume that the family $\mathscr{F}$ is a minimal interior cover of $\{x\}$, and the family $\mathscr{B}$ satisfies that $\mathscr{F}$ is a minimal interior cover of $B \cap A$ for each $B \in \mathscr{B}$, i.e., $B \in \mathscr{H}(\mathscr{F})$. Hence, $(\bigcup \mathscr{B})^\circ \subset V(\mathscr{F}) \in \mathscr{V}$. It follows that $x \in V(\mathscr{F}) \subset U$. So $\mathscr{V}$ is a base at A for X.

Claim 2. $\mathscr{V}$ is point-countable at A.

Let $x \in A$. If $x \in V(\mathscr{F}) \in \mathscr{V}$, then there exists $K \in \mathscr{H}(\mathscr{F}) \cap \mathscr{Q}$ such that $x \in K$. Since $(\mathscr{Q})_x$ is countable, to complete the proof of Claim 2, we only need to prove the following claim: suppose that $K \cap A \neq \varnothing$, then $K \in \mathscr{H}(\mathscr{F})$ for at most countably many $\mathscr{F} \in \Phi$.

To the contrary, assume that $K \in \mathscr{H}(\mathscr{F})$ for uncountably many $\mathscr{F} \in \Phi$. It follows from $\Phi = \bigcup_{n \in \mathbb{N}} \{\mathscr{F} \subset \mathscr{Q}^{<\omega} : |\mathscr{F}| = n \text{ and } A \cap \bigcap \mathscr{F} \neq \varnothing\}$ that we can choose $m \in \mathbb{N}$ and an uncountable subset Φ' of Φ such that $|\mathscr{F}| = m$ and $K \in \mathscr{H}(\mathscr{F})$ for every $\mathscr{F} \in \Phi'$. According to the Zorn lemma, suppose that $\mathcal{M}$ is a maximal subset of $\mathscr{Q}$ satisfying $\{\mathscr{F} \in \Phi' : \mathcal{M} \subset \mathscr{F}\}$ is uncountable. Then $0 \leq |\mathcal{M}| < m$ and $K \cap A \not\subset (\bigcup \mathcal{M})^\circ$. Pick a point $y \in (K \cap A) \backslash (\bigcup \mathcal{M})^\circ$, then $y \in \overline{X \backslash \bigcup \mathcal{M}}$. Since X is first-countable at A, there exists a sequence L in $X \backslash \bigcup \mathcal{M}$ converging to y. Let $\Phi'' = \{\mathscr{F} \in \Phi' : \mathcal{M} \subset \mathscr{F}\}$. For each $\mathscr{F} \in \Phi''$, it follows from $y \in K \cap A \subset (\bigcup \mathscr{F})^\circ$ that $L \cap (\bigcup \mathscr{F})^\circ \neq \varnothing$. We may assume that $L \subset A$ or $L \cap A = \varnothing$.

If $L \subset A$, since $\mathscr{Q}$ is point-countable at A and Φ'' is uncountable, there exists $Q \in \mathscr{Q}$ such that $Q \cap L \neq \varnothing$ and $\{\mathscr{F} \in \Phi'' : Q \in \mathscr{F}\}$ is uncountable. It follows from $L \subset X \backslash \bigcup \mathcal{M}$ that $Q \notin \mathcal{M}$ and $\{\mathscr{F} \in \Phi'' : \mathcal{M} \cup \{Q\} \subset \mathscr{F}\}$ is uncountable. This implies that $\mathcal{M}$ is not maximal, which is a contradiction.

Now, we assume that $L \cap A = \varnothing$. For each $\mathscr{F} \in \Phi''$, there is $P_{\mathscr{F}} \in \mathscr{F}$ such that $P_{\mathscr{F}} \cap L \neq \varnothing$; thus, $P_{\mathscr{F}} \notin \mathcal{M}$. It follows from $A \cap \bigcap \mathscr{F} \neq \varnothing$ that $P_{\mathscr{F}} \cap A \neq \varnothing$, so we can fix a point $x_{\mathscr{F}} \in P_{\mathscr{F}} \cap A$. By the definition of $\mathscr{Q}$, $\{x_{\mathscr{F}}\} = P_{\mathscr{F}} \cap A$ and $x_{\mathscr{F}} \in \overline{P_{\mathscr{F}} \backslash A}$. Let $T = \{x_{\mathscr{F}} : \mathscr{F} \in \Phi''\}$. Then, $T \subset A \cap \overline{X \backslash A} \subset \partial A$. It follows from $|\partial A| \leqslant \omega$ that T is countable. Let $\mathscr{Q}' = \{P_{\mathscr{F}} : \mathscr{F} \in \Phi''\}$. By the point-countability of $\mathscr{Q}$, the family $\mathscr{Q}'$ is countable. Since Φ'' is uncountable, there exists $R \in \mathscr{Q}'$ such that $\{\mathscr{F} \in \Phi'' : R \in \mathscr{F}\}$ is uncountable, so then $\{\mathscr{F} \in \Phi'' : \mathcal{M} \cup \{R\} \subset \mathscr{F}\}$ is uncountable. This implies that $\mathcal{M}$ is not maximal, which is a contradiction.

According to Claims 1 and 2, X has the point-countable base $\mathscr{V}$ at A for X. □

By Theorem 4.3 and Lemmas 4.8 and 4.9, we obtain the following result.

Theorem 4.10. *Suppose that A is a subset of a space X satisfying $|\partial A| \leqslant \omega$. Then the followings are equivalent*:
(1) *X has a point-countable base at A for X*;
(2) *X is the image of a metric space under a countably bi-quotient and s-mapping at A.*

Remark 4.11. It is obvious that $\partial X = \emptyset$ for a space X; thus, the followings are equivalent by Lemma 4.9 and Theorem 4.10.
(1) X has a point-countable base.
(2) X is the image of a metric space under a countably bi-quotient and s-mapping [18].
(3) X is a first-countable space and has a point-countable cs^*-network for X [1, Corollary 2.1.7].

Corollary 4.12. *The followings are equivalent for a space X with $|\partial NI(X)| \leqslant \omega$.*
(1) *X has a base which is point-countable at $NI(X)$.*
(2) *X is a countably bi-quotient and almost s-image of a metric space.*

5 Conclusion

In this article, we study some questions related to the almost s-images (resp., almost compact images) of metric spaces. The following conclusions are obtained.

Conclusion 5.1. A space X is a quotient and almost compact image of a metric space if and only if X is a sequential space having a cs^*-network which is point-regular at nonisolated points.

Conclusion 5.2. A space X is a pseudo-open (or bi-quotient) and almost compact image of a metric space if and only if X is a first-countable space having a cs^*-network which is point-regular at nonisolated points.

Conclusion 5.3. A space X is an almost-open (or a strictly countably bi-quotient) and almost s-image of a metric space if and only if X has a base that is point-countable at nonisolated points.

Conclusion 5.4. There exists a bi-quotient and almost compact image of a metric space satisfying no base which is point-countable at nonisolated points.

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