Research Article

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The dimension-free estimate for the truncated maximal operator

https://doi.org/10.1515/math-2022-0529
received January 15, 2022; accepted November 8, 2022

Abstract: We mainly study the dimension-free L^p-inequality of the truncated maximal operator

$$M_n^a f(x) = \sup_{t>0} \frac{1}{|B_a^1|} \left| \int_{B_a^1} f(x-ty)\mathrm{d}y \right|,$$

where $B_a^1 = \{x : a \le |x| \le 1\}$. When $0 \le a < 1$, we prove that $\|M_n^a\|_{L^p(\mathbb{R}^n)} \le C(p)\|f\|_{L^p(\mathbb{R}^n)}$ for $p > n/(n-1)$. When $a = 1$, we prove that $\|M_n^1\|_{L^p(\mathbb{R}^n)} \le C(p)\|f\|_{L^p(\mathbb{R}^n)}$ for $p \ge 2$.

Keywords: maximal function, dimension-free estimate, Fourier multiplier

MSC 2020: 42B20, 42B25

1 Introduction

Let $f \in \mathcal{S}(\mathbb{R}^n)$ be a rapidly decreasing function defined on $\mathbb{R}^n$. For $0 \le a < 1$, define the truncated maximal operator

$$M_n^a f(x) = \sup_{t>0} \frac{1}{|B_a^1|} \left| \int_{B_a^1} f(x-ty)\mathrm{d}y \right|,$$

where $B_a^1 \coloneqq \{x : a \le |x| \le 1\}$ and $|B_a^1|$ denote the volume of B_a^1. Conveniently, we denote the unite sphere S^{n-1} by B_1^1. Define

$$M_n^1 f(x) = \sup_{t>0} \frac{1}{|B_1^1|} \left| \int_{B_1^1} f(x-ty)\mathrm{d}\sigma(y) \right|,$$

where $\mathrm{d}\sigma(y)$ denotes the usual measure of the unite sphere and $|B_1^1|$ denotes its total mass. Obviously, M_n^0 is the Hardy-Littlewood maximal operator and M_n^1 is the sphere maximal operator. We mainly study the dimension-free L^p-boundedness of the truncated maximal operators M_n^a for $0 \le a \le 1$. Stein first studied the sphere maximal operator M_n^1. In [1], he pointed out that if $p > n/(n-1)$, then

$$\|M_n^1 f\|_{L^p(\mathbb{R}^n)} \le C(p,n)\|f\|_{L^p(\mathbb{R}^n)},$$

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where $C(p, n)$ denotes a constant depending on p and n. The case for $n = 2$ was proved by Bourgain [2]. Later many authors tried to generalize the spherical maximal function to other hyper-surface. In [3], Sogge and Stein pointed out that if the Gaussian curvature of $\mathcal{S}$ does not vanish of infinite order at each point of $\mathcal{S}$, then there exists a $2 < P_0(\mathcal{S}) < \infty$ such that the corresponding surface maximal operator is strongly L^p-bounded for $p > p_0(\mathcal{S})$. Moreover, when the surface has at least one non-vanishing principal curvature everywhere in $\mathbb{R}^n$, then the maximal operator maps L^p to L^p for $p > 2$ [4]. More related works can be seen in [5–14] and the references therein.

When $a = 0$, M_n^a is the classical Hardy-Littlewood maximal operator. According to the result of [1], Stein [15] obtained that there exists a constant $C(p)$ independent of n such that

$$\|M_n^0 f\|_{L^p(\mathbb{R}^n)} \le C(p)\|f\|_{L^p(\mathbb{R}^n)} \tag{1}$$

for $p > 1$. Analogously, many authors turned to focus on generalizing the dimension-free estimate for the maximal function to the general case. Bourgain [16] studied the maximal operator over central symmetric convex sets and obtained a similar result as (1) for $p = 2$. Later, Bourgain [17] and Carbery [18] generalized this result for all $p > 3/2$ in different ways independently. For some special convex set such as q-balls, $1 \le q < \infty$, Müller [19] extended the L^p bound to every $p > 1$. Recently, Bourgain [20] showed that the strong type constant can be bounded by a constant, which is independent of the dimension for the Hardy-Littlewood maximal operator over cubes for all $p > 1$. In this article, we will study the dimension-free estimate of the truncated maximal operator M_n^a, generalizing the results of Hardy-Littlewood maximal operators and sphere maximal operators proved in [1].

Theorem 1.1. *Let $f \in \mathcal{S}(\mathbb{R}^n)$. If $0 \le a < 1$, then there exists a constant $C(p)$ independent of n and a such that*

$$\|M_n^a f\|_{L^p(\mathbb{R}^n)} \le (1-a)^{-1}C(p)\|f\|_{L^p(\mathbb{R}^n)},$$

for $p > 1$. When $p > n/(n-1)$,

$$\|M_n^a f\|_{L^p(\mathbb{R}^n)} \le C(p)\|f\|_{L^p(\mathbb{R}^n)}.$$

If $a = 1$, then

$$\|M_n^1 f\|_{L^p(\mathbb{R}^n)} \le C(p)\|f\|_{L^p(\mathbb{R}^n)}, \tag{2}$$

for $p \ge 2$.

2 The proof of main theorem for the case $a = 1$

By interpolation theorem, we only need to prove the case $p = 2$. This result follows from the general L^2-estimate on convolution maximal operators obtained by Bourgain [16].

Lemma 2.1. *Let $K \in \mathcal{S}'(\mathbb{R}^n)$ be a tempered distribution. Suppose $\widehat{K} \in L^\infty(\mathbb{R}^n)$ and is differentiable. Define the following quantities*:

$$\alpha_j = \sup_{2^j \le |\xi| \le 2^{j+1}} |\widehat{K}(\xi)| \quad \text{and} \quad \beta_j = \sup_{2^j \le |\xi| \le 2^{j+1}} |\langle \nabla\widehat{K}(\xi), \xi\rangle|.$$

for $j \in \mathbb{Z}$. Then,

$$\left\| \sup_{t>0} |f * K_t| \right\|_{L^2(\mathbb{R}^n)} \le C\Gamma(K)\|f\|_{L^2(\mathbb{R}^n)},$$

where

$$\Gamma(K) = \sum_{j\in\mathbb{Z}} \alpha_j^{1/2} (\alpha_j + \beta_j)^{1/2}.$$

Let $M = \frac{1}{|B_n^1|}\mathrm{d}\sigma(y)$ and $m(\xi)$ is the Fourier transform of M, where $\mathrm{d}\sigma(y)$ is the usual measure of unite sphere. Then,

$$m(\xi) = \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\int_{-1}^{1} e^{-2\pi i s|\xi|}(1-s^2)^{\frac{n-3}{2}}\mathrm{d}s,$$

and

$$M_n^1 f(x) = \sup_{t>0}\left|\int_{\mathbb{R}^n} f * M_t(x)\right|.$$

Let $P_t(t>0)$ be the Poisson-semigroup on $\mathbb{R}^n$ satisfying $\widehat{P_t}(\xi) = e^{-t|\xi|}$. To derive (2), we take $\widehat{K}(\xi) = m(\xi) - e^{-n^{-\frac{1}{2}}|\xi|}$.
Stein [15] showed that there exists a constant C_p independent of dimension n, such that

$$\left\|\sup_{t>0}|P_t * f|\right\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}$$

for all $p > 1$. So it is sufficient to prove that

$$\left\|\sup_{t>0}|K_t * f|\right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{3}$$

To use Lemma 2.1, we will study the asymptotic properties of $K(\cdot)$. Suppose $n \ge 3$, we give three estimates about $m(\xi)$.

Firstly, when $|\xi|$ is large enough, we have

$$\begin{aligned} |m(\xi)| &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{1}{2\pi|\xi|}\left|\int_{-1}^{1} e^{-2\pi i s|\xi|}\frac{\mathrm{d}}{\mathrm{d}s}(1-s^2)^{\frac{n-3}{2}}\mathrm{d}s\right| \\ &\le \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\cdot\frac{1}{2\pi|\xi|}\int_{-1}^{1}\left|\frac{\mathrm{d}}{\mathrm{d}s}(1-s^2)^{\frac{n-3}{2}}\right|\mathrm{d}s \\ &\le Cn^{\frac{1}{2}}|\xi|^{-1}. \end{aligned} \tag{4}$$

When $|\xi|$ is small enough, we obtain

$$\begin{aligned} |m(\xi)-1| &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\left|\int_{-1}^{1}(e^{-2\pi i s|\xi|}-1)(1-s^2)^{\frac{n-3}{2}}\mathrm{d}s\right| \\ &\le \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\cdot 2\pi|\xi|\int_{-1}^{1}|s|(1-s^2)^{\frac{n-3}{2}}\mathrm{d}s \\ &\le Cn^{-\frac{1}{2}}|\xi|. \end{aligned} \tag{5}$$

Finally, we obtain

$$\begin{aligned} |\langle\nabla m(\xi),\xi\rangle| &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\left|\int_{-1}^{1} e^{-2\pi i s|\xi|}(-2\pi i s|\xi|)(1-s^2)^{\frac{n-3}{2}}\mathrm{d}s\right| \\ &= \frac{\Gamma\left(\frac{n}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right)}\left|\int_{-1}^{1} e^{-2\pi i s|\xi|}\frac{\mathrm{d}}{\mathrm{d}s}\left(s(1-s^2)^{\frac{n-3}{2}}\right)\mathrm{d}s\right| \\ &\le Cn^{\frac{1}{2}}\int_{0}^{1}\left|\frac{\mathrm{d}}{\mathrm{d}s}\left(s(1-s^2)^{\frac{n-3}{2}}\right)\right|\mathrm{d}s \le C. \end{aligned} \tag{6}$$

It is easy to know that

$$|e^{-n^{-\frac{1}{2}}|\xi|} - 1| \le n^{-\frac{1}{2}}|\xi|,$$

$$|e^{-n^{-\frac{1}{2}}|\xi|}| \le n^{\frac{1}{2}}|\xi|^{-1},$$

and

$$|\langle \nabla e^{-n^{-\frac{1}{2}}|\xi|}, \xi\rangle| \le C.$$

Combining these two inequalities with (4)–(6), we have

$$|\widehat{K}(\xi)| \le Cn^{\frac{1}{2}}|\xi|^{-1},$$

$$|\widehat{K}(\xi)| \le Cn^{-\frac{1}{2}}|\xi|,$$

and

$$|<\nabla\widehat{K}(\xi), \xi>| \le C.$$

According to three inequalities mentioned earlier and Lemma 2.1, we obtain

$$\left\| \sup_{t>0} |f * K_t| \right\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^2(\mathbb{R}^n)}.$$

3 The proof of main theorem for the case $0 < a < 1$

For $0 \le a < 1$, we have

$$M_n^a f(x) = \sup_{t>0} \frac{1}{|B_a^1|} \left| \int\limits_{B_a^1} f(x - ty)\mathrm{d}y \right| \le \frac{|B_0^1|}{|B_a^1|} \sup_{t>0} \frac{1}{|B_0^1|} \int\limits_{B_0^1} |f(x - ty)|\mathrm{d}y \le \frac{1}{1-a} M_n^0(|f|)(x). \tag{7}$$

It follows from (1) and (7) that

$$\|M_a f\|_{L^p(\mathbb{R}^n)} \le \frac{1}{1-a} C(p)\|f\|_{L^p(\mathbb{R}^n)}.$$

The general case has been proved. It remains to consider the special case $p > n/(n-1)$. We introduce the following operator:

$$M_{n,m}^a f(x) = \sup_{t>0} \frac{\left|\int_{a\le|y|\le 1} f(x - ty)|y|^m \mathrm{d}y\right|}{\int_{a\le|y|\le 1} |y|^m \mathrm{d}y}.$$

Noting that

$$\frac{\left|\int_{a\le|y|\le 1} f(x - ty)|y|^m \mathrm{d}y\right|}{\int_{a\le|y|\le 1} |y|^m \mathrm{d}y} = \frac{\left|\int_a^1 \int_{S^{n-1}} f(x - t\rho\theta)\rho^{m+n-1} d\theta d\rho\right|}{\int_a^1 \int_{S^{n-1}} \rho^{n+m-1} d\theta d\rho}. \tag{8}$$

It implies

$$M_{n,m}^a f(x) \le M_n^1(|f|)(x).$$

From this, it follows that

$$\|M^a_{n,m}f\|_{L^p(\mathbb{R}^n)} \le C(p,n)\|f\|_{L^p(\mathbb{R}^n)}. \tag{9}$$

For $\tau \in O(n)$, define

$$M^a_{k,\tau}f(x) = \sup_{t>0} \frac{\left|\int_{a\le|y_1|\le1} f(x - t\tau(y_1,0))|y_1|^{n-k}\mathrm{d}y_1\right|}{\int_{a\le|y_1|\le1}|y_1|^{n-k}\mathrm{d}y_1},$$

where $(y_1, 0) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. Then

$$M^a_{k,\tau}f(x) = \sup_{t>0} \frac{\left|\int_{a\le|y_1|\le1} f_\tau(\tau^{-1}x - t(y_1,0))|y_1|^{n-k}\mathrm{d}y_1\right|}{\int_{a\le|y_1|\le1}|y_1|^{n-k}\mathrm{d}y_1} = M^a_{n,k}f_\tau(\tau^{-1}x),$$

where $f_\tau(\cdot) = f(\tau\,\cdot)$. Hence,

$$\|M^a_{k,\tau}f\|_{L^p(\mathbb{R}^n)} \le A(p,k)\|f\|_{L^p(\mathbb{R}^n)}. \tag{10}$$

Since

$$\begin{aligned}\frac{1}{|B^1_a|}\int_{B^1_a} f(x-ty)\mathrm{d}y &= \frac{\int_{a\le|y|\le1} f(x-ty)\mathrm{d}y}{\int_{a\le|y|\le1}\mathrm{d}y}\\ &= \frac{\int_{O(n)}\int_{a\le|y_1|\le1} f(x-t\tau(y_1,0))|y_1|^{n-k}\mathrm{d}y_1\mathrm{d}\tau}{\int_{O(n)}\int_{a\le|y_1|\le1}|y_1|^{n-k}\mathrm{d}y_1\mathrm{d}\tau},\end{aligned}$$

we obtain

$$M^a_n f(x) \le \int_{O(n)} M^a_{k,\tau}(|f|)(x)\mathrm{d}\tau, \tag{11}$$

where $\mathrm{d}\tau$ is the normalized Harr measure on the orthogonal group $O(n)$.

It follows from (10) and (11) that

$$\|M^a_n f\|_{L^p(\mathbb{R}^n)} \le A(p,k)\|f\|_{L^p(\mathbb{R}^n)}. \tag{12}$$

Since $n > p/(p-1)$, we take k to be the smallest integer greater than $p/(p-1)$. Our theorem follows from (12).

Funding information: The research was supported by the Hebei Province introduced overseas student support projects (Grant No. C20190365).

Author contributions: All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflict of interest: The authors state no conflict of interests.

Code availability: Not applicable

Data availability statement: Not applicable

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