Research Article

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On 7-valent symmetric graphs of order $2pq$ and 11-valent symmetric graphs of order $4pq$

https://doi.org/10.1515/math-2022-0528
received October 20, 2021; accepted November 7, 2022

Abstract: A graph is said to be symmetric if its automorphism group is transitive on its arcs. This article is one of a series of articles devoted to characterizing prime-valent arc-transitive graphs of square-free order or twice square-free order. In this article, we determine all 7-valent symmetric graphs of order $2pq$ and 11-valent symmetric graphs of order $4pq$.

Keywords: symmetric graph, normal quotient, automorphism group

MSC 2020: 05C25, 05E18

1 Introduction

For a simple, connected, and undirected graph Γ, the vertex set and arc set of Γ are denoted by $V\Gamma$ and $A\Gamma$, respectively. Let G be a subgroup of the full automorphism group $\mathrm{Aut}\Gamma$ of Γ. Then, Γ is called *G-vertex-transitive* and *G-arc-transitive* if G is transitive on $V\Gamma$ and $A\Gamma$, respectively. An arc-transitive graph is also called *symmetric*. It is well known that Γ is G-arc-transitive if and only if G is transitive on $V\Gamma$ and the stabilizer $G_\alpha := \{g \in G | \alpha^g = \alpha\}$ for some $\alpha \in V\Gamma$ is transitive on the neighbor set $\Gamma(\alpha)$ of α in Γ.

For a group G and a subset $S = S^{-1} := \{s^{-1} | s \in S\}$ of G, the *Cayley graph* $\mathrm{Cay}(G, S)$ is a graph with vertex set G and edge set $\{\{g, sg\} | g \in G, s \in S\}$. It is well known that the right multiplication of G, say $R(G)$, and the set $\mathrm{Aut}(G, S) := \{\alpha \in \mathrm{Aut}(G) | S^\alpha = S\}$ are groups of automorphisms of $\mathrm{Cay}(G, S)$. The Cayley graph $\mathrm{Cay}(G, S)$ is called *normal* if the right multiplication of G is normal in $\mathrm{Aut}(\mathrm{Cay}(G, S))$. The following Cayley graphs of dihedral groups are denoted by CD^k_{2pq}.

Example 1.1. Let $G = \langle a, b | a^{pq} = b^2 = 1, a^b = a^{-1}\rangle \cong \mathrm{D}_{2pq}$, and let k be a solution of the equation

$$x^6 + x^5 + \cdots + x + 1 \equiv 0 \pmod{pq}.$$

Set

$$\mathrm{CD}^k_{2pq} = \mathrm{Cay}\Big(G, \Big\{b, ab, a^{k+1}b, \ldots, a^{k^5+k^4+\cdots+k+1}b\Big\}\Big).$$

The study of graphs with square-free order has a long history, see, e.g., [1–4]. In recent work [5], the authors gave a characterization for connected prime-valent arc-transitive graphs of square-free order. This article is devoted to classifying 7-valent arc-transitive graphs of order $2pq$, which gives supplementary proof of Lemma 2.9 in [5, Lemma 2.9]. The first result of this article is the following theorem.

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Theorem 1.2. *Let* Γ *be a* 7*-valent symmetric graph of order* $2pq$, *where* $q > p \geq 3$ *are primes. Then one of the following statements holds*:
(1) $\Gamma \cong \mathrm{CD}^k_{2pq}$ *and* $\mathrm{Aut}\Gamma \cong \mathrm{D}_{2pq} : \mathbb{Z}_7$, *where* $p \mid q-1$. *Up to isomorphism, there is only one such graph for* $p = 7$ *and there are exactly six such graphs for* $p > 7$.
(2) Γ *lies in* Table 1.

The method used in this article for classifying symmetric graphs of square-free order is also applicable to classifying symmetric graphs of twice square-free order. See the following two theorems.

Theorem 1.3. *Let* Γ *be a connected symmetric graph of order* $4p$ *with valency* 11, *where p is a prime, then* $p = 3$ *and* $\Gamma = K_{12}$, *the complete graph of order* 12.

Theorem 1.4. *Let* Γ *be a connected symmetric graph of order* $4pq$ *with valency* 11, *where* $p > q \geq 3$ *are distinct primes, then* $\Gamma \cong \mathcal{G}_{60}$, $\mathcal{G}_{532}$, *or* $\mathcal{G}^i_{276}$ *for* $1 \leq i \leq 4$, *with their automorphism groups* $\mathrm{Aut}\Gamma$ *and vertex stabilizers* $(\mathrm{Aut}\Gamma)_\alpha$ *listed in* Table 2, *where* α *is a vertex.*

2 Preliminaries

We now give some necessary preliminary results. The first one is a property of the Fitting subgroup, see [6, P. 30, Corollary].

Lemma 2.1. *Let F be the Fitting subgroup of a group G. If G is soluble, then* $F \neq 1$ *and the centralizer* $C_G(F) \leq F$.

We shall need information of maximal subgroups of PSL(2, r) and PGL(2, r), where r is an odd prime, refer to [7, Section 239] and [8, Theorem 2].

Lemma 2.2. *Let* $G = \mathrm{PSL}(2, r)$ *or* $\mathrm{PGL}(2, r)$ *and let M be a maximal subgroup of G, where* $r \geq 5$ *is a prime.*
(1) *If* $G = \mathrm{PSL}(2, r)$, *then* $M \in \{D_{r-1}, D_{r+1}, Z_r : Z_{(r-1)/2}, A_4, S_4, A_5\}$.
(2) *If* $G = \mathrm{PGL}(2, r)$, *then* $M \in \{D_{2(r-1)}, D_{2(r+1)}, Z_r : Z_{r-1}, S_4, \mathrm{PSL}(2, r)\}$.

By [9], we have the next lemma.

Lemma 2.3. *Let* $\Gamma = \mathrm{Cay}(G, S)$ *be a normal Cayley graph on G. Then*, $(\mathrm{Aut}\Gamma)_1 = \mathrm{Aut}(G, S)$, *where* 1 *is the identity of G.*

For a graph Γ and a positive integer s, an *s-arc* of Γ is a sequence $\alpha_0, \alpha_1, \ldots, \alpha_s$ of vertices such that α_{i-1} and α_i are adjacent for $1 \leq i \leq s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc. Then, Γ is called (G, s)*-arc-transitive* with $G \leq \mathrm{Aut}\Gamma$ if G is transitive on the set of s-arcs of Γ. A (G, s)-arc-transitive

Table 1: Connected 7-valent symmetric graphs of order $2pq$

Row	Γ	(p, q)	$\mathrm{Aut}\Gamma$	$(\mathrm{Aut}\Gamma)_\alpha$	Transitivity	Remark
1	C^1_{78}	(3,13)	PGL(2,13)	D_{28}	1-transitive	No bipartite
2	C^2_{78}	(3,13)	PSL(2,13)	D_{14}	1-transitive	No bipartite
3	C_{310}	(5,31)	$\mathrm{PSL}(5,2).\mathbb{Z}_2$	$\mathbb{Z}_2^6 : (\mathrm{SL}(2,2) \times \mathrm{SL}(3,2))$	3-transitive	Bipartite
4	C_{30}	(3,5)	S_8	$\mathbb{Z}_2^3 : \mathrm{SL}(3,2)$	2-transitive	Bipartite

Table 2: Connected 11-valent symmetric graphs of order $4pq$

Graph	AutΓ	$(\text{Aut}\Gamma)_\alpha$	(q,p)	Bipartite?
$\mathcal{G}_{60}$	PGL(2,11)	F_{22}	(3,5)	No
$\mathcal{G}^1_{276}$	PGL(2,23)	D_{44}	(3, 23)	Yes
$\mathcal{G}^i_{276}$, $2 \le i \le 4$	PSL(2,23)	D_22	(3, 23)	No
$\mathcal{G}_{532}$	$J_1 \times \mathbb{Z}_2$	PSL(2,11)	(7, 19)	Yes

graph is called (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a graph Γ is simply called s-*transitive* if it is $(\text{Aut}\Gamma, s)$-transitive.

The following lemma is about the stabilizers of arc-transitive 7-valent graphs, refer to [10, Corollary 2.2] and [11, Theorem 3.4].

Lemma 2.4. *Let Γ be a 7-valent (G, s)-transitive graph, where $G \le \text{Aut}\Gamma$ and $s \ge 1$. Let $\alpha \in V\Gamma$. Then, one of the following statements holds*:
(a) *If G_α is soluble, then $s \le 3$ and $|G_\alpha||252$. Furthermore, the couple (s, G_α) lies in the following table.*

s	1	2	3
G_α	$\mathbb{Z}_7$, F_{14}, F_{21}, $F_{14} \times \mathbb{Z}_2$, $F_{21} \times \mathbb{Z}_3$	F_{42}, $F_{42} \times \mathbb{Z}_2$, $F_{42} \times \mathbb{Z}_3$	$F_{42} \times \mathbb{Z}_6$

(b) *If G_α is insoluble, then $|G_\alpha||2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$.*

From [12, pp. 134–136], we can obtain the following two lemmas by checking the orders of nonabelian simple groups. The arguments in the proof of Lemmas 2.5 and 2.6 are heavily relying on the classification of finite simple groups.

Lemma 2.5. *Let $q > p \ge 3$ be primes, and let T be a nonabelian simple group of order $2^i \cdot 3^j \cdot 5^k \cdot 7 \cdot p \cdot q$, where $2 \le i \le 25$, $0 \le j \le 4$, and $0 \le k \le 2$. Then, T is listed in* Table 3.

Table 3: Simple group T with order dividing $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$

T	$\|T\|$	(p, q)	T	$\|T\|$	(p, q)
M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	(3, 11), (5, 11)	PSL(2, 2^6)	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$	(3, 13), (5, 13)
M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	(11, 23)	PSL(2, 2^9)	$2^9 \cdot 3^3 \cdot 7 \cdot 19 \cdot 73$	(19, 73)
M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	(11, 23)	PSL(2, 27)	$2^2 \cdot 3^3 \cdot 7 \cdot 11$	(3, 11)
J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$	(11, 19)	PSL(2, 125)	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 31$	(5, 31)
J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	(3, 5)	PSL(2, 49)	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	(3, 7), (5, 7)
HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	(5, 11)	PSU(3, 5)	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	(3, 5)
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	(3, 5)	PSL(3, 8)	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	(3, 19)
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	(3, 5)	$D_4(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	(3, 5)
A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	(3, 5)	${}^3D_4(2)$	$2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$	(7, 13)
A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	(3, 5)	PSp(8, 2)	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7$	(3, 5)
A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	(3, 11), (5, 11)	PSL(4, 4)	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	(3, 17), (5, 17)
A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	(3, 11)	PSL(5, 2)	$2^{10} \cdot 3^2 \cdot 5 \cdot 7 \cdot 31$	(3, 31), (5, 31)
Sz(8)	$2^6 \cdot 5 \cdot 7 \cdot 13$	(5, 13)	PSp(4, 8)	$2^{12} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 13$	(7, 13)
PSU(3, 8)	$2^9 \cdot 3^4 \cdot 7 \cdot 19$	(3, 19)	${}^2D_4(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	(3, 17), (5, 17)
PSp(6, 2)	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	(3, 5)	$G_2(4)$	$2^{12} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 13$	(3, 13), (5, 13)
PSL(4, 2)	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	(3, 5)	PSL(3, 16)	$2^{12} \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17$	(13, 17)
PSL(3, 4)	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	(3, 5)	PSL(2, q)	$\frac{q(q+1)(q-1)}{2}$	

Proof. If T is a sporadic simple group, by [12, pp. 135–136], $T = \mathrm{M}_{22}, \mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{J}_1, \mathrm{J}_2$, or HS. If $T = \mathrm{A}_n$ is an alternating group, since 2^{10} does not divide $|T|$, we have $n \leq 13$, it then follows that $T = \mathrm{A}_7, \mathrm{A}_8, \mathrm{A}_9, \mathrm{A}_{10}, \mathrm{A}_{11}$, or A_{12} in Table 3.

Suppose now $T = X(q)$ is a simple group of Lie type, where X is one type of Lie groups, and $q = p^f$ is a prime power. If $p \geq 3$, as $|T|$ contains at most five 3-factors, three 5-factors, and two 7-factors, it easily follows from [12, p. 135] that the only possibility is $T = \mathrm{PSL}(2, q)$, $\mathrm{PSL}(2, 27)$, $\mathrm{PSL}(2, 125)$, $\mathrm{PSL}(2, 49)$, or $\mathrm{PSU}(3, 5)$. Similarly, if $p = 2$, then we have $T = \mathrm{Sz}(8)$, $\mathrm{PSU}(3, 8)$, $\mathrm{PSp}(6, 2)$, $\mathrm{PSL}(4, 2)$, $\mathrm{PSL}(3, 4)$, $\mathrm{PSL}(3, 8)$, $\mathrm{PSL}(2, 2^6)$, $\mathrm{PSL}(2, 2^9)$, $D_4(2)$, ${}^3D_4(2)$, $\mathrm{PSp}(8, 2)$, $\mathrm{PSL}(4, 4)$, $\mathrm{PSL}(5, 2)$, $\mathrm{PSp}(4, 8)$, ${}^2D_4(2)$, $G_2(4)$, or $\mathrm{PSL}(3, 16)$. ☐

Lemma 2.6. *Let T be a nonabelian simple group and let $p > q$ be two distinct odd primes. Suppose that* $11 \| |T|$ *and* $|T| \mid 2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot p \cdot q$, *then T lies in* Table 4.

Proof. Assume T is a sporadic simple group. Then, by checking the order of sporadic simple group in [12], we have Part 1 of the table.

Assume $T = A_n$ is an alternating group with $n \geq 5$. Since $11 \| |T|$, $n \geq 11$; and since $|T|$ has at most seven distinct prime divisors, $n \leq 18$. Then, we have Part 2 of the table.

Assume from now on that T is a simple group of Lie type over a field $GF(r)$ of order $r = t^e$, where t is a prime. Note that the order of T is not divisible by 2^{19}, 3^{10}, 5^6, 7^4, 11^3, and s^2, where $s > 11$ is a prime.

Assume first that T is a simple exceptional group. By [12], we can easily rule out $F_4(r)$, $E_6(r)$, ${}^2E_6(r)$, $E_7(r)$, and $E_8(r)$ as $r^{19} \| |T|$ if T is one of them. Since $11 \nmid |{}^2F_4(2)|$, $T \neq {}^2F_4(2)$. If $T = {}^2F_4(r)$ with $r = 2^{2m+1} \geq 2^3$, then $r^{12} \| |T|$, and hence $2^{36} \| |T|$, a contradiction. If $T = {}^3D_4(r)$, then $r^{12} \| |T|$, and hence $T = {}^3D_4(2)$. However $11 \nmid |{}^3D_4(2)|$, a contradiction. If $T = G_2(r)$, then $r^6 \| |T|$, and hence the possibilities are $G_2(2)$, $G_2(4)$, $G_2(8)$, and $G_2(3)$. However, a computation shows that 11 does not divide the orders of these four groups, a contradiction. If $T = {}^2B_2(r)$ with $r = 2^{2m+1} \geq 2^3$ (noting that ${}^2B_2(2)$ is solvable), then $r^2 \| |T|$, and hence the possibilities are ${}^2B_2(2^3)$, ${}^2B_2(2^5)$, ${}^2B_2(2^7)$, and ${}^2B_2(2^9)$. However 11 does not divide the orders of these four groups, a contradiction. If $T = {}^2G_2(r)$ with $r = 3^{2m+1} \geq 3^3$ (noting that ${}^2G_2(3) \cong \mathrm{PSL}(2, 8){\cdot}3$ is not a simple group, and $11 \nmid |\mathrm{PSL}(2, 8)|$), then $r^3 \| |T|$, and hence $3^9 \| |T|$. Then, $T = {}^2G_2(3^3)$. However, $11 \nmid |{}^2G_2(3^3)|$, a contradiction. To summary, we have shown that T is not a simple exceptional group.

Assume next that T is a classical group. Note that $r^{n(n-1)/2} \| |PGL(n, r)|$ and $|\mathrm{PSU}_n(r)|$, $r^{m^2} \| |\mathrm{PS}p_{2m}(r)|$ and $|P\Omega_{2m+1}(r)|$, and $r^{m(m-1)/2} \| |P\Omega^{\pm}_{2m}(r)|$. Considering the isomorphisms between classical groups (see [12]), the possibilities of T are as follows:

Table 4: Simple group T with order dividing $2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot p \cdot q$

Part	T	$\|T\|$	T	$\|T\|$
1	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	M_{12}	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	M_{23}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$	J_1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
	HS	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$	McL	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
	Suz	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	Co_3	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
	Co_2	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$	Fi_{22}	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
2	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	A_{12}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$
	A_{13}	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	A_{14}	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$
	A_{15}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	A_{16}	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$
	A_{17}	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	A_{18}	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$
3	PSL(2,11)	$2^2 \cdot 3 \cdot 5 \cdot 11$	$\mathrm{PSL}(2, 11^2)$	$2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$
	$\mathrm{PSL}(2, 2^5)$	$2^5 \cdot 3 \cdot 11 \cdot 31$	$\mathrm{PSL}(2, 2^{10})$	$2^{10} \cdot 3 \cdot 5^2 \cdot 11 \cdot 31 \cdot 41$
	$\mathrm{PSL}(2, 3^5)$	$2^2 \cdot 3^5 \cdot 11^2 \cdot 61$	$\mathrm{PSU}_5(2)$	$2^{10} \cdot 3^5 \cdot 5 \cdot 11$
	$\mathrm{PSU}_6(2)$	$2^{15} \cdot 3^6 \cdot 5 \cdot 7 \cdot 11$	$\mathrm{PSL}(2, p)$	$p(p-1)(p+1)/2$

$\mathrm{PSL}(2,r)$ with r divides one of $\{2^{18}, 3^9, 5^5, 7^3, 11^2, p(p>11)\}$,
$\mathrm{PSL}_3(2^k)$ for $1 \le k \le 6$, $\mathrm{PSL}_3(3)$, $\mathrm{PSL}_3(3^2)$, $\mathrm{PSL}_3(3^3)$, $\mathrm{PSL}_3(5)$, $\mathrm{PSL}_3(7)$,
$\mathrm{PSL}_4(2^k)$ for $1 \le k \le 3$, $\mathrm{PSL}_4(3)$, $\mathrm{PSL}_5(2)$, $\mathrm{PSL}_6(2)$,
$\mathrm{PSU}_3(2^k)$ for $2 \le k \le 6$, $\mathrm{PSU}_3(3)$, $\mathrm{PSU}_3(3^2)$, $\mathrm{PSU}_3(3^3)$, $\mathrm{PSU}_3(5)$, $\mathrm{PSU}_3(7)$,
$\mathrm{PSU}_4(2^k)$ for $1 \le k \le 3$, $\mathrm{PSU}_4(3)$, $\mathrm{PSU}_5(2)$, $\mathrm{PSU}_6(2)$,
$P\Omega_7(3)$, $P\Omega_9(2)$, $P\Omega_8^+(2)$, $P\Omega_8^-(2)$, $\mathrm{PS}p_6(3)$, $\mathrm{PS}p_8(2)$,
$\mathrm{PS}p_4(2^k)$ for $2 \le k \le 4$, $\mathrm{PS}p_4(3)$, $\mathrm{PS}p_4(3^2)$, $\mathrm{PS}p_4(5)$, $\mathrm{PS}p_6(2)$, $\mathrm{PS}p_6(4)$.

Then, computation shows that T is in Part 3 of Table 4. □

The next lemma is about the vertex stabilizer in an arc-transitive group of automorphisms of symmetric graph of valency 11, see [10] and [13].

Lemma 2.7. *Let Γ be a connected G-arc-transitive graph with valency 11 and α a vertex of Γ. Then, one of the following statements holds*:
(1) *If G_α is soluble, then $|G_\alpha| \mid 1{,}100$ and G_α is one of*

$$Z_{11}, D_{22}, F_{55}, Z_2 \times D_{22}, Z_5 \times D_{55}, F_{110}, Z_2 \times F_{110}, Z_5 \times F_{110}, Z_{110} \times F_{110}.$$

(2) *If G_α is insoluble, then $|G_\alpha| \mid 2^{16} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$, and the pairs $(G_\alpha, |G_\alpha|)$ lie in* Table 5.

A typical method for studying vertex-transitive graphs is taking normal quotients. Let Γ be a G-vertex-transitive graph, where $G \le \mathrm{Aut}\Gamma$. Suppose that G has a normal subgroup N, which is intransitive on $V\Gamma$. Let $V\Gamma_N$ be the set of N-orbits on $V\Gamma$. The *normal quotient graph* Γ_N of Γ induced by N is defined as the graph with vertex set $V\Gamma_N$, and B is adjacent to C in Γ_N if and only if there exist vertices $\beta \in B$ and $\gamma \in C$ such that β is adjacent to γ in Γ. In particular, if $\mathrm{val}(\Gamma) = \mathrm{val}(\Gamma_N)$, then Γ is called a *normal cover* of Γ_N.

A graph Γ is called *G-locally primitive* if, for each $\alpha \in V\Gamma$, the stabilizer G_α acts primitively on $\Gamma(\alpha)$. Obviously, an arc-transitive pentavalent graph is locally primitive. The following theorem gives a basic method for studying vertex-transitive locally primitive graphs, see [14, Theorem 4.1] and [15, Lemma 2.5].

Theorem 2.8. *Let Γ be a G-vertex-transitive locally primitive graph, where $G \le \mathrm{Aut}\Gamma$, and let $N \lhd G$ have at least three orbits on $V\Gamma$. Then, the following statements hold*:
(*i*) *N is semi-regular on $V\Gamma$, $G/N \le \mathrm{Aut}\Gamma_N$, and Γ is a normal cover of Γ_N*;
(*ii*) *$G_\alpha \cong (G/N)_\gamma$, where $\alpha \in V\Gamma$ and $\gamma \in V\Gamma_N$*;
(*iii*) *Γ is (G,s)-transitive if and only if Γ_N is $(G/N,s)$-transitive, where $1 \le s \le 5$ or $s = 7$.*

For the case where N has at most two orbits on $V\Gamma$, the next fact is a consequence of the connectivity of the graph, which is well known.

Lemma 2.9. *Let Γ be a connected G-arc-transitive graph of odd prime valency d. Let $1 \ne N$ be a normal subgroup of G. Suppose that N have at most two orbits on $V\Gamma$ and $N_\alpha \ne 1$, where α is a vertex of Γ. Then, N_α is transitive on the neighbors $\Gamma(\alpha)$ of α, particularly, $d \mid |N_\alpha|$.*

By Li and Feng [16, Theorem 3.6], we have the following lemma.

Table 5: Insoluble vertex stabilizer of arc-transitive graph with valency 11

G_α	$\|G_\alpha\|$	G_α	$\|G_\alpha\|$	G_α	$\|G_\alpha\|$
PSL(2,11)	$2^2 \cdot 3 \cdot 5 \cdot 11$	M_{11}	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	A_{11}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$
S_{11}	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	$A_5 \times \mathrm{PSL}(2,11)$	$2^4 \cdot 3^2 \cdot 5 \cdot 11$	$A_6 \times M_{11}$	$2^7 \cdot 3^4 \cdot 5^2 \cdot 11$
$M_{10} \times M_{11}$	$2^8 \cdot 3^4 \cdot 5^2 \cdot 11$	$(A_{10} \times A_{11}) : Z_2$	$2^{15} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$	$A_{10} \times A_{11}$	$2^{14} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$
$S_{10} \times S_{11}$	$2^{16} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11$				

Lemma 2.10. *Let n be a square-free integer and Γ a 7-valent one-regular graph of order n. Then, $n = 2 \cdot 7^t \cdot p_1 p_2 \cdots p_s \geq 13$, where $t \leq 1$, $s \geq 1$, and p_i's are distinct primes such that $7|(p_i - 1)$. Furthermore, Γ is isomorphic to one of $\mathcal{CD}_n^l$ and there are exactly 6^{s-1} such non-isomorphic graphs of order n.*

For reduction, we need some information of 7-valent symmetric graphs of order $2p$, stated in the following lemma, see [2, Table 1].

Lemma 2.11. *Let p be a prime and let Γ be a 7-valent symmetric graph of order $2p$. Then, Γ is isomorphic to one of the following graphs:*
(1) *The complete bipartite graph $\mathrm{K}_{7,7}$ for $p = 7$ with $\mathrm{Aut}\Gamma \cong \mathrm{S}_7 \wr \mathrm{S}_2$.*
(2) *The graph $\mathrm{G}(2p, 7)$ for $p > 7$ with $\mathrm{AutG}(2p, 7) \cong \mathrm{D}_{2p} : \mathbb{Z}_7$.*

Remark of Lemma 2.11. We define the graph $\mathrm{G}(2p, 7)$ in the following. Let A and A' be two disjoint copies of $\mathbb{Z}_p$. For each element i of $\mathbb{Z}_p$, we shall denote the corresponding elements of A and A' by i and i', respectively. Let r be a positive integer dividing $p - 1$, where p is prime, and let $H(p, r)$ denote the unique subgroup of $\mathbb{Z}_p^*$ of order r. We define the graph $G(2p, r)$ to have vertex-set $A \cup A'$ and edge-set $\{xy' : x, y \in \mathbb{Z}_p, \text{ and } y - x \in H(p, r)\}$.

We need some classification results on symmetric graphs of valency 11. The following two lemmas are obtained from [2], [17], and [18].

Lemma 2.12. *Let Γ be a connected symmetric graph of order $2r$ and valency 11, where r is an odd prime. Suppose that $\mathrm{Aut}\Gamma$ is insolvable, then Γ is the complete bipartite graph $K_{11,11}$.*

Lemma 2.13. *Let Γ be a connected symmetric graph of order $2m$ and valency 11, where m is an odd square-free integer, then one of the following statements holds:*
(1) *Γ is a normal Cayley graph on D_{2m} and $\mathrm{Aut}\Gamma = D_{2m} : Z_{11}$;*
(2) *$\mathrm{Aut}\Gamma = J_1$, $\mathrm{Aut}\Gamma_\alpha = \mathrm{PSL}(2, 11)$, and $m = 7 \cdot 19$, moreover, Γ is not bipartite;*
(3) *$\mathrm{Aut}\Gamma = \mathrm{PSL}(2, r)$ or $\mathrm{PGL}(2, r)$ where $r \equiv \pm 1 (\mathrm{mod}\ 11)$ is a prime.*

3 Examples

In this section, we give some examples of 7-valent symmetric graphs of order $2pq$ with $q > p \geq 3$ distinct primes.

For a given small permutation group X, one may determine all graphs that admit X as an arc-transitive automorphism group by using Magma program [19]. It is then easy to have the following result.

Example 3.1. There are two connected 7-valent symmetric graphs of order 78, which admit $\mathrm{PSL}(2, 13)$ or $\mathrm{PGL}(2, 13)$ as an arc-transitive automorphism group. These two graphs are denoted by $\mathcal{C}_{78}^1$ and $\mathcal{C}_{78}^2$, which satisfy the conditions in Rows 1 and 2 of Table 1.

Example 3.2. There is a unique connected 7-valent symmetric graph of order 310, which admits $\mathrm{PSL}(5, 2)$. $\mathbb{Z}_2$ as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}_{310}$, which satisfies the conditions in Row 3 of Table 1.

Example 3.3. There is a unique connected 7-valent symmetric graph of order 30, which admits S_8 as an arc-transitive automorphism group. This graph is denoted by $\mathcal{C}_{30}$, which satisfies the conditions in Row 4 of Table 1.

4 The proof of Theorem 1.2

Now, we prove the main result of this article. Let Γ be a 7-valent symmetric graph of order $2pq$. Set $\mathsf{A} = \mathrm{Aut}\Gamma$. By Lemma 2.4, $|\mathsf{A}_\alpha| | 2^{24} \cdot 3^4 \cdot 5^2 \cdot 7$, and hence $|\mathsf{A}| | 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$. Let R be the soluble radical of A and let F be the Fitting subgroup of A (recall that the Fitting subgroup F of A is defined to be the product of all normal nilpotent subgroups of A). We divide our discussion into the following three cases.

Case 1. $R = 1$

Let N be a minimal normal subgroup of A and let $C = \mathbf{C}_{\mathsf{A}}(N)$. Since $R = 1$, we have that $N = T^d$, where T is a nonabelian simple group and $d \geq 1$. Furthermore, since $|\mathsf{A}| | 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$, we have $|N| | 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

Assume that N has t orbits on $V\Gamma$. If $t \geq 3$, then by Theorem 2.8, $N_\alpha = 1$ and so $|N| = |T|^d \mid 2pq$, which is a contradiction as T is a nonabelian simple group. Hence, $N_\alpha \neq 1$, N has at most two orbits on $V\Gamma$ and pq divides $|N : N_\alpha|$. Since Γ is connected, $N \lhd \mathsf{A}$, and $N_\alpha \neq 1$, we have $1 \neq N_\alpha^{\Gamma(\alpha)} \lhd \mathsf{A}_\alpha^{\Gamma(\alpha)}$, it follows that 7 divides $|N_\alpha|$, we thus have that $7pq$ divides $|T|$.

We first show that $d = 1$. If not, $d \geq 2$, then $7^2 \mid |T|^2 = |N|$ as $|N| = |T|^d \mid 2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$. It follows that $p = 7$ or $q = 7$. If $p = 7$, then $q > 7$ and $q^2 \mid |T|^2$, a contradiction. If $q = 7$, then $p = 3$ or 5. It can conclude that $|T| \mid 2^{12} \cdot 3^2 \cdot 5 \cdot 7$. Note that $21 \mid |T|$ or $35 \mid |T|$. By checking the nonabelian simple group of order less than $2^{12} \cdot 3^2 \cdot 5 \cdot 7$ (e.g., [12]), we have that $T \cong \mathsf{A}_7$, A_8, or $\mathrm{PSL}(3, 4)$, and so $d = 2$, $N = \mathsf{A}_7^2$, A_8^2, or $\mathrm{PSL}(3, 4)^2$. On the other hand, $C \lhd \mathsf{A}$, $C \cap N = 1$ and $\langle C, N\rangle = C \times N$. Because $|C \times N|$ divides $2^{25} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot p \cdot q$ and $|N| = |T|^2 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2$ or $2^9 \cdot 3^4 \cdot 5^2 \cdot 7^2$, C is a $\{2, p\}$-group, and hence soluble, where $p = 3$ or $p = 5$. So $C = 1$ as $R = 1$. This implies $\mathsf{A} = \mathsf{A}/C \leq \mathrm{Aut}(N) \cong \mathrm{Aut}(T) \wr \mathsf{S}_2$. By Magma [19], no such graph exists. Thus, $d = 1$ and $N = T \trianglelefteq \mathsf{A}$ is a nonabelian simple group.

We next show that $C = 1$. If not, then C is insoluble as $R = 1$ and $C \trianglelefteq \mathsf{A}$. The same argument as for the case N leads to $7 \mid |C_\alpha|$. Since $\langle C, N\rangle = C \times N$ and C, $N \trianglelefteq \mathsf{A}$, we have $N_\alpha \times C_\alpha \leq \mathsf{A}_\alpha$. On the other hand, $7 \mid |N_\alpha|$, it concludes that $7^2 \mid |\mathsf{A}_\alpha|$, a contradiction with Lemma 2.4. Hence, A is almost simple and $\mathsf{A} \leq \mathrm{Aut}(T)$. Thus, we have $\mathrm{soc}(\mathsf{A}) = T$ as a nonabelian simple group and satisfies the following condition.

Condition $(*)$: $|T|$ lies in Table 3 such that $7pq||T|$ and $|T||2^{25} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot p \cdot q$.

Assume first that $T = \mathrm{M}_{22}$, M_{24}, J_1, J_2, HS, PSU(3, 8), PSp(6, 2), PSp(8, 2), PSp(4, 8), PSL(3, 4), PSL(2, 2^9), PSL(2, 27), PSL(2, 125), PSL(2, 49), PSU(3, 5), PSL(3, 16), A_9, or A_{10}. Note that $|T : T_\alpha| = pq$ or $2pq$. By Atlas [20], T has no subgroup of index pq or $2pq$, a contradiction.

Assume that $T = \mathrm{M}_{23}$, A_7, A_{11}, A_{12}, Sz(8), PSL(4, 2), PSL(4, 4), PSL(3, 8), or PSL(2, 2^6). Note that $T \leq \mathsf{A} \leq \mathrm{Aut}(T)$. We can exclude all these cases by using Magma [19].

Assume that $T = \mathrm{PSL}(5, 2)$. Then, $(p, q) = (3, 31)$ or $(5, 31)$. For the former case, T has no subgroup of index 93 or 186, a contradiction. For the latter case, by Example 3.2, Γ is isomorphic to $\mathcal{C}_{310}$. Assume that $T = \mathsf{A}_8$. Then, $(p, q) = (3, 5)$. By Example 3.3, Γ is isomorphic to $\mathcal{C}_{30}$.

Assume that $T = \mathrm{PSL}(2, q)$. Then, $T \leq \mathsf{A} \leq \mathrm{Aut}(T) = \mathrm{PGL}(2, q)$ and $|\mathsf{A} : T| \leq 2$. If A_α is insoluble, then T_α is also insoluble as $|\mathsf{A}_\alpha : T_\alpha| \leq 2$. By Lemma 2.2, $T_\alpha = \mathsf{A}_5$, which is impossible as $7 \mid |T_\alpha|$. Thus, A_α is soluble, and by Lemma 2.4, A_α divides 252, and so $|T_\alpha||252$. It implies that the order of T divides $504 \cdot p \cdot q$. Note that $|\mathrm{PSL}(2, q)| = \frac{q(q-1)(q+1)}{2}$ and $(\frac{q+1}{2}, \frac{q-1}{2}) = 1$. If $p|\frac{q-1}{2}$, then $q + 1$ divides 504. It follows that $q = 5, 7, 11, 13, 17, 23, 41, 71, 83, 167, 251$, or 503. However, PSL(2, q) does not satisfy the Condition (*) for $q = 5, 7, 11, 17$, or 23. Thus, $q = 13, 41, 71, 83, 167, 251$, or 503 for this case. If $p|\frac{q+1}{2}$, then $q - 1$ divides 504. It follows that $q = 5, 7, 13, 19, 29, 37, 43, 73$, or 127. However, PSL(2, q) does not satisfy the Condition (*) for $q = 5, 7, 19, 37$, or 73. Thus, $q = 13, 29, 43$, or 127 for this case. Therefore, for $T = \mathrm{PSL}(2, q)$, T is one of the following groups:

T	Order	T	Order
PSL(2, 13)	$2^2 \cdot 3 \cdot 7 \cdot 13$	PSL(2, 29)	$2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 29$
PSL(2, 41)	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 41$	PSL(2, 43)	$2^2 \cdot 3 \cdot 7 \cdot 11 \cdot 43$
PSL(2, 71)	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 71$	PSL(2, 83)	$2^2 \cdot 3 \cdot 7 \cdot 41 \cdot 83$
PSL(2, 127)	$2^7 \cdot 3^2 \cdot 7 \cdot 127$	PSL(2, 167)	$2^3 \cdot 3 \cdot 7 \cdot 83 \cdot 167$
PSL(2, 251)	$2^2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 251$	PSL(2, 503)	$2^3 \cdot 3^2 \cdot 7 \cdot 251 \cdot 503$

Assume that $q = 29, 41, 71, 127$, or 251. Note that $|T : T_\alpha| = pq$ or $2pq$. By Lemma 2.2, T has no subgroup of index pq or $2pq$, a contradiction. Assume that $q = 43, 83$, or 167. Note that $\mathsf{A} = \mathsf{PGL}(2, q)$ or $\mathsf{PSL}(2, q)$. We can exclude all the cases by Magma [19]. Assume that $q = 503$. Then, $|T_\alpha| = 504$ or 252. It implies that T_α is soluble and so as A_α. By Lemma 2.4, $|\mathsf{A}_\alpha| \mid 252$, and therefore, $\mathsf{A} = \mathsf{PSL}(2, 503)$, $|\mathsf{A}_\alpha| = 252$. Again by Lemma 2.4, $\mathsf{A}_\alpha \cong \mathsf{F}_{42} \times \mathbb{Z}_6$, which is impossible by Lemma 2.2. Assume, finally, that $q = 13$. Then, $T = \mathsf{PSL}(2, 13)$ and $\mathsf{A} = \mathsf{PSL}(2, 13)$ or $\mathsf{PGL}(2, 13)$. By example 3.1, Γ is isomorphic to C^1_{78} or C^2_{78}. This completes the proof of this case.

Case 2. $R \neq 1$ **and** A **is soluble**

Then, $R = \mathsf{A}$, and by Lemma 2.1, $F \neq 1$ and $\mathbf{C}_{\mathsf{A}}(F) \leq F$. As $|V\Gamma| = 2pq$, A has no nontrivial normal Sylow s-subgroup, where $s \neq 2, p$, or q. So $F = \mathbf{O}_2(\mathsf{A}) \times \mathbf{O}_p(\mathsf{A}) \times \mathbf{O}_q(\mathsf{A})$, where $\mathbf{O}_2(\mathsf{A})$, $\mathbf{O}_p(\mathsf{A})$, and $\mathbf{O}_q(\mathsf{A})$ denote the largest normal 2-, p-, and q-subgroups of A, respectively.

For each $r \in \{2, p, q\}$, since $q > p \geq 3$, $\mathbf{O}_r(\mathsf{A})$ has at least three orbits on $V\Gamma$, by Proposition 2.8, $\mathbf{O}_r(\mathsf{A})$ is semi-regular on $V\Gamma$. Therefore, $|\mathbf{O}_2(\mathsf{A})| \leq 2$, $|\mathbf{O}_p(\mathsf{A})| \leq p$, $|\mathbf{O}_q(\mathsf{A})| \leq q$, $F \leq \mathbb{Z}_{2pq}$ is abelian, and $\mathbf{C}_R(F) = F$.

If $|F| = 2$, by Proposition 2.8, the normal quotient graph Γ_F is a 7-valent A/F-arc-transitive graph of odd order pq, not possible. Thus, there exists a prime $r \in \{p, q\}$ such that $r \mid |F|$, and so $\mathbf{O}_r(\mathsf{A}) = r$. By Theorem 2.8, $\Gamma_{\mathbf{O}_r(\mathsf{A})}$ is a 7-valent $\mathsf{A}/\mathbf{O}_r(\mathsf{A})$-arc transitive graph of order $2s$ with $s \in \{p, q\}$ and $\mathsf{A}/\mathbf{O}_r(\mathsf{A})$ is soluble. Then, by Lemma 2.11, $\Gamma_{\mathbf{O}_r(\mathsf{A})}$ is isomorphic to $\mathsf{K}_{7,7}$ or $\mathsf{G}(2p, 7)$. For the former case, by [21, Theorem 1.1], $p = 7$ and $\Gamma_{\mathbf{O}_r(\mathsf{A})} \cong \mathsf{CD}^k_{14q}$ as described in Theorem 1.2 (1). For the latter case, by Lemma 2.11, $\Gamma_{\mathbf{O}_r(\mathsf{A})} \cong \mathsf{G}(2p, 7)$ and $\mathsf{Aut}\Gamma_{\mathbf{O}_r(\mathsf{A})} \cong \mathsf{D}_{2s} : \mathbb{Z}_7$ is arc-regular on $A\Gamma$. Hence, $\mathsf{A}/\mathbf{O}_r(\mathsf{A}) \cong \mathsf{D}_{2s} : \mathbb{Z}_7$, it implies that Γ is an 7-valent arc-regular graph of order $2pq$. By Lemma 2.10, $\Gamma \cong \mathsf{CD}^k_{2pq}$ as in Theorem 1.2 (1).

Case 3. $R \neq 1$ **and** A **is insoluble**

Let N be a minimal soluble normal subgroup of A. Then, $N \cong \mathbb{Z}_r^d$ has at least three orbits on $V\Gamma$, where r is a prime. It follows from Theorem 2.8 that N is semi-regular on $V\Gamma$, and so $d = 1$, $r \in \{p, q\}$. Furthermore, Γ_N is A/N-arc-transitive graph of order $\frac{2pq}{r} = 2t$ and A/N is insoluble, where $t \in \{p, q\}$. Since Γ_N is A/N-arc-transitive and A/N is insoluble, by Lemma 2.11, Γ_N is isomorphic to $\mathsf{K}_{7,7}$. Thus, Γ is a normal $\mathbb{Z}_t$-cover of $\mathsf{K}_{7,7}$, where $t \neq 7$. By [21, Theorem 1.1], no such graph Γ exists.

Thus, we complete the proof of Theorem 1.2.

5 The proof of Theorems 1.3 and 1.4

In this section, we prove Theorems 1.3 and 1.4. Let Γ be a connected symmetric graph of order $4n$ and valency 11, where $n = p \cdot q$ with $p, q \geq 3$ two distinct primes, and let α be a vertex of Γ. Set $A = \mathsf{Aut}\Gamma$ and let R be the largest solvable normal subgroup of A.

Lemma 5.1. *A is insolvable.*

Proof. Suppose for a contradiction that A is solvable. Let H be the Fitting subgroup of A. Then, H is nilpotent and H is the product of all its Sylow r-subgroups, where r is a prime dividing $|H|$. Clearly, H_r is characteristic in H, and hence, normal in A. If H_r has at most two orbits on $V\Gamma$, then $2n = |V\Gamma|/2$ divides $|H_r|$, a contradiction. Therefore, H_r has at least three orbits on $V\Gamma$. Considering the quotient graph Γ_{H_r}, by Lemma 2.8, we have H_r is semi-regular on $V\Gamma$, and hence $|H_r|| 4n$, and Γ_{H_r} is a connected A/H_r-arc-transitive graph of valency 11. This implies $|H_2| = 1$ or 2 as there is no symmetric graph of odd order and odd valency, and $|H_r|$ is a prime if r is odd. Then, H is cyclic. Let $C = C_A(H)$. Then, $C \leq H$ by Lemma 2.1, and hence, $C = H$. Thus, $A/H = A/C \leq \mathsf{Aut}(H)$ is abelian. Since $A_\alpha \cong A_\alpha/(A_\alpha \cap H) \cong HA_\alpha/H \leq A/H$ is abelian, A_α is abelian, and hence $A_\alpha \cong Z_{11}$ by Lemma 2.7. Thus, Γ is an arc-regular graph (i.e., $\mathsf{Aut}\Gamma$ is regular on the arc set of Γ). However, there is no arc-regular graph of order four times an odd square-free integer, see [22], a contradiction. This proves the lemma. □

Lemma 5.2. *Assume that A is insolvable and $R = 1$. Then, $T \trianglelefteq A \leq \mathrm{Aut}(T)$ for some nonabelian simple group T, and T has at most two orbits on $V\Gamma$ and $11||T_\alpha|$.*

Proof. Let $N \neq 1$ be a minimal normal subgroup of A. Then, $N = T^m$ for some nonabelian simple group T, where $m \geq 1$ be a positive integer. Since T is nonabelian simple group, $4||T|$, and so $4^m||N|$. If N has at least three orbits on $V\Gamma$, then $N_\alpha = 1$ by Lemma 2.8. Considering the quotient graph Γ_N, and $4||V\Gamma| = 4n$, we obtain the quotient graph Γ_N is of odd order and valency 11, which is impossible. Therefore, N has at most two orbits on $V\Gamma$.

If $N_\alpha = 1$, then the order of N divides $|V\Gamma| = 4n$. Since $4||T|$, we have $N = T$. Note that $|T|$ has at least three prime divisors. Thus, $n = pq$ and $|T| = 4pq$. By [23], we have $N \cong A_5$. Note that now N is regular on $V\Gamma$. Thus, $\Gamma = \mathrm{Cay}(N, S)$ is a normal Cayley graph for some subset $S \subseteq N\backslash\{1\}$. Then, by Lemma 2.3, $A_\alpha \cong A_1 = \mathrm{Aut}(N, S) \leq \mathrm{Aut}(N) \cong S_5$, and so $11 \nmid |A_\alpha|$, contradicting to Lemma 2.7.

Therefore, $N_\alpha \neq 1$. Then, by Lemma 2.9, $11||N_\alpha|$. If A has another minimal normal subgroup, namely M, then $11||M_\alpha|$ by an argument similar to N. It follows $11^2||M_\alpha \times N_\alpha|||A_\alpha|$, a contradiction. Therefore, N is the unique minimal normal subgroup of A.

It remains to show $N = T$. Since N has at most two orbits on $V\Gamma$, $n||N|$. This implies $4 \cdot 11 \cdot n||T|$. Thus, $(4 \cdot n)^m \cdot 11^m||N|||A| = 4n|A_\alpha|$. Then, $(4 \cdot n)^{m-1} \cdot 11^m||A_\alpha|$. Since $11^2 \nmid |A_\alpha|$ by Lemma 2.7, we have $m = 1$. Thus, $N = T$, as required. □

We further determine graphs in the case where A is insolvable and $R = 1$.

Lemma 5.3. *Assume that A is insolvable and $R = 1$.*
(1) *If n is a prime, then $\Gamma = K_{12}$.*
(2) *If $n = pq$, where $p > q \geq 3$ are two distinct primes, then (Γ, A, A_α) is listed in the first three rows of* Table 2.

Proof. By Lemma 5.3, $T \trianglelefteq A \leq \mathrm{Aut}(T)$ for a nonabelian simple group, T has at most two orbits on $V\Gamma$, and $11||T_\alpha|$. By Lemma 2.7, $|A| = |V\Gamma||A_\alpha||2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot n$. Thus, $44||T|$ and $|T||2^{18} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot n$. Therefore, such simple groups T are determined by Lemma 2.6 and are listed in Table 4.

We first deal with the case where $T = \mathrm{PSL}(2, p)$. Assume that $T = \mathrm{PSL}(2, p)$. Then, $A = \mathrm{PSL}(2, p)$ or $\mathrm{PGL}(2, p)$. By the information of maximal subgroups of $\mathrm{PSL}(2, p)$ and $\mathrm{PGL}(2, p)$ in Lemma 2.2, we know A_α is solvable. Then, by Lemma 2.7, there are only three possibilities for A_α, that are Z_{11}, D_{22}, and $D_{22} \times Z_2$. In particular, A_α is a $\{2, 11\}$-group. Since $|A| = 4n|A_\alpha|$, where n has at most two prime divisors, we obtain $|A|$ that has at most four prime divisors, so does $|T|$. By [23], $T = \mathrm{PSL}(2, 23)$. Noting $|PSL(2, 23)| = 2^3 \cdot 3 \cdot 11 \cdot 23$ and $p > q$, we have $p = 23$ and $q = 3$. Then, by Lemma 2.7, pair $(A, A_\alpha) = (\mathrm{PGL}(2, 23), D_{44})$ or $(\mathrm{PSL}(2, 23), D_{22})$. Computation with Magma [19] shows that, up to graph isomorphism, there is only one such graph Γ for pair $(\mathrm{PGL}(2, 23), D_{44})$, say $\mathcal{G}^1_{276}$, with automorphism group $\mathrm{PGL}(2, 23)$; and there are three graphs for pair $(\mathrm{PSL}(2, 23), D_{22})$, say $\mathcal{G}^i_{276}$ for $2 \leq i \leq 4$, with automorphism group $\mathrm{PSL}(2, 23)$. Then, (Γ, A, A_α) is as the second and third rows of Table 2.

Now, we prove parts (1) and (2) of the lemma.

(1). Suppose that n is a prime.

Clearly, $n \neq 2$. If $n = 3$, then $|V\Gamma| = 12$, which implies that $\Gamma = K_{12}$. Actually, $\Gamma = K_{12}$ arises when $T = M_{11}, M_{12}, A_{12}, PSL(2, 11)$ because each of them has a 2-transitive permutation representation of degree 12 (see [24]).

Therefore, we assume that $n \geq 5$. Since T has at most two orbits on $V\Gamma$, $|T : T_\alpha| = 2n$ or $4n$. By Atlas [20] or direct computation in Magma [19], it is easy to check whether a simple group T in Table 4 has a subgroup of index $2n$ or $4n$ and of order divisible by 11. For example, let $T = M_{11}$, then Atlas [20] tells us that a maximal subgroup of M_{11} has index 11, 12, 55, 60, and 165 and the maximal subgroup of index 11 is $M_{10} = A_6 . 2$. Therefore, the only possibility for T_α is A_6. However, this contradicts $11||T_\alpha|$. Therefore, we can rule out the case $T = M_{11}$. Other simple groups can be ruled out similarly.

(2). Suppose that $n = pq$, where $p > q \geq 3$ are two distinct primes.

We may assume that $T \neq \mathrm{PSL}(2,p)(p > 11)$. Assume $T \cong M_{11}$ with order $2^4 \cdot 3^2 \cdot 5 \cdot 11$. Then, $A = M_{11}$ as $\mathrm{Out}(M_{11}) = 1$. Then, $pq = 5 \cdot 3$ and A_α is of order $2^2 \cdot 3 \cdot 11$, contradicting Lemma 2.7. By an argument similar to M_{11}, we can rule out M_{23}, M_{24}, J_1, Co_2, and Co_3, those simple groups with outer automorphism group 1.

Assume $N \cong M_{12}$ with order $2^6 \cdot 3^3 \cdot 5 \cdot 11$. Then, $A \cong M_{12}$ or $M_{12}.\,Z_2$ as $\mathrm{Out}(M_{12}) = 2$, refer to [20]. Then, $pq = 5 \cdot 3$ and A_α is of order $2^4 \cdot 3^2 \cdot 11$ or $2^5 \cdot 3^2 \cdot 11$. By Lemma 2.7, this is impossible. Similarly, we can rule out other simple groups with nontrivial out automorphism groups in Table 4, except two groups PSL(2, 11) and PSL(2, 11^2).

Assume $T = \mathrm{PSL}(2,11)$ with order $2^2 \cdot 3 \cdot 5 \cdot 11$. Then, $A = \mathrm{PSL}(2,11)$ or $\mathrm{PGL}(2,11)$ as $\mathrm{Out}(\mathrm{PSL}(2,11)) \cong Z_2$. In this case, $pq = 5 \cdot 3$, and hence, $|V\Gamma| = 60$. Computation in Magma shows that there is a unique such graph Γ up to graph isomorphism, which is $\mathcal{G}_{60}$, its automorphism group and the vertex of stabilizer are PGL(2, 11) and D_{22}, respectively. This is the first row of Table 2.

Assume $T = \mathrm{PSL}(2,11^2)$. Then, $A = \mathrm{PSL}(2,11^2).\,o$, where $o \leq \mathrm{Out}(\mathrm{PSL}(2,11^2)) \cong Z_2^2$. Note that $|PSL(2,11^2)| = 2^3 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61$. By Lemma 2.7, 11 is the largest prime divisor of A_α, thus $pq = 61 \cdot 11$ and so $|V\Gamma| = 4 \cdot 11 \cdot 61$. If $o = 1$, then $A = \mathrm{PSL}(2,11^2)$ with $|A_\alpha| = 2 \cdot 3 \cdot 5 \cdot 11$, contradicting Lemma 2.7. If $o = \mathrm{Out}(\mathrm{PSL}(2,11^2)) \cong Z_2^2$, then $|A_\alpha| = 2^3 \cdot 3 \cdot 5 \cdot 11$, also contradicting Lemma 2.7. Therefore, $o \cong Z_2$. Then, $|A_\alpha| = 2^2 \cdot 3 \cdot 5 \cdot 11$, and so $A_\alpha = \mathrm{PSL}(2,11)$ by Lemma 2.7. However, no such graph exists by computation with Magma [19]. □

At last, we complete the proof of Theorems 1.3 and 1.4 by dealing with the case where $R \neq 1$.

Lemma 5.4. *Assume that A is insolvable and $R \neq 1$. Then, $p = 19$, $q = 7$, and $(\Gamma, A, A_\alpha) = (\mathcal{G}_{532}, Z_2 \times J_1, \mathrm{PSL}(2,11))$, as the fourth row of* Table 2.

Proof. Since $R \neq 1$, A has a minimal normal subgroup $N \cong Z_r^m \neq 1$ contained in R, where r is a prime. If N has at most two orbits on $V\Gamma$, then $2n = |V\Gamma|/2 \,\|\, |N|$, a contradiction. Therefore, N has at least three orbits on $V\Gamma$. Considering the quotient graph Γ_N, by Lemma 2.8, we have N as semi-regular on $V\Gamma$, and hence, $|N|\,\|\,4n$ and Γ_N is a connected A/N-arc-transitive graph of valency 11. Put v a vertex of $V\Gamma_N$.

Case 1. Suppose that n is a prime.

Then, we have $r = 2$. Then, $N = Z_2$ because if $N = Z_2^2$, then Γ_N is a symmetric graph of odd order and valency 11, which is impossible. Note that A/R is insolvable as A is insolvable. By Lemma 2.12, $\Gamma_N = K_{11,11}$, and so Γ is a normal Z_2-cover of $K_{11,11}$. However, there is no such graph Γ by [21, Theorem 1.1]. This proves Theorem 1.3.

Case 2. Suppose that $n = pq$, where $p > q \geq 3$ are two distinct primes.

Then, $r \in \{p, q, 2\}$ as $|N|\,\|\,4pq$. If $r = q$, then $|V\Gamma_N| = 4p$. By Theorem 1.3 $\Gamma_N = K_{12}$, and so $p = 3$, contradicting $p > q \geq 3$. Therefore, $r \neq q$.

Assume that $r = p$. Then, $|V\Gamma_N| = 4q$. By Theorem 1.3 $\Gamma_N = K_{12}$, and so $q = 3$. Note that a subgroup $H \leq \mathrm{Aut}\Gamma_N = S_{12}$ is arc-transitive if and only if H is 2-transitive on 12 points. By the classification of 2-transitive permutation groups, see, e.g., [24], possibilities for $(A/N, (A/N)_v)$ are as follows:

$$(\mathrm{PSL}(2,11), Z_{11}:Z_5), (\mathrm{PGL}(2,11), Z_{11}:Z_{10}), (M_{11}, \mathrm{PSL}(2,11)), (A_{12}, A_{11}), (S_{12}, S_{11}).$$

If $(A/N, (A/N)_v) \neq (\mathrm{PSL}(2,11), Z_{11}:Z_5)$, then A/N acts 2-arc-transitively on $\Gamma_N = K_{12}$. Note that 2-arc-transitive cyclic cover of complete graph was determined in [25, Theorem 1.1], and from their result we can obtain a contradiction.

Therefore, $(A/N, (A/N)_v) = (\mathrm{PSL}(2,11), Z_{11}:Z_5)$. Then, $A = Z_p.\,PSL(2,11)$. Since the Schur multiplier of PSL(2, 11) is isomorphic to Z_2, see Atlas [20], we have $A = NA' = N \times A' = Z_p \times \mathrm{PSL}(2,11)$. Let $K = \mathrm{PSL}(2,11) \trianglelefteq A$. Note that $|PSL(2,11)| = 2^2 \cdot 3 \cdot 5 \cdot 11$. Therefore, $K_\alpha \neq 1$. Then, Lemma 2.7 implies that K has at most two orbits on $V\Gamma$. By Lemma 2.9, $11\,\|\,|K_\alpha|$. Then $p = 5$. Since K has at most two orbits on $V\Gamma$, $|K_\alpha| = |K|/(2pq) = 44$ or $|K|/(4pq) = 22$. By Atlas [20], PSL(2, 11) has no subgroup of order 44 but has subgroups isomorphic to D_{22} of order 22. Therefore, $K_\alpha = D_{22}$ and K is transitive on $V\Gamma$, and hence Γ is K-arc-transitive. By computation in Magma [19], $\Gamma = \mathcal{G}_{60}$ with automorphism group PGL(2, 11), contradicting the assumption that $R \neq 1$.

Assume last that $r = 2$. Then, $N = Z_2$ and Γ_N satisfies Lemma 2.13. Since A/N is insolvable, the case (1) of Lemma 2.13 is impossible.

Suppose that case (2) of Lemma 2.13 happens, that is, $p = 19$, $q = 7$, $\mathrm{Aut}\Gamma_N = J_1$, and $(\mathrm{Aut}\Gamma_N)_v = \mathrm{PSL}(2, 11)$. Now, $A/N \le \mathrm{Aut}\Gamma_N = J_1$. Note that $|J_1| = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$. Since A/N acts arc-transitively on Γ_N, $|A/N|$ is divisible by $11 \cdot |V\Gamma_N| = 11 \cdot 2 \cdot 7 \cdot 19$. By the information of maximal subgroups of J_1 in Atlas [20], we have $A/N = J_1$. Then, $A = Z_2. J_1$. Since the Schur multiplier of J_1 is 1, also refer to Atlas [20], we have $A = Z_2 \times J_1$. Let $K = J_1 \trianglelefteq A$. Then, $K_\alpha \neq 1$, and Lemma 2.8 implies that K has at most two orbits on $V\Gamma$. If K has two orbits on $V\Gamma$, then $|K_\alpha| = |K|/2pq = 660$; however, J_1 has no subgroup of order 660 by Atlas [20]. Thus, K is transitive on $V\Gamma$, and hence Γ is arc-transitive by Lemma 2.9. Computation in Magma [19] shows that there is a unique such graph Γ, which is $\mathcal{G}_{532}$, with automorphism group $Z_2 \times J_1$.

Suppose that case (3) of Lemma 2.13 happens, then $\mathrm{Aut}\Gamma_N = \mathrm{PSL}(2, r)$ or $\mathrm{PGL}(2, r)$, where r is a prime such that $r \equiv \pm 1 (\mathrm{mod}\ 11)$. By Lemma 2.2, we have $\mathrm{PSL}(2, r) \le A/N$ as A/N is insolvable. In addition, by Lemma 2.2, we have $(A/N)_v = Z_{11}, D_{22}$ or $Z_2 \times D_{22}$. Then, $\mathrm{PSL}(2, r)$ is a simple group with at most four prime divisors and $11 \,|\, |PSL(2, r)|$, and hence $r = 23$ by [23]. Then, we obtain $p = 23$ and $q = 3$. If $A/N = \mathrm{PSL}(2, 23)$, then $|(A/N)_v| = |PSL(2, 23)|/(2 \cdot 3 \cdot 23) = 44$; however, $\mathrm{PSL}(2, 23)$ has no subgroup of order 44 by Lemma 2.2, a contradiction. If $A/N = \mathrm{PGL}(2, 23)$, then $|(A/N)_v| = |A_\alpha| = 88$; however, $\mathrm{PGL}(2, 23)$ has no subgroup of order 88, see Lemma 2.2, which is a contradiction. □

Acknowledgement: The authors are grateful to the constructive comments of referees.

Funding information: This work was partially supported by the National Natural Science Foundation of China (12061089, 11861076, 11701503, and 11761079), and the Natural Science Foundation of Yunnan Province (202201AT070022, 2018FB003, and 2019FB139).

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: Not applicable.

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Appendix A

Magma codes used in Example3.1

```
TU:=[];
j:=0;
G:=PSL(2,13);
H:=Subgroups(G:OrderEqual:=14);
for t in [1..#H] do
HH:=H[t]‘subgroup;
A:=CosetAction(G,HH);
O:=Orbits(A(HH));
for i in [1..#O] do
  OO:=SetToSequence(O[i]);  GA:=OrbitalGraph(A(G),1,OO[1]);

if (IsConnected(GA) eq true) and (Valence(GA) eq 7) and
(not existst:t in TU IsIsomorphic(GA,t) eq true) then
    Append( TU,GA);
    j:=j+1;
  end if;
end for;
end for;
j;
```