

Research Article

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The structural properties of the Gompertz-two-parameter-Lindley distribution and associated inference

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Abstract: In this article, we propose a Gompertz-two-parameter-Lindley distribution by mixing the frailty parameter of the Gompertz distribution with a two-parameter Lindley distribution. The structural properties of the model, such as shape properties, cumulative distribution, quantile functions, moment, moment generating function, failure rate function, mean residual function, and stochastic orders, were derived. Moreover, the unknown parameters are estimated by the profile log likelihood algorithm, and their performance is examined by simulation studies. Finally, a real data example is used to demonstrate the application of the proposed model.

Keywords: Gompertz-Lindley distribution, failure rate function, mean residual life function, maximum likelihood estimation, Fisher information matrix

MSC 2020: 62E10, 62F10, 62P10

1 Introduction

The Gompertz distribution [1] is a fundamental and important model in survival analysis. Let $X|\theta$ be a random variable having the Gompertz distribution with frailty parameter θ and scale parameter λ , with conditional probability density function (PDF):

$$f(x|\theta) = \lambda\theta \exp\{\lambda x - \theta(e^{\lambda x} - 1)\}, x > 0, \theta, \lambda > 0.$$

Lenart [2], Ghitany et al. [3], and Lenart and Missov [4] have studied the structural properties, estimations, and goodness-of-fit tests for the Gompertz distribution, respectively.

It is widely known that the data collected by a data analyst are often heterogeneous; that is, some observations in the data follow the Gompertz distribution and the rest follow another distribution. Hence, the failure rate function (FRF) in reality can be increasing, decreasing, bathtub shaped, upside bathtub shaped, or a combination of these shapes. Many mixture models of Gompertz distributions [5–14] have been proposed for data fitting.

As an important extension, the Gompertz-Lindley distribution [9] is a useful model to accommodate the heterogeneity in the data. Generally, the Gompertz-Lindley distribution is obtained by mixing the Gompertz distribution with a Lindley distribution. Assume that the frailty parameter θ has a Lindley distribution [15] with shape parameter α , with PDF

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$$g(\theta) = \frac{\alpha^2}{\alpha + 1}(1 + \theta) \exp\{-\alpha\theta\}, \quad \theta > 0, \alpha > 0.$$

Then the unconditional PDF of the Gompertz-Lindley distribution is given by

$$\begin{aligned} f(x) &= \int_0^{\infty} f(x|\theta)g(\theta)d\theta \\ &= \frac{\lambda\alpha^2 e^{\lambda x}}{\alpha + 1} \int_0^{\infty} (\theta + \theta^2) \exp\{-\theta(e^{\lambda x} + \alpha - 1)\}d\theta \\ &= \frac{\lambda\alpha^2 e^{\lambda x}}{(\alpha + 1)(e^{\lambda x} + \alpha - 1)^2}, \quad x > 0, \alpha, \lambda > 0. \end{aligned}$$

For a number of datasets, Shanker et al. [16] found that the two-parameter Lindley distribution provides closer fits than the one-parameter Lindley distribution. The PDF of a two-parameter Lindley distribution with shape parameters α and β is given by

$$p(\theta) = \frac{\alpha^2}{\alpha + \beta}(1 + \beta\theta)e^{-\alpha\theta}, \quad \theta > 0, \alpha > 0, \beta \geq 0.$$

To obtain a more flexible model, we mixed the frailty parameter of the Gompertz distribution by a two-parameter Lindley distribution giving rise to various different shaped FRFs. The new distribution could provide an elastic method for data fitting.

By using the two-parameter Lindley distribution to replace the Lindley distribution, the resulting PDF of a random variable X is

$$\begin{aligned} f(x) &= \frac{\lambda\alpha^2 e^{\lambda x}}{\alpha + \beta} \int_0^{\infty} (\theta + \beta\theta^2) \exp\{-\theta(e^{\lambda x} + \alpha - 1)\}d\theta \\ &= \frac{\lambda\alpha^2 e^{\lambda x}(e^{\lambda x} + \alpha + 2\beta - 1)}{(\alpha + \beta)(e^{\lambda x} + \alpha - 1)^3}, \quad x > 0, \alpha, \lambda > 0, \beta \geq 0. \end{aligned}$$

We refer to the random variable X as the Gompertz-two-parameter-Lindley distribution with parameters α , β , and λ will be denoted by GTPL(α, β, λ). The structural properties and associated inference of the proposed model are considered.

2 Structural properties of the model

2.1 Shape properties

Theorem 2.1. For all $\alpha, \lambda > 0, \beta \geq 0$, the PDF $f(x)$ of GTPL(α, β, λ) is

- (i) decreasing if $0 < \alpha \leq 2$;
- (ii) unimodal if $\alpha \geq 3$;
- (iii) decreasing (unimodal) if $\beta \geq \frac{\alpha(2-\alpha)}{2(\alpha-3)} \left(\beta < \frac{\alpha(2-\alpha)}{2(\alpha-3)} \right)$ and $2 < \alpha < 3$, with $f(0) = \frac{\lambda(\alpha+2\beta)}{\alpha(\alpha+\beta)}$, $f(\infty) = 0$.

Proof. The first derivative of $f(x)$ is

$$f'(x) = \frac{\alpha^2 \lambda^2 e^{\lambda x}}{(\alpha + \beta)(e^{\lambda x} + \alpha - 1)^4} \cdot (-(e^{\lambda x})^2 - 4\beta e^{\lambda x} + (\alpha - 1)(\alpha + 2\beta - 1)).$$

Let $\zeta(t) = -t^2 - 4\beta t + (\alpha - 1)(\alpha + 2\beta - 1)$, where $t = e^{\lambda x}$. Since $\zeta(t)$ is a unimodal function in t , $\zeta(1) = \alpha(\alpha - 2) + 2\beta(\alpha - 3)$ is negative (changes sign from positive to negative) when $0 < \alpha \leq 2$ ($\alpha \geq 3$).

Moreover, let $\xi(\beta) = \alpha(\alpha - 2) + 2\beta(\alpha - 3)$, $2 < \alpha < 3$, $\beta = \frac{\alpha(\alpha-2)}{2(\alpha-3)}$ be the solution of equation $\xi(\beta) = 0$. Since $\xi(\beta)$ is a decreasing function in β , $\zeta(t)$ is negative (changes sign from positive to negative) when $\beta \geq \frac{\alpha(\alpha-2)}{2(\alpha-3)} \left(\beta < \frac{\alpha(\alpha-2)}{2(\alpha-3)} \right)$ and $2 < \alpha < 3$. $\square$

Remark 2.1. Let $\beta = 0$ and $\beta = 1$, the GTPL(α, β, λ) reduces to the extended exponential distribution with two parameters [17] and the Gompertz-Lindley distribution [9], respectively.

Remark 2.2. For all $\lambda > 0$, when $0 < \alpha \leq 2$ or $2 < \alpha < 3$ and $\beta \geq \frac{\alpha(\alpha-2)}{2(\alpha-3)}$, the mode of x is equal to 0. On the other hand, when $\alpha \geq 3$ or $2 < \alpha < 3$ and $\beta < \frac{\alpha(\alpha-2)}{2(\alpha-3)}$, the mode of x is the solution of $f'(x) = 0$. Since $t_0 = e^{\lambda x_0} = -2\beta + (4\beta^2 + (\alpha - 1)(\alpha + 2\beta - 1))^{1/2}$ is the solution of the quadratic equation $-t_0^2 - 4\beta t_0 + (\alpha - 1)(\alpha + 2\beta - 1) = 0$, the mode of x is equal to $\frac{\ln(-2\beta + (4\beta^2 + (\alpha - 1)(\alpha + 2\beta - 1))^{1/2})}{\lambda}$.

2.2 Cumulative distribution and quantile functions

The cumulative distribution function (CDF) of the GTPL(α, β, λ) is given by

$$\begin{aligned} F(x) &= P(X \leq x) = \int_0^x \frac{\lambda \alpha^2}{\alpha + \beta} \frac{e^{\lambda u} (e^{\lambda u} + \alpha + 2\beta - 1)}{(e^{\lambda u} + \alpha - 1)^3} du \\ &= \frac{\alpha^2}{\alpha + \beta} \int_0^x (e^{\lambda u} + \beta - 1)^{-2} de^{\lambda u} + \frac{2\beta \alpha^2}{\alpha + \beta} \int_0^x (e^{\lambda u} + \alpha - 1)^{-3} de^{\lambda u} \\ &= 1 - \frac{\alpha^2 (e^{\lambda x} + \alpha + \beta - 1)}{(\alpha + \beta)(e^{\lambda x} + \alpha - 1)^2}, \quad x > 0, \quad \alpha, \lambda > 0, \quad \beta \geq 0. \end{aligned}$$

The q th quantile of $x_{q\text{GTPL}}(\alpha, \beta, \lambda)$ is the solution of $q = F(x_q)$, where $0 < q < 1$. Hence, $x_q = F^{-1}(q)$ is equivalent to

$$x_q = \lambda^{-1} \ln \frac{(\alpha(\alpha^2 + 4\beta(\alpha + \beta)(1 - q))^{1/2} + \alpha)}{2(\alpha + \beta)(1 - q)}.$$

Clearly, x_q is increasing in β and decreasing in α and λ .

The quantiles of the GTPL(α, β, λ) are given by

$$\begin{aligned} Q_1 &= \lambda^{-1} \ln \left(\frac{2\alpha\beta(\alpha^2 + 3\beta(\alpha + \beta))^{1/2} + \alpha}{3\beta(\alpha + \beta)} - \alpha + 1 \right), \\ Q_2 &= \lambda^{-1} \ln \left(\frac{2\alpha\beta(\alpha^2 + 2\beta(\alpha + \beta))^{1/2} + \alpha}{2\beta(\alpha + \beta)} - \alpha + 1 \right), \\ Q_3 &= \lambda^{-1} \ln \left(\frac{2\alpha\beta(\alpha^2 + \beta(\alpha + \beta))^{1/2} + \alpha}{\beta(\alpha + \beta)} - \alpha + 1 \right). \end{aligned}$$

2.3 Failure rate and mean residual functions

The FRF $h(x)$ of the GTPL distribution is given by

$$h(x) = \frac{f(x)}{1 - F(x)} = \frac{\lambda e^{\lambda x} (e^{\lambda x} + \alpha + 2\beta - 1)}{(e^{\lambda x} + \alpha - 1)(e^{\lambda x} + \alpha + \beta - 1)}, \quad x > 0, \quad \alpha, \lambda > 0, \quad \beta \geq 0.$$

Theorem 2.2. For all $\alpha, \lambda > 0$, and $\beta \geq 0$, $h(x)$ is

(i) decreasing if $0 < \alpha \leq \alpha_0$, where $1 < \alpha_0 < 1 + \beta$ such that

$$\alpha_0^3 + (3\beta - 1)\alpha_0^2 + 2\beta(\beta - 2)\alpha_0 - 2\beta^2 = 0;$$

(ii) unimodal if $\alpha_0 < \alpha < 1 + \beta$;

(iii) increasing if $\alpha \geq 1 + \beta$, with $h(0) = \frac{\lambda(\alpha + 2\beta)}{\alpha(\alpha + \beta)}$ and $h(\infty) = \lambda$.

Proof. The first derivative of $h(x)$ is

$$h'(x) = \frac{\lambda^2 e^{\lambda x}}{(e^{\lambda x} + \alpha - 1)^2 (e^{\lambda x} + \alpha + \beta - 1)^2} \zeta(t),$$

where $\zeta(t) = (\alpha - \beta - 1)t^2 + 2(\alpha - 1)(\alpha + \beta - 1)t + (\alpha - 1)(\alpha + \beta - 1)(\alpha + 2\beta - 1)$ and $e^{\lambda x} = t > 1$. Note that $\zeta(t) > 0$ if $\alpha \geq 1 + \beta$ and $\zeta(t) < 0$ if $\alpha \leq 1$.

For $1 < \alpha \leq 1 + \beta$, $\zeta(t)$ has a unique maximum at point $t_0 = \frac{(1 - \alpha)(\alpha + \beta - 1)}{\alpha - \beta - 1}$. Since $\zeta(\infty) = -\infty$ and $\zeta(1) = \alpha^3 + (3\beta - 1)\alpha^2 + 2\beta(\beta - 2)\alpha - 2\beta^2$, it follows that $\zeta(t) < 0$ if $\zeta(1) < 0$. Since the equation $\alpha_0^3 + (3\beta - 1)\alpha_0^2 + 2\beta(\beta - 2)\alpha_0 - 2\beta^2 = 0$ has a unique zero point,

$$\alpha_0 = (-q/2 + ((q/2)^2 + (q/3)^3)^{1/2})^{1/3} + (-q/2 - ((q/2)^2 + (q/3)^3)^{1/2})^{1/3} - \beta + 1/3$$

on the interval $(1, 1 + \beta)$, where $p = -\beta^2 - 2\beta - \frac{1}{3}$ and $q = \frac{1}{3}(2\beta^2 - 2\beta - \frac{2}{9})$. Then, $h'(x)$ has the same sign as $\zeta(t)$. $\square$

As an alternative, the mean residual life function (MRLF) is defined as follows:

$$\begin{aligned} \mu(x) &= E(X - x | X > x) \\ &= \frac{1}{1 - F(x)} \int_x^\infty \frac{e^{\lambda y} + \alpha + \beta - 1}{(e^{\lambda y} + \alpha - 1)^2} dy \\ &= \frac{(e^{\lambda x} + \alpha - 1)^2}{\lambda(1 - \alpha)(e^{\lambda x} + \alpha + \beta - 1)} \left(\frac{\alpha + \beta - 1}{\alpha - 1} \ln \frac{e^{\lambda x}}{e^{\lambda x} + \alpha - 1} + \frac{\beta}{e^{\lambda x} + \alpha - 1} \right), \quad x > 0. \end{aligned}$$

Bryson and Siddiqui [18] showed that increasing (decreasing) FRFs imply decreasing (increasing) mean residual life functions. However, Olcay [19] indicated that this unique property may not hold for nonmonotone FRFs. That is, if $h(x)$ is unimodal, then $\mu(x)$ is increasing (anti-unimodal) if $f(0)\mu(x) \geq 1$ ($f(0)\mu(x) < 1$).

Theorem 2.3. For all $\alpha, \lambda > 0$ and $\beta \geq 0$, $\mu(x)$ is

(i) increasing if $0 < \alpha \leq \alpha_1$, where $\alpha_0 < \alpha_1 < 1 + \beta$ such that

$$(1 - \alpha_1)^2(\alpha_1 + \beta)^2 - (\alpha_1 + 2\beta)(\alpha_1(\alpha_1 + \beta - 1)\ln \alpha_1 + \beta(1 - \alpha_1)) = 0;$$

(ii) anti-unimodal if $\alpha_1 < \alpha < 1 + \beta$;

(iii) decreasing if $\alpha \geq 1 + \beta$, with $\mu(0) = \mu$ and $\mu(\infty) = \lambda^{-1}$.

Proof. For $0 < \alpha \leq \alpha_1$ ($\alpha \geq 1 + \beta$), Theorem 2.2 (i) (Theorem 2.2 (iii)) shows that $h(x)$ is decreasing (increasing) and hence by [18], $\mu(x)$ is increasing (decreasing).

For $\alpha_1 < \alpha < 1 + \beta$, Theorem 2.2 (ii) shows that $h(x)$ is unimodal. Since

$$f(0)\mu(x) = \frac{\alpha + 2\beta}{(1 - \alpha)^2(\alpha + \beta)^2} (\alpha(\alpha + \beta - 1)\ln \alpha + \beta(1 - \alpha)),$$

it follows that $f(0)\mu(x) \geq 1$ ($f(0)\mu(x) < 1$) if $\alpha_0 < \alpha \leq \alpha_1$ ($\alpha_1 < \alpha < 1 + \beta$), where $\alpha_0 < \alpha_1 < 1 + \beta$ such that

$$(1 - \alpha_1)^2(\alpha_1 + \beta)^2 - (\alpha_1 + 2\beta)(\alpha_1(\alpha_1 + \beta - 1)\ln \alpha_1 + \beta(1 - \alpha_1)) = 0.$$

Hence, $\mu(x)$ is increasing (anti-unimodal), by [4]. $\square$

Remark 2.3. The equation $(1 - \alpha_1)^2(\alpha_1 + \beta)^2 - (\alpha_1 + 2\beta)(\alpha_1(\alpha_1 + \beta - 1)\ln \alpha_1 + \beta(1 - \alpha_1)) = 0$ has a unique zero point α_1 on the interval $(\alpha_0, 1 + \beta)$.

2.4 Limit behavior

Proposition 2.1. The asymptotic CDF, PDF, and FRF of the GTPL distribution as $x \rightarrow 0$ are given by

$$\begin{aligned} 1 - F(x) &\sim \frac{\alpha^2}{\alpha + \beta} \frac{\lambda x + \alpha + \beta}{(\lambda x + \alpha)^2}, \\ f(x) &\sim \frac{\lambda \alpha^2}{\alpha + \beta} \frac{\lambda x + \alpha + 2\beta}{(\lambda x + \alpha)^3}, \\ h(x) &\sim \frac{\lambda(\lambda x + \alpha + 2\beta)}{(\lambda x + \alpha)(\lambda x + \alpha + \beta)}. \end{aligned}$$

Proof. When $x \rightarrow 0$, we have $e^{\lambda x} \rightarrow 1$ and $e^{\lambda x} - 1 \sim \lambda x$. □

Proposition 2.2. The asymptotic CDF, PDF, and FRF of the GTPL distribution as $x \rightarrow \infty$ are given by

$$1 - F(x) \sim \frac{\alpha^2}{\alpha + \beta} e^{-\lambda x}, \quad f(x) \sim \frac{\lambda \alpha^2}{\alpha + \beta} e^{-\lambda x}, \quad h(x) \sim \lambda.$$

Proof. When $x \rightarrow \infty$, we have $e^{\lambda x} + C \sim e^{\lambda x}$, where C is a constant. □

2.5 Moments and associated measures

We derive the r th raw moment (about the origin) of the GTPL(α , β , λ) distribution. We consider the following two cases:

(i) $\alpha \neq 1$: For positive integer r , we have

$$\mu_r = E(X^r) = \frac{r\alpha^2}{\alpha + \beta} \int_0^\infty \frac{x^{r-1}(e^{\lambda x} + \alpha + \beta)}{(e^{\lambda x} + \alpha - 1)^2} dx = \frac{\alpha^2 \Gamma(r+1)}{\lambda^r \bar{\alpha}^2 (\alpha + \beta)} ((\bar{\alpha} - \beta)L_r(\bar{\alpha}) + \beta L_{r-1}(\bar{\alpha})),$$

where $\bar{\alpha} = 1 - \alpha$ and

$$L_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - z} dt, \quad s > 0, \quad z < 1,$$

is the polynomial function. In particular, $L_1(z) = -\ln(1 - z)$ and $L_0(z) = z/(1 - z)$. Moreover, the polynomial function satisfies

$$\begin{aligned} L_{s-1}(z) - L_s(z) &= \frac{z^2}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{(e^t - z)^2} dt, \\ L_{s-1}(z) - 3L_s(z) + 2L_{s+1}(z) &= \frac{2z^3}{\Gamma(s+1)} \int_0^\infty \frac{t^s}{(e^t - z)^3} dt. \end{aligned}$$

(ii) $\alpha = 1$: For all positive integers r ,

$$\mu_r = E(X^r) = \frac{\lambda}{1 + \beta} \int_0^{\infty} \frac{x^r (e^{\lambda x} + 2\beta)}{e^{2\lambda x}} dx = \frac{r!(2^r + \beta)}{(2\lambda)^r(1 + \beta)}.$$

Then, the first four raw moments are given by

$$\begin{aligned} \mu &= \begin{cases} \frac{\alpha((\beta - \bar{\alpha})\alpha \ln \alpha + \beta\bar{\alpha})}{\lambda\bar{\alpha}^2(\alpha + \beta)}, & \alpha \neq 1, \\ \frac{2 + \beta}{2\lambda(1 + \beta)}, & \alpha = 1, \end{cases} \\ \mu_2 &= \begin{cases} \frac{2\alpha^2((\bar{\alpha} - \beta)L_2(\bar{\alpha}) - \beta \ln \alpha)}{\lambda^2\bar{\alpha}^2(\alpha + \beta)}, & \alpha \neq 1, \\ \frac{4 + \beta}{2(1 + \beta)\lambda^2}, & \alpha = 1, \end{cases} \\ \mu_3 &= \begin{cases} \frac{6\alpha^2((\bar{\alpha} - \beta)L_3(\bar{\alpha}) + \beta L_2(\bar{\alpha}))}{\lambda^3\bar{\alpha}^2(\alpha + \beta)}, & \alpha \neq 1, \\ \frac{24 + 3\beta}{4(1 + \beta)\lambda^3}, & \alpha = 1, \end{cases} \\ \mu_4 &= \begin{cases} \frac{24\alpha^2((\bar{\alpha} - \beta)L_4(\bar{\alpha}) + \beta L_3(\bar{\alpha}))}{\lambda^4\bar{\alpha}^2(\alpha + \beta)}, & \alpha \neq 1, \\ \frac{48 + 3\beta}{2(1 + \beta)\lambda^4}, & \alpha = 1. \end{cases} \end{aligned}$$

Therefore, the variance of the GTPL distribution is

$$\sigma^2 = \begin{cases} \frac{\alpha^2(2\bar{\alpha}^2(\alpha + \beta)(\bar{\alpha} - \beta)L_2(\bar{\alpha}) - \alpha^2(\bar{\alpha} - \beta)^2(\ln \alpha)^2 - (2\ln \alpha + \bar{\alpha})\bar{\alpha}\beta^2)}{\lambda^2\bar{\alpha}^4(\alpha + \beta)^2}, & \alpha \neq 1, \\ \frac{\beta^2 + 6\beta + 4}{4(\lambda + \lambda\beta)^2}, & \alpha = 1. \end{cases}$$

The skewness and kurtosis of the GTPL distribution can be expressed as follows:

$$\begin{aligned} \text{Skewness} &= \frac{E(X - \mu)^3}{\sigma^3} = \frac{\mu_3 - 3\mu_2\mu + 2\mu^3}{\sigma^3}, \\ \text{Kurtosis} &= \frac{E(X - \mu)^4}{\sigma^4} = \frac{\mu_4 - 4\mu_3\mu + 6\mu_2\mu^2 - 3\mu^4}{\sigma^4}. \end{aligned}$$

Using Maclaurin's series expansion of an exponential function, the moment generating function of the GTPL distribution is

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r).$$

For $\alpha \neq 1$, we have

$$M_X(t) = \frac{\alpha^2}{\bar{\alpha}^2(\alpha + \beta)} \sum_{r=1}^{\infty} \frac{t^r \Gamma(r+1)((\bar{\alpha} - \beta)L_r(\bar{\alpha}) + \beta L_{r-1}(\bar{\alpha}))}{\lambda^r r!}.$$

For $\alpha = 1$, we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r (2^r + \beta)}{(1 + \beta)(2\lambda)^r}.$$

2.6 Stochastic orders

Many stochastic orders exist and have various applications [20]. Here, the likelihood ratio order $\leq_{lr}$, the usual stochastic order $\leq_{st}$, the failure rate order $\leq_{fr}$, and the mean residual life order $\leq_{mrl}$ are considered. A random variable X is said to be smaller than a random variable Y :

- (i) $X \leq_{st} Y$ if $F_X(x) \geq F_Y(x)$ for all x ;
- (ii) $X \leq_{fr} Y$ if $h_X(x) \geq h_Y(x)$ for all x ;
- (iii) $X \leq_{mrl} Y$ if $\mu_X(x) \leq \mu_Y(x)$ for all x ;
- (iv) $X \leq_{lr} Y$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x .

Theorem 2.4. Let $X \sim \text{GTPL}(\alpha, \beta, \lambda)$ and $Y \sim \text{GTPL}(\alpha, \beta, \lambda)$. For all $\lambda > 0$, $\beta \geq 0$ and $\alpha_1 \leq \alpha_2$, $X \leq_{lr} Y$ (and hence $X \leq_{fr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$).

Proof. Since

$$\frac{d}{dx} \ln \frac{f_X(x)}{f_Y(x)} = (\alpha_2 - \alpha_1) \lambda e^{\lambda x} (e^{\lambda x} + \alpha_1 + 2\beta - 1)^{-1} (e^{\lambda x} + \alpha_2 + 2\beta - 1)^{-1} - 3(e^{\lambda x} + \alpha_1 - 1)^{-1} (e^{\lambda x} + \alpha_2 - 1)^{-1} \leq 0,$$

and $\alpha_1 \leq \alpha_2$, it follows that $\frac{f_X(x)}{f_Y(x)}$ is decreasing in x . Then $X \leq_{lr} Y$. Note that $X \leq_{lr} Y$ implies $X \leq_{fr} Y$, $X \leq_{mrl} Y$, and $X \leq_{st} Y$. $\square$

3 Estimation

3.1 Maximum likelihood estimation

Let $x_1, x_2, \dots, x_n$ be a random sample from the GTPL distribution with parameters $\theta = (\alpha, \beta, \lambda)$. The log-likelihood function is

$$l(\theta) = \sum_{i=1}^n \ln f(x_i; \theta) = 2n \ln \alpha + n \ln \lambda + \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n \ln(e^{\lambda x_i} + \alpha + 2\beta - 1) - n \ln(\alpha + \beta) - 3 \sum_{i=1}^n \ln(e^{\lambda x_i} + \alpha - 1).$$

The maximum likelihood estimators (MLEs) $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ of the parameters $\theta = (\alpha, \beta, \lambda)$ are the simultaneous solutions of the following equations:

$$\begin{cases} \frac{\partial l(\theta)}{\partial \alpha} = \frac{2n}{\alpha} + \sum_{i=1}^n \frac{1}{e^{\lambda x_i} + \alpha + 2\beta - 1} - \frac{n}{\alpha + \beta} - 3 \sum_{i=1}^n \frac{1}{e^{\lambda x_i} + \alpha - 1} = 0 \\ \frac{\partial l(\theta)}{\partial \beta} = 2 \sum_{i=1}^n \frac{1}{e^{\lambda x_i} + \alpha + 2\beta - 1} - \frac{n}{\alpha + \beta} = 0 \\ \frac{\partial l(\theta)}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} + \alpha + 2\beta - 1} - 3 \sum_{i=1}^n \frac{x_i e^{\lambda x_i}}{e^{\lambda x_i} + \alpha - 1} = 0. \end{cases}$$

Since solutions $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ have no closed form, we use the profile likelihood method [21] to obtain the MLEs.

3.2 Interval estimation

For interval estimation and tests of hypotheses about the parameters $\theta = (\alpha, \beta, \lambda)$, we need to calculate the following Fisher information matrix:

$$I(\theta) = (I_{ij}(\theta)) = \left(E \left(-\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ln f(x; \theta) \right) \right), \quad i, j = 1, 2, 3.$$

After some derivations, we have

$$\begin{aligned} I(\alpha) = I_{11}(\theta) &= \begin{cases} \frac{2}{\alpha^2} - \frac{1}{(\alpha + \beta)^2} + \frac{\alpha^2}{4\beta(\alpha + \beta)} \left(\frac{\beta - \alpha}{\alpha^2} - \frac{1}{2\beta^2} \ln \frac{\alpha}{\alpha + 2\beta} \right) - \frac{2\alpha + 3\beta}{2\alpha^2(\alpha + \beta)}, & \beta \neq 0, \\ \frac{1}{3\alpha^2}, & \beta = 0, \end{cases} \\ I(\beta) = I_{22}(\theta) &= \begin{cases} \frac{\alpha^2}{\beta(\alpha + \beta)} \left(\frac{1}{\alpha^2} - \frac{1}{\alpha\beta} - \frac{1}{2\beta^2} \ln \frac{\alpha}{\alpha + 2\beta} \right) - \frac{1}{(\alpha + \beta)^2}, & \beta \neq 0, \\ \frac{1}{3\alpha^2}, & \beta = 0, \end{cases} \\ I(\lambda) = I_{33}(\theta) &= \begin{cases} \frac{1}{\lambda^2} - V_1 + V_2, & \beta \neq 0, \\ \frac{1}{\lambda^2} - \frac{2\alpha}{3\lambda^2\bar{\alpha}} \left(\frac{\bar{\alpha}}{\alpha} + (1 - \lambda) \ln \alpha - \lambda L_2(\bar{\alpha}) \right), & \beta = 0, \end{cases} \\ I_{12}(\theta) = I_{21}(\theta) &= \begin{cases} \frac{\alpha^2}{4\beta^3(\alpha + \beta)} \ln \frac{\alpha + 2\beta}{\alpha} - \frac{\alpha - \beta}{2\beta^2(\alpha + \beta)} - \frac{1}{(\alpha + \beta)^2}, & \beta \neq 0, \\ -\frac{1}{3\alpha^2}, & \beta = 0, \end{cases} \\ I_{13}(\theta) = I_{31}(\theta) &= \begin{cases} W_1 - 3W_2, & \beta \neq 0, \\ \frac{\alpha}{\lambda} \left(\frac{\ln \alpha}{3\bar{\alpha}^2} + \frac{\bar{\alpha}}{3\alpha\bar{\alpha}^2} - \frac{1}{3\alpha^2} \right), & \beta = 0, \end{cases} \\ I_{23}(\theta) = I_{32}(\theta) &= \begin{cases} 2W_1, & \beta \neq 0, \\ \frac{\alpha}{\lambda} \left(\frac{1}{3\bar{\alpha}^2} - \frac{\ln \alpha}{3\bar{\alpha}^2} - \frac{\bar{\alpha}}{3\alpha\bar{\alpha}^2} \right), & \beta = 0, \end{cases} \end{aligned}$$

where

$$\begin{aligned} W_1 &= \frac{\alpha^2}{\lambda(\alpha + \beta)} \left(\frac{A_1 L_2(\bar{\alpha} - 2\beta)}{\bar{\alpha} - 2\beta} + \frac{A_2 L_2(\bar{\alpha})}{\bar{\alpha}} - \frac{A_3 (\ln \alpha + L_2(\bar{\alpha}))}{\bar{\alpha}^2} + \frac{A_4 (\bar{\alpha}/\alpha + 3 \ln \alpha + 2L_2(\bar{\alpha}))}{\bar{\alpha}^3} \right), \\ W_2 &= \frac{\alpha^2}{\lambda(\alpha + \beta)} \left(\frac{1}{6\alpha^2} - \frac{\ln \alpha + \bar{\alpha}/\alpha}{6\bar{\alpha}^2} + \frac{\beta}{6\alpha^3} - \frac{\beta}{6} \left(\frac{\ln \alpha}{\bar{\alpha}^3} + \frac{1}{\alpha\bar{\alpha}^2} - \frac{1}{2\alpha^2\bar{\alpha}} \right) \right), \\ V_1 &= -\frac{\alpha^2(\alpha + 2\beta - 1)}{\lambda^2(\alpha + \beta)} \left(A_1 \cdot \frac{2L_3(1 - \alpha - 2\beta)}{1 - \alpha - 2\beta} + A_2 \cdot \frac{2L_3(\bar{\alpha})}{\bar{\alpha}} + A_3 \cdot \frac{2(L_2(\bar{\alpha}) - L_3(\bar{\alpha}))}{\bar{\alpha}^2} \right. \\ &\quad \left. + A_4 \cdot \frac{(-\ln \bar{\alpha} - 3L_2(\bar{\alpha}) + 2L_3(\bar{\alpha}))}{\bar{\alpha}^3} \right), \\ V_2 &= -\frac{\alpha^2}{\lambda^2\bar{\alpha}(\alpha + \beta)} \left(\frac{\bar{\alpha}}{\alpha} - L_2(\bar{\alpha}) - \beta \frac{2\alpha^2 \ln \alpha + 2\alpha\bar{\alpha} - \bar{\alpha}^2}{2\alpha^2\bar{\alpha}} + \frac{\beta}{2\bar{\alpha}} \left(\frac{\bar{\alpha}}{\alpha} + 3 \ln \alpha + 2L_2(\bar{\alpha}) \right) \right), \\ A_1 &= -\frac{(\alpha - 1)^2 + 4\beta(\alpha + \beta - 1)}{8\beta^3}, A_2 = -\frac{(\alpha - 1)^2 + 4\beta(\alpha + \beta - 1)}{4\beta^3}, A_3 = -\frac{(\alpha - 1)^2 + 4\beta(\alpha - 1)}{4\beta^2}, A_4 = \frac{(\alpha - 1)^2}{2\beta}. \end{aligned}$$

Under mild regularity conditions [22], the MLE $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ of $\theta = (\alpha, \beta, \lambda)$ is consistent and asymptotically normal:

$$n^{-1/2}(\hat{\theta} - \theta) \xrightarrow{D} N_3(0, I^{-1}(\theta)).$$

Hence, $\hat{\alpha} \xrightarrow{D} N(\alpha, 1/nI(\alpha))$, $\hat{\beta} \xrightarrow{D} N(\beta, 1/nI(\beta))$ and $\hat{\lambda} \xrightarrow{D} N(\lambda, 1/nI(\lambda))$. The asymptotic $100(1 - \delta)\%$ confidence intervals for α , β , and λ are given by

$$\hat{\alpha} \pm z_{\delta/2} S.E.(\hat{\alpha}), \quad \hat{\beta} \pm z_{\delta/2} S.E.(\hat{\beta})$$

and

$$\hat{\lambda} \pm z_{\delta/2} S.E.(\hat{\lambda}),$$

where $z_{\delta/2}$ is the upper $\delta/2$ quantile of the standard normal distribution.

3.3 Simulation study

Here, we assess the performance of the MLEs $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$. We also report the coverage probabilities and average lengths of the confidence intervals for the parameters $\theta = (\alpha, \beta, \lambda)$. We generate data from the GTPL distribution:

$$X_i = F^{-1}(U_i) = \lambda^{-1} \ln \left(\frac{\alpha((\alpha^2 + 4\beta(\alpha + \beta)(1 - U_i))^{-1/2} + \alpha)}{2(\alpha + \beta)(1 - U_i)} - \alpha + 1 \right), \quad i = 1, 2, \dots, n,$$

where U_i are i.i.d. uniform(0,1) random variables. For different parameter settings, we consider $(\alpha, \beta, \lambda) = (1, 1, 1)$, $(2, 2, 1)$, $(5, 6, 3)$ and sample size $n = 30, 50, 100, 200$. In each simulation, we first resampled the observed value $(x_1, x_2, \dots, x_n)$ 10,000 times from the GTPL distribution and then calculated the average estimates (AEs), the mean squared errors (MSEs), the coverage probabilities (CPs), and the average lengths (ALs). Specifically, the AE, MSE, CP, and AL are defined as follows:

$$\begin{aligned} AE(\hat{\theta}) &= \sum_{i=1}^N \hat{\theta}_i / N, \\ MSE(\hat{\theta}) &= \sum_{i=1}^N (\hat{\theta}_i - \theta)^2 / N, \\ CP(\theta) &= \sum_{i=1}^N I\{\theta \in (\theta_L, \theta_U)\} / N, \\ AL(\theta) &= \sum_{i=1}^N (\theta_U - \theta_L) / N, \end{aligned}$$

where θ_U and θ_L are the upper and lower limits of the interval, respectively.

From the system equations, we find that the parameter β is a function of α and λ ,

$$\beta = \frac{1}{\frac{4}{\alpha} - \frac{6}{n} \sum_{i=1}^n \frac{1}{e^{\lambda x_i} + \alpha - 1}} - \alpha.$$

Since the MLEs do not have explicit solutions, some numerical techniques are required to obtain the MLEs. However, it is widely known that the EM algorithm is sensitive to the initial values of the parameters. Moreover, the choice of parameters directly affects the convergence efficiency and the global optimal solution. As an alternative, the MLEs of the parameters can be obtained by directly maximizing the log-likelihood function using numerical optimization algorithms, for example, the numerical maximization function Optim, using the BFGS method, in R language. Furthermore, Ghitany et al. [9] used the differential evolution algorithm (DEoptim package) to obtain the exact MLEs of the Gompertz-Lindley distribution. As an extension, we provide the following profile log likelihood algorithm for estimating the parameters in the GTPL distribution.

Algorithm 1: Profile log likelihood algorithm

Step 1. Given the lower and upper bounds of three parameter values (α, β, λ) , generate a regular equally spaced sequence for each of α, β, λ . Next, construct a grid of all possible combinations of values of (α, β, λ) .

Step 2. Given an initial value of $\beta = \beta^*$. Construct combinations of all possible values of $(\alpha, \beta^*, \lambda)$ as in Step 1. Choose the combination of $(\alpha_{\text{opt1}}, \beta^*, \lambda_{\text{opt1}})$, which gives the largest likelihood for $\beta = \beta^*$. The likelihood curve for this combination with $\alpha = \alpha_{\text{opt1}}$ and $\lambda = \lambda_{\text{opt1}}$ is maximized with respect to β , which can provide an initial estimate $\hat{\beta}_0$.

Step 3. Construct combinations of all possible values of $(\alpha_{\text{opt1}}, \hat{\beta}_0, \lambda)$ as in Step 1. Choose the combination of $(\alpha_{\text{opt1}}, \hat{\beta}_0, \lambda_{\text{opt2}})$, which gives the largest likelihood for $\lambda = \lambda_{\text{opt2}}$. Next, construct combinations of all possible values of $(\alpha, \hat{\beta}_0, \lambda_{\text{opt2}})$ as in Step 1, and choose the combination of $(\alpha_{\text{opt2}}, \hat{\beta}_0, \lambda_{\text{opt2}})$, which gives the largest likelihood for $\alpha = \alpha_{\text{opt2}}$.

Step 4. Let $\hat{\alpha}_0 = \alpha_{\text{opt2}}$, construct combinations of all possible values of $(\hat{\alpha}_0, \hat{\beta}_0, \lambda)$ as in Step 1. Choose the combination of $(\hat{\alpha}_0, \hat{\beta}_0, \lambda_{\text{opt3}})$, which gives the largest likelihood for $\lambda = \lambda_{\text{opt3}}$. Steps 3 and 4 gives an initial estimate for $\hat{\alpha}_0 = \alpha_{\text{opt2}}, \hat{\lambda}_0 = \lambda_{\text{opt3}}$.

Step 5. Let $\theta = (\alpha, \beta, \lambda)$. Following Steps 2–4, we can use $\beta^* = \beta_0$ to update θ . Generally, for $k = 0, 1, 2, \dots, (\hat{\alpha}_k, \hat{\beta}_k, \hat{\lambda}_k)$ can be updated to $(\hat{\alpha}_{k+1}, \hat{\beta}_{k+1}, \hat{\lambda}_{k+1})$.

Step 6. Given $\varepsilon > 0$, while $|\hat{\theta}_{k+1} - \hat{\theta}_k| > \varepsilon$, set $k = k + 1$ and repeat Step 5 until convergence is achieved

Simulation studies are presented in Table 1. When the sample size n becomes larger, the average estimates of the parameters approach the preset values, and MSEs of the AEs decrease progressively. Moreover, CPs of the confidence intervals for α and λ are close to the confidence level of 95%, and the ALs decrease as the sample size increases. Note that the confidence interval of β has lower coverage probabilities for small to moderate sample sizes, which requires some modified approaches to improve the poor coverage performance of the confidence interval of β .

Table 1: Performance of MLEs and 95% confidence intervals

α	β	λ	n	AE			MSE			CP			AL		
				$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	α	β	λ	α	β	λ
1	1	1	30	1.21	0.86	1.18	0.34	0.19	0.26	0.94	0.85	0.93	0.95	4.56	0.78
			50	1.15	0.93	1.12	0.25	0.12	0.17	0.95	0.88	0.94	0.74	3.53	0.61
			100	1.09	0.99	1.07	0.15	0.07	0.09	0.95	0.91	0.95	0.52	2.50	0.43
			200	1.05	1.00	1.04	0.07	0.03	0.04	0.95	0.93	0.95	0.37	1.76	0.30
2	2	1	30	2.18	1.62	1.11	0.58	1.11	0.13	0.94	0.87	0.94	1.90	9.12	0.66
			50	2.15	1.85	1.06	0.48	1.07	0.08	0.95	0.89	0.94	1.47	7.06	0.51
			100	2.10	1.99	1.03	0.34	0.78	0.05	0.95	0.92	0.95	1.04	4.99	0.36
			200	2.07	2.00	1.02	0.21	0.16	0.03	0.95	0.94	0.95	0.74	3.53	0.26
5	6	3	30	5.12	5.70	3.09	0.81	0.75	0.24	0.94	0.86	0.94	4.71	27.07	1.55
			50	5.10	5.73	3.07	0.77	0.67	0.18	0.94	0.89	0.94	3.65	20.97	1.20
			100	5.09	5.80	3.04	0.68	0.55	0.12	0.94	0.91	0.95	2.58	14.83	0.85
			200	5.07	5.89	3.02	0.57	0.40	0.08	0.95	0.93	0.95	1.83	10.48	0.60

4 Real data analysis

In this section, we illustrate the application of the Gompertz-two-parameter-Lindley distribution with a real data example [23]. The dataset consists of 217 survival times (in years) for female patients with hepatocellular carcinoma. A summary of descriptive statistics of this dataset is presented in Table 2, and its histogram and boxplot are shown in Figure 1. For comparison, this dataset is also applied to evaluate some well-known models such as the log-normal distribution, the Gompertz distribution, the gamma distribution, the Weibull distribution, and the Gompertz-Lindley distribution.

Table 3 reports the MLEs of α , β , and λ ; the estimated log-likelihood function, and the Cramer-von Mises (CvM) goodness-of-fit tests of the six models. Among the six models, the proposed GTPL distribution has the highest log-likelihood, the smallest test statistic and the highest p -value of the CvM goodness-of-fit tests. Thus, we can conclude that the GTPL distribution provides the best fit for this dataset.

Table 2: Descriptive statistics for the survival data

Minimum	First quartile	Median	Third quartile	Maximum	Mean	Variance
0.01	0.55	1.16	2.86	8.35	2.01	4.17

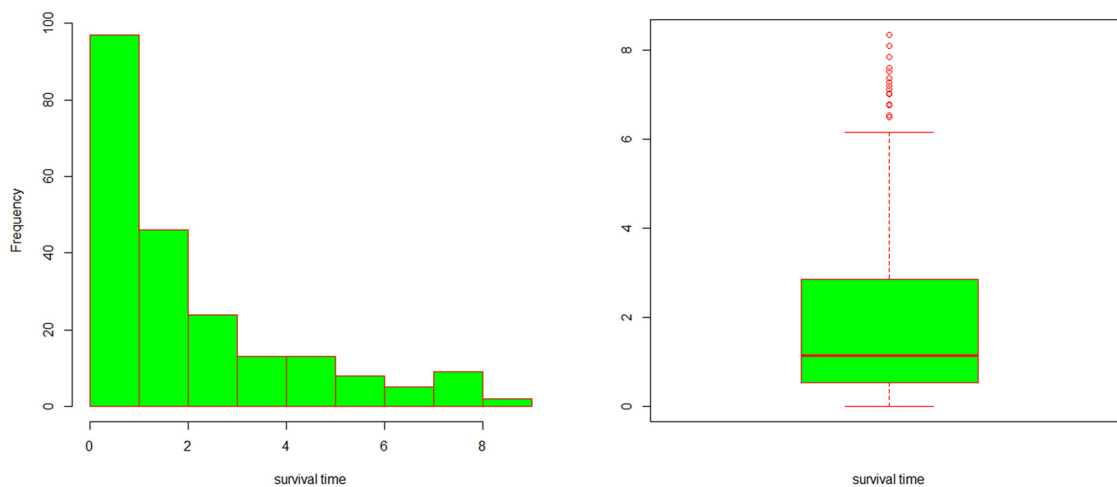


Figure 1: Histogram and box plot for the survival data.

Table 3: Fitted distributions and CvM goodness-of-fit tests for the survival data

Model	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{l}(\theta)$	CvM	p value
Log-normal	1.331	0.066	—	−384.212	0.245	0.211
Gompertz	0.001	379.343	—	−375.631	0.205	0.260
Gamma	0.459	0.923	—	−367.934	0.114	0.504
Weibull	1.962	0.949	—	−367.903	0.100	0.552
Gompertz-Lindley	0.393	1.069	—	−368.256	0.086	0.650
GTPL	0.429	0.754	0.010	−367.569	0.071	0.738

5 Conclusion

In this article, we propose a Gompertz-two-parameter-Lindley distribution by mixing the Gompertz distribution and the two-parameter Lindley distribution. The structural properties of the model reveal that the proposed model has at least three advantages. First, compared with the distribution in the literature, the new distribution has three distributions, which could provide a flexible method for data fitting. Second, the FRF of the proposed distribution has several shapes, which are sufficient for most practical applications. Finally, the new distribution has a mixture structure, which enables some special algorithms for statistical inference. When analyzing lifetime data, we recommend using the GTPL distribution. Future research on accurate parameter inference and prediction of future observations will be two important research directions.

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