Research Article

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On nonnil-coherent modules and nonnil-Noetherian modules

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Abstract: In this article, we introduce two new classes of modules over a ϕ-ring that generalize the classes of coherent modules and Noetherian modules. We next study the possible transfer of the properties of being nonnil-Noetherian rings, ϕ-coherent rings, and nonnil-coherent rings in the amalgamated algebra along an ideal.

Keywords: nonnil-coherent ring, ϕ-coherent ring, ϕ-submodule, nonnil-coherent module, nonnil-Noetherian ring, nonnil-Noetherian module

MSC 2020: 13A15, 13C05, 13E15, 13F05

1 Introduction

All rings considered in this article are assumed to be commutative with non-zero identity and prime nilradical. We use $\mathrm{Nil}(R)$ to denote the set of nilpotent elements of R and $Z(R)$, the set of zero-divisors of R. A ring with $\mathrm{Nil}(R)$ being divided prime (i.e., $\mathrm{Nil}(R) \subset xR$ for all $x \in R\setminus\mathrm{Nil}(R)$) is called a *$\phi$-ring*. El Khalfi et al. [1], and Chhiti et al. [2] studied when the amalgamation algebra along an ideal is a ϕ-ring. Let $\mathcal{H}$ be the set of all rings with divided prime nilradical. A ring R is called a *strongly ϕ-ring* if $R \in \mathcal{H}$ and $Z(R) = \mathrm{Nil}(R)$. Let R be a ring and M be an R-module; we define

$$\phi\text{-tor}(M) = \{x \in M | sx = 0 \quad \text{for some} \quad s \in R\setminus\mathrm{Nil}(R)\}.$$

If ϕ-tor$(M) = M$, then M is called a *ϕ-torsion module*, and if ϕ-tor$(M) = 0$, then M is called a *ϕ-torsion-free module*. It is worth noting that in the language of torsion theory, the class $\mathfrak{T}$ of all ϕ-torsion modules is a (hereditary) torsion class, whereas $\mathfrak{T}$ is closed under (submodules,) direct sums, epimorphic images, and extensions. An ideal I of R is said to be *nonnil* if $I \nsubseteq \mathrm{Nil}(R)$. An R-module M is said to be *ϕ-divisible* if $M = sM$ for all $s \in R\setminus\mathrm{Nil}(R)$.

Among the many recent generalizations of the concept of a coherent ring in the literature, we can find the following: due to Bacem and Ali [3], a ϕ-ring R is called *ϕ-coherent* if $R/\mathrm{Nil}(R)$ is a coherent domain [3, Corollary 3.1]. A ϕ-ring R is said to be *nonnil-coherent* if every finitely generated nonnil ideal is finitely presented, which is equivalent to saying that R is ϕ-coherent and $(0 : r)$ is a finitely generated ideal of R for each $r \in R\setminus\mathrm{Nil}(R)$, where $(0 : r) = \{x \in R | rx = 0\}$ [4, Proposition 1.3]. In [5], an R-module M is said to be coherent if M is a finitely generated R-module and every finitely generated submodule of M is a finitely presented R-module. In [6], an R-module M is said to be Noetherian if every submodule of M is finitely

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generated. In [7], Badawi introduced and studied a new class of ϕ-rings, which are said to be nonnil-Noetherian. A ϕ-ring R is said to be *nonnil-Noetherian* if every nonnil ideal of R is finitely generated, which is equivalent to saying that $R/\mathrm{Nil}(R)$ is a Noetherian domain ([7, Theorem 2.4]). In 2015, Yousefian Darani [8] introduced a new class of modules that is closely related to the class of Noetherian modules. An R-module M with $\mathrm{Nil}(M) := \mathrm{Nil}(R)M$, a divided prime submodule (i.e., $\mathrm{Nil}(M)$, is a prime submodule of M and comparable with each submodule of M) is said to be *nonnil-Noetherian* if every nonnil submodule N of M (i.e., $N \nsubseteq \mathrm{Nil}(M)$) is finitely generated. In 2020, Yousefian Darani and Rahmatinia [9] introduced and studied ϕ-Noetherian modules as a new class of Noetherian modules. A module M is said to be *ϕ-Noetherian* if $\mathrm{Nil}(M)$ is divided prime and each submodule that properly contains $\mathrm{Nil}(M)$ is finitely generated.

Let R be a ring and E an R-module. Then $R \propto E$, the trivial ring extension of R by E, is the ring whose additive structure is that of the external direct sum $R \oplus E$ and whose multiplication is defined by $(a, e)(b, f) := (ab, af + be)$ for all $a, b \in R$ and all $e, f \in E$ (this construction is also known by other terminologies and other notations, such as the idealization $R(+)E$) (see [5,10–12]).

Let A and B be two rings, let J be an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) | a \in A, j \in J\},$$

called the amalgamation of A with B along J with respect to f (introduced and studied by D'Anna et al. [13,14]). This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana [15] and denoted by $A \bowtie I$).

This article consists of five sections including an Introduction. In Section 2, we introduce and study a new class of modules over a ϕ-ring R which are called nonnil-coherent modules. Let M be an R-module and N be a submodule of M. Then, N is said to be a ϕ-submodule of M if M/N is a ϕ-torsion module (see Definition 2.1). Using Definition 2.1, an R-module M is said to be nonnil-coherent if M is finitely generated and each finitely generated ϕ-submodule of M is finitely presented (see Definition 2.4). We give some properties that characterize these modules. In Section 3, we introduce and study another definition of nonnil-Noetherian modules that is different from the definition of [8,9]. An R-module M is said to be nonnil-Noetherian if M is a finitely generated module and every ϕ-submodule of M is finitely generated (see Definition 3.1). Next, we give some properties that characterize these modules. In Section 4, we study the possible transfer of the properties of nonnil-coherent rings and nonnil-Noetherian rings in trivial ring extensions. In the last section, we study the possible transfer of the properties of being ϕ-coherent rings and nonnil-Noetherian rings in an amalgamation algebra along an ideal.

For any undefined terminology and notation, the reader is referred to [5,6,16,17]. Throughout this article, if S is a multiplicative subset of a ring R, then we assume that $S \cap \mathrm{Nil}(R) = \varnothing$.

2 On nonnil-coherent modules

In this section, we introduce and study a new class of modules over a ϕ-ring R, which are called nonnil-coherent modules. Recall that in [5], an R-module M is said to be coherent if M is finitely generated and every finitely generated submodule is finitely presented.

Recall that an R-module M is said to be ϕ-torsion if, for all $x \in M$, there exists $s \in R \backslash \mathrm{Nil}(R)$ such that $sx = 0$.

Definition 2.1. Let $R \in \mathcal{H}$ and M be an R-module. A submodule N of M is said to be a ϕ-submodule if M/N is a ϕ-torsion module.

Example 2.2. A nonnil submodule is not in general a ϕ-submodule. For example, set $R := \mathbb{Z}$, which is a ϕ-ring, and $M = \mathbb{C}[X]$ as an R-module. It is easy to see that every nonzero subgroup N of M is a nonnil submodule, in particular, the subgroup $N = \mathbb{Q}[X]$ is a nonnil submodule of M. But for any nonzero $s \in \mathbb{Z}$, we obtain $si \notin N$. Hence, N is never a ϕ-submodule of M. Therefore, we deduce that the class of nonnil-submodules of an R-module is different from the class of ϕ-submodules of that R-module.

There is a natural question: If R is a ϕ-ring, then is every submodule N of an R-module M (with $\mathrm{Nil}(M)$ being prime divided) such that N contains properly $\mathrm{Nil}(M)$ a ϕ-submodule of M? The following example shows that the answer to this question is negative.

Example 2.3. Let $R = \mathbb{Z}$, $M = \mathbb{C}$, and $N = \mathbb{Q}$. Then, $\mathrm{Nil}(M) = 0$ is a divided prime submodule of M and N properly contains $\mathrm{Nil}(M)$, but $\mathbb{C}/\mathbb{Q}$ is never a torsion abelian group. Therefore, $\mathbb{Q}$ is not a ϕ-subgroup of $\mathbb{C}$.

Definition 2.4 allows us to generalize the definition of coherent modules over a ϕ-ring.

Definition 2.4. Let $R \in \mathcal{H}$. An R-module M is said to be nonnil-coherent if M is finitely generated and every finitely generated ϕ-submodule of M is finitely presented. In particular, every coherent module over a ϕ-ring is nonnil-coherent.

Remark 2.5. Note that for a ϕ-torsion R-module M, we have

$$M \text{ is nonnil-coherent } \Leftrightarrow M \text{ is coherent}.$$

Recall from [18] that an R-module F is said to be ϕ-flat if $f \otimes_R F$ is an R-monomorphism for any R-monomorphism f, where $\mathrm{Coker}(f)$ is a ϕ-torsion R-module. Recall in [3] that a ϕ-ring is said to be nonnil-coherent if every finitely generated nonnil ideal is finitely presented.

Now, we are able to give a new characterization of nonnil-coherent rings.

Theorem 2.6. *The following are equivalent for a ϕ-ring R:*
(1) *R is a nonnil-coherent ring.*
(2) *R is a nonnil-coherent R-module.*
(3) *Every finitely generated free R-module is nonnil-coherent.*
(4) *Every finitely presented module is nonnil-coherent.*
(5) *Every finitely generated ϕ-submodule of a finitely presented R-module is finitely presented.*
(6) *Any direct product of ϕ-flat R-modules is ϕ-flat.*
(7) *R^I is ϕ-flat for any index set I.*

Proof. (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (1) This follows from [3, Theorem 2.4].

(4) $\Rightarrow$ (5) Straightforward.

(5) $\Rightarrow$ (1) This follows immediately from the fact that every nonnil ideal of R is a ϕ-submodule of R.

(1) $\Rightarrow$ (2) Assume that R is a nonnil-coherent ring and let I be a finitely generated ideal of R such that R/I is ϕ-torsion. If $I \subset \mathrm{Nil}(R)$, then, for any $r \in R\backslash\mathrm{Nil}(R)$, there exists $s \in R\backslash\mathrm{Nil}(R)$ such that $sr \in I \subset \mathrm{Nil}(R)$ since R/I is a ϕ-torsion R-module, a desired contradiction since $\mathrm{Nil}(R)$ is a prime ideal of R. Therefore, I is a nonnil ideal. As R is a nonnil-coherent ring, I is a finitely presented ideal. Therefore, R is a nonnil-coherent R-module.

(2) $\Rightarrow$ (1) Let I be a finitely generated nonnil ideal of R. Since R is a nonnil-coherent module and R/I is ϕ-torsion, I is finitely presented. Therefore, R is a nonnil-coherent ring.

(6) $\Rightarrow$ (3) Let F be a finitely generated free R-module and N be a finitely generated ϕ-submodule of F. Then, F and F/N are finitely presented R-modules. Since R^I is a ϕ-flat module for any index set I, by [18, Theorem 3.2] we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_R R^I & \longrightarrow & F \otimes_R R^I & \longrightarrow & F/N \otimes_R R^I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\cong} & & \downarrow{\cong} & & \\ 0 & \longrightarrow & N^I & \longrightarrow & F^I & \longrightarrow & (F/N)^I & \longrightarrow & 0 \end{array}$$

Since the two right vertical arrows are isomorphisms by [17, Lemma I.13.2], we obtain $N \otimes_R R^I \cong N^I$, and so N is a finitely presented R-module by [17, Lemma I.13.2]. Therefore, F is a nonnil-coherent R-module.

(3) ⇒ (4) Let M be a finitely presented R-module. Then, $M \cong F/N$, where F is a finitely generated free R-module and N is a finitely generated submodule of F. Let X be a finitely generated ϕ-submodule of M. Then, $X \cong L/N$ such that L is a finitely generated submodule of F with $N \subset L$. Since F is a nonnil-coherent module and $M/X \cong F/L$ is ϕ-torsion, L is a finitely presented R-module. Now, it follows immediately from [19, (4.54) Lemma] that X is finitely presented. Therefore, M is a nonnil-coherent module. □

The following theorem characterizes when a finitely generated submodule of a nonnil-coherent module is nonnil-coherent.

Theorem 2.7. *Let $R \in \mathcal{H}$ and M be a nonnil-coherent R-module. If N is a finitely generated ϕ-submodule of M, then N is a nonnil-coherent module.*

Before proving Theorem 2.7, we need the following lemma.

Lemma 2.8. [20, Proposition 2.4] *Let $R \in \mathcal{H}$ and $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules and R-homomorphisms. Then, M is ϕ-torsion if and only if M' and M'' are ϕ-torsion modules.*

Proof of Theorem 2.7. Let M be a nonnil-coherent R-module and N be a finitely generated ϕ-submodule of of M. We claim that N is a nonnil-coherent R-module. Let X be a finitely generated ϕ-submodule of N. Then, the following sequence $0 \to N/X \to M/X \to M/N \to 0$ is exact. Since M/N and N/X are ϕ-torsion modules, so is M/X by Lemma 2.8. Therefore, X is finitely presented, and so N is a nonnil-coherent module. □

Corollary 2.9. *If R is a nonnil-coherent ring, then any finitely generated nonnil ideal of R is a nonnil-coherent R-module.*

Proof. This follows from Theorem 2.7. □

Theorem 2.10. *Let $R \in \mathcal{H}$ and $0 \to P \to N \to M \to 0$ be an exact sequence of R-modules and R-homomorphisms, where P is a finitely generated R-module. If N is a nonnil-coherent module, then so is M.*

Proof. We can set $M = N/P$. Let X/P be a finitely generated ϕ-submodule of M. Since N is a nonnil-coherent module and X is a finitely generated ϕ-submodule of N, it follows that X is finitely presented. We claim that X/P is a finitely presented R-module. Actually it follows from [19, (4.54) Lemma] that X/P is finitely presented, and so M is a nonnil-coherent module. □

Corollary 2.11 is a consequence of Theorem 2.10.

Corollary 2.11. *Every factor module M/N of a nonnil-coherent module M by a finitely generated submodule N is also a nonnil-coherent module. In particular, every factor module of a nonnil-coherent ring R by a finitely generated ideal I of R is a nonnil-coherent R-module.*

Proof. Straightforward. □

Corollary 2.12. *Let $R \in \mathcal{H}$ and M and N be nonnil-coherent modules. Let $f : M \longrightarrow N$ be an R-homomorphism. Then:*

(1) *If* $\mathrm{Im}(f)$ *is a ϕ-torsion R-module and* $\ker(f)$ *is finitely generated, then* $\ker(f)$ *is a nonnil-coherent module.*

(2) *If* $\ker(f)$ *is finitely generated, then* $\mathrm{Im}(f)$ *is a nonnil-coherent module.*

(3) *If* $\mathrm{Coker}(f)$ *is a* ϕ*-torsion* R*-module and* $\mathrm{Im}(f)$ *is finitely generated, then* $\mathrm{Im}(f)$ *is a nonnil-coherent module.*
(4) *If* $\mathrm{Im}(f)$ *is finitely generated, then* $\mathrm{Coker}(f)$ *is a nonnil-coherent module.*

Proof. By the following two exact sequences $0 \to \ker(f) \to M \to \mathrm{Im}(f) \to 0$ and $0 \to \mathrm{Im}(f) \to N \to \mathrm{Coker}(f) \to 0$, the proof is finished using Theorems 2.7 and 2.10. ◻

Theorem 2.13. *Let* $R \in \mathcal{H}$ *and* $0 \to P \xrightarrow{f} N \xrightarrow{g} M \to 0$ *be an exact sequence of* R*-modules and* R*-homomorphisms. If* P *and* M *are nonnil-coherent modules, then so is* N*.*

Proof. Let X be a finitely generated ϕ-submodule of N. Then, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker(g_{|X}) & \xrightarrow{f} & X & \xrightarrow{g} & g(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \\ 0 & \longrightarrow & P & \xrightarrow{f} & N & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

Since X is a finitely generated module, so is $g(X)$. Let $x \in M$. Then, $g(n) = x$ for some $n \in N$. Since N/X is a ϕ-torsion module, $sn \in X$ for some $s \in R\backslash\mathrm{Nil}(R)$, and so $sx \in g(X)$. Therefore, $M/g(X)$ is ϕ-torsion. As M is nonnil-coherent, $g(X)$ is a finitely presented R-module. Therefore, $\ker(g_{|X})$ is a finitely generated R-module since X is finitely generated. Let $x \in P$. Then, there exists $t \in R\backslash\mathrm{Nil}(R)$ such that $tf(x) \in X$, and so $tf(x) \in \ker(g_{|X})$ since $g(tf(x)) = 0$. We can consider f as an embedding, and so $P/\ker(g_{|X})$ is a ϕ-torsion module. Then, $\ker(g_{|X})$ is finitely presented since P is a nonnil-coherent module, and so X is a finitely presented R-module. Therefore, N is a nonnil-coherent module. ◻

Corollary 2.14. *Let* $R \in \mathcal{H}$ *and* $\{M_i\}_{i=1}^n$ *be a family of nonnil-coherent modules. Then,* $\oplus_{i=1}^n M_i$ *is a nonnil-coherent module.*

Proof. We prove this by induction on n. Consider the following exact sequence $0 \to M_1 \to \oplus_{i=1}^n M_i \to \oplus_{i=2}^n M_i \to 0$ and apply Theorem 2.13. ◻

Corollary 2.15. *Let* $R \in \mathcal{H}$ *and let* M *and* N *be nonnil-coherent submodules of a nonnil-coherent* R*-module* L*. If* $M + N$ *is a* ϕ*-torsion* R*-module and* $M \cap N$ *is finitely generated, then* $M + N$ *and* $M \cap N$ *are nonnil-coherent modules.*

Proof. We use the exact sequence $0 \to M \cap N \to M \oplus N \to M + N \to 0$ and Theorems 2.7 and 2.10. ◻

Corollary 2.16. *Let* $R \in \mathcal{H}$ *and* I *be a finitely generated nonnil ideal of* R*. Then,* R *is a nonnil-coherent ring if and only if* I *and* R/I *are nonnil-coherent* R*-modules.*

Proof. Assume that R is a nonnil-coherent ring and let I be a finitely generated nonnil ideal of R. By Corollary 2.11, R/I is a nonnil-coherent R-module, and so I is a nonnil-coherent R-module by Theorem 2.7.

Conversely, assume that I and R/I are nonnil-coherent R-modules for any finitely generated nonnil ideal I of R. Then, R is a nonnil-coherent ring by Theorem 2.13. ◻

Next, Theorem 2.17 gives an analog of the well-known behavior of [5, Theorem 2.2.6].

Theorem 2.17. *Let $R \in \mathcal{H}$ and S be a multiplicative subset of R. If M is a nonnil-coherent R-module, then $S^{-1}M$ is a nonnil-coherent $(S^{-1}R)$-module.*

Proof. It is clear that $S^{-1}R \in \mathcal{H}$ and $S^{-1}M$ is a finitely generated $(S^{-1}R)$-module. Let N be an R-module such that $S^{-1}N$ is a ϕ-torsion $(S^{-1}R)$-module. Then, N is a ϕ-torsion R-module. Indeed, let $n \in N$. Then, there exist $a \in R \setminus \mathrm{Nil}(R)$ and $s \in S$ such that $\frac{a}{s}\frac{n}{1} = \frac{0}{1}$. Thus, $(ta)n = 0$ for some $t \in R \setminus \mathrm{Nil}(R)$, and so N is a ϕ-torsion R-module. Let X be a finitely generated $(S^{-1}R)$-submodule of $S^{-1}M$ such that $\frac{S^{-1}M}{X}$ is ϕ-torsion. Then, we can set $X = S^{-1}K$, where K is a finitely generated submodule of M. Therefore, $S^{-1}(M/K)$ is ϕ-torsion, and so M/K is a ϕ-torsion R-module. Hence, K is a finitely presented R-module. Thus, X is a finitely presented $(S^{-1}R)$-module. Therefore, $S^{-1}M$ is a nonnil-coherent $(S^{-1}R)$-module. □

Next, we pay attention to the localization of nonnil-coherent rings. Using Theorem 2.17, we obtain immediately:

Corollary 2.18. *If R is a nonnil-coherent ring and S is a multiplicative subset of R, then $S^{-1}R$ is a nonnil-coherent ring.*

Proof. Straightforward. □

Theorem 2.19. *Let $f : R \to T$ be a finite surjective homomorphism of ϕ-rings (i.e., T is a finitely generated R-module). Let M be a finitely generated T-module which is a nonnil-coherent R-module. Then, M is a nonnil-coherent T-module.*

Proof. Let X be a finitely generated T-submodule of M. Then, X is a finitely generated R-module since f is finite. If M/X is a ϕ-torsion T-module, then M/X is a ϕ-torsion R-module, and so X is a finitely presented R-module. Therefore, X is a finitely presented T-module since $X \cong T \otimes_R X$. Hence, M is a nonnil-coherent T-module. □

Theorem 2.20. *Let $R \in \mathcal{H}$ and I be a finitely generated nil ideal of R. Let M be an (R/I)-module. Then, M is a nonnil-coherent R-module if and only if M is a nonnil-coherent (R/I)-module.*

In order to prove Theorem 2.20, we need the following lemmas.

Lemma 2.21. [5, Theorem 2.1.8] *Let R be a ring and I be a finitely generated ideal of R. Let M be an (R/I)-module. Then, M is a finitely presented R-module if and only if M is a finitely presented (R/I)-module.*

Lemma 2.22. *Let $R \in \mathcal{H}$ and I be a nil ideal of R. Then, $R/I \in \mathcal{H}$.*

Proof. Note that $\mathrm{Nil}(R/I) = \mathrm{Nil}(R)/I$ and $\frac{R/I}{\mathrm{Nil}(R/I)} \cong R/\mathrm{Nil}(R)$ is an integral domain, and so $\mathrm{Nil}(R/I)$ is a prime ideal of R/I. If $\bar{x} \in (R/I) \setminus \mathrm{Nil}(R/I)$, then $x \in R \setminus \mathrm{Nil}(R)$, and so $\mathrm{Nil}(R) \subset Rx$. Therefore, $\mathrm{Nil}(R/I) \subset (R/I)\bar{x}$, as desired. □

Proof of Theorem 2.20. Assume that M is a nonnil-coherent R-module. Since $R/I \in \mathcal{H}$ by Lemma 2.22, M is a nonnil-coherent (R/I)-module by Theorem 2.19.

Conversely, assume that M is a nonnil-coherent (R/I)-module. Then, M is a finitely generated (R/I)-module, and so M is a finitely generated R-module. Let X be a finitely generated R-submodule of M such that M/X is a ϕ-torsion R-module. Thus, M/X is a ϕ-torsion (R/I)-module, and so X is a finitely presented (R/I)-module. By Lemma 2.21, X is a finitely presented R-module. Therefore, M is a nonnil-coherent R-module. □

Corollary 2.23. *Let $R \in \mathcal{H}$ and I be a finitely generated nil ideal of R. Then, R/I is a nonnil-coherent ring if and only if R/I is a nonnil-coherent R-module.*

Proof. Straightforward. □

Corollary 2.24. *Let R be a nonnil-coherent ring and I be a finitely generated nil ideal of R. Then, R/I is a nonnil-coherent ring.*

Proof. This follows immediately from Corollaries 2.11 and 2.23. □

Corollary 2.25. *Let $R \in \mathcal{H}$ and I be a finitely generated nil ideal of R. If R/I is a nonnil-coherent ring and I is a nonnil-coherent R-module, then R is a nonnil-coherent ring.*

Proof. This follows directly from Theorem 2.13 and Corollary 2.23. □

3 On nonnil-Noetherian modules

We introduce a new definition of nonnil-Noetherian modules which is different from that in [8]. In [6], an R-module M is said to be Noetherian if every submodule of M is finitely generated.

Definition 3.1. Let $R \in \mathcal{H}$. An R-module M is said to be *nonnil-Noetherian* if every ϕ-submodule of M is finitely generated. In particular, every Noetherian module over a ϕ-ring is nonnil-Noetherian.

Remark 3.2.
(1) Note that for a ϕ-torsion R-module M, we have

$$M \text{ is nonnil-Noetherian} \quad \Leftrightarrow \quad M \text{ is Noetherian}.$$

(2) The definition of nonnil-Noetherian modules in Definition 3.1 is different from that of nonnil-Noetherian modules in [8] by Example 2.2 and that of ϕ-Noetherian modules in [9] by Example 2.3. Although the term "non-Noetherian module" used in [8] is the same as in Definition 3.1, we will still use it in the spirit of [7] and following Theorem 3.3.

Recall that in [7], a ϕ-ring R is said to be nonnil-Noetherian if every nonnil ideal of R is finitely generated, equivalently $R/\mathrm{Nil}(R)$ is a Noetherian domain. The following theorem allows us to see that each nonnil-Noetherian ring is a nonnil-Noetherian module over itself.

Theorem 3.3. *Let R be a ϕ-ring. Then, R is a nonnil-Noetherian ring if and only if R is a nonnil-Noetherian module over itself.*

Proof. Assume that R is a nonnil-Noetherian ring and let I be an ideal of R such that R/I is ϕ-torsion. Then, I is a nonnil ideal of R, and so I is finitely generated since R is nonnil-Noetherian. Therefore, R is a nonnil-Noetherian module over itself.

Conversely, assume that R is a nonnil-Noetherian module over itself and let I be a nonnil ideal of R. Then, R/I is ϕ-torsion, and so I is finitely generated. Therefore, R is a nonnil-Noetherian ring. □

According to [3, Corollary 3.1], a ϕ-ring R is said to be ϕ-coherent if $R/\mathrm{Nil}(R)$ is a coherent domain. From [7, Theorem 2.4], a ϕ-ring R is nonnil-Noetherian if and only if $R/\mathrm{Nil}(R)$ is a Noetherian domain. Therefore, every nonnil-Noetherian ring is ϕ-coherent. The following theorem characterizes when a nonnil-Noetherian ring is nonnil-coherent.

Theorem 3.4. *The following statements are equivalent for a nonnil-Noetherian ring R:*
(1) *R is nonnil-coherent.*
(2) *Rs is a finitely presented ideal of R for any $s \in R\backslash \mathrm{Nil}(R)$.*
(3) *Every nonnil ideal of R is finitely presented.*

Proof. (1) ⇒ (3) ⇒ (2) They are straightforward.

(2) ⇒ (1) Assume that Rs is finitely presented for every $s \in R\backslash \mathrm{Nil}(R)$. Using the exact sequence $0 \to (0 : s) \to R \to Rs \to 0$, we obtain that $(0 : s)$ is a finitely generated ideal of R. Since R is assumed to be nonnil-Noetherian, and so ϕ-coherent, R is a nonnil-coherent ring by [4, Proposition 1.3]. □

Recall that a ϕ-ring R is called a *strongly ϕ-ring* if $Z(R) = \mathrm{Nil}(R)$. Strongly ϕ-rings are abundant. Indeed, these rings can be generated from the following pullback introduced by Chang and Kim recently [21]. Let D be a domain with K as its quotient field. Let $K[X]$ be the polynomial ring over K, $n \geq 2$ be an integer and $K[\theta] = K[X]/\langle X^n\rangle$, where $\theta := X + \langle X^n\rangle$. Denote by $i : D \hookrightarrow K$ the natural embedding map and $\pi : K[\theta] \twoheadrightarrow K$ a ring homomorphism satisfying $\pi(f) = f(0)$. Consider the pullback of i and π as follows:

$$\begin{array}{ccc} R_n := D + \theta K[\theta] & \longrightarrow & K[\theta] \\ \downarrow & & \downarrow \pi \\ D & \overset{i}{\hookrightarrow} & K \end{array}$$

Then, $R_n = D + \theta K[\theta] = \{f \in K[\theta] | f(0) \in D\}$ is a strongly ϕ-ring.

Corollary 3.5. *If R is a nonnil-Noetherian strongly ϕ-ring, then R is a nonnil-coherent ring.*

Proof. If R is a strongly ϕ-ring, then every principal nonnil ideal is free. Therefore, R is a nonnil-coherent ring if it is nonnil-Noetherian by Theorem 3.4. □

Theorem 3.6. *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence. If M' and M'' are nonnil-Noetherian modules, then so is M. In addition, if M' is a ϕ-submodule of M, then the converse holds.*

Proof. Assume that M' and M'' are nonnil-Noetherian. Let N be a ϕ-submodule of M. Then, $g(N)$ is a ϕ-submodule of M''. Indeed, if $x \in M''$, then $g(m) = x$ for some $m \in M$, and so there exists $s \in R\backslash \mathrm{Nil}(R)$ such that $sm \in N$. Thus, $sx \in g(N)$. Therefore, $g(N)$ is a finitely generated submodule of M''. Set $g(N) = \sum_{i=1}^{t} Rg(n_i)$, where each $n_i \in N$. Let $n \in N$. Then, $g(n) = \sum_{i=1}^{t} r_i g(n_i)$ with $r_i \in R$. Thus, $n - \sum_{i=1}^{t} r_i n_i \in \ker(g) = \mathrm{Im}(f)$, and so $n = f(y) + \sum_{i=1}^{t} r_i n_i$ for some $y \in M'$. In addition, M' is finitely generated since it is nonnil-Noetherian. Thus, $M' = \sum_{i=t+1}^{t+l} Rn_i$ for some $n_{t+1}, n_{t+2}, \ldots, n_{t+l} \in M$, and so there exists $r_{t+1}, r_{t+2}, \ldots, r_{t+l} \in R$ such that $f(y) = \sum_{i=t+1}^{t+l} r_i n_i$. Hence, $n = \sum_{i=1}^{t+l} r_i n_i$. Therefore, N is finitely generated, and so M is a nonnil-Noetherian module.

Assume that M is a nonnil-Noetherian module and M' is a ϕ-submodule of M. Let X be a ϕ-submodule of M'. Then, $0 \to M'/X \to M/X \to M'' \to 0$ is exact with M'/X and M'' ϕ-torsion, and so X is a ϕ-submodule of M. Thus, X is a finitely generated submodule of M'. Therefore, M' is a nonnil-Noetherian module. Let N be a submodule of M such that $M' \subset N$ and N/M' is a ϕ-submodule of $M'' \cong M/M'$. We claim that N is a ϕ-submodule of M. If $x \in M$, then $s(x + M') = M'$ for some $s \in R\backslash \mathrm{Nil}(R)$ since M/M' is ϕ-torsion, and so $sx \in M' \subset N$. Thus, N is a ϕ-submodule of M. Therefore, N is a finitely generated submodule of M, and so N/M' is a finitely generated submodule of M''. Therefore, M'' is a nonnil-Noetherian module. □

Corollary 3.7. *Let $R \in \mathcal{H}$ and M be a nonnil-Noetherian R-module. Then, every ϕ-submodule of M is nonnil-Noetherian.*

Proof. This follows immediately from Theorem 3.6. □

Corollary 3.8. *Let $R \in \mathcal{H}$ and $\{M_i\}_{1 \le i \le n}$ be a family of nonnil-Noetherian modules. Then, $\oplus_{i=1}^{n} M_i$ is a nonnil-Noetherian module.*

Proof. We prove this by induction on n. Consider the following exact sequence $0 \to M_1 \to \oplus_{i=1}^{n} M_i \to \oplus_{i=2}^{n} M_i \to 0$ and apply Theorem 3.6. □

Corollary 3.9. *If R is a nonnil-Noetherian ring, then every finitely generated ϕ-torsion module is nonnil-Noetherian (and so is Noetherian).*

Proof. If M is a finitely generated ϕ-torsion R-module, then $M \cong R^{(n)}/N$, where $n \in \mathbb{N}$ and N is a submodule of $R^{(n)}$. Since M is ϕ-torsion, N is a ϕ-submodule of $R^{(n)}$. Using the exact sequence $0 \to N \to R^{(n)} \to M \to 0$ and Theorem 3.6, we can deduce that M is nonnil-Noetherian. □

Corollary 3.10. *Let $R \in \mathcal{H}$ and I be a finitely generated nonnil ideal of R. Then, R is a nonnil-Noetherian ring if and only if I and R/I are nonnil-Noetherian R-modules.*

Proof. This follows immediately from Theorem 3.6. □

Theorem 3.11. *Let $R \in \mathcal{H}$. If M is a nonnil-Noetherian R-module, then every factor module of M is nonnil-Noetherian.*

Proof. Let M be a nonnil-Noetherian module and N be a submodule of M. We claim that M/N is a nonnil-Noetherian module. Let P/N be a ϕ-submodule of M/N, where P is a submodule of M containing N. Since $\frac{M/N}{P/N} \cong \frac{M}{P}$ is a ϕ-torsion R-module, P is finitely generated, and so P/N is a finitely generated submodule of M/N. Therefore, M/N is nonnil-Noetherian. □

Corollary 3.12. *If R is a nonnil-Noetherian ring and I is an ideal of R, then R/I is a nonnil-Noetherian R-module.*

Proof. This follows immediately from Theorem 3.11. □

Corollary 3.13. *Let R be a nonnil-Noetherian ring and M be an R-module. Then, M is a nonnil-Noetherian module if and only if M is a finitely generated R-module.*

Proof. If M is a nonnil-Noetherian module, then it is easy to see that M is a finitely generated module. Conversely, if M is a finitely generated module, then M is a factor of $R^{(n)}$, where $n \in \mathbb{N}$. Since $R^{(n)}$ is a nonnil-Noetherian module by Corollary 3.8, M is a nonnil-Noetherian module by Theorem 3.11. □

Corollary 3.14. *A ring R is nonnil-Noetherian if and only if every ϕ-submodule of a finitely generated R-module is finitely generated.*

Proof. Straightforward. □

Theorem 3.15 establishes that every finitely generated ϕ-torsion module over a nonnil-Noetherian ring is finitely presented.

Theorem 3.15. *Let R be a nonnil-Noetherian ring and M be a finitely generated ϕ-torsion R-module. Then, M is finitely presented.*

Proof. Let M be a finitely generated ϕ-torsion R-module. Then, there exist $n \in \mathbb{N}$ and a sequence $0 \to N \to R^{(n)} \to M \to 0$. Since $R^{(n)}$ is a nonnil-Noetherian R-module by Corollary 3.8 and M is a ϕ-torsion module, N is a finitely generated module. Therefore, M is a finitely presented module. □

Theorem 3.16 establishes that the class of nonnil-Noetherian modules is closed under localizations.

Theorem 3.16. *Let R be a ϕ-ring and S be a multiplicative subset of R. If M is a nonnil-Noetherian R-module, then $S^{-1}M$ is a nonnil-Noetherian $(S^{-1}R)$-module.*

Proof. Let M be a nonnil-Noetherian R-module and $S^{-1}N$ be a ϕ-submodule of $S^{-1}M$, where N is a submodule of M. Then, N is a ϕ-submodule of M, and so N is a finitely generated R-module. Thus, $S^{-1}N$ is a finitely generated $(S^{-1}R)$-module. Therefore, $S^{-1}M$ is a nonnil-Noetherian $(S^{-1}R)$-module. □

Corollary 3.17. *If R is a nonnil-Noetherian ring and S is a multiplicative subset of R, then $S^{-1}R$ is a nonnil-Noetherian ring.*

Proof. This follows immediately from Theorem 3.16. □

We end this section by the following theorem.

Theorem 3.18. *Let R be a nonnil-Noetherian ring and I be a nil ideal of R. Then, R/I is a nonnil-Noetherian ring.*

Proof. Let J/I be a nonnil ideal of R/I. Then, $\frac{R/I}{J/I} \cong R/J$ is a ϕ-torsion R-module, and so J is a nonnil ideal of R. As R is nonnil-Noetherian, J is a finitely generated ideal of R, and so J/I is a finitely generated ideal of R/I. Therefore, R/I is nonnil-Noetherian. □

4 Transfer of nonnil-coherence and nonnil-Noetherianity in trivial ring extensions

Now, we study the transfer of nonnil-coherent rings in the trivial ring extensions. From [1, Corollary 2.4], a trivial ring extension $R \propto M$ is a ϕ-ring if and only if R is a ϕ-ring and M is a ϕ-divisible module (i.e., $sM = M$ for all $s \in R\backslash \mathrm{Nil}(R)$).

Let M be an R-module and $r \in R$. Set $(0 :_M r) := \{m \in M | rm = 0\}$. It is easy to verify that $(0 :_M r)$ is a submodule of M such that $(0 : r)M \subset (0 :_M r)$. Therefore, $(0 : r) \propto (0 :_M r)$ is an ideal of $R \propto M$ by [22, Theorem 3.1].

The following theorem characterizes when a trivial ring extension is a nonnil-coherent ring.

Theorem 4.1. *Let $A \in \mathcal{H}$, M be a ϕ-divisible A-module, and set $R := A \propto M$. Then, the following statements are equivalent:*
(1) *R is a nonnil-coherent ring.*
(2) *A is a nonnil-coherent ring and $(0 : r) \propto (0 :_M r)$ is a finitely generated ideal of R for each $r \in A\backslash \mathrm{Nil}(A)$.*
(3) *A is a nonnil-coherent ring and $R(r, 0)$ is finitely presented for all $r \in A\backslash \mathrm{Nil}(A)$.*

Before proving Theorem 4.1, we need the following lemmas:

Lemma 4.2. *Let $A \in \mathcal{H}$ and M be a ϕ-divisible A-module. Let J be an ideal of $R := A \propto M$. Then, J is a nonnil ideal of R if and only if there exists a unique nonnil ideal I of A such that $J = I \propto M$.*

Proof. Assume that J is a nonnil ideal of R. Then, $0 \propto M \subset \mathrm{Nil}(R \propto M) \subset J$, and so $J = I \propto M$ for a unique nonnil ideal I of R by [22, Theorem 3.1].

Conversely, assume that $J = I \propto M$ for a unique nonnil ideal I of A. Then, it is clear that J is a nonnil ideal of R. □

Lemma 4.3. *Let $A \in \mathcal{H}$ and M be a ϕ-divisible A-module. Let $J = I \propto M$ be a nonnil ideal of $R = A \propto M$. Then, J is a finitely generated nonnil ideal of R if and only if I is a finitely generated nonnil ideal of A.*

Proof. Assume that I is a finitely generated nonnil ideal of A. Then, $I = \sum_{i=1}^{n} Aa_i$, where each $a_i \in A$, and we may assume that $a_1 \in A\setminus\mathrm{Nil}(A)$. First, it is easy to see that $\sum_{i=1}^{n} R(a_i, 0) \subset J$. Conversely, let $(\alpha, \beta) \in J$. Then $\alpha = \sum_{i=1}^{n} r_i a_i$ for some $r_i \in A$. Since M is ϕ-divisible, $\beta = a_1 v_1$ for some $v_1 \in M$, and so $(\alpha, \beta) = \sum_{i=1}^{n} (a_i, 0)(r_i, v_i)$, where $v_i = 0$ for all $2 \le i \le n$. Therefore, $J \subset \sum_{i=1}^{n} R(a_i, 0)$, and so $J = \sum_{i=1}^{n} R(a_i, 0)$ is a finitely generated nonnil ideal. The converse is straightforward. □

Lemma 4.4. *Let $A \in \mathcal{H}$ and M be a ϕ-divisible A-module. Let r be a non-nilpotent element of A and $u \in M$. Then,*

$$((0,0) : (r,u)) = (0 : r) \propto (0 :_M r).$$

Proof. Let $(r, u) \in A\setminus\mathrm{Nil}(A) \propto M$ and $(\alpha, \beta) \in ((0, 0) : (r, u))$. Since M is ϕ-divisible, $u = rv$ for some $v \in M$, and so $(r, u) = (r, 0)(1, v)$

$$\begin{aligned}(\alpha, \beta) \in ((0,0) : (r,u)) &\Leftrightarrow (\alpha, \beta)(r, u) = (0, 0)\\ &\Leftrightarrow (\alpha, \beta)(r, 0)(1, v) = (0, 0)\\ &\Leftrightarrow (\alpha r, \alpha r v + \beta r) = (0, 0)\\ &\Leftrightarrow (\alpha, \beta) \in (0 : r) \propto (0 :_M r).\end{aligned}$$

Therefore, $((0, 0) : (r, u)) = (0 : r) \propto (0 :_M r)$. □

Lemma 4.5. [3, Theorem 2.1] *A ϕ-ring R is nonnil-coherent if and only if $(0 : r)$ is a finitely generated ideal for every non-nilpotent element $r \in R$, and the intersection of two finitely generated nonnil ideals of R is a finitely generated nonnil ideal of R.*

Proof of Theorem 4.1. (1) ⇒ (2) Assume that R is a nonnil-coherent ring. Let I and J be finitely generated nonnil ideals of A. Then, $I \propto M$ and $J \propto M$ are finitely generated nonnil ideals of R by Lemma 4.3. Since R is a nonnil-coherent ring, $(I \propto M) \cap (J \propto M) = (I \cap J) \propto M$ is a finitely generated nonnil ideal of R by Lemma 4.5. Therefore, $I \cap J$ is a finitely generated nonnil ideal of A by Lemma 4.3. Let $r \in A\setminus\mathrm{Nil}(A)$. Then, $(0 : r) \propto (0 :_M r)$ is a finitely generated ideal of R by Lemma 4.4, and so $(0 : r)$ is a finitely generated ideal of A. Therefore, A is a nonnil-coherent ring by Lemma 4.5.

(2) ⇒ (1) Assume that A is a nonnil-coherent ring and $(0 : r) \propto (0 :_M r)$ is a finitely generated ideal of R for each $r \in A\setminus\mathrm{Nil}(A)$. Let $I \propto M$ and $J \propto M$ be finitely generated nonnil ideals of R. Then, I and J are finitely generated nonnil ideals of A. Since A is a nonnil-coherent ring, $I \cap J$ is a finitely generated nonnil ideal of A, and so $(I \propto M) \cap (J \propto M) = (I \cap J) \propto M$ is a finitely generated nonnil ideal of R by Lemma 4.3. Let $(r, u) \in R\setminus\mathrm{Nil}(R)$. Then, $((0, 0) : (r, u)) = (0 : r) \propto (0 :_M r)$ is a finitely generated ideal of R by hypothesis. Therefore, R is a nonnil-coherent ring by Lemma 4.5.

(2) ⇔ (3) Let $r \in A\setminus\mathrm{Nil}(A)$ and $u \in M$. Then, the following sequence $0 \to ((0, 0) : (r, u)) \to R \to R(r, 0) \to 0$ is exact. Therefore, by Lemma 4.4, $(0 : r) \propto (0 :_M r)$ is a finitely generated ideal of R if and only if $R(r, 0)$ is finitely presented. □

Corollary 4.6. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = \mathrm{Nil}(A)$. Then, R is a nonnil-coherent ring if and only if A is a nonnil-coherent ring and $(0 :_M r)$ is a finitely generated A-submodule of M for every $r \in A\setminus\mathrm{Nil}(A)$.*

Proof. Let $r \in A \setminus \mathrm{Nil}(A)$. Since $Z(A) = \mathrm{Nil}(A)$, it follows that $(0 : r) = 0$. Therefore, $((0,0) : (r,u)) = 0 \propto (0 :_M r)$. Now the assertion follows immediately from Theorem 4.1. □

Corollary 4.7. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = \mathrm{Nil}(A)$ and M is a Noetherian A-module. Then, R is a nonnil-coherent ring if and only if A is a nonnil-coherent ring.*

Proof. This follows immediately from Theorem 4.1. □

For a ring R and an R-module M, set $Z_R(M) := \{r \in R | rm = 0 \text{ for some nonzero } m \in M\}$.

Corollary 4.8. *Let $R = A \propto M$ be a ϕ-ring such that $Z(A) = \mathrm{Nil}(A) = Z_A(M)$. Then, R is a nonnil-coherent ring if and only if A is a nonnil-coherent ring.*

Proof. It is easy to see that $(0 : r) = 0$ and $(0 :_M r) = 0$ for each $r \in A \setminus \mathrm{Nil}(A)$. Now the proof follows directly from Theorem 4.1. □

Example 4.9.
(1) $\mathbb{Z} \propto \mathbb{Q}$ is a nonnil-coherent ring.
(2) $\mathbb{Z}/4\mathbb{Z} \propto \mathbb{Z}/2\mathbb{Z}$ is a nonnil-coherent ring.

The following theorem studies the transfer of being a ϕ-coherent ring in trivial extensions.

Theorem 4.10. *Let $A \in \mathcal{H}$ and M be a ϕ-divisible A-module. Then, $A \propto M$ is a ϕ-coherent ring if and only if A is a ϕ-coherent ring.*

Proof. First, note that $\mathrm{Nil}(A \propto M) = \mathrm{Nil}(A) \propto M$, and so $\frac{A \propto M}{\mathrm{Nil}(A \propto M)} \cong A/\mathrm{Nil}(A)$. Therefore, $A \propto M$ is a ϕ-coherent ring if and only if A is a ϕ-coherent ring. □

Recently, Qi and Zhang [4] provided for the first time an example of a ϕ-coherent ring, which is not nonnil-coherent. Now, we give a concrete example by using Corollary 4.6 and Theorem 4.10.

Example 4.11. Let $E = \oplus_{i=1}^{\infty} \mathbb{Q}/\mathbb{Z}$. Then, E is a divisible abelian group. Therefore, $R = \mathbb{Z} \propto E$ is a ϕ-ring. Since

$$(0 :_E 2) = \left\{ \left(\frac{a_i}{b_i} + \mathbb{Z} \right)_{i \in \mathbb{N}^*} | a_i \in \mathbb{Z} \quad \text{and} \quad gcd(a_i, b_i) = 1, b_i \in \{1, 2\} \ \forall i \in \mathbb{N}^* \right\},$$

which is an infinitely generated abelian group. Therefore, R is not a nonnil-coherent ring by Corollary 4.6. Note that R is an example of a ϕ-coherent ring, which is not nonnil-coherent by Theorem 4.10.

Now, we study the transfer of nonnil-Noetherian rings in the trivial ring extensions.

Theorem 4.12. *Let $A \in \mathcal{H}$ and M be a ϕ-divisible R-module. Then, $A \propto M$ is a nonnil-Noetherian ring if and only if A is a nonnil-Noetherian ring.*

Proof. $A \propto M$ is nonnil-Noetherian ring if and only if $\frac{A \propto M}{\mathrm{Nil}(A) \propto M} \cong \frac{A}{\mathrm{Nil}(A)}$ is a Noetherian domain and A is a nonnil-Noetherian ring. □

We give some examples of nonnil-Noetherian extension rings $A \propto M$ that are nonnil-coherent.

Example 4.13. If $R = A \propto M$ is a ϕ-ring such that $Z(A) = \mathrm{Nil}(A) = Z_A(M)$, then for all $r \in A\backslash\mathrm{Nil}(A)$, we obtain $(0 : r) = 0$ and $(0 :_M r) = 0$, and so $(A \propto M)(r, 0) \cong A \propto M$. Therefore, it follows from Theorem 3.4 that $A \propto M$ is nonnil-coherent if it is nonnil-Noetherian.

Example 4.14. Let A be a strongly ϕ-ring and M be a ϕ-torsion-free A-module. If $A \propto M$ is a nonnil-Noetherian ring, then $A \propto M$ is a nonnil-coherent ring.

Proof. Let $(\alpha, m) \in A \propto M$ such that $(\alpha, m)(r, 0) = (0, 0)$. Then, $\alpha r = 0$, and $rm = 0$ and so $(\alpha, m) = (0, 0)$. Thus, $(A \propto M)(r, 0)$ is a finitely generated free ideal. Hence, if $A \propto M$ is a nonnil-Noetherian ring, then $A \propto M$ is a nonnil-coherent ring by Theorem 3.4. □

Recall that every nonnil-Noetherian ring is ϕ-coherent. The following Example 4.15 gives a ϕ-coherent ring that is not nonnil-Noetherian.

Example 4.15. Let $R := (\mathbb{Z} + X\mathbb{Q}[[X]]) \propto qf(\mathbb{Q}[[X]])$. Then, R is a ϕ-coherent ring that is not nonnil-Noetherian.

Proof. First, it is easy to see that R is a ϕ-ring by [1, Corollary 2.4]. By [23, Theorem 3], $\mathbb{Z} + X\mathbb{Q}[[X]]$ is a coherent domain, and so R is a ϕ-coherent ring by Theorem 4.10. By [23, Theorem 3], $\mathbb{Z} + X\mathbb{Q}[[X]]$ is not a Noetherian domain, and so is not nonnil-Noetherian. Therefore, R is never a nonnil-Noetherian ring by Theorem 4.12. □

Remark 4.16. Note that the ring R in Example 4.15 is nonnil-coherent since R is a strongly ϕ-ring, and a ϕ-ring A is nonnil-coherent if and only if A is ϕ-coherent and $(0 : a)$ is a finitely generated ideal of A for each $a \in A\backslash\mathrm{Nil}(A)$ ([4, Proposition 1.3]). Using Examples 4.11 and 4.15, we can deduce that the converse of the following implications are not true in general:

$$\text{nonnil-Noetherian} \;\Rightarrow\; \text{nonnil-coherent} \;\Rightarrow\; \phi\text{-coherent}.$$

5 On transfer nonnil-Noetherian and ϕ-coherent rings in the amalgamation algebra along an ideal

In this section, we study the transfer of nonnil-Noetherian rings in the amalgamation algebra along an ideal. El Khalfi et al. [1] studied when the amalgamation algebra along an ideal is a ϕ-ring, a ϕ-chained ring, and a ϕ-pseudo-valuation ring.

Our next result characterizes when the amalgamation of a ring is a nonnil-Noetherian ring. Before starting this section, we need the following theorems.

Theorem 5.1. [1, Proposition 2.20] *Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then,*

$$\mathrm{Nil}(A \bowtie^f J) = \{(a, f(a) + j) | a \in \mathrm{Nil}(A) \quad \textit{and} \quad j \in J \cap \mathrm{Nil}(B)\}.$$

Theorem 5.2. [1, Theorem 2.1] *Let $f : A \to B$ be a ring homomorphism and J be a nonnil ideal of B. Set $N(J) := J \cap \mathrm{Nil}(B)$. The following statements are equivalent:*

(1) $R = A \bowtie^f J \in \mathcal{H}$.
(2) *A is an integral domain, $f^{-1}(J) = 0$, and $N(J)$ is a divided prime ideal of $f(A) + J$.*

Theorem 5.3 studies the transfer of being a nonnil-Noetherian ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nonnil ideal J.

Theorem 5.3. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \longrightarrow B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for all $a \in A$. If $A \bowtie^f J$ is a ϕ-ring, then the following statements are equivalent*:
(1) *$A \bowtie^f J$ is a nonnil-Noetherian ring.*
(2) *$A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a Noetherian domain.*
(3) *$f^{-1}(J) = \{0\}$, A, and $\bar{f}(A) + J/N(J)$ are Noetherian domains.*

Before proving Theorem 5.3, we establish the following lemmas.

Lemma 5.4. *With the notations of* Theorem 5.3, *we obtain $\bar{f}^{-1}(J/N(J)) = f^{-1}(J)$.*

Proof. Straightforward. □

Lemma 5.5. *Let $f : A \longrightarrow B$ be a ring homomorphism and J be a nonzero ideal of B. Let J' be a subideal of J and I be an ideal of A such that $f(I) \subset J'$. Define $\bar{\bar{f}} : A/I \longrightarrow B/J'$ by $\bar{\bar{f}}(\bar{a}) = \overline{f(a)}$, where $\bar{a} := a + I$ and $\overline{f(a)} := f(a) + J'$. Then, we have the following ring isomorphism*:

$$\frac{A \bowtie^f J}{I \bowtie^f J'} \cong \frac{A}{I} \bowtie^{\bar{\bar{f}}} \frac{J}{J'}.$$

Proof. Define

$$\varphi : \begin{array}{c} A \bowtie^f J \longrightarrow \frac{A}{I} \bowtie^{\bar{\bar{f}}} \frac{J}{J'} \\ (a, f(a) + j) \longmapsto (\bar{a}, \overline{f(a)} + \bar{j}). \end{array}$$

It is easy to see that φ is a surjective ring homomorphism and for all $(a, f(a) + j) \in A \bowtie^f J$, $(\bar{a}, \overline{f(a)} + \bar{j}) = (\bar{0}, \bar{0})$ if and only if $a \in I$ and $j \in J'$ and $(a, f(a) + j) \in I \bowtie^f J'$. Therefore $\frac{A \bowtie^f J}{I \bowtie^f J'} \cong \frac{A}{I} \bowtie^{\bar{\bar{f}}} \frac{J}{J'}$. □

Proof of Theorem 5.3. (1) ⇒ (2) Assume that $A \bowtie^f J$ is a nonnil-Noetherian ring. Since $A \bowtie^f J \in \mathcal{H}$, A is an integral domain by Theorem 5.2. Therefore, $\mathrm{Nil}(A \bowtie^f J) = 0 \times N(J)$. As $A \bowtie^f J$ is a nonnil-Noetherian ring, $\frac{A \bowtie^f J}{0 \times N(J)}$ is a Noetherian domain. Therefore, $A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a Noetherian domain by Lemma 5.5.

(2) ⇒ (1) This follows immediately from Lemma 5.5.

(2) ⇒ (3) Assume that $A \bowtie^{\bar{f}} J/N(J)$ is a Noetherian domain. By [13, Proposition 5.2] and Lemma 5.4, $f^{-1}(J) = 0$ and $\bar{f}(A) + J/N(J)$ is an integral domain. By [13, Proposition 5.6], A and $\bar{f}(A) + J/N(J)$ are Noetherian domains, as desired.

(3) ⇒ (2) By Lemma 5.4, we have $\bar{f}^{-1}(J/N(J)) = 0$. By [13, Proposition 5.1], $\bar{f}(A) + J/N(J) \cong A \bowtie^{\bar{f}} J/N(J)$, which is a Noetherian domain, as desired. □

Recall from [1, Corollary 2.6] that a polynomial ring $R[X]$ is a ϕ-ring if and only if R is an integral domain.

Theorem 5.6. *Let R be an integral domain. Then, $R[X]$ is a nonnil-Noetherian ring if and only if $R[X]$ is a Noetherian domain.*

Proof. By [1, Corollary 2.6], $R[X]$ is a ϕ-ring and $R[X] \cong R \bowtie^j J$, where $J = XR[X]$ and $j : R \hookrightarrow R[X]$. Since $J \not\subset \mathrm{Nil}(R[X])$, it follows that $R[X]$ is a nonnil-Noetherian ring if and only if $R \bowtie^j J$ is a Noetherian domain by Theorem 5.3. □

Corollary 5.7. *Let A be a ring and J be a nonnil ideal of A. Assume that $A \bowtie J \in \mathcal{H}$. Then, $A \bowtie J$ is never a nonnil-Noetherian ring.*

Proof. Assume, on the contrary, that $A \bowtie J$ is a nonnil-Noetherian ring. Then, $A \bowtie J/N(J)$ is a Noetherian domain, and so A is a Noetherian domain with $J = N(J)$ by [13, Remark 5.3]. Therefore, $J \subset \mathrm{Nil}(A)$, a desired contradiction. □

Theorem 5.8 studies the transfer of being a nonnil-Noetherian ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Theorem 5.8. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nil ideal of B. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, $A \bowtie^f J$ is a nonnil-Noetherian ring if and only if A is a nonnil-Noetherian ring.*

Proof. Note that $J \subset \mathrm{Nil}(B)$, and thus $N(J) = J$. So $\mathrm{Nil}(A \bowtie^f J) = \mathrm{Nil}(A) \bowtie^f J$. Therefore, $A \bowtie^f J$ is a nonnil-Noetherian ring if and only if $\frac{A \bowtie^f J}{\mathrm{Nil}(A) \bowtie^f J}$ is a Noetherian domain, and $\frac{A}{\mathrm{Nil}(A)}$ is a Noetherian domain, and A is a nonnil-Noetherian ring. □

Example 5.9. $R = \mathbb{Z}[X] \propto qf(\mathbb{Z}[X])$ is a nonnil-Noetherian ring that is not a Noetherian ring.

Now, we study the transfer of being ϕ-coherent rings in the amalgamation algebra along an ideal.

Theorem 5.10. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \longrightarrow B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for any $a \in A$. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, the following statements are equivalent:*

(1) *$A \bowtie^f J$ is a ϕ-coherent ring.*

(2) *$A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a coherent domain.*

(3) *$f^{-1}(J) = \{0\}$ and $\bar{f}(A) + J/N(J)$ is a coherent domain.*

Proof. (1) ⇒ (2) Assume that $A \bowtie^f J$ is a ϕ-coherent ring. Since $A \bowtie^f J \in \mathcal{H}$, it follows that A is an integral domain by Theorem 5.2, and so $\mathrm{Nil}(A \bowtie^f J) = 0 \times N(J)$. As $A \bowtie^f J$ is a ϕ-coherent ring, $\frac{A \bowtie^f J}{0 \times N(J)}$ is a coherent domain. Therefore, $A \bowtie^{\bar{f}} \frac{J}{N(J)}$ is a coherent domain by Lemma 5.5.

(2) ⇒ (1) This follows directly from Lemma 5.5.

(2) ⇒ (3) Assume that $A \bowtie^{\bar{f}} J/N(J)$ is a coherent domain. From [13, Proposition 5.2] and Lemma 5.4, $f^{-1}(J) = 0$ and $\bar{f}(A) + J/N(J)$ is an integral domain. From [13, Proposition 5.1], $\bar{f}(A) + J/N(J) \cong A \bowtie^{\bar{f}} J/N(J)$, as desired.

(3) ⇒ (2) By Lemma 5.4 we have $\bar{f}^{-1}(J/N(J)) = 0$ and from [13, Proposition 5.1], we obtain $\bar{f}(A) + J/N(J) \cong A \bowtie^{\bar{f}} J/N(J)$, which is a coherent domain, as desired. □

Corollary 5.11. *Let R be an integral domain. Then, $R[X]$ is a ϕ-coherent ring if and only if $R[X]$ is a coherent domain.*

Proof. By [1, Corollary 2.6], we have that $R[X]$ is a ϕ-ring and $R[X] \cong R \bowtie^j J$, where $J = XR[X]$ and $j : R \hookrightarrow R[X]$. Since $J \not\subset \mathrm{Nil}(R[X])$, it follows that $R[X]$ is a ϕ-coherent ring if and only if $R \bowtie^j J$ is a coherent domain by Theorem 5.10. □

Corollary 5.12 studies the transfer of being a nonnil-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nonnil ideal J.

Corollary 5.12. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nonnil ideal of B. Define $\bar{f} : A \longrightarrow B/N(J)$ by $\bar{f}(a) = f(a) + N(J)$ for any $a \in A$. Assume $A \bowtie^f J$ is a ϕ-ring. Then, the following statements are equivalent:*
(1) *$A \bowtie^f J$ is a nonnil-coherent ring,*
(2) *The following conditions hold:*
  (a) $f^{-1}(J) = \{0\}$.
  (b) *$\bar{f}(A) + J/N(J)$ is a coherent domain.*
  (c) *$(A \bowtie^f J)(r, f(r) + j)$ is a finitely presented ideal for any non-nilpotent element $(r, f(r) + j)$ of $A \bowtie^f J$.*

Proof. This follows immediately from [4, Proposition 1.3] and Theorem 5.10 □

Theorem 5.13 studies the transfer of being a ϕ-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Theorem 5.13. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nil ideal of B. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, $A \bowtie^f J$ is a ϕ-coherent ring if and only if A is a ϕ-coherent ring.*

Proof. Since $J \subset \mathrm{Nil}(B)$, we have $N(J) = J$. It is easy to see that $\mathrm{Nil}(A \bowtie^f J) = \mathrm{Nil}(A) \bowtie^f J$. Therefore, $A \bowtie^f J$ is a ϕ-coherent ring, $\frac{A \bowtie^f J}{\mathrm{Nil}(A) \bowtie^f J}$ is a coherent domain, $\frac{A}{\mathrm{Nil}(A)}$ is a coherent domain, and A is a ϕ-coherent ring. □

Corollary 5.14 studies the transfer of being a nonnil-coherent ring between a ϕ-ring A and an amalgamation algebra $A \bowtie^f J$ along a nil ideal J.

Corollary 5.14. *Let A and B be two rings and $f : A \longrightarrow B$ be a ring homomorphism. Let J be a nil ideal of B. Assume that $A \bowtie^f J$ is a ϕ-ring. Then, the following are equivalent:*
(1) *$A \bowtie^f J$ is a nonnil-coherent ring.*
(2) *A is a ϕ-coherent ring and $(A \bowtie^f J)(r, f(r) + j)$ is a finitely presented ideal for any non-nilpotent element $(r, f(r) + j)$ of $A \bowtie^f J$.*

Proof. This follows immediately from [4, Proposition 1.3] and Theorem 5.13. □

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