Research Article

WonTae Hwang, Jong-Do Park*, and Kyunghwan Song

On the reciprocal sum of the fourth power of Fibonacci numbers

https://doi.org/10.1515/math-2022-0525
received December 17, 2021; accepted October 24, 2022

Abstract: Let f_n be the nth Fibonacci number with $f_1 = f_2 = 1$. Recently, the exact values of $\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k^s}\right)^{-1} \right\rfloor$ have been obtained only for $s = 1, 2$, where $\lfloor x \rfloor$ is the floor function. It has been an open problem for $s \geq 3$. In this article, we consider the case of $s = 4$ and show that

$$\left\lfloor \left(\sum_{k=n}^{\infty}\frac{1}{f_k^4}\right)^{-1} \right\rfloor = f_n^4 - f_{n-1}^4 + \frac{2(-1)^n}{5}f_{2n-1} - \left\{\frac{n+2}{5}\right\},$$

where $\{x\} \coloneqq x - \lfloor x \rfloor$.

Keywords: recurrence relation, Fibonacci number, Lucas number, Catalan's identity, Pisano period

MSC 2020: Primary 11B37, 11B39, 11B50, Secondary 11Y55, 65Q30

1 Introduction and statement of main results

Let $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$. The number f_n is called the nth Fibonacci number. The Fibonacci sequence is important and widely used in many mathematical areas, especially related to number theory and combinatorics. For more detailed information, see [1]. Also, many generalizations of the Fibonacci numbers have been introduced in [2–6]. For the recent results in view of the classical identities such as Binet's formula, the generating function, Catalan's identity, Cassini's identity, and some binomial sums, see [7–10].

By contrast, the study on the reciprocal of the sum of convergent series is a relatively new research field, see the references [6,11–21]. For example, the reciprocal sum of Fibonacci numbers was studied in 2008 [17]. By Binet's formula, f_n can be written as the explicit form $f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ for any nonnegative integer n, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ are two solutions of $x^2 - x - 1 = 0$. Since f_n is comparable to α^n for sufficiently large n, we see that

$$F(s) \coloneqq \sum_{k=1}^{\infty}\frac{1}{f_k^s}$$

converges for all $s \geq 1$. It follows that

* **Corresponding author: Jong-Do Park,** Department of Mathematics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 02447, Korea, e-mail: mathjdpark@khu.ac.kr
WonTae Hwang: Department of Mathematics, and Institute of Pure and Applied Mathematics, Jeonbuk National University, Baekje-daero, Deokjin-gu, Jeonju-si, Jeollabuk-do, 54896 Republic of Korea, e-mail: hwangwon@jbnu.ac.kr
Kyunghwan Song: Department of Mathematics, Jeju National University, 102 Jejudaehak-ro Jeju, 63243, Republic of Korea, e-mail: khsong@jejunu.ac.kr

$$\lim_{n\to\infty}\left(\sum_{k=n}^{\infty}\frac{1}{f_k^s}\right)^{-1}=\infty.$$

Motivated by the above observation, in [17], the floor functions of $(\sum_{k=n}^{\infty}1/f_k)^{-1}$ and $(\sum_{k=n}^{\infty}1/f_k^2)^{-1}$ have been obtained as follows: for any $n\in\mathbb{N}$,

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k}\right)^{-1}\right\rfloor=\begin{cases}f_{n-2}, & n\geq 2 \text{ is even};\\ f_{n-2}-1, & n\geq 3 \text{ is odd},\end{cases}\tag{1.1}$$

and

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k^2}\right)^{-1}\right\rfloor=\begin{cases}f_{n-1}f_n-1, & n\geq 2 \text{ is even};\\ f_{n-1}f_n, & n\geq 1 \text{ is odd},\end{cases}\tag{1.2}$$

where $\lfloor\cdot\rfloor$ is the floor function. In this context, it is natural to consider the following problem.

Problem 1.1. For an integer $s\geq 3$, find explicit formulas for $\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k^s}\right)^{-1}\right\rfloor$.

The above problem has been an open problem.

In this article, we give an answer for Problem 1.1 for $s=4$. It turns out that the formula itself is more complicated in the sense that it includes some unexpected terms, and the method of the proof is essentially different from those of (1.1) and (1.2). For all $n\in\mathbb{N}$, define

$$g_n := f_n^4-f_{n-1}^4+\frac{2(-1)^n}{5}f_{2n-1}+\frac{2\sqrt{5}}{75}.\tag{1.3}$$

In Section 6, we explain how we can find g_n.

The following is the main theorem of this article.

Theorem 1.2. *Let* $c_n=\frac{1}{f_{2n-1}}$. *Then,*

(*i*) $g_n<\left(\sum_{k=n}^{\infty}\frac{1}{f_k^4}\right)^{-1}<g_n+c_n$ *when* n *is even;*

(*ii*) $g_n-c_n<\left(\sum_{k=n}^{\infty}\frac{1}{f_k^4}\right)^{-1}<g_n$ *when* n *is odd.*

As a consequence of Theorem 1.2, we have the following.

Theorem 1.3. *For any positive integer n, we have*

$$\left\lfloor\left(\sum_{k=n}^{\infty}\frac{1}{f_k^4}\right)^{-1}\right\rfloor=f_n^4-f_{n-1}^4+\frac{2(-1)^n}{5}f_{2n-1}-\left\{\frac{n+2}{5}\right\},$$

where $\{x\}:=x-[x]$ *is the decimal part of* x.

In Section 2, we introduce the basic properties of the Fibonacci numbers and Lucas numbers, which are defined as $L_0=2$, $L_1=1$, and $L_n=L_{n-1}+L_{n-2}$ for all $n\geq 2$. These Lucas numbers are used for the proof of our main result. In Section 3, we introduce the basic identities, which are used directly to prove our main theorem. The key idea is that the principal part

$$(g_{n+2}-g_n)f_n^4f_{n+1}^4-(f_n^4+f_{n+1}^4)g_ng_{n+2}$$

can be expressed as a polynomial with respect to one variable L_{2n+1} (see Proposition 3.3). In Section 4, we prove Theorem 1.2 using the identities proved in Section 3. In Section 5, we use the periodicity of f_n modulo 5 to prove Theorem 1.3. In Section 6, we explain how we obtain the formula g_n.

Remark 1.4. Our main theorem is the first result regarding Problem 1.1. We expect that our result and method can be used to solve the problem for any $s \geq 3$ and any linear recurrence sequences.

2 Fibonacci numbers and Lucas numbers

The Fibonacci numbers satisfy the following well-known useful identities.

Lemma 2.1. [1] *For any positive integers n and k, we have*
(*i*) $f_n^2 = f_{n+k}f_{n-k} + (-1)^{n+k}f_k^2$,
(*ii*) $f_{2n+1} = f_n^2 + f_{n+1}^2$.

It is more efficient to use the Lucas numbers when we compute the Fibonacci numbers. The *Lucas numbers* are defined as follows:

$$L_n = L_{n-1} + L_{n-2} \text{ for all } n \geq 3 \text{ with } L_0 = 2, L_1 = 1, L_2 = 3.$$

Then, the Lucas numbers are related to the Fibonacci numbers as follows:

$$f_{2n} = f_n L_n, \quad L_n = f_{n-1} + f_{n+1} \tag{2.1}$$

for any positive integer n.

Now, we explain the basic properties of Fibonacci numbers and Lucas numbers.

Lemma 2.2. [1,22] *For any $m, n \in \mathbb{N}$, we have*
(*i*) $f_m f_n = \frac{1}{5}(L_{m+n} - (-1)^n L_{m-n})$.
(*ii*) $L_m L_n = L_{m+n} + (-1)^n L_{m-n}$.
(*iii*) $f_{n+4m} + f_n = L_{2m} f_{n+2m}$.
(*iv*) $f_{n+4m} - f_n = f_{2m} L_{n+2m}$.
(*v*) $L_n^2 = 5f_n^2 + 4(-1)^n$.
(*vi*) $L_{2n} = L_n^2 - 2(-1)^n$.

Now, we obtain the equivalent forms of g_n defined as in (1.3). Note that

$$g_n = (f_n^2 + f_{n-1}^2)(f_n + f_{n-1})(f_n - f_{n-1}) + \frac{2(-1)^n}{5}f_{2n-1} + \gamma = f_{2n-1}f_{n+1}f_{n-2} + \frac{2(-1)^n}{5}f_{2n-1} + \gamma,$$

where $\gamma = \frac{2\sqrt{5}}{75}$. By Lemma 2.2 (i), we have

$$f_{n+1}f_{n-2} = \frac{1}{5}(L_{2n-1} - (-1)^{n-2}L_3) = \frac{1}{5}L_{2n-1} - \frac{4(-1)^n}{5}.$$

By the first relation (2.1), we have

$$g_n = \frac{1}{5}f_{2n-1}L_{2n-1} - \frac{2(-1)^n}{5}f_{2n-1} + \gamma = \frac{1}{5}f_{4n-2} - \frac{2(-1)^n}{5}f_{2n-1} + \gamma.$$

3 Basic identities

In Section 4, we compute

$$\frac{1}{g_n} - \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4} + \frac{1}{g_{n+2}}\right)$$

for the proof of Theorem 1.2. In this section, we compute

$$F_0 := (g_{n+2} - g_n)f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)g_n g_{n+2}.$$

In Section 2, we obtained the following two equivalent forms of g_n:

$$g_n = \frac{1}{5}f_{2n-1}(L_{2n-1} - 2(-1)^n) + \gamma \tag{3.1}$$

and

$$g_n = \frac{1}{5}f_{4n-2} - \frac{2(-1)^n}{5}f_{2n-1} + \gamma. \tag{3.2}$$

In fact, three terms $g_{n+2} - g_n$, $f_n^4 f_{n+1}^4$, and $f_n^4 + f_{n+1}^4$ in the definition of F_0 can be expressed in terms of L_{2n+1}.

Proposition 3.1. *For any positive integer n, we have*
(*i*) $g_{n+2} - g_n = \frac{1}{5}(3L_{2n+1}^2 - 2(-1)^n L_{2n+1} + 6)$.
(*ii*) $f_n^4 f_{n+1}^4 = \frac{1}{625}(L_{2n+1} - (-1)^n)^4$.
(*iii*) $f_n^4 + f_{n+1}^4 = \frac{1}{25}(3L_{2n+1}^2 + 4(-1)^n L_{2n+1} + 18)$.

Proof. Here, we use the form (3.2) of g_n.

(i) By Lemma 2.2 (iv) and (vi), we have

$$\begin{aligned} g_{n+2} - g_n &= \frac{1}{5}(f_{4n+6} - f_{4n-2}) - \frac{2}{5}(-1)^n(f_{2n+3} - f_{2n-1}) \\ &= \frac{3}{5}L_{4n+2} - \frac{2}{5}(-1)^n L_{2n+1} \\ &= \frac{1}{5}(3L_{2n+1}^2 - 2(-1)^n L_{2n+1} + 6). \end{aligned}$$

(ii) By Lemma 2.2 (i), we have

$$f_n^4 f_{n+1}^4 = (f_{n+1}f_n)^4 = \left(\frac{1}{5}(L_{2n+1} - (-1)^n)\right)^4.$$

(iii) By Lemmas 2.1 (ii) and 2.2 (v), we have

$$f_n^4 + f_{n+1}^4 = (f_n^2 + f_{n+1}^2)^2 - 2f_n^2 f_{n+1}^2 = f_{2n+1}^2 - 2(f_n f_{n+1})^2 = \frac{1}{25}(3L_{2n+1}^2 + 4(-1)^n L_{2n+1} + 18).$$

However, $g_n g_{n+2}$ can be written in terms of L_{2n+1} and f_{2n+1}. □

Proposition 3.2. *For any positive integer n, we have*

$$g_n g_{n+2} = \frac{1}{125}(L_{2n+1}^2 + 9)(L_{2n+1}^2 - 6(-1)^n L_{2n+1} - 1) + A,$$

where

$$A := \frac{\gamma}{5}f_{2n+1}(7L_{2n+1} - 6(-1)^n) + \gamma^2 \quad \text{with } \gamma = \frac{2\sqrt{5}}{75}. \tag{3.3}$$

Proof. By (3.1), it follows that

$$g_n g_{n+2} = \frac{1}{25}f_{2n-1}f_{2n+3}(L_{2n-1} - 2(-1)^n)(L_{2n+3} - 2(-1)^n) + \gamma^2 + \frac{\gamma}{5}\{f_{2n-1}(L_{2n-1} - 2(-1)^n) + f_{2n+3}(L_{2n+3} - 2(-1)^n)\}.$$

By Lemmas 2.1 (i) and 2.2 (v), we have

$$f_{2n-1}f_{2n+3} = f_{2n+1}^2 + 1 = \frac{1}{5}(L_{2n+1}^2 + 9).$$

By Lemmas 2.2 (ii) and (vi), we have

$$L_{2n-1}L_{2n+3} = L_{4n+2} - 7 = L_{2n+1}^2 - 5.$$

Note that

$$L_{2n-1} + L_{2n+3} = 3L_{2n+1}.$$

Combining the above two identities, we have

$$(L_{2n-1} - 2(-1)^n)(L_{2n+3} - 2(-1)^n) = L_{2n-1}L_{2n+3} - 2(-1)^n(L_{2n-1} + L_{2n+3}) + 4 = L_{2n+1}^2 - 6(-1)^nL_{2n+1} - 1.$$

By (2.1) and Lemma 2.2 (iii), we have

$$\begin{aligned} f_{2n-1}(L_{2n-1} - 2(-1)^n) + f_{2n+3}(L_{2n+3} - 2(-1)^n) &= (f_{4n-2} + f_{4n+6}) - 2(-1)^n(f_{2n-1} + f_{2n+3}) \\ &= 7f_{2n+1}L_{2n+1} - 6(-1)^nf_{2n+1} \\ &= f_{2n+1}(7L_{2n+1} - 6(-1)^n). \end{aligned}$$

If we combine the above formulas, the proof is done. □

To simplify our formulas, we write

$$x := L_{2n+1}.$$

Then, Propositions 3.1 and 3.2 can be summarized as follows:

$$\begin{aligned} g_{n+2} - g_n &= \frac{1}{5}(3x^2 - 2(-1)^nx + 6), \\ f_n^4f_{n+1}^4 &= \frac{1}{625}(x - (-1)^n)^4, \\ f_n^4 + f_{n+1}^4 &= \frac{1}{25}(3x^2 + 4(-1)^nx + 18), \\ g_ng_{n+2} &= \frac{1}{125}(x^2 + 9)(x^2 - 6(-1)^nx - 1) + A. \end{aligned}$$

Proposition 3.3. *Let* $F_0 = (g_{n+2} - g_n)f_n^4f_{n+1}^4 - (f_n^4 + f_{n+1}^4)g_ng_{n+2}$. *Then,*

$$F_0 = \frac{1}{3{,}125}\{14x^4 + 190(-1)^nx^3 + 146x^2 + 982(-1)^nx + 168\} - \frac{1}{25}\{3x^2 + 4(-1)^nx + 18\}A,$$

where $x = L_{2n+1}$ *and* A *is the form defined as in* (3.3).

Here, the term A cannot be written in terms of $x = L_{2n+1}$. Instead, we obtain the bound of A as follows.

Lemma 3.4. *In fact, A satisfies the inequality*

$$\frac{2}{375}x(7x - 6(-1)^n) + \frac{4}{1{,}125} \le A \le \frac{2}{375}(x + 1)(7x - 6(-1)^n) + \frac{4}{1{,}125}.$$

Proof. Note that $L_{2n+1}^2 < 5f_{2n+1}^2 = L_{2n+1}^2 + 4 < (L_{2n+1} + 1)^2$. □

4 Proof of Theorem 1.2

In this section, we prove the main theorem using the identities of the previous section. The key idea is that all important terms can be written as the function of the variable $x = L_{2n+1}$.

Theorem 4.1. (Theorem 1.2 (i)) *Let $n \in \mathbb{N}$ be even and let $c_n = \frac{1}{f_{2n-1}}$.*

$$g_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} < g_n + c_n.$$

Proof. Note that

$$\frac{1}{g_n} - \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4} + \frac{1}{g_{n+2}}\right) = \frac{F_0}{g_n g_{n+2} f_n^4 f_{n+1}^4}, \tag{4.1}$$

where $F_0 = (g_{n+2} - g_n)f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)g_n g_{n+2}$, which is the same as in Section 3.

(i) By Lemma 3.4, we have

$$A \le \frac{2}{375}(x+1)(7x-6) + \frac{4}{1,125}.$$

Now, we will show that F_0 is positive for any even $n \in \mathbb{N}$. More precisely, Proposition 3.3 gives the inequality

$$\begin{aligned} F_0 &= \frac{1}{3,125}(14x^4 + 190x^3 + 146x^2 + 982x + 168) - \frac{1}{25}(3x^2 + 4x + 18)A \\ &\ge \frac{2}{28,125}(762x^3 + 315x^2 + 4,429x + 1,044) > 0. \end{aligned}$$

Since $F_0 > 0$ for all $n \in \mathbb{N}$, identity (4.1) gives the inequality

$$\frac{1}{g_n} > \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2}}$$

for all $n \in \mathbb{N}$. If we apply the above inequality repeatedly, it follows that

$$\frac{1}{g_n} > \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2}} > \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \left(\frac{1}{f_{n+2}^4} + \frac{1}{f_{n+3}^4}\right) + \frac{1}{g_{n+4}} > \cdots > \sum_{k=n}^{\infty} \frac{1}{f_k^4},$$

which proves the left inequality of our theorem.

(ii) Note that

$$\frac{1}{g_n + c_n} - \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4} + \frac{1}{g_{n+2} + c_{n+2}}\right) = \frac{F}{(g_n + c_n)(g_{n+2} + c_{n+2})f_n^4 f_{n+1}^4}, \tag{4.2}$$

where

$$F := ((g_{n+2} + c_{n+2}) - (g_n + c_n))f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)(g_n + c_n)(g_{n+2} + c_{n+2}).$$

Now, we will show that F is negative for all $n \in \mathbb{N}$. We write $F = F_0 - F_c$, where

$$F_0 = (g_{n+2} - g_n)f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)g_n g_{n+2}$$

and

$$F_c := (c_n - c_{n+2})f_n^4 f_{n+1}^4 + (f_n^4 + f_{n+1}^4)(c_n g_{n+2} + c_{n+2} g_n + c_n c_{n+2}).$$

By Lemma 3.4, we have

$$A \ge \frac{2}{375}x(7x-6) + \frac{4}{1,125}.$$

It follows that

$$F_0 = \frac{1}{3,125}(14x^4 + 190x^3 + 146x^2 + 982x + 168) - \frac{1}{25}(3x^2 + 4x + 18)A \le \frac{2}{5,625}(165x^3 + 69x^2 + 947x + 144).$$

Since $c_n > c_{n+2}$ and $g_{n+2} > g_n$, we have

$$F_c > (f_n^4 + f_{n+1}^4)(c_n g_{n+2} + c_{n+2} g_n) > (f_n^4 + f_{n+1}^4)(c_n g_n + c_{n+2} g_{n+2}).$$

Note that

$$c_n g_n = \frac{1}{5}(L_{2n-1} - 2) + \frac{\alpha}{f_{2n-1}} > \frac{1}{5}(L_{2n-1} - 2).$$

Since we have

$$c_n g_n + c_{n+2} g_{n+2} > \frac{1}{5}(L_{2n-1} + L_{2n+3} - 4) = \frac{1}{5}(3L_{2n+1} - 4) = \frac{1}{5}(3x - 4),$$

we obtain

$$F_c > \frac{1}{125}(3x^2 + 4x + 18)(3x - 4).$$

It follows that

$$F = F_0 - F_c < \frac{1}{5{,}625}(-75x^3 + 138x^2 + 184x + 3{,}528) < 0.$$

Since $F < 0$ for all $n \in \mathbb{N}$, identity (4.2) gives the inequality

$$\frac{1}{g_n + c_n} < \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2} + c_{n+2}}$$

for all $n \in \mathbb{N}$. If we apply the above inequality repeatedly, it follows that

$$\begin{aligned}\frac{1}{g_n + c_n} &< \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2} + c_{n+2}}\\ &< \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \left(\frac{1}{f_{n+2}^4} + \frac{1}{f_{n+3}^4}\right) + \frac{1}{g_{n+4} + c_{n+4}}\\ &< \cdots < \sum_{k=n}^{\infty} \frac{1}{f_k^4},\end{aligned}$$

which proves the right inequality of Theorem 1.2(i). □

Theorem 4.2. (Theorem 1.2 (ii)) *Let $n \in \mathbb{N}$ be odd and let $c_n = \frac{1}{f_{2n-1}}$.*

$$g_n - c_n < \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} < g_n.$$

Proof. (i) By Lemma 3.4, we have

$$A \geq \frac{2}{375}x(7x + 6) + \frac{4}{1{,}125}$$

for any odd n. It follows that

$$\begin{aligned}F_0 &= \frac{1}{3{,}125}(14x^4 - 190x^3 + 146x^2 - 982x + 168) - \frac{1}{25}(3x^2 - 4x + 18)A\\ &\leq \frac{2}{5{,}625}(-165x^3 + 69x^2 - 947x + 144) < 0.\end{aligned}$$

Similarly, as the proof of the previous theorem, we obtain

$$\frac{1}{g_n} < \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2}}$$

for any odd n. It completes the proof of the right inequality of our theorem.

(ii) Note that

$$\frac{1}{g_n - c_n} - \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4} + \frac{1}{g_{n+2} - c_{n+2}}\right) = \frac{F'}{(g_n - c_n)(g_{n+2} - c_{n+2})f_n^4 f_{n+1}^4}, \tag{4.3}$$

where

$$F' := ((g_{n+2} - c_{n+2}) - (g_n - c_n))f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)(g_n - c_n)(g_{n+2} - c_{n+2}).$$

We write $F' = F_0 + F_c'$, where

$$F_0 = (g_{n+2} - g_n)f_n^4 f_{n+1}^4 - (f_n^4 + f_{n+1}^4)g_n g_{n+2}$$

and

$$F_c' := (c_n - c_{n+2})f_n^4 f_{n+1}^4 + (f_n^4 + f_{n+1}^4)(c_n g_{n+2} + c_{n+2} g_n - c_n c_{n+2}).$$

By Lemma 3.4, we have

$$A \le \frac{2}{375}(x + 1)(7x + 6) + \frac{4}{1{,}125}.$$

It follows that

$$\begin{aligned} F_0 &= \frac{1}{3{,}125}(14x^4 - 190x^3 + 146x^2 - 982x + 168) - \frac{1}{25}(3x^2 - 4x + 18)A \\ &\ge \frac{2}{28{,}125}(-888x^3 + 375x^2 - 5{,}041x + 396). \end{aligned}$$

By Lemma 2.2 (iv), we have

$$c_n - c_{n+2} = \frac{f_{2n+3} - f_{2n-1}}{f_{2n-1}f_{2n+3}} = \frac{5L_{2n+1}}{L_{2n+1}^2 + 9} = \frac{5x}{x^2 + 9} \ge \frac{5x}{(x + 1)^2} \quad \text{for } x \ge 4.$$

Note that

$$c_n g_{n+2} + c_{n+2} g_n > c_n g_n + c_{n+2} g_{n+2} > \frac{1}{5}(3L_{2n+1} + 4) = \frac{1}{5}(3x + 4).$$

It follows that

$$F_c' > \frac{5x}{(x + 1)^2} \cdot \frac{1}{625}(x + 1)^4 + \frac{1}{25}(3x^2 - 4x + 18)\left(\frac{1}{5}(3x + 4) - 1\right) = \frac{1}{125}(10x^3 - 13x^2 + 59x - 18).$$

Thus, we have

$$F' = F_0 + F' > \frac{1}{28{,}125}(474x^3 - 2{,}175x^2 + 3{,}193x - 3{,}528) > 0$$

for $x \ge 4$. Similarly as the proof of the previous theorem, we obtain

$$\frac{1}{g_n - c_n} > \left(\frac{1}{f_n^4} + \frac{1}{f_{n+1}^4}\right) + \frac{1}{g_{n+2} - c_{n+2}}$$

for any odd n. It completes the proof of the left inequality of Theorem 1.2 (ii). □

Corollary 4.3. $\lim_{n\to\infty}\left(\left(\sum_{k=n}^{\infty}\frac{1}{f_k^4}\right)^{-1} - g_n\right) = 0.$

Proof. It follows from squeeze property, Theorems 4.1 and 4.2. □

5 Proof of Theorem 1.3

There are many previous results related to the periodicity modulo $m \in \mathbb{N}$ of various linear recurrence sequences such as the case for the Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ [23], and other generalized cases for three-step Fibonacci sequence [24], (a, b)-Fibonacci sequence [25], and so on. We focus on the length of the period modulo m of the Fibonacci sequence, which is called the *Pisano period* $\pi(m)$ after Fibonacci's real name, Leonard Pisano. For example,

$$\pi(2) = 3, \quad \pi(3) = 8, \quad \pi(4) = 6, \quad \pi(5) = 20, \quad \pi(6) = 24, \ldots.$$

Now, we compute the value of $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} \right\rfloor$ using the periodicity of the Fibonacci sequence modulo 5.

Lemma 5.1. [23] *The Pisano period* $\pi(5)$ *is* 20. *Precisely, the Fibonacci sequence* $\{f_0, f_1, f_2, f_3, \ldots,\}$ *modulo* 5 *has the repeating pattern*

$$\{0, 1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1\}$$

of length 20. *It is clear that* $\pi(5) = 20$, *since* $f_{20} \equiv 0 \pmod 5$ *and* $f_{21} \equiv 1 \pmod 5$.

Proof of Theorem 1.3. By Theorem 1.2 and the inequality $\left| \frac{2\sqrt{5}}{75} \pm \frac{1}{f_{2n-1}} \right| < \frac{1}{5}$ for $n \geq 4$, we have

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4} \right)^{-1} \right\rfloor = f_n^4 - f_{n-1}^4 + \left\lfloor \frac{2}{5}(-1)^n f_{2n-1} \right\rfloor. \tag{5.1}$$

Now, we divide into the integer part and the decimal part of

$$\frac{2}{5}(-1)^n f_{2n-1} = \left\lfloor \frac{2}{5}(-1)^n f_{2n-1} \right\rfloor + \left\{ \frac{2}{5}(-1)^n f_{2n-1} \right\}.$$

By Lemma 5.1, the sequence $\{f_{2n-1}\}_{n=1}^{\infty}$ modulo 5 has the repeating pattern

$$\{1, 2, 0, 3, 4, 4, 3, 0, 2, 1\}$$

of length 10.

(i) If $n = 4, 6, 8, 10, 12 \ldots$ then the sequence $\{2f_7, 2f_{11}, 2f_{15}, 2f_{19}, 2f_{23} \ldots\}$ modulo 5 has the repeating pattern

$$\{1, 3, 0, 2, 4\}$$

of length 5. It means that

$$\left\{ \frac{2}{5} f_{2n-1} \right\} = \left\{ \frac{n+2}{5} \right\} \tag{5.2}$$

when n is even.

(ii) If $n = 5, 7, 9, 11, 13 \ldots$, then the sequence $\{2f_9, 2f_{13}, 2f_{17}, 2f_{21}, 2f_{25}, \ldots\}$ modulo 5 has the repeating pattern

$$\{3, 1, 4, 2, 0\}$$

of length 5. It means that

$$\left\{ -\frac{2}{5} f_{2n-1} \right\} = \left\{ \frac{n+2}{5} \right\} \tag{5.3}$$

when n is odd.

By (5.2) and (5.3), we have

$$\frac{2}{5}(-1)^n f_{2n-1} = \left\lfloor \frac{2}{5}(-1)^n f_{2n-1} \right\rfloor + \left\{ \frac{n+2}{5} \right\} \tag{5.4}$$

for all $n \geq 4$. Combining (5.1) and (5.4), we conclude that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4} \right)^{-1} \right\rfloor = f_n^4 - f_{n-1}^4 + \frac{2(-1)^n}{5} f_{2n-1} - \left\{ \frac{n+2}{5} \right\} \tag{5.5}$$

for $n \geq 4$. We can easily show that identity (5.5) holds for $n \leq 3$ by direct computation. It completes the proof of Theorem 1.3. □

6 How to obtain the formula g_n

In this final section, we need to explain how we guess the formula

$$g_n = f_n^4 - f_{n-1}^4 + \frac{2}{5}(-1)^n f_{2n-1} + \frac{2\sqrt{5}}{75}.$$

By using a computer software program, we have constructed the following table for the values of

$$h_n := \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4} \right)^{-1} - (f_n^4 - f_{n-1}^4).$$

Table 1: The values of h_n for $3 \leq n \leq 6$

n	$\left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1}$	$f_n^4 - f_{n-1}^4$	$\left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} - (f_n^4 - f_{n-1}^4)$
3	13.032567 ⋯	15	−1.967432
4	70.269868 ⋯	65	5.269869
5	530.455702 ⋯	544	−13.544298
6	3506.661126 ⋯	3,471	35.661126

One can see that h_n is positive for even n and negative for odd n in Table 1. Since $|h_n|$ goes to infinity as $n \to \infty$, we must calculate h_n. Some h_n's can be written as follows:

$$\begin{aligned}
h_3 &= -1.9674 \cdots = -2 + 0.0325 \cdots \\
h_4 &= 5.2698 \cdots = 5 + 0.2698 \cdots \\
h_5 &= -13.5442 \cdots = -14 + 0.4557 \cdots \\
h_6 &= 35.6611 \cdots = 35 + 0.6611 \cdots \\
h_7 &= -93.1409 \cdots = -94 + 0.8590 \cdots \\
h_8 &= 244.0598 \cdots = 243 + 1.0598 \cdots .
\end{aligned}$$

The above observation gives us the following expectation:

$$h_n = d_n + \frac{1}{5}(n-3) + \gamma_n,$$

where $d_n \in \mathbb{Z}$ and γ_n approaches $\frac{2}{75}\sqrt{5} = 0.0596284793999943 \cdots$. See the following values of d_n in Table 2.

Table 2: The values of d_n for $1 \leq n \leq 18$

n	d_n	n	d_n	n	d_n
1	−1	7	−94	13	−30,012
2	0	8	243	14	78,565
3	−2	9	−640	15	−205,694
4	5	10	1,671	16	538,505
5	−14	11	−4,380	17	−1,409,834
6	35	12	11,461	18	3,690,983

Finally, we obtain the explicit form of d_n, which is the main part of this section.

Theorem 6.1. *For any positive integer n, we have*

$$d_n = \frac{2(-1)^n}{5} f_{2n-1} - \frac{1}{5}(n-3).$$

By Theorem 6.1, it follows that

$$g_n = f_n^4 - f_{n-1}^4 + \frac{2(-1)^n}{5} + \gamma,$$

where $\gamma = \frac{2\sqrt{5}}{75}$. From the above table, the sequence $\{d_n\}$ satisfies the relation

$$d_n = -2d_{n-1} + 2d_{n-2} + d_{n-3} - 1, \quad n \geq 6 \tag{6.1}$$

with

$$d_3 = -2, \quad d_4 = 5, \quad d_5 = -14. \tag{6.2}$$

If we define $e_n := d_n - d_{n-1}$, then the relation (6.1) is reduced to

$$e_n = -3e_{n-1} - e_{n-2} - 1$$

with

$$e_4 = 7, \quad e_5 = -19.$$

Note that

$$\begin{aligned} e_6 &= -(3e_5 + e_4 + 1) = -(f_4 e_5 + f_2 e_4 + f_1 f_2) \\ e_7 &= 8e_5 + 3e_4 + 2 = f_6 e_5 + f_4 e_4 + f_2 f_3 \\ e_8 &= -(21e_5 + 8e_4 + 6) = -(f_8 e_5 + f_6 e_4 + f_3 f_4). \end{aligned}$$

One can easily prove the following by induction.

Proposition 6.2. *For any $n \geq 6$, we have*

$$e_n = (-1)^{n+1}\{f_{2n-8}e_5 + f_{2n-10}e_4 + f_{n-5}f_{n-4}\}.$$

Lemma 6.3. *Let $m \in \mathbb{N}$.*

(i) $\sum_{k=1}^m f_{4k-1} = f_{2m}f_{2m+1}$, $\sum_{k=1}^m f_{4k+1} = f_{2m}f_{2m+3}$.

(ii) $f_1^2 + f_2^2 + \cdots + f_m^2 = f_m f_{m+1}$.

(iii) $f_1^2 + f_3^2 + \cdots + f_{2n-1}^2 = \frac{1}{5}(f_{4n} + 2n)$.

(iv) $f_2^2 + f_4^2 + \cdots + f_{2n}^2 = \frac{1}{5}(f_{4n+2} - 2n - 1)$.

(v) $5f_n^2 + f_{2n} = 2f_{2n+1} - 2(-1)^n$.

Proof. The formulas (i), (ii), (iii), and (iv) are known. Here, we prove (v) only. By Lemma 2.2 (v), we have

$$5f_n^2 + f_{2n} = L_n^2 + f_{2n} - 4(-1)^n.$$

Since $L_n = f_{n-1} + f_{n+1}$ and $f_{2n} = f_n L_n$, we have

$$5f_n^2 + f_{2n} = 2(f_{n+1}f_{n-1} + f_{n+1}^2) - 4(-1)^n.$$

By Lemma 2.1 (i) and (ii), we have

$$5f_n^2 + f_{2n} = 2(f_n^2 + f_{n+1}^2) - 2(-1)^n = 2f_{2n+1} - 2(-1)^n.$$ □

Lemma 6.4. *Let m be any positive integer. Then,*

(i) $\sum_{k=1}^m (-1)^k f_{2k} = (-1)^m f_m f_{m+1}$.

(ii) $\sum_{k=1}^{m}(-1)^k f_{2k+2} = (-1)^m f_m f_{m+3}$.

(iii) $\sum_{k=1}^{m}(-1)^k f_k f_{k+1} = \begin{cases} f_2^2 + f_4^2 + \cdots + f_m^2, & m \text{ is even} \\ (f_2^2 + f_4^2 + \cdots + f_{m-1}^2) - f_m f_{m+1}, & m \text{ is odd} \end{cases}.$

Proof. Since the proofs are similar, we prove (i) only. If m is even, then by Lemma 6.3 (i), we have

$$\sum_{k=1}^{m}(-1)^k f_{2k} = (-f_2 + f_4) + (-f_6 + f_8) + \cdots + (-f_{2m-2} + f_{2m}) = f_3 + f_7 + \cdots + f_{2m-1} = f_m f_{m+1}.$$

If m is odd, then by Lemma 6.3 (i), we have

$$\sum_{k=1}^{m}(-1)^k f_{2k} = (f_3 + f_7 + \cdots + f_{2m-3}) - f_{2m} = f_{m-1} f_m - f_m L_m = -f_m f_{m+1}.$$

It completes the proof of (i). □

Proof of Theorem 6.1. Note that for any positive integer $n \geq 6$, we have

$$d_n = d_5 + \sum_{k=1}^{n-5} e_{k+5} = -14 + 7\sum_{k=1}^{n-5}(-1)^k f_{2k} - 19\sum_{k=1}^{n-5}(-1)^k f_{2k+2} + \sum_{k=1}^{n-5}(-1)^k f_k f_{k+1}.$$

(i) If n is odd, then by Lemma 6.4, we have

$$d_n = -14 + 7f_{n-5}f_{n-4} - 19f_{n-5}f_{n-2} + (f_2^2 + f_4^2 + \cdots + f_{n-5}^2) = -14 - f_{n-5}(19f_{n-2} - 7f_{n-4}) + (f_2^2 + f_4^2 + \cdots + f_{n-5}^2).$$

A direct computation shows that

$$f_{n-5}(19f_{n-2} - 7f_{n-4}) = f_{n-5}(f_{n+4} - f_{n-1}) = f_{n-5}(f_{n+3} + f_{n+1} + f_{n-3} + f_{n-4}) = f_{n-1}^2 + f_{n-2}^2 + f_{n-4}^2 - 14 + f_{n-5}f_{n-4}.$$

It follows that

$$d_n = -(f_{n-1}^2 + f_{n-2}^2 + f_{n-4}^2) - f_{n-5}f_{n-4} + (f_2^2 + f_4^2 + \cdots + f_{n-5}^2).$$

By Lemma 6.3 (ii), we have

$$f_1^2 + f_2^2 + \cdots + f_{n-5}^2 = f_{n-5}f_{n-4}.$$

By Lemma 6.3 (iii) and (v), we have

$$d_n = -(f_1^2 + f_3^2 + \cdots + f_{n-2}^2) - f_{n-1}^2 = -\frac{1}{5}(f_{2n-2} + n - 1) - \frac{1}{5}(2f_{2n-1} - f_{2n-2} - 2) = -\frac{2}{5}f_{2n-1} - \frac{1}{5}(n-3).$$

(ii) If n is even, then by Lemma 6.4, we have

$$\begin{aligned} d_n &= -14 + 7\sum_{k=1}^{n-5}(-1)^k f_{2k} - 19\sum_{k=1}^{n-5}(-1)^k f_{2k+2} + \sum_{k=1}^{n-5}(-1)^k f_k f_{k+1} \\ &= -14 + f_{n-5}(19f_{n-2} - 7f_{n-4}) + (f_2^2 + f_4^2 + \cdots + f_{n-6}^2) - f_{n-5}f_{n-4}. \end{aligned}$$

Similarly as in the case when n is odd, we obtain

$$f_{n-5}(19f_{n-2} - 7f_{n-4}) = f_{n-1}^2 + f_{n-2}^2 + f_{n-4}^2 + 14 + f_{n-5}f_{n-4}.$$

By Lemma 6.3 (iv) and (v), we conclude that

$$\begin{aligned} d_n &= f_{n-1}^2 + f_{n-2}^2 + f_{n-4}^2 + (f_2^2 + f_4^2 + \cdots + f_{n-6}^2) \\ &= (f_2^2 + f_4^2 + \cdots + f_{n-2}^2) + f_{n-1}^2 \\ &= \frac{1}{5}(f_{2n-2} - n + 1) + \frac{1}{5}(2f_{2n-1} - f_{2n-2} + 2) \\ &= \frac{2}{5}f_{2n-1} - \frac{1}{5}(n-3). \end{aligned}$$

It completes the proof of Theorem 6.1. □

We will finish this section with proving Theorem 6.1 again using the theory of linear recurrence relations. Note that the characteristic equation of a nonhomogeneous linear recurrence relation (6.1) is $(x-1)(x^2+3x+1)=0$ and has three distinct roots $-\alpha^2$, $-\beta^2$, and 1, where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. It is easy to show that (6.1) has a particular solution $d_n = -\frac{1}{5}n$. Thus, there exist constants A, B, and C such that

$$d_n = A(-\alpha^2)^n + B(-\beta^2)^n + C - \frac{1}{5}n.$$

From the conditions (6.2), the constants A, B, and C satisfy

$$\alpha^6 A + \beta^6 B - C = \frac{7}{5}, \quad \alpha^8 A + \beta^8 B + C = \frac{29}{5}, \quad \alpha^{10} A + \beta^{10} B - C = 13.$$

Solving the system of linear equations, we obtain

$$A = -\frac{2(18\beta^2 - 47)}{5\alpha^6(\alpha^2+1)(\alpha^2-\beta^2)}, \quad B = -\frac{2(18\alpha^2 - 47)}{5\beta^6(\beta^2+1)(\beta^2-\alpha^2)}, \quad C = \frac{-7\alpha^2\beta^2 + 29\alpha^2 + 29\beta^2 - 65}{5(\alpha^2+1)(\beta^2+1)}.$$

The forms of A, B, and C are too complicated, but we can simplify them. Since α and β are roots of $x^2 - x - 1 = 0$, we obtain

$$A = \frac{2}{5\alpha(\alpha-\beta)}, \quad B = -\frac{2}{5\beta(\alpha-\beta)}, \quad C = \frac{3}{5}.$$

It follows that

$$d_n = \frac{2}{5}(-1)^n \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha - \beta} + \frac{3}{5} - \frac{1}{5}n.$$

This is exactly the same as the form in Theorem 6.1 by Binet's formula.

7 Conclusion and future work

In this article, we find an explicit formula for $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} \right\rfloor$. It is given by the following expression:

$$f_n^4 - f_{n-1}^4 + \frac{2(-1)^n}{5} f_{2n-1} - \left\{\frac{n+2}{5}\right\}.$$

To obtain the desired formula, we start with guessing the leading terms $f_n^4 - f_{n-1}^4$ by observing

$$\lim_{n\to\infty} \frac{\left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} - (f_n^4 - f_{n-1}^4)}{f_n^4} = 0.$$

After that, we note that the values of the sequence $\left(\sum_{k=n}^{\infty} \frac{1}{f_k^4}\right)^{-1} - (f_n^4 - f_{n-1}^4)$ are proportional to f_n^2 as n increases, and we obtain the remaining term by direct computation. We give the detailed process in Section 6.

This work might help in finding the formula for $\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{f_k^s}\right)^{-1} \right\rfloor$ for $s = 3$ or $s \geq 5$ and might also be useful for the problem of obtaining the formula for the reciprocal of the sums of the sequences that have homogeneous and non-homogeneous recurrence relations with second-order constant coefficients and some other related problems.

Funding information: This paper was supported by research funds for newly appointed professors of Jeonbuk National University in 2021. This work was supported by NRF-2018R1D1A1B07050044 from the

National Research Foundation of Korea. Kyunghwan Song was supported by NRF-2020R1I1A1A01070546 from the National Research Foundation of Korea.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state that there is no conflict of interest.

Data availability statement: No data were used to support this study.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Vol. 1. Second edition, John Wiley and Sons, Inc., Hoboken, NJ, 2018.
[2] M. Çetin, C. Kızılateş, F. Yeşil Baran, and N. Tuglu, *Some identities of harmonic and hyperharmonic Fibonacci numbers*, Gazi Univ. J. Sci. **34** (2021), no. 2, 493–504.
[3] A. F. Horadam, *A generalized Fibonacci sequence*, Amer. Math. Monthly **68** (1961), no. 5, 455–459.
[4] C. Kızılateş, *New families of Horadam numbers associated with finite operators and their applications*, Math. Methods Appl. Sci. **44** (2021), 14371–14381.
[5] C. Kızılateş and N. Tuglu, *A new generalization of convolved* (p, q)*-Fibonacci and* (p, q)*-Lucas polynomials*, J. Math. Comput. Sci. **7** (2017), 995–1005.
[6] N. Tuglu, C. Kızılateş, and S. Kesim, *On the harmonic and hyperharmonic Fibonacci numbers*, Adv. Difference Equations **2015** (2015), 297.
[7] J. L. Ramírez, *On convolved generalized Fibonacci and Lucas polynomials*, Appl. Math. Comput. **229** (2014), 208–213.
[8] D. Tasci and M. C. Firengiz, *Incomplete Fibonacci and Lucas p-numbers*, Math. Comput. Model. **52** (2010), no. 9–10, 1763–1770.
[9] J. Yang and Z. Zhang, *Some identities of the generalized Fibonacci and Lucas sequences*, Appl. Math. Comput. **339** (2018), 451–458.
[10] O. Yayenie, *A note on generalized Fibonacci sequences*, Appl. Math. Comput. **217** (2011), no. 12, 5603–5611.
[11] G. Choi and Y. Choo, *On the reciprocal sums of products of Fibonacci and Lucas numbers*, Filomat. **32** (2018), no. 8, 2911–2920.
[12] G. Choi and Y. Choo, *On the reciprocal sums of square of generalized bi-periodic Fibonacci numbers*, Miskolc Math. Notes. **19** (2018), 201–209.
[13] H.-H. Lee and J.-D. Park, *The limit of reciprocal sum of some subsequential Fibonacci numbers*, AIMS Math. **6** (2021), no. 11, 12379–12394.
[14] H.-H. Lee and J.-D. Park, *Asymptotic behavior of reciprocal sum of two products of Fibonacci numbers*, J. Inequal. Appl. **2020** (2020), 91.
[15] H.-H. Lee and J.-D. Park, *Asymptotic behavior of the inverse of tails of Hurwitz zeta function*, J. Korean Math. Soc. **57** (2020), no. 6, 1535–1549.
[16] R. Liu and A. Y. Z. Wang, *Sums of products of two reciprocal Fibonacci numbers*, Adv. Difference Equations **2016** (2016), 136.
[17] H. Ohtsuka and S. Nakamura, *On the sum of reciprocal Fibonacci numbers*, Fibonacci Q. **46–47** (2008/2009), no. 2, 153–159.
[18] A. Y. Z. Wang and F. Zhang, *The reciprocal sums of even and odd terms in the Fibonacci sequence*, J. Inequal. Appl. **2015** (2015), 376.
[19] L. Xin and L. Xiaoxue, *A reciprocal sum related to the Riemann* ζ*-function*, J. Math. Inequal. **11** (2017), no. 1, 209–215.
[20] G. J. Zhang, *The infinite sum of reciprocal of the Fibonacci numbers*, J. Math. Res. Exposition **31** (2011), no. 6, 1030–1034.
[21] W. Zhang and T. Wang, *The infinite sum of reciprocal Pell numbers*, Appl. Math. Comput. **218** (2012), no. 10, 6164–6167.
[22] S. Rabinowitz, *Algorithmic manipulation of Fibonacci identities*, In: G. E. Bergum, A. N. Philippou, A. F. Horadam (eds), Applications of Fibonacci Numbers, Springer, Dordrecht, 1996.
[23] D. D. Wall, *Fibonacci series modulo m*, Amer. Math. Monthly **67** (1960), 525–532.
[24] E. Özkan, H. Aydin, and R. Dikici, *3-step Fibonacci series modulo m*, Appl. Math. Comput. **143** (2003), no. 1, 165–172.
[25] M. Renault, *The period, rank, and order of the* (a, b)*-Fibonacci sequence mod m*, Math. Mag. **86** (2013), no. 5, 372–380.