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Research Article

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N-Tuples of weighted noncommutative Orlicz space and some geometrical properties

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Abstract: In this article, we present a new concept named the N-tuples weighted noncommutative Orlicz space $\oplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)}(\widetilde{\mathcal{M}}, \tau)$, where $L^{(\Phi_j)}(\widetilde{\mathcal{M}}, \tau)$ is the noncommutative Orlicz space. Based on the maximum principle, the Riesz-Thorin interpolation theorem of $\oplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)}(\widetilde{\mathcal{M}}, \tau)$ is given. As applications, we obtain the Clarkson inequality and some other geometrical properties which include the uniform convexity and uniform smoothness of noncommutative Orlicz spaces $L^{(\Phi_s)}(\widetilde{\mathcal{M}}, \tau)$, $0 < s \leq 1$.

Keywords: noncommutative Orlicz spaces, τ-measurable operator, von Neumann algebra, Orlicz function, Riesz-Thorin interpolation

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1 Preliminaries

In the world of analysis, one of the important tools for modern mathematicians is the interpolation of operators. The first theorem regarding the interpolation of operators was proven by Marcel Riesz in 1927 [1]. In 1939, his student Olof Thorin proved a generalization of Riesz's theorem. Although sometimes referred to as the Riesz convexity theorem, due to the way he initially stated it in [1], this theorem is usually known as the Riesz-Thorin interpolation theorem. In 1936, in order to study the uniform convexity of L^p space, Clarkson gave some important inequalities which are named the Clarkson inequality [2]. In [3], the author used the noncommutative Riesz-Thorin interpolation theorem to obtain the Clarkson inequality of noncommutative L^p space.

The principal objective of this article is to investigate the Riesz-Thorin interpolation theorem on noncommutative Orlicz spaces, which yields the Clarkson inequality of noncommutative L^p spaces. As applications, some geometrical properties such as uniform convexity and uniform smoothness of noncommutative Orlicz space $L^{(\Phi_s)}(\widetilde{\mathcal{M}}, \tau)$, $0 < s \leq 1$ are given.

The theory of noncommutative Orlicz spaces associated with a trace was introduced by Muratov [4] and Kunze [5]. Let $\mathcal{M}$ be a semi-finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ with a normal semi-finite faithful trace τ. A densely defined closed linear operator $A : \mathcal{D}(A) \to \mathcal{H}$ with domain $\mathcal{D}(A) \subseteq \mathcal{H}$ is called affiliated with $\mathcal{M}$ if and only if $U^*AU = A$ for all unitary operators U belonging to the commutant $\mathcal{M}'$ of $\mathcal{M}$. Clearly, if $A \in \mathcal{M}$, then A is affiliated with $\mathcal{M}$. If A is a (densely defined closed) operator affiliated

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with $\mathcal{M}$ and $A = U|A|$ the polar decomposition, where $|A| = (A^*A)^{\frac{1}{2}}$ and U is a partial isometry, then A is said to be τ-measurable if and only if there exists a number $\lambda \geq 0$ such that $\tau(e_{(\lambda,\infty)}(|A|)) < \infty$, where $e_{[0,\lambda]}$ is the spectral projection of $|A|$ and τ is the trace of normal faithful and semifinite. The collection of all τ-measurable operators is denoted by $\widetilde{\mathcal{M}}$. The spectral decomposition implies that a von Neumann algebra $\mathcal{M}$ is generated by its projections. Recall that an element $A \in \mathcal{M}_+$ is a linear combination of mutually orthogonal projections if $A = \sum_{k=1}^{n} \alpha_k e_k$ with $\alpha_k \in \mathbb{R}_+$ and projection $e_k \in \mathcal{M}$ such that $e_k e_j = 0$ whenever $k \neq j$ [3].

Next, we recall the definition and some basic properties of noncommutative Orlicz spaces.

A function $\Phi : [0, \infty) \to [0, \infty]$ is called an Orlicz function if and only if $\Phi(u) = \int_0^{|u|} p(t)\mathrm{d}t$, where the right derivative p of Φ satisfies that p is right-continuous and nondecreasing, $p(t) > 0$ whenever $t > 0$ and $p(0) = 0$ with $\lim_{t\to\infty} p(t) = \infty$ [6].

If $A \in \widetilde{\mathcal{M}}$ and Φ is an Orlicz function, one can define a corresponding space, which is named the noncommutative Orlicz space, as follows:

$$L^{\Phi}(\widetilde{\mathcal{M}}, \tau) = \{A \in \widetilde{\mathcal{M}} : \tau(\Phi(\lambda|A|)) < \infty \text{ for some } \lambda > 0\}.$$

The following Luxemburg norm could be equipped for these spaces

$$\|A\|_{(\Phi)} = \inf\left\{\lambda > 0 : \tau\left(\Phi\left(\frac{|A|}{\lambda}\right)\right) \leq 1\right\}.$$

In the case of $\Phi(A) = |A|^p$, $1 \leq p < \infty$, $L^{(\Phi)}(\widetilde{\mathcal{M}}, \tau)$ is nothing but the noncommutative space $L^p(\widetilde{\mathcal{M}}, \tau) = \{A \in \widetilde{\mathcal{M}} : \tau(|A|^p) < \infty\}$ [7] and the Luxemburg norm generated by this function is expressed by the formula:

$$\|A\|_p = (\tau(|A|^p))^{\frac{1}{p}}.$$

One can define another norm on $L^{\Phi}(\widetilde{\mathcal{M}}, \tau)$ as follows:

$$\|A\|_{\Phi} = \sup\{\tau(|AB|) : B \in L^{\Psi}(\widetilde{\mathcal{M}}, \tau) \text{ and } \tau(\Psi(B)) \leq 1\},$$

where $\Psi : [0, \infty) \to [0, \infty]$ is defined by $\Psi(u) = \sup\{uv - \Phi(v) : v \geq 0\}$. Here we call Ψ the complementary function of Φ. In this article, we use $L^{(\Phi)}(\widetilde{\mathcal{M}}, \tau)$ and $L^{\Phi}(\widetilde{\mathcal{M}}, \tau)$ to denote the noncommutative Orlicz spaces which are equipped with Luxemberg norm and Orlicz norm, respectively.

For more information on the theory of noncommutative Orlicz spaces we refer the reader to [4,5] and [7–10].

2 Riesz-Thorin interpolation theorem of noncommutative Orlicz spaces

In this section, we first present a new concept named N-tuple noncommutative Orlicz spaces and then give some norm inequalities. In order to research the Riesz-Thorin interpolation theorem, an equivalent definition of the Luxemburg norm must be given. As a corollary, the Clarkson inequality of noncommutative L^p space could be obtained. The main ideas of proof in this article are derived from literature [3] and [11].

Let $\mathcal{N} = \mathcal{M} \oplus \mathcal{M} \oplus \cdots \oplus \mathcal{M}$ be the nth von Neumann algebra direct sum of $\mathcal{M}$ with itself. We know that $\mathcal{N}$ acts on the direct sum Hilbert space $\mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ coordinatewise:

$$(A_1, A_2, \ldots, A_n)(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} A_j x_j,$$

where $A_j \in \mathcal{M}$, $x_j \in \mathcal{H}$, and $j = 1, 2, \ldots n$. Then $\mathcal{N}_+ = \mathcal{M}_+ \oplus \mathcal{M}_+ \oplus \cdots \oplus \mathcal{M}_+$.

We define $\upsilon : \mathcal{N}_+ \to \mathbb{C}$ by $\upsilon(A_1, A_2, \ldots, A_n) = \sum_{j=1}^{n} \lambda_j \tau(A_j)$, where $\lambda_j \geq 0$ and τ is normal semifinite faithful trace on $\mathcal{M}$, then υ is a weighted normal faithful normal trace on $\mathcal{N}$.

Definition 2.1. Let $\Phi = (\Phi_1, \Phi_2, \ldots \Phi_n)$ be n-tuple of N-functions. For each $p \geq 1, \lambda_j \geq 0$, and n-tuple of weights $\lambda = (\lambda_1, \ldots, \lambda_n)$, consider the following direct sum space:

$$\bigoplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)} = \left\{A = (A_1, A_2, \ldots, A_n) : A_j \in L^{(\Phi_j)}(\widetilde{\mathcal{M}}, \tau), 1 \leq j \leq n\right\}$$

with norm $\|\cdot\|_{(\Phi),p,\lambda}$ defined as follows:

$$\|A\|_{(\Phi),p,\lambda} = \begin{cases} \left[\sum_{j=1}^{n} \lambda_j \|A_j\|_{(\Phi_j)}^p\right]^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \max_j \|A_j\|_{(\Phi_j)}, & p = \infty, \end{cases} \tag{1}$$

or the norm $\|\cdot\|_{\Phi,p,\lambda}$ defined in the same way as before in which $\|\cdot\|_{(\Phi_j)}$ is replaced by the Orlicz norm $\|\cdot\|_{\Phi_j}$. If Ψ_j is the complementary N-function of Φ_j, denote by $\oplus_{j=1}^{n} L_{q,\lambda}^{(\Psi_j)}$ which is equipped with $\|\cdot\|_{(\Psi),q,\lambda}$ and $\oplus_{j=1}^{n} L_{q,\lambda}^{\Psi_j}$ which is equipped with $\|\cdot\|_{\Psi,q,\lambda}$ for the same weights $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $q = \frac{p}{p-1}$.

Lemma 2.1. *Assuming $A \in \oplus_{j=1}^{n} L_{p,\lambda}^{(\Phi_j)}$ and $B \in \oplus_{j=1}^{n} L_{q,\lambda}^{\Psi_j}$, where $1 \leq p < \infty$, we have*

(1) *If $\|A_j\|_{(\Phi_j),p,\lambda} \leq 1$, then $\upsilon(\Phi(A)) \leq \|A\|_{(\Phi),p,\lambda} \cdot \delta_1$, where $\delta_1 = \left(\sum_{j=1}^{n} \lambda_j\right)^{\frac{1}{q}}$.*

(2) *If $\|A_j\|_{(\Phi_j),p,\lambda} > 1$, then $\upsilon(\Phi(A)) > \delta_2$, where $\delta_2 = \left[\sum_{j=1}^{n} \lambda_j^p \|A_j\|_{(\Phi_j)}^p\right]^{\frac{1}{p}}$.*

(3) *(Hölder inequality) $\upsilon(AB) \leq \|A\|_{(\Phi),p,\lambda} \cdot \|B\|_{\Psi,q,\lambda}$.*

Proof. (1) If $\|A_j\|_{(\Phi_j),p,\lambda} \leq 1$, by Proposition 3.4 of [8] and classical Hölder inequality, we then have

$$\begin{aligned} \upsilon(\Phi(A)) &= \sum_{j=1}^{n} \lambda_j \tau(\Phi_j(A_j)) \\ &= \sum_{j=1}^{n} \lambda_j^{\frac{1}{q}} \cdot \lambda_j^{\frac{1}{p}} \tau(\Phi_j(A_j)) \\ &\leq \left[\sum_{j=1}^{n} \lambda_j \tau(\Phi_j(A_j))^p\right]^{\frac{1}{p}} \cdot \left(\sum_{j=1}^{n} \lambda_j\right)^{\frac{1}{q}} \\ &\leq \left[\sum_{j=1}^{n} \lambda_j \|A_j\|_{(\Phi_j)}^p\right]^{\frac{1}{p}} \cdot \delta_1 \\ &= \|A\|_{(\Phi),p,\lambda} \cdot \delta_1. \end{aligned}$$

(2) If $\|A_j\|_{(\Phi_j),p,\lambda} > 1$, by Proposition 3.4 of [8], we have

$$[\upsilon(\Phi(A))]^p = \left[\sum_{j=1}^{n} \lambda_j \tau(\Phi_j(A_j))\right]^p > \left[\sum_{j=1}^{n} \lambda_j \|A_j\|_{(\Phi_j)}\right]^p \geq \sum_{j=1}^{n} \lambda_j^p \|A_j\|_{(\Phi_j)}^p,$$

which means that

$$\upsilon(\Phi(A)) > \left[\sum_{j=1}^{n} \lambda_j^p \|A_j\|_{(\Phi_j)}^p\right]^{\frac{1}{p}} = \delta_2.$$

(3) By Theorem 3.3 of [8] and classical Hölder inequality, one obtain that

$$\begin{aligned}\upsilon(AB) &= \sum_{j=1}^{n}\lambda_j|\tau(A_jB_j)| \\ &\leq \sum_{j=1}^{n}\lambda_j^{\frac{1}{p}+\frac{1}{q}}\|A_j\|_{(\Phi_j)}\|B_j\|_{\Psi_j} \\ &\leq \left(\sum_{j=1}^{n}\lambda_j\|A_j\|_{(\Phi_j)}^{p}\right)^{\frac{1}{p}}\cdot\left(\sum_{j=1}^{n}\lambda_j\|B_j\|_{\Psi_j}^{q}\right)^{\frac{1}{q}} \\ &= \|A\|_{(\Phi),p,\lambda}\cdot\|B\|_{\Psi,q,\lambda}.\end{aligned}$$ □

Remark 1. If Φ is 1-tuple of N-function and $\lambda = 1$, Lemma 2.1 is exactly Theorem 3.3 and Proposition 3.4 of [8].

Theorem 2.1. *If $A \in \oplus_{j=1}^{n}L_{p,\lambda}^{(\Phi_j)}$, then for $1 \leq p < \infty$, the weighted norm $\|\cdot\|_{(\Phi),p,\lambda}$ is given by*

$$\|A\|_{(\Phi),p,\lambda} = \sup\left\{\upsilon(AB) : B \in \bigoplus_{j=1}^{n}L_{p,\lambda}^{\Psi_j},\ \|B\|_{\Psi,q,\lambda} \leq 1\right\}.$$

Proof. Assume $\|B\|_{\Psi,q,\lambda} \leq 1$. On one side, by (3) of Lemma 2.1, we have

$$\upsilon(AB) \leq \|A\|_{(\Phi),p,\lambda}\cdot\|B\|_{\Psi,q,\lambda} \leq \|A\|_{(\Phi),p,\lambda}.$$

On the other side, we may take, for simplicity, that $A_j \geq 0$, $\|A_j\|_{(\Phi_j)} = 1$, and $\sum_{j=1}^{n}\lambda_j = 1$, then $\|A\|_{(\Phi),p,\lambda} = 1$.

By Proposition 3.4 of [8], for any $\varepsilon > 0$ one obtain that

$$\tau[\Phi_j((1+\varepsilon)A_j)] \geq \|(1+\varepsilon)A_j\|_{(\Phi_j)} = 1+\varepsilon.$$

Let $\{e_{jn}\}_{n=1}^{\infty}$ be orthogonal projections of A_j and $0 < \tau(e_{jn}) < \infty$. If we define the operator $A_{jm} = A_j(e_{j1}+e_{j2}+\cdots+e_{jm})(m \leq n)$, where $A_j = \sum_{k=1}^{n}\alpha_k e_{jk}$ and $e_{jk} = 0$ for $k > n$, then $A_{jm} \uparrow A_j$ as $m \to \infty$, and there exists an m_0 such that for $m \geq m_0$,

$$\upsilon[\Phi((1+\varepsilon)A_m)] = \sum_{j=1}^{n}\lambda_j\tau[\Phi_j((1+\varepsilon)A_{jm})] \geq 1+\frac{\varepsilon}{2}.$$

Recalling that p is the left derivative of Φ, if we set

$$B_{jm} = \frac{p((1+\varepsilon)A_{jm})}{1+\tau(\Psi_j(p((1+\varepsilon)A_{jm})))},$$

then $B_m \in \oplus_{j=1}^{n}E_{q,\lambda}^{\Psi_j}$ and B_{jm} is bounded for each m. Moreover, by Definition 1.7 of [7] and 1.9 of [6] (Young's inequality) we have

$$\begin{aligned}\tau(A_{jm}\cdot B_{jm}) &= \frac{\tau(A_{jm}\cdot p((1+\varepsilon)\lambda_jA_{jm}))}{1+\tau(\Psi_j(p((1+\varepsilon)\lambda_jA_{jm})))} \\ &\leq \frac{\tau(\Phi_j(A_{jm}))+\tau(\Psi_j(p((1+\varepsilon)A_{jm})))}{1+\tau(\Psi_j(p((1+\varepsilon)\lambda_jA_{jm})))} \\ &\leq \frac{1+\tau(\Psi_j(p((1+\varepsilon)A_{jm})))}{1+\tau(\Psi_j(p((1+\varepsilon)\lambda_jA_{jm})))} \\ &= 1,\end{aligned}$$

one can obtain $\|B_{jm}\|_{\Psi_j} \leq 1$, which implies that

$$\|B_m\|_{\Psi,q,\lambda} = \left(\sum_{j=1}^{n}\lambda_j\|B_{jm}\|_{\Psi_j}^{q}\right)^{\frac{1}{q}} \leq 1,$$

since $\sum_{j=1}^{n}\lambda_j = 1$.

However, one has

$$\begin{aligned}\sup\{\upsilon(AB), \|B\|_{\Psi,q,\lambda} \le 1\} &= \sup\left\{\sum_{j=1}^{n}\lambda_j\tau(A_jB_j) : B_j \in \bigoplus_{j=1}^{n} L_{q,\lambda}^{\Psi_j}, \|B\|_{\Psi,q,\lambda} \le 1\right\}\\
&\ge \sup_{m\ge m_0}\left\{\sum_{j=1}^{n}\lambda_j\tau(A_jB_{jm}) : B_{jm} \in \bigoplus_{j=1}^{n} L_{q,\lambda}^{\Psi_j}, \|B_{jm}\|_{\Psi_j,q,\lambda} \le 1\right\}\\
&\ge \frac{1}{1+\varepsilon}\sup_{m\ge m_0}\left\{\sum_{j=1}^{n}\lambda_j\tau((1+\varepsilon)A_{jm}B_{jm})\right\}\\
&= \frac{1}{1+\varepsilon}\sup_{m\ge m_0}\left\{\sum_{j=1}^{n}\lambda_j\frac{\tau(\Phi_j(1+\varepsilon)A_{jm}) + \tau(\Psi_j(p(1+\varepsilon)A_{jm}))}{1+\tau(\Psi_j(p(1+\varepsilon)A_{jm}))}\right\}\\
&> \frac{1}{1+\varepsilon}.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary we obtain the desired inequality. □

Definition 2.2. [12] Let Φ_1 and Φ_2 be a pair of N-functions, and $0 \le s \le 1$ be fixed. Then Φ_s is the uniquely defined inverse of $\Phi_s^{-1}(u) = [\Phi_1^{-1}(u)]^{1-s}[\Phi_2^{-1}(u)]^s$ for $u \ge 0$, where Φ_i^{-1} is the uniquely inverse of the N-function Φ_i, Φ_s is called an intermediate function.

Theorem 2.2. *Let $\Phi_i = (\Phi_{i1}, \Phi_{i2}, \ldots \Phi_{in})$, $Q_i = (Q_{i1}, Q_{i2}, \ldots Q_{in})$, $i = 1, 2$ be n-tuples of N-functions and $0 \le r_1, r_2, t_1, t_2 \le \infty$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ be given positive numbers. Next let $\Phi_s = (\Phi_{s1}, \Phi_{s2}, \ldots \Phi_{sn})$, $Q_s = (Q_{s1}, Q_{s2}, \ldots Q_{sn})$ be the associated intermediate N-functions,*

$$\frac{1}{r_s} = \frac{1-s}{r_1} + \frac{s}{r_2},\ \frac{1}{t_s} = \frac{1-s}{t_1} + \frac{s}{t_2},\ 0 \le s \le 1.$$

If $T : \oplus_{j=1}^{n} L_{r_i,\lambda}^{(\Phi_{ij})} \to \oplus_{j=1}^{n} L_{t_i,\lambda}^{(Q_{ij})}$ is a bounded linear operator with bounds K_1, K_2, such that $\|TA\|_{(Q_i),t_i,\lambda} \le K_i\|A\|_{(\Phi_i),r_i,\lambda}$, $A \in \oplus_{j=1}^{n} L_{r_i,\lambda}^{(\Phi_{ij})}$, $i = 1, 2$, then T is also defined on $\oplus_{j=1}^{n} L_{r_s,\lambda}^{(\Phi_{sj})}$ into $\oplus_{j=1}^{n} L_{t_s,\lambda}^{(Q_{sj})}$ for all $0 \le s \le 1$ and one has the bound

$$\|TA\|_{(Q_s),t_s,\lambda} \le K_1^{1-s}K_2^{s}\|A\|_{(\Phi_s),r_s,\lambda},$$

where $A \in \oplus_{j=1}^{n} L_{r_s,\lambda}^{(\Phi_{sj})}$.

Proof. Let $A = (A_1, A_2, \ldots, A_n) \in \oplus_{j=1}^{n} L_{r_s,\lambda}^{(\Phi_{sj})}$, $B = (B_1, B_2, \ldots, B_n) \in \oplus_{j=1}^{n} L_{t_s,\lambda}^{\Psi_{sj}}$, and polar decompositions $A_k = U_k|A_k|$, $B_k = V_k|B_k|$ where $|A_k| = \sum_{j=1}^{n}\alpha_j e_{kj}$, $|B_k| = \sum_{j=1}^{n}\beta_j e'_{kj}$. For convenience, we assume that $\|A\|_{(\Phi_s),r_s,\lambda} \le 1$, $\|B\|_{\Psi_s,t_s,\lambda} \le 1$.

For $z \in \mathbb{C}$ and $k = 1, 2, \ldots, n$, we define

$$A(z) = (A_1(z), A_2(z), \ldots, A_n(z))$$

and

$$B(z) = (B_1(z), B_2(z), \ldots, B_n(z)),$$

where

$$A_k(z) = U_k\Phi_{sk}[(\Phi_{1k}^{-1})^{1-z}(\Phi_{2k}^{-1})^{z}](|A_k|), \quad B_k(z) = V_k\Psi_{sk}[(\Psi_{1k}^{-1})^{1-z}(\Psi_{2k}^{-1})^{z}](|B_k|).$$

Then,

$$\begin{aligned}A_k(z) &= U_k\Phi_{sk}\left[\left(\Phi_{1k}^{-1}\left(\sum_{j=1}^{n}\alpha_j e_{kj}\right)\right)^{1-z}\left(\Phi_{2k}^{-1}\left(\sum_{j=1}^{n}\alpha_j e_{kj}\right)\right)^{z}\right]\\
&= \sum_{j=1}^{n}\Phi_{sk}[(\Phi_{1k}^{-1}(\alpha_j))^{1-z}(\Phi_{2k}^{-1}(\alpha_j))^{z}]U_k e_{kj}.\end{aligned}$$

Hence, $z \mapsto A(z)$ is an analytic function on $\mathbb{C}$ with value in $\widetilde{\mathcal{M}}$. The same reduction applies to B.

Now we could define a bounded entire function

$$H(z) = K_1^{z-1}K_2^{-z}\upsilon(B(z)TA(z)).$$

If $z = it$ for $t \in \mathbb{R}$, we have

$$\begin{aligned}A_k(it) &= \sum_{j=1}^{n}\Phi_{sk}[(\Phi_{1k}^{-1}(\alpha_j))^{1-it}(\Phi_{2k}^{-1}(\alpha_j))^{it}]U_k e_{kj}\\ &= \sum_{j=1}^{n}\Phi_{sk}\left[\left(\frac{\Phi_{2k}^{-1}(\alpha_j)}{\Phi_{1k}^{-1}(\alpha_j)}\right)^{it}\right]U_k e_{kj}\cdot\sum_{j=1}^{n}\Phi_{sk}[\Phi_{1k}^{-1}(\alpha_j)]U_k e_{kj}\\ &= \left[\Phi_{sk}\left(\frac{\Phi_{2k}^{-1}}{\Phi_{1k}^{-1}}(|A_k|)\right)\right]^{it}\cdot\Phi_{sk}(\Phi_{1k}^{-1}(|A_k|)).\end{aligned}$$

Hence,

$$|A_k(it)|^2 = A_k(it)^*A_k(it) = [\Phi_{sk}(\Phi_{1k}^{-1}(|A_k|))]^2,$$

which means

$$|A_k(it)| = \Phi_{sk}(\Phi_{1k}^{-1}(|A_k|)).$$

Hence, for any $1 \le k \le n$ we have $\tau(\Phi_{1k}(A_k(it))) = \tau(\Phi_{sk}(A_k))$, which implies that

$$\|A_j(it)\|_{(\Phi_{1j})} = \|A_j\|_{(\Phi_{sj})}$$

and

$$\begin{aligned}\upsilon(\Phi_1(A(it))) &= \sum_{j=1}^{n}\lambda_j\tau[\Phi_{1j}[\Phi_{sk}(\Phi_{1j}^{-1}(|A_j|))]]\\ &= \lambda_1\tau(\Phi_{s1}(|A_1|)) + \lambda_2\tau(\Phi_{s2}(|A_2|)) + \ldots + \lambda_n\tau(\Phi_{sn}(|A_n|))\\ &= \upsilon(\Phi_s(|A|)).\end{aligned}$$

We obtain that

$$\|A(it)\|_{(\Phi_1),r_s,\lambda} = \|A\|_{(\Phi_s),r_s,\lambda} \le 1.$$

Similarly $\|B(it)\|_{\Psi_1,t_s,\lambda} = \|B\|_{\Psi_s,t_s,\lambda} \le 1$. Thus by (3) of Lemma 2.1 and the assumption on T, we have

$$|\upsilon(B(it)TA(it))| \le K_1\|B(it)\|_{\Psi_1,t_s,\lambda}\|A(it)\|_{(\Phi_1),r_s,\lambda} \le K_1.$$

It then follows that $|H(it)| \le 1$ for any $t \in \mathbb{R}$. In the same way, we could know that $|H(1+it)| \le 1$.

Therefore, by the maximum principle, for any $\theta \in \mathbb{C}$, we obtain

$$|H(\theta)| = |K_1^{\theta-1}K_2^{-\theta}\upsilon(B(\theta)TA(\theta))| \le 1. \qquad \square$$

Hence,

$$|\upsilon(BTA)| \le K_1^{1-\theta}K_2^{\theta}.$$

By Theorem 2.1 we could obtain that

$$\|TA\|_{(Q_s),r_s,\lambda} \le K_1^{1-\theta}K_2^{\theta}\|A\|_{(\Phi_s),r_s,\lambda}.$$

Theorem 2.3. *Let Φ be an N-function and Φ_s be the inverse which satisfies that $\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s = [\Phi^{-1}(u)]^{1-s}u^{\frac{s}{2}}$ where $0 < s \le 1$ and $\Phi_0(u) = u^2$. If $L^{(\Phi)}(\widetilde{\mathcal{M}},\tau)$ is the noncommutative Orlicz space, then we have for $A, B \in L^{(\Phi_s)}(\widetilde{\mathcal{M}},\tau)$:*

$$\left(\|A+B\|_{(\Phi_s)}^{\frac{2}{s}} + \|A-B\|_{(\Phi_s)}^{\frac{2}{s}}\right)^{\frac{s}{2}} \le 2^{\frac{s}{2}}\left(\|A\|_{(\Phi_s)}^{\frac{2}{2-s}} + \|B\|_{(\Phi_s)}^{\frac{2}{2-s}}\right)^{\frac{2-s}{2}}.$$

Proof. Let $\Phi_1 = (\Phi, \Phi)$ be the 2-vector of N-functions, $\lambda = (1, 1)$, $1 \leq r_1 \leq \infty$ and set

$$\overset{2}{\underset{j=1}{\oplus}} L_{r_1}^{(\Phi)}(\widetilde{\mathcal{M}}, \tau) = \left\{(A, B) : A, B \in L^{(\Phi)}(\widetilde{\mathcal{M}}, \tau), \|(A, B)\|_{(\Phi_1), r_1} < \infty\right\},$$

where

$$\|(A, B)\|_{(\Phi_1), r_1} = \begin{cases} [\|A\|_{(\Phi)}^{r_1} + \|B\|_{(\Phi)}^{r_1}]^{\frac{1}{r_1}}, & 1 \leq r_1 < \infty, \\ \max\{\|A\|_{(\Phi)}, \|B\|_{(\Phi)}\}, & r_1 = \infty. \end{cases} \tag{2}$$

Take $Q_1 = \Phi_1 = (\Phi, \Phi)$ and $Q_2 = \Phi_2 = (\Phi_0, \Phi_0)$ where $\Phi_0(u) = u^2$.

Set $r_1 = 1$, $r_2 = t_2 = 2$, and $t_1 = +\infty$. Define the linear operator $T : \oplus_{j=1}^2 L_{r_i}^{(\Phi_i)} \to \oplus_{j=1}^2 L_{t_i}^{(Q_i)}$ by the equation $T(A, B) = (A + B, A - B)$, we then have

$$\|T(A, B)\|_{(Q_1), t_1} = \max\{\|A + B\|_{(\Phi)}, \|A - B\|_{(\Phi)}\} \leq \|A\|_{(\Phi)} + \|B\|_{(\Phi)} = K_1\|(A, B)\|_{(\Phi_1), r_1}.$$

Hence, $K_1 = 1$ and since $\|\cdot\|_{(\Phi_0)} = \|\cdot\|_2$, we find

$$\|T(A, B)\|_{(Q_2), t_2} = [\|A + B\|_2^2 + \|A - B\|_2^2]^{\frac{1}{2}} = \sqrt{2}[\|A\|_2^2 + \|B\|_2^2]^{\frac{1}{2}} = K_2\|(A, B)\|_{(\Phi_2), r_2}.$$

Thus, $K_2 = \sqrt{2}$. Let r_s and t_s be given by

$$\frac{1}{r_s} = \frac{1 - s}{r_1} + \frac{s}{r_2}, \frac{1}{t_s} = \frac{1 - s}{t_1} + \frac{s}{t_2},$$

then we have $r_s = \frac{2}{2 - s}$, $t_s = \frac{2}{s}$.

By Theorem 2.2,

$$\|T(A, B)\|_{(Q_s), t_s} \leq 2^{\frac{s}{s}}\|(A, B)\|_{(\Phi_s), r_s},$$

since $K_1^{1-s}K_2^s = 2^{\frac{s}{2}}$.

Hence, we have

$$\|(A, B)\|_{(Q_s), r_s} = \left[\|A\|_{(\Phi_s)}^{\frac{2}{2-s}} + \|B\|_{(\Phi_s)}^{\frac{2}{2-s}}\right]^{\frac{2-s}{2}}$$

and

$$\|T(A, B)\|_{(Q_s), t_s} = \left(\|A + B\|_{(\Phi_s)}^{\frac{2}{s}} + \|A - B\|_{(\Phi_s)}^{\frac{2}{s}}\right)^{\frac{s}{2}}.$$ □

The following corollary is the Clarkson inequality of noncommutative L^p space and the proof is similar to the P42 of [11].

Corollary 2.1. *Suppose that* $1 < p < \infty$ *and* $q = \frac{p}{p-1}$. *Then for* $A, B \in L^p(\widetilde{\mathcal{M}}, \tau)$, *we have*

$$(\|A + B\|_p^q + \|A - B\|_p^q)^{\frac{1}{q}} \leq 2^{\frac{1}{q}}(\|A\|_p^p + \|B\|_p^p)^{\frac{1}{p}}, \quad 1 < p \leq 2,$$

and

$$(\|A + B\|_p^p + \|A - B\|_p^p)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}(\|A\|_p^q + \|B\|_p^q)^{\frac{1}{q}}, \quad 2 \leq p \leq \infty.$$

Proof. If $1 < p \leq 2$, let $1 < \alpha < p \leq 2$ and $\Phi(u) = |u|^\alpha$, $\Phi_0(u) = |u|^2$, $s = \frac{2(p-\alpha)}{p(2-\alpha)}$. Then $0 < s \leq 1$ and $\Phi_s^{-1}(u) = |u|^{\frac{1}{p}}$ or $\Phi_s(u) = |u|^p$. Hence, $\|\cdot\|_{(\Phi_s)} = \|\cdot\|_{(p)}$ and since $\lim_{\alpha\downarrow 1}\frac{2}{s} = \frac{p}{p-1} = q$, $\lim_{\alpha\downarrow 1}\frac{2-s}{2} = \frac{1}{p}$ by Theorem 2.3 we obtain the first inequality.

Similarly, let $2 \leq p < \beta < \infty$ and $\Phi(u) = |u|^\beta$, $\Phi_0(u) = |u|^2$, $s = \frac{2(\beta - p)}{p(\beta - 2)}$. Then $0 \leq s \leq 1$ and $\Phi_s(u) = |u|^p$, $\lim_{\beta\uparrow\infty}\frac{2}{s} = p$; $\lim_{\beta\uparrow\infty}\frac{2-s}{2} = \frac{1}{q}$, by the Theorem 2.3 we obtain the second inequality. □

3 Uniform convexity and uniform smoothness

In this section, we present some geometrical properties of noncommutative Orlicz spaces which include uniform convexity and uniform smoothness.

Definition 3.1. [13] Let X be a Banach space. We define its modulus of convexity by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X, \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\right\}, \quad 0 < \varepsilon < 2$$

and its modulus of smoothness by

$$\rho_X(t) = \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in X, \|x\| = \|y\| = 1\right\}, \quad t > 0.$$

X is said to be uniformly convex if $\delta_X(\varepsilon) > 0$ for every $2 \geq \varepsilon > 0$, and uniformly smooth if $\lim_{t\to 0}\frac{\rho_X(t)}{t} = 0$.

Theorem 3.1. *Let Φ be an N-function and Φ_s be the inverse which satisfies that $\Phi_s^{-1}(u) = [\Phi^{-1}(u)]^{1-s}[\Phi_0^{-1}(u)]^s = [\Phi^{-1}(u)]^{1-s}u^{\frac{s}{2}}$ where $0 < s \leq 1$ and $\Phi_0(u) = u^2$, then we have for $0 < \varepsilon \leq 2$,*

$$\delta_{L^{(\Phi_s)}}(\varepsilon) \geq 1 - \frac{1}{2}\left[2^{\frac{2}{s}} - \varepsilon^{\frac{2}{s}}\right]^{\frac{s}{2}}$$

and

$$\rho_{L^{(\Phi_s)}(t)} \leq \left(1 + t^{\frac{2}{2-s}}\right)^{\frac{2-s}{2}} - 1.$$

Proof. First, if $\|A - B\|_{(\Phi_s)} = \varepsilon$, then Theorem 2.3 implies for $A, B \in L^{(\Phi_s)}(\widetilde{\mathcal{M}}, \tau)$,

$$\left(\|A + B\|_{(\Phi_s)}^{\frac{2}{s}} + \varepsilon^{\frac{2}{s}}\right)^{\frac{s}{2}} \leq 2^{\frac{s}{2}} \cdot 2^{\frac{2-s}{2}} = 2.$$

Hence,

$$1 - \frac{1}{2}\|A + B\|_{(\Phi_s)} \geq 1 - \frac{1}{2}\left[2^{\frac{2}{s}} - \varepsilon^{\frac{2}{s}}\right]^{\frac{s}{2}}.$$

Taking infimum of $\|A\|_{(\Phi_s)} = \|B\|_{(\Phi_s)} = 1$ we can obtain the desired result and $L^{(\Phi_s)}(\widetilde{\mathcal{M}}, \tau)$ is uniformly convex if $0 < \varepsilon \leq 2$ and reflexive.

Second, if $\|A\|_{(\Phi_s)} = \|B\|_{(\Phi_s)} = 1$, then since $\frac{2}{s} \geq 2$,

$$\begin{aligned}\left[\frac{1}{2}(\|A + tB\|_{(\Phi_s)} + \|A - tB\|_{(\Phi_s)})\right]^{\frac{2}{s}} &\leq \frac{1}{2}\left[\|A + tB\|_{(\Phi_s)}^{\frac{2}{s}} + \|A - tB\|_{(\Phi_s)}^{\frac{2}{s}}\right] \\ &\leq \frac{1}{2}\left[2^{\frac{s}{2}}\left(\|A\|_{(\Phi_s)}^{\frac{2}{2-s}} + \|tB\|_{(\Phi_s)}^{\frac{2}{2-s}}\right)^{\frac{2-s}{2}}\right]^{\frac{2}{s}} \\ &= \frac{1}{2}\left[2^{\frac{s}{2}}\left(1 + t^{\frac{2}{2-s}}\right)^{\frac{2-s}{2}}\right]^{\frac{2}{s}} \\ &= \left(1 + t^{\frac{2}{2-s}}\right)^{\frac{2-s}{s}}.\end{aligned}$$

Hence,

$$\frac{1}{2}(\|A + tB\|_{(\Phi_s)} + \|A - tB\|_{(\Phi_s)}) - 1 \leq \left(1 + t^{\frac{2}{2-s}}\right)^{\frac{2-s}{2}} - 1.$$

Taking the supremum on the left we can obtain the conclusion. Since $t > 0$, we have that $L^{(\Phi_s)}(\widetilde{\mathcal{M}}, \tau)$ is uniformly smooth. □

From corollary 2.1, we can easily obtain the following result which appeared in [3].

Corollary 3.1. *Suppose that* $1 < p < \infty$, $q = \frac{p}{p-1}$, $0 < \varepsilon$, *and* $t > 0$. *Then for* $A, B \in L^p(\widetilde{\mathcal{M}}, \tau)$, *we have*

(1) *If* $1 < p < 2$, *then*

$$\delta_{L^p}(\varepsilon) \geq \frac{\varepsilon^q}{q \cdot 2^q} \quad \textit{and} \quad \rho_{L^p(t)} \leq \frac{t^p}{p}.$$

(2) *If* $2 < p < \infty$, *then*

$$\delta_{L^p}(\varepsilon) \geq \frac{\varepsilon^p}{p \cdot 2^p} \quad \textit{and} \quad \rho_{L^p(t)} \leq \frac{t^q}{q}.$$

(3) $L^p(\widetilde{\mathcal{M}}, \tau)$ *is uniformly convex and uniformly smooth. Consequently, it is reflexive.*

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