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Research Article

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The transitivity of primary conjugacy in regular ω-semigroups

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Abstract: The conjugacy relation plays an important role in group theory and the conjugacy relation of groups has been generalized to semigroups in various methods by several authors. If a and b are elements of a semigroup S, then a is called *primarily conjugate* to b if $a = uv$ and $b = vu$ for some $u, v \in S^1$. In general, primary conjugacy is reflexive and symmetric, but not transitive. Finding the classes of semigroups in which the primary conjugacy is transitive is an open problem raised by Araújo et al. in the literature. In this article, among other things we prove that the primary conjugacy is transitive in regular ω-semigroups.

Keywords: regular ω-semigroup, primary conjugacy, transitive

MSC 2020: 20M10, 20M18

1 Introduction and preliminaries

The conjugacy relation of groups has many applications in the theory of groups, such as defining normal subgroups and studying the normalizers of elements in groups. It is an interesting topic to extend the concept of conjugation in groups to semigroups. Many authors have generalized the conjugacy relation of groups to some kinds of semigroups by various methods and obtained a series of results (see [1–15]). To state some necessary results on conjugacy relations of semigroups, we need to recall the notions of various kinds of conjugacy relations of semigroups.

Let G be a group. Then the conjugacy relation ~ on G is defined as follows: For all $a, b \in G$, $a \sim b$ if and only if there exists $g \in G$ such that $b = g^{-1}ag$. It is easy to see that

$$a \sim b \iff (\exists g \in G)ag = gb \iff (\exists u, v \in G)a = uv, b = vu.$$

Inspired by this fact and the text [16], Lallement [17] has considered the conjugacy relations $\sim_l$ and $\sim_p$ on semigroups. Let S be a semigroup and $a, b \in S$. Recall that

$$a \sim_l b \iff (\exists g \in S^1)ag = gb$$

and

$$a \sim_p b \iff (\exists u, v \in S^1)a = uv, b = vu.$$

From [1], the relation $\sim_p$ is called the *primary conjugacy* on S. According to Lallement [17], if S is a free semigroup, then $\sim_p = \sim_l$, and in this case, $\sim_p$ is an equivalence. In a general semigroup, $\sim_p$ is reflexive and symmetric, but not transitive. For this reason, the texts [1] and [2] investigated the transitive closure $\sim_p^*$ of $\sim_p$ and obtained some useful results. On the other hand, the relation $\sim_l$ is reflexive and transitive, but not symmetric. In 1984, Otto [3] defined the relation $\sim_o$ on a semigroup S as follows:

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For all $a, b \in S$,

$$a \sim_o b \iff (\exists g, h \in S^1) ag = gb, bh = ha.$$

It is easy to see that $\sim_o$ is an equivalence on S and $\sim_p \subseteq \sim_o$ as $(uv)u = u(vu)$ and $(vu)v = v(uv)$ for all $u, v \in S^1$. Moreover, if e and f are two idempotents in S, we have $e(ef) = ef = (ef)f$ and $f(fe) = fe = (fe)e$, and so $e \sim_o f$. Thus,

$$e \sim_o f \text{ for all idempotents } e \text{ and } f \text{ in } S. \tag{1.1}$$

However, if S has a zero, the $\sim_o$ becomes the universal relation on S. To overcome this defect, Araújo et al. [4] defined the relation $\sim_c$ on a semigroup S as follows: For all $a, b \in S$,

$$a \sim_c b \iff (\exists g \in \mathbb{P}(a)) (\exists h \in \mathbb{P}(b))\ ag = gb,\ bh = ha,$$

where $\mathbb{P}(0) = \{0\}$ and

$$\mathbb{P}(x) = \{g \in S^1 | \forall m \in S^1 (mx \neq 0 \Rightarrow (mx)g \neq 0)\}$$

for all $x \in S$ with $x \neq 0$. In view of the text [4], if S contains no zero, then $\sim_c = \sim_o$, and in general, $\sim_c$ is contained in $\sim_o$ properly. More recently, some new conjugacy relations on semigroups have been introduced and investigated, such as trace conjugacy $\sim_{tr}$ for epigroups [5], i-conjugacy $\sim_i$ for inverse semigroups [6], n-conjugacy $\sim_n$ for general semigroups [8] and so on. Observe that $\sim_i = \sim_n$ in inverse semigroups by [8, Theorem 2.6].

As mentioned above, the primary conjugacy $\sim_p$ is not transitive in general semigroups. In fact, the primary conjugacy is not transitive in many well-known classes of semigroups, such as E-unitary inverse semigroups [5], polycyclic monoids [9], free inverse monoids [10], left or right patience sorting monoids [11], symmetric inverse semigroups [12,18], etc. In view of this fact, the study of $\sim_p$ is divided into two aspects: One is to consider the descriptions of $\sim_p$ and the transitive closure $\sim_p^*$ of $\sim_p$ for some interesting classes of semigroups (in particular, for various classes of (partial) transformation semigroups), the other is to find natural classes of semigroups in which $\sim_p$ is transitive. For instance, Ganyushkin and Kormysheva [12] obtained the characterizations of $\sim_p^*$ for finite symmetric inverse semigroups, Kudryavtseva [7] studied $\sim_p^*$ for the class of regular epigroups and its some special classes, and Kudryavtseva and Mazorchuk [2] gave the descriptions of $\sim_p^*$ for the full transformation semigroup and partial transformation semigroup on a finite set, symmetric inverse semigroup on a countable set and Brauer-type semigroups.

On the other hand, many authors have found some natural classes of semigroups in which $\sim_p$ is transitive. For example, Otto [3] has proved that the primary conjugacy $\sim_p$ is always transitive in monoids defined by special presentations (see also in Zhang [13]), and Kudryavtseva [7] has shown that $\sim_p$ is transitive in completely regular semigroups. As a consequence, we have the following result which will be used several times in the next sections. Recall that *Clifford semigroups* are just semigroups that are both completely regular and inverse. It is well known that each $\mathcal{H}$-class is a subgroup in a Clifford semigroup and a Clifford semigroup is a strong semilattice of its $\mathcal{H}$-classes (see Howie [19]).

Lemma 1.1. *The primary conjugacy $\sim_p$ is transitive in Clifford semigroups.*

Recently, Araújo et al. [5] have introduced the class $\mathcal{W}$ of semigroups which consists of semigroups S such that the subsemigroup $S^2 = \{ab | a, b \in S\}$ is completely regular, and proved that $\sim_p$ is transitive in every epigroup in $\mathcal{W}$. As a consequence, it is shown that $\sim_p$ is transitive in the variants of completely regular semigroups. However, Example 4.19 in [5] indicates that there is an epigroup that has a variant in which $\sim_p$ is not transitive. Based on this fact, the paper [5] has raised the following question (see Problem 6.18 in [5]): Is it true that $\sim_p$ is transitive in the variants of the members in $\mathcal{W}$? In 2020, Borralho and Kinyon [14] provided an affirmative answer to this question (see Corollary 11 in [14]). In the same year, Borralho [15] has also shown that for semigroups satisfying $xy \in \{yx, (xy)^n\}$ for some fixed integer $n > 1$, the primary conjugacy $\sim_p$ is transitive.

From the aforementioned statements, we can see that it is an interesting research topic to find new classes of semigroups in which primary conjugacy is transitive. In fact, the four kinds of conjugacy relations of abstract semigroups including $\sim_p$, $\sim_o$, $\sim_c$, $\sim_{tr}$ have already been studied systematically by Araújo et al. in [5] where the task to find new classes of semigroups in which primary conjugacy is transitive is also proposed (see Problem 6.3 in [5]). Specifically, they have raised the following question.

Problem 1.2. Find other classes of semigroups in which primary conjugacy is transitive. Describe the (E-unitary) inverse semigroups in which primary conjugacy is transitive. Ultimately, classify the class of semigroups in which primary conjugacy is transitive.

The aim of this article is to continue the study of primary conjugacy relations of semigroups around the above problem. It is well known that the class of regular ω-semigroups is an important class of inverse monoids. Recall that a *regular ω-semigroup* is a regular monoid in which the idempotents form a chain that is isomorphic to the chain $\mathbb{N}^0$ (the set of non-negative integers) with respect to the partial order

$$0 > 1 > 2 > 3 > 4 > \cdots .$$

By using the classification and construction of regular ω-semigroups given in Kočin [20], Munn [21] and Reilly [22] (see also in Howie [19], Petrich [23] and McAlister and Medeiros [24]), among other things we shall prove that the primary conjugacy $\sim_p$ is transitive in any regular ω-semigroup.

In the remainder of this section, we shall recall the classification and construction of regular ω-semigroups which are given in Kočin [20], Munn [21] and Reilly [22]. We first recall the notion of the Bruck-Reilly extension of a monoid determined by a morphism θ. Let T be a monoid with identity 1 and let θ be a morphism from T into H_1 the "group of units" of T (i.e. the $\mathcal{H}$-class containing 1). Then we can make $\mathbb{N}^0 \times T \times \mathbb{N}^0$ into a monoid with identity $(0, 1, 0)$ by defining

$$(m, a, n)(p, b, q) = (m - n + t, (a\theta^{t-n})(b\theta^{t-p}), q - p + t)$$

where $t = \max\{n, p\}$ and θ^0 is interpreted as the identity map of T. From Howie [19], this semigroup is denoted by $\mathrm{BR}(T, \theta)$ and called the *Bruck-Reilly extension of T determined by θ*. Obviously, $\mathrm{BR}(T, \theta)$ contains no zero whence $\sim_c = \sim_o$ in $\mathrm{BR}(T, \theta)$. Now we can state the constructions of regular ω-semigroups.

Lemma 1.3. [20,21] *Let S be a regular ω-semigroup. If S has no kernel, then S is a Clifford semigroup. If S has kernel K and $S = K$ (or equivalently, S is simple), then S is a Bruck-Reilly extension* $\mathrm{BR}(T, \theta)$ *of T determined by θ where T is a Clifford monoid whose idempotents form a chain.*

Lemma 1.4. [20,21] *Let $G_0, \ldots, G_{l-1}$ be a set of pairwise disjoint groups for some non-negative integer l and let K be a simple regular ω-semigroup, disjoint from each G_i, with group of units G. Write $G_l = G$. For each i such that $0 \leq i \leq l - 1$, let γ_i be a morphism of G_i into G_{i+1}. For all integers i, j with $0 \leq i < j \leq l$, define $\alpha_{i,j}$ to be $\gamma_i\gamma_{i+1} \cdots \gamma_{j-1}$ and let $\alpha_{i,i}$ be the identity automorphism of G_i for all integers i with $0 \leq i \leq l - 1$. Let*

$$S = G_0 \bigcup G_1 \bigcup \cdots \bigcup G_{l-1} \bigcup K.$$

Define a multiplication "$\circ$" on S, extending that of K and of each G_i, as follows:

$$a_i \circ b_j = (a_i\alpha_{i,t})(b_j\alpha_{j,t}), \quad a_i \circ x = (a_i\alpha_{i,l})x, \quad x \circ a_i = x(a_i\alpha_{i,l}), \quad x \circ y = xy,$$

where $a_i \in G_i$, $b_j \in G_j$, $t = \max\{i, j\}$, and $x, y \in K$, $i, j = 0, 1, \ldots, l - 1$. Then S is a regular ω-semigroup with kernel $K \neq S$, and the identity of G_0 is the identity of the whole semigroup S. Conversely, every such semigroup can be constructed in this way.

2 Primary conjugacy in Bruck-Reilly extensions

In this section, we shall explore some technical results of the primary conjugacy in Bruck-Reilly extensions. We begin our discussions by giving the lemma below.

Lemma 2.1. *Let T be a monoid and $S = \mathrm{BR}(T, \theta)$ the Bruck-Reilly extension of T determined by θ. Then $(m, a, m) \sim_p (n, b, n)$ in S if and only if $a \sim_p b$ in T for all $(m, a, m), (n, b, n) \in S$.*

Proof. Suppose that $(m, a, m), (n, b, n) \in S$ and $(m, a, m) \sim_p (n, b, n)$. Then there are $(u, c, v), (s, d, t) \in S$ such that

$$(m, a, m) = (u, c, v)(s, d, t) = (u - v + p, (c\theta^{p-v})(d\theta^{p-s}), t - s + p),$$
$$(n, b, n) = (s, d, t)(u, c, v) = (s - t + q, (d\theta^{q-t})(c\theta^{q-u}), v - u + q),$$

where $p = \max\{v, s\}$ and $q = \max\{t, u\}$. This implies that

$$u - v = t - s, \quad a = (c\theta^{p-v})(d\theta^{p-s}), \quad b = (d\theta^{q-t})(c\theta^{q-u}),$$

and so $v - s = u - t$. If $v - s \geq 0$, then $p = v$ and $q = u$. Otherwise, we have $p = s$ and $q = t$. In either case, we have $p - v = q - u$ and $p - s = q - t$, and hence $a \sim_p b$ in T.

Conversely, let $a \sim_p b$ in T. Then $a = uv$ and $b = vu$ with $u, v \in T$. This implies that

$$(m, a, n) = (m, u, n)(n, v, m) \text{ and } (n, b, n) = (n, v, m)(m, u, n),$$

which gives that $(m, a, m) \sim_p (n, b, n)$ in S. □

Lemma 2.2. *Let T be a monoid, $S = \mathrm{BR}(T, \theta)$ the Bruck-Reilly extension of T determined by θ and (m_1, a, n_1), $(m_2, b, n_2) \in S$ such that $(m_1, a, n_1) \sim_p (m_2, b, n_2)$. Then $m_1 - n_1 = m_2 - n_2$, say p, and at least one of the following conditions is satisfied:*

(i) *There exist $q \in \mathbb{Z}$, $c, d \in T$ such that $p \leq q \leq 0$, $a = c(d\theta^{q-p})$, and $b = d(c\theta^{-q})$.*

(ii) *There exist $q \in \mathbb{Z}$, $c, d \in T$ such that*

$$0 \leq q \leq \min\{m_1, n_1\}, \quad q \geq m_1 - m_2, \quad a = c\left(d\theta^{n_1-q}\right), \quad b = \left(d\theta^{m_1-q}\right)c.$$

(iii) *There exist $q \in \mathbb{Z}$, $c, d \in T$ such that*

$$0 \leq q \leq \min\{m_1, n_1\}, \quad q \geq m_1 - m_2, \quad a = \left(c\theta^{m_1-q}\right)d, \quad b = d\left(c\theta^{n_1-q}\right).$$

(iv) *There exist $q \in \mathbb{Z}$, $c, d \in T$ such that $0 \leq q \leq p$, $a = (c\theta^{p-q})d$ and $b = (d\theta^{q})c$.*

Proof. Suppose that $(m_1, a, n_1) \sim_p (m_2, b, n_2)$. Then there are $(m_3, c, n_3), (m_4, d, n_4) \in S$ such that

$$(m_1, a, n_1) = (m_3, c, n_3)(m_4, d, n_4) = \left(m_3 - n_3 + t, \left(c\theta^{t-n_3}\right)\left(d\theta^{t-m_4}\right), n_4 - m_4 + t\right),$$
$$(m_2, b, n_2) = (m_4, d, n_4)(m_3, c, n_3) = \left(m_4 - n_4 + s, \left(d\theta^{s-n_4}\right)\left(c\theta^{s-m_3}\right), n_3 - m_3 + s\right),$$

where $t = \max\{n_3, m_4\}$, $s = \max\{n_4, m_3\}$. This implies that

$$\begin{aligned} m_1 &= m_3 - n_3 + t, \quad n_1 = n_4 - m_4 + t, \quad a = \left(c\theta^{t-n_3}\right)\left(d\theta^{t-m_4}\right), \\ m_2 &= m_4 - n_4 + s, \quad n_2 = n_3 - m_3 + s, \quad b = \left(d\theta^{s-n_4}\right)\left(c\theta^{s-m_3}\right), \end{aligned} \tag{2.1}$$

and so $m_1 - n_1 = m_3 - n_3 + m_4 - n_4 = m_2 - n_2$. Denote

$$m_1 - n_1 = m_3 - n_3 + m_4 - n_4 = m_2 - n_2 = p. \tag{2.2}$$

If $m_4 \leq n_3$ and $m_3 \leq n_4$, then $t = n_3$ and $s = n_4$, and so $m_1 = m_3$, $m_2 = m_4$ and $n_2 = n_3 - m_3 + n_4$ by (2.1). Let $q = n_3 - n_2$. Then

$$0 \geq m_3 - n_4 = q = n_3 - n_2 \geq m_4 - n_2 = m_2 - n_2 = p.$$

Observe that $t - m_4 = n_3 - m_2 = n_3 - n_2 - p = q - p$ by (2.2) and $s - m_3 = n_4 - m_3 = -q$, it follows that $a = c(d\theta^{q-p})$ and $b = d(c\theta^{-q})$ by (2.1) again. In this case, (i) holds.

If $m_4 \leq n_3$ and $m_3 \geq n_4$, then $t = n_3$ and $s = m_3$, and so $m_1 = m_3$, $n_2 = n_3$ and $n_1 = n_4 - m_4 + n_3$ by (2.1).

This implies that

$$n_4 = n_1 - n_3 + m_4 = n_1 - n_2 + m_4 = m_1 - m_2 + m_4 \tag{2.3}$$

by (2.2). Let $q = n_4$. Then by the facts that $0 \le m_4 \le n_3$ and $m_3 \ge n_4 \ge 0$,

$$0 \le q = n_4 \le m_3 = m_1,\ q = n_1 + m_4 - n_3 \le n_1,\ q = m_1 - m_2 + m_4 \ge m_1 - m_2.$$

Since $t - m_4 = n_3 - m_4 = n_1 - n_4 = n_1 - q$ by (2.3) and $s - n_4 = m_3 - n_4 = m_1 - q$, we have $a = c(d\theta^{n_1-q})$ and $b = (d\theta^{m_1-q})c$ by (2.1) again. In this case, (ii) holds.

If $m_4 \ge n_3$ and $m_3 \le n_4$, then $t = m_4$ and $s = n_4$, and so $m_2 = m_4$, $n_1 = n_4$ and $m_1 = m_3 - n_3 + m_4$ by (2.1). This yields that

$$m_3 = m_1 - m_4 + n_3 = m_1 - m_2 + n_3. \tag{2.4}$$

Let $q = m_3$. Then by the facts $m_4 \ge n_3 \ge 0$ and $0 \le m_3 \le n_4$,

$$q = m_3 \le n_4 = n_1,\ q = m_1 + n_3 - m_4 \le m_1,\ q = m_1 - m_2 + n_3 \ge m_1 - m_2.$$

As $t - n_3 = m_4 - n_3 = m_1 - m_3 = m_1 - q$ by (2.4) and $s - m_3 = n_4 - m_3 = n_1 - q$, we obtain $a = (c\theta^{m_1-q})d$ and $b = d(c\theta^{n_1-q})$ by (2.1) again. In this case, (iii) is true.

If $m_4 \ge n_3$ and $m_3 \ge n_4$, then $t = m_4$ and $s = m_3$, and so $n_2 = n_3$, $n_1 = n_4$ and $m_1 = m_3 - n_3 + m_4$ by (2.1). This implies that

$$m_3 - n_1 = m_1 + n_3 - m_4 - n_1 = m_1 - n_1 + n_3 - m_4 = n_3 - m_4 + p$$

by (2.2). Let $q = m_3 - n_1$. Then $0 \le m_3 - n_4 = q = m_3 - n_1 = n_3 - m_4 + p \le p$ by the facts $m_4 \ge n_3$ and $m_3 \ge n_4 = n_1$. Observe that

$$t - n_3 = m_4 - n_3 = m_1 - m_3 = n_1 + p - m_3 = p - q$$

by the fact that $m_1 = m_3 - n_3 + m_4$ and (2.2), and $s - n_4 = m_3 - n_4 = m_3 - n_1 = q$, it follows that $a = (c\theta^{p-q})d$ and $b = (d\theta^q)c$ by (2.1) again. In this case, (iv) is valid. □

Lemma 2.3. *Let T be a Clifford monoid with the identity 1 in which the idempotents form a chain, and $S = \mathrm{BR}(T, \theta)$ be the Bruck-Reilly extension of T determined by θ. Then $\sim_p$ on S is transitive.*

Proof. Let $(m_1, a, n_1), (m_2, b, n_2)$, and (m_3, c, n_3) be in S such that $(m_1, a, n_1) \sim_p (m_2, b, n_2)$ and $(m_2, b, n_2) \sim_p (m_3, c, n_3)$. Then by Lemma 2.2, we can denote

$$m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p, \tag{2.5}$$

and have 16 cases including

(i, i), (i, ii), (i, iii), (i, iv), (ii, i), (ii, ii), (ii, iii), (ii, iv),
(iii, i), (iii, ii), (iii, iii), (iii, iv), (iv, i), (iv, ii), (iv, iii), (iv, iv)

to consider. In the following statements, we shall check all the cases one by one. We first observe that

$$(x\theta^k)^{-1}(x\theta^k) = (x^{-1}\theta^k)(x\theta^k) = (x^{-1}x)\theta^k = 1 = (xx^{-1})\theta^k = (x\theta^k)(x^{-1}\theta^k) = (x\theta^k)(x\theta^k)^{-1}$$

for any positive integer k and $x \in T$.

Case 1–(i, i). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$p \le q_1 \le 0, \quad a = u\left(v\theta^{q_1-p}\right), \quad b = v\left(u\theta^{-q_1}\right), \quad p \le q_2 \le 0, \quad b = s\left(t\theta^{q_2-p}\right), \quad c = t\left(s\theta^{-q_2}\right).$$

In this case, by (2.5) we have

$$n_3 + q_2 \ge n_3 + p = m_3, \quad m_1 - q_2 \ge m_1, \quad n_3 + q_1 \ge n_3 + p = m_3, \quad m_1 - q_1 \ge m_1. \tag{2.6}$$

(1.1) $q_1 \ne p, q_1 \ne 0, q_2 \ne p, q_2 \ne 0$. In this case, we have $2p < q_1 + q_2 < 0$, and so $p < q_1 + q_2 - p < -p$. If $p < q_1 + q_2 - p < 0$, then

$$0 < p - q_1 - q_2 < -p, \quad a = u\left(s\theta^{q_1-p}\right)\left(c\theta^{q_1+q_2-p-p}\right)\left[\left(s^{-1}\theta^{q_1-p}\right)u^{-1}\right]\theta^{-p},$$

and $m_1 + p - q_1 - q_2 - m_1 = p - q_1 - q_2 > 0$ and

$$(n_3 + q_1 + q_2 - p) - m_3 = n_3 - m_3 + q_1 + q_2 - p = q_1 + q_2 - p - p > p - p = 0$$

by (2.5). This implies that

$$(m_1, a, n_1) = \left(m_1, u\left(s\theta^{q_1-p}\right), n_3 + q_1 + q_2 - p\right)\left(m_3, c\left[\left(s^{-1}\theta^{q_1-p}\right)u^{-1}\right]\theta^{p-q_1-q_2}, m_1 + p - q_1 - q_2\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(s^{-1}\theta^{q_1-p}\right)u^{-1}\right]\theta^{p-q_1-q_2}, m_1 + p - q_1 - q_2\right)\left(m_1, u\left(s\theta^{q_1-p}\right), n_3 + q_1 + q_2 - p\right),$$

which gives that $(m_1, a, n_1) \sim_p (m_3, c, n_3)$.

If $0 < q_1 + q_2 - p < -p$, then we have

$$c = t\left(v\theta^{-q_2}\right)\left(a\theta^{-(q_1+q_2)}\right)\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{-p},$$

and $m_1 - q_1 - q_2 - m_1 = -q_1 - q_2 > 0$ and

$$(n_3 + q_1 + q_2) - m_3 = q_1 + q_2 + n_3 - m_3 = q_1 + q_2 - p > 0,$$

by (2.5). This implies that

$$(m_1, a, n_1) = \left(m_1, a\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{q_1+q_2-p}, n_3 + q_1 + q_2\right)\left(m_3, t\left(v\theta^{-q_2}\right), m_1 - q_1 - q_2\right),$$
$$(m_3, c, n_3) = \left(m_3, t\left(v\theta^{-q_2}\right), m_1 - q_1 - q_2\right)\left(m_1, a\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{q_1+q_2-p}, n_3 + q_1 + q_2\right).$$

From now on, for each $x \in T$, we shall denote the idempotent in the $\mathcal{H}$-class containing x by 1_x. Thus for all $x, y \in T$, it follows that

$$1_x x = x = x1_x,\ 1_x = xx^{-1} = x^{-1}x \ \text{ and } \ 1_x y = y1_x.$$

If $q_1 + q_2 - p = 0$, then we have

$$a = u\left(s\theta^{q_1-p}\right)(c\theta^{-p})\left[\left(s^{-1}\theta^{q_1-p}\right)u^{-1}\right]\theta^{-p},\ c = t\left(v\theta^{-q_2}\right)(a\theta^{-p})\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{-p}$$

and so $1_u a = a$ and $1_t c = c$. This implies that

$$a\left(v^{-1}\theta^{-q_2}\right)1_u = a\left(v^{-1}\theta^{-q_2}\right), \quad c\left(s^{-1}\theta^{q_1-p}\right)1_t = c\left(s^{-1}\theta^{q_1-p}\right).$$

Since the idempotents of T form a chain, we have $1_u 1_t = 1_t$ or $1_u 1_t = 1_u$. If $1_u 1_t = 1_u$, then

$$a\left(v^{-1}\theta^{-q_2}\right)t^{-1}t\left(v\theta^{-q_2}\right) = a\left(v^{-1}\theta^{-q_2}\right)1_u 1_t\left(v\theta^{-q_2}\right) = a\left(v^{-1}\theta^{-q_2}\right)1_u\left(v\theta^{-q_2}\right) = a\left(v^{-1}\theta^{-q_2}\right)\left(v\theta^{-q_2}\right) = a.$$

This implies that

$$(m_1, a, n_1) = \left(m_1, a\left(v^{-1}\theta^{-q_2}\right)t^{-1}, m_3\right)\left(m_3, t\left(v\theta^{-q_2}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, t\left(v\theta^{-q_2}\right), n_1\right)\left(m_1, a\left(v^{-1}\theta^{-q_2}\right)t^{-1}, m_3\right).$$

If $1_u 1_t = 1_t$, then

$$c\left(s^{-1}\theta^{q_1-p}\right)u^{-1}u\left(s\theta^{q_1-p}\right) = c\left(s^{-1}\theta^{q_1-p}\right)1_t 1_u\left(s\theta^{q_1-p}\right) = c\left(s^{-1}\theta^{q_1-p}\right)1_t\left(s\theta^{q_1-p}\right) = c\left(s^{-1}\theta^{q_1-p}\right)\left(s\theta^{q_1-p}\right) = c.$$

This implies that

$$(m_1, a, n_1) = \left(m_1, u\left(s\theta^{q_1-p}\right), n_3\right)\left(m_3, c\left(s^{-1}\theta^{q_1-p}\right)u^{-1}, m_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left(s^{-1}\theta^{q_1-p}\right)u^{-1}, m_1\right)\left(m_1, u\left(s\theta^{q_1-p}\right), n_3\right).$$

(1.2) $q_1 = p, q_1 \neq 0, q_2 \neq p, q_2 \neq 0$. In this case, we have

$$a = uv, \quad b = v(u\theta^{-p}), \quad b = s\left(t\theta^{q_2-p}\right), \quad c = t\left(s\theta^{-q_2}\right),$$

and so $a = us(c\theta^{q_2-p})(s^{-1}u^{-1})\theta^{-p}$. This implies by (2.6) that

$$(m_1, a, n_1) = (m_1, us, n_3 + q_2)\big(m_3, c(s^{-1}u^{-1})\theta^{-q_2}, m_1 - q_2\big),$$
$$(m_3, c, n_3) = \big(m_3, c(s^{-1}u^{-1})\theta^{-q_2}, m_1 - q_2\big)(m_1, us, n_3 + q_2).$$

(1.3) $q_1 \neq p, q_1 = 0, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = u(v\theta^{-p}), \quad b = vu, \quad b = s\big(t\theta^{q_2-p}\big), \quad c = t\big(s\theta^{-q_2}\big),$$

whence $c = t(v\theta^{-q_2})(a\theta^{-q_2})[(v^{-1}\theta^{-q_2})t^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \Big(m_1, a\Big[\big(v^{-1}\theta^{-q_2}\big)t^{-1}\Big]\theta^{q_2-p}, n_3 + q_2\Big)\Big(m_3, t\big(v\theta^{-q_2}\big), m_1 - q_2\Big),$$
$$(m_3, c, n_3) = \Big(m_3, t\big(v\theta^{-q_2}\big), m_1 - q_2\Big)\Big(m_1, a\Big[\big(v^{-1}\theta^{-q_2}\big)t^{-1}\Big]\theta^{q_2-p}, n_3 + q_2\Big).$$

(1.4) $q_1 \neq p, q_1 \neq 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$a = u\big(v\theta^{q_1-p}\big), \quad b = v\big(u\theta^{-q_1}\big), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = u(s\theta^{q_1-p})(c\theta^{q_1-p})[(s^{-1}\theta^{q_1-p})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \Big(m_1, u\big(s\theta^{q_1-p}\big), n_3 + q_1\Big)\Big(m_3, c\Big[\big(s^{-1}\theta^{q_1-p}\big)u^{-1}\Big]\theta^{-q_1}, m_1 - q_1\Big),$$
$$(m_3, c, n_3) = \Big(m_3, c\Big[\big(s^{-1}\theta^{q_1-p}\big)u^{-1}\Big]\theta^{-q_1}, m_1 - q_1\Big)\Big(m_1, u\big(s\theta^{q_1-p}\big), n_3 + q_1\Big).$$

(1.5) $q_1 \neq p, q_1 \neq 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$a = u\big(v\theta^{q_1-p}\big), \quad b = v\big(u\theta^{-q_1}\big), \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = (tv)(a\theta^{-q_1})(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, a(v^{-1}t^{-1})\theta^{q_1-p}, n_3 + q_1\big)(m_3, tv, m_1 - q_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1 - q_1)\big(m_1, a(v^{-1}t^{-1})\theta^{q_1-p}, n_3 + q_1\big).$$

(1.6) $q_1 = p, q_1 = 0$. In this case, we obtain

$$q_2 = p = 0, \quad m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$. It is easy to see that $q_1 = p, q_1 = 0$ if and only if $q_2 = p, q_2 = 0$. Thus, all the cases that there exist at least three items are equal to zero in $q_1 - p, q_1, q_2 - p, q_2$ reduce to this case.

(1.7) $q_1 = p, q_1 \neq 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = (us)c(s^{-1}u^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, us, m_3)(m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1),$$
$$(m_3, c, n_3) = (m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1)(m_1, us, m_3).$$

(1.8) $q_1 = p, q_1 \neq 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $1_t c = 1_t ts = ts = c$, $u^{-1}a = u^{-1}uv = 1_u v$, $t^{-1}c = t^{-1}tc = 1_t s$ and

$$1_u b = 1_u v(u\theta^{-p}) = u^{-1}uv(u\theta^{-p}) = u^{-1}a(u\theta^{-p}), 1_t b = 1_t s(t\theta^{-p}) = t^{-1}c(t\theta^{-p}).$$

Since the idempotents of T form a chain, we have $1_u 1_t = 1_t$ or $1_u 1_t = 1_u$. If $1_u 1_t = 1_t$, then

$$1_t u^{-1}a(u\theta^{-p}) = 1_t 1_u b = 1_u 1_t b = 1_u t^{-1}c(t\theta^{-p}),$$
$$1_t 1_t u^{-1}a(u\theta^{-p}) = 1_t 1_u t^{-1}c(t\theta^{-p}) = 1_t t^{-1}c(t\theta^{-p}).$$

Multiplying by t from the left on both sides of the second equation above, we have

$$(tu^{-1})a(u\theta^{-p}) = t1_t t^{-1}c(t\theta^{-p}) = tt^{-1}c(t\theta^{-p}) = 1_t c(t\theta^{-p}) = c(t\theta^{-p}),$$

and hence $c = tu^{-1}a(u\theta^{-p})(t^{-1}\theta^{-p}) = tu^{-1}a(ut^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, a(ut^{-1})\theta^{-p}, n_3)(m_3, tu^{-1}, m_1),$$
$$(m_3, c, n_3) = (m_3, tu^{-1}, m_1)(m_1, a(ut^{-1})\theta^{-p}, n_3).$$

Similarly, we can prove the case that $1_u 1_t = 1_u$.
(1.9) $q_1 \neq p, q_1 = 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $1_u a = a$, $t^{-1}c = 1_t(s\theta^{-p})$, $u^{-1}a = 1_u(v\theta^{-p})$ and

$$1_u(b\theta^{-p}) = 1_u(v\theta^{-p})(u\theta^{-p}) = u^{-1}a(u\theta^{-p}), \quad 1_t(b\theta^{-p}) = 1_t(s\theta^{-p})(t\theta^{-p}) = t^{-1}c(t\theta^{-p}),$$
$$1_u t^{-1}c(t\theta^{-p}) = 1_u 1_t(b\theta^{-p}) = 1_t 1_u(b\theta^{-p}) = 1_t u^{-1}a(u\theta^{-p}).$$

Since the idempotents of T form a chain, we have $1_u 1_t = 1_t$ or $1_u 1_t = 1_u$. If $1_u 1_t = 1_u$, then

$$1_u 1_u t^{-1}c(t\theta^{-p}) = 1_u 1_t u^{-1}a(u\theta^{-p}) = 1_u u^{-1}a(u\theta^{-p}) = u^{-1}a(u\theta^{-p}),$$

Multiplying by u from the left on both sides of the equation above, we have

$$(ut^{-1})c(t\theta^{-p}) = uu^{-1}a(u\theta^{-p}) = 1_u a(u\theta^{-p}) = a(u\theta^{-p}),$$

and hence $a = (ut^{-1})c(t\theta^{-p})(u^{-1}\theta^{-p}) = (ut^{-1})c(tu^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, ut^{-1}, m_3)(m_3, c(tu^{-1})\theta^{-p}, n_1),$$
$$(m_3, c, n_3) = (m_3, c(tu^{-1})\theta^{-p}, n_1)(m_1, ut^{-1}, m_3).$$

Similarly, we can prove the case that $1_u 1_t = 1_t$.
(1.10) $q_1 \neq p, q_1 = 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = tva(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3)(m_3, tv, m_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1)(m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3).$$

Case 2–(i, ii). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$m_2 - n_2 = p \leq q_1 \leq 0, a = u\left(v\theta^{q_1-p}\right), b = v\left(u\theta^{-q_1}\right),$$

$$0 \leq q_2 \leq m_2 \leq n_2, q_2 \geq m_2 - m_3, b = s\left(t\theta^{n_2-q_2}\right), c = \left(t\theta^{m_2-q_2}\right)s.$$

In this case,

$$n_1 - m_1 = n_3 - m_3 = -p \geq 0, n_3 + q_1 \geq n_3 + p = m_3, m_1 - q_1 \geq m_1.$$

(2.1) $q_1 \neq p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = u\left(t^{-1}\theta^{n_2+q_1-q_2}\right)\left(c\theta^{q_1-p}\right)\left[\left(t\theta^{n_2+q_1-q_2}\right)u^{-1}\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \left(m_1, u\left(t^{-1}\theta^{n_2+q_1-q_2}\right), n_3 + q_1\right)\left(m_3, c\left[\left(t\theta^{n_2+q_1-q_2}\right)u^{-1}\right]\theta^{-q_1}, m_1 - q_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(t\theta^{n_2+q_1-q_2}\right)u^{-1}\right]\theta^{-q_1}, m_1 - q_1\right)\left(m_1, u\left(t^{-1}\theta^{n_2+q_1-q_2}\right), n_3 + q_1\right).$$

(2.2) $q_1 = p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s,$$

whence $a = u(t^{-1}\theta^{m_2-q_2})c[(t\theta^{m_2-q_2})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, u\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right)\left(m_1, u\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right).$$

(2.3) $q_1 \neq p, q_1 = 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s,$$

whence $c = (t\theta^{m_2-q_2})va[v^{-1}(t^{-1}\theta^{m_2-q_2})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, a\left[v^{-1}\left(t^{-1}\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right)\left(m_3, \left(t\theta^{m_2-q_2}\right)v, m_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left(t\theta^{m_2-q_2}\right)v, m_1\right)\left(m_1, a\left[v^{-1}\left(t^{-1}\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right).$$

(2.4) $q_2 = n_2$. In this case,

$$q_2 = n_2 = m_2,\ q_1 = p = 0,\ m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0,\ a = uv,\ b = vu,\ b = st,\ c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(2.5) $q_1 \neq p, q_1 \neq 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$a = u\left(v\theta^{q_1-p}\right), \quad b = v\left(u\theta^{-q_1}\right), \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = tv(a\theta^{-q_1})(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, a(v^{-1}t^{-1})\theta^{q_1-p}, n_3 + q_1\right)(m_3, tv, m_1 - q_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1 - q_1)\left(m_1, a(v^{-1}t^{-1})\theta^{q_1-p}, n_3 + q_1\right).$$

(2.6) $q_1 = p, q_1 = 0$. In this case, we obtain

$$m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0, \quad a = uv, \quad b = vu$$

and $b = s(t\theta^{m_2-q_2})$, $c = (t\theta^{m_2-q_2})s$. By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(2.7) $q_1 = p, q_1 \neq 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$a = uv, \quad b = v(u\theta^{-p}), \quad b = s(t\theta^{-p}), \quad c = ts.$$

In view of (1.8), we obtain $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(2.8) $q_1 \neq p, q_1 = 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = tva(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3)(m_3, tv, m_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1)(m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3).$$

Case 3–(i, iii). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$m_2 - n_2 = p \leq q_1 \leq 0, \quad a = u\left(v\theta^{q_1-p}\right), \quad b = v\left(u\theta^{-q_1}\right),$$
$$0 \leq q_2 \leq m_2 \leq n_2, \quad q_2 \geq m_2 - m_3, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right).$$

In this case,

$$n_1 - m_1 = n_3 - m_3 = -p \geq 0, n_3 + q_1 \geq n_3 + p = m_3, m_1 - q_1 \geq m_1.$$

(3.1) $q_1 \neq p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = u\left(s\theta^{n_2+q_1-q_2}\right)\left(c\theta^{q_1-p}\right)\left[\left(s^{-1}\theta^{n_2+q_1-q_2}\right)u^{-1}\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \big(m_1, u\big(s\theta^{n_2+q_1-q_2}\big), n_3 + q_1\big)\big(m_3, c\big[\big(s^{-1}\theta^{n_2+q_1-q_2}\big)u^{-1}\big]\theta^{-q_1}, m_1 - q_1\big),$$
$$(m_3, c, n_3) = \big(m_3, c\big[\big(s^{-1}\theta^{n_2+q_1-q_2}\big)u^{-1}\big]\theta^{-q_1}, m_1 - q_1\big)\big(m_1, u\big(s\theta^{n_2+q_1-q_2}\big), n_3 + q_1\big).$$

(3.2) $q_1 = p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = \big(s\theta^{m_2-q_2}\big)t, \quad c = t\big(s\theta^{n_2-q_2}\big),$$

whence $a = u(s\theta^{m_2-q_2})c[(s^{-1}\theta^{m_2-q_2})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, u\big(s\theta^{m_2-q_2}\big), m_3\big)\big(m_3, c\big[\big(s^{-1}\theta^{m_2-q_2}\big)u^{-1}\big]\theta^{-p}, n_1\big),$$
$$(m_3, c, n_3) = \big(m_3, c\big[\big(s^{-1}\theta^{m_2-q_2}\big)u^{-1}\big]\theta^{-p}, n_1\big)\big(m_1, u\big(s\theta^{m_2-q_2}\big), m_3\big).$$

(3.3) $q_1 \neq p, q_1 = 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = \big(s\theta^{m_2-q_2}\big)t, \quad c = t\big(s\theta^{n_2-q_2}\big),$$

whence $c = (s^{-1}\theta^{m_2-q_2})va[v^{-1}(s\theta^{m_2-q_2})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, a\big[v^{-1}\big(s\theta^{m_2-q_2}\big)\big]\theta^{-p}, n_3\big)\big(m_3, \big(s^{-1}\theta^{m_2-q_2}\big)v, m_1\big),$$
$$(m_3, c, n_3) = \big(m_3, \big(s^{-1}\theta^{m_2-q_2}\big)v, m_1\big)\big(m_1, a\big[v^{-1}\big(s\theta^{m_2-q_2}\big)\big]\theta^{-p}, n_3\big).$$

(3.4) $q_2 = n_2$. In this case, we obtain

$$q_2 = n_2 = m_2, \quad m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(3.5) $q_1 \neq p, q_1 \neq 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$a = u\big(v\theta^{q_1-p}\big), \quad b = v\big(u\theta^{-q_1}\big), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = u(s\theta^{q_1-p})(c\theta^{q_1-p})[(s^{-1}\theta^{q_1-p})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, u\big(s\theta^{q_1-p}\big), n_3 + q_1\big)\big(m_3, c\big[\big(s^{-1}\theta^{q_1-p}\big)u^{-1}\big]\theta^{-q_1}, m_1 - q_1\big),$$
$$(m_3, c, n_3) = \big(m_3, c\big[\big(s^{-1}\theta^{q_1-p}\big)u^{-1}\big]\theta^{-q_1}, m_1 - q_1\big)\big(m_1, u\big(s\theta^{q_1-p}\big), n_3 + q_1\big).$$

(3.6) $q_1 = p, q_1 = 0$. In this case, we obtain

$$m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0, \quad a = uv, \quad b = vu, \quad b = \big(s\theta^{m_2-q_2}\big)t, \quad c = t\big(s\theta^{m_2-q_2}\big).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(3.7) $q_1 = p, q_1 \neq 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = usc(s^{-1}u^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, us, m_3)(m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1),$$
$$(m_3, c, n_3) = (m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1)(m_1, us, m_3).$$

(3.8) $q_1 \neq p, q_1 = 0, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$a = u(v\theta^{-p}), \quad b = vu, \quad b = st, \quad c = t(s\theta^{-p}).$$

It follows that $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ from (1.9).

Case 4–(i, iv). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$p \leq q_1 \leq 0, \quad a = u\big(v\theta^{q_1-p}\big), \quad b = v\big(u\theta^{-q_1}\big), \quad 0 \leq q_2 \leq p, \quad b = \big(s\theta^{p-q_2}\big)t, \quad c = \big(t\theta^{q_2}\big)s.$$

In this case,

$$m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p = 0, q_1 = q_2 = 0, a = uv, b = vu, b = st, c = ts,$$

and so $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by Lemmas 1.1 and 2.1.

Case 5–(ii, i). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \le q_1 \le m_1 \le n_1, \quad q_1 \ge m_1 - m_2, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u,$$

and $p \le q_2 \le 0$, $b = s(t\theta^{q_2-p})$, $c = t(s\theta^{-q_2})$. In this case,

$$n_2 - m_2 = n_1 - m_1 = n_3 - m_3 = -p \ge 0, \quad n_3 + q_2 \ge n_3 + p = m_3, \quad m_1 - q_2 \ge m_1.$$

(5.1) $q_1 \ne n_1, q_1 \ne m_1, q_2 \ne p, q_2 \ne 0$. In this case, we obtain

$$c = t\left(v\theta^{m_1-q_1-q_2}\right)\left(a\theta^{-q_2}\right)\left[\left(v^{-1}\theta^{m_1-q_1-q_2}\right)t^{-1}\right]\theta^{-p}.$$

This yields that

$$\begin{aligned}(m_1, a, n_1) &= \left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1-q_2}\right)t^{-1}\right]\theta^{q_2-p}, n_3 + q_2\right)\left(m_3, t\left(v\theta^{m_1-q_1-q_2}\right), m_1 - q_2\right),\\ (m_3, c, n_3) &= \left(m_3, t\left(v\theta^{m_1-q_1-q_2}\right), m_1 - q_2\right)\left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1-q_2}\right)t^{-1}\right]\theta^{q_2-p}, n_3 + q_2\right).\end{aligned}$$

(5.2) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1, \quad p = m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = 0, \quad q_2 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(5.3) $q_1 \ne n_1, q_1 = m_1, q_2 \ne p, q_2 \ne 0$. In this case, we obtain

$$a = u(v\theta^{-p}), \quad b = vu, \quad b = s\left(t\theta^{q_2-p}\right), \quad c = t\left(s\theta^{-q_2}\right),$$

whence $c = t(v\theta^{-q_2})(a\theta^{-q_2})[(v^{-1}\theta^{-q_2})t^{-1}]\theta^{-p}$. This yields that

$$\begin{aligned}(m_1, a, n_1) &= \left(m_1, a\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{q_2-p}, n_3 + q_2\right)\left(m_3, t\left(v\theta^{-q_2}\right), m_1 - q_2\right),\\ (m_3, c, n_3) &= \left(m_3, t\left(v\theta^{-q_2}\right), m_1 - q_2\right)\left(m_1, a\left[\left(v^{-1}\theta^{-q_2}\right)t^{-1}\right]\theta^{q_2-p}, n_3 + q_2\right).\end{aligned}$$

(5.4) $q_1 \ne n_1, q_1 \ne m_1, q_2 = p, q_2 \ne 0$. In this case, we obtain

$$p = q_2 \ne 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = (v^{-1}\theta^{m_1-q_1})sc[s^{-1}(v\theta^{m_1-q_1})]\theta^{-p}$. This yields that

$$\begin{aligned}(m_1, a, n_1) &= \left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)s, m_3\right)\left(m_3, c\left[s^{-1}\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),\\ (m_3, c, n_3) &= \left(m_3, c\left[s^{-1}\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)s, m_3\right).\end{aligned}$$

(5.5) $q_1 \ne n_1, q_1 \ne m_1, q_2 \ne p, q_2 = 0$. In this case, we obtain

$$p \ne 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = t(v\theta^{m_1-q_1})a[(v^{-1}\theta^{m_1-q_1})t^{-1}]\theta^{-p}$. This yields that

$$\begin{aligned}(m_1, a, n_1) &= \left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1}\right)t^{-1}\right]\theta^{-p}, n_3\right)\left(m_3, t\left(v\theta^{m_1-q_1}\right), m_1\right),\\ (m_3, c, n_3) &= \left(m_3, t\left(v\theta^{m_1-q_1}\right), m_1\right)\left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1}\right)t^{-1}\right]\theta^{-p}, n_3\right).\end{aligned}$$

(5.6) $q_1 \ne n_1, q_1 = m_1, q_2 = p, q_2 \ne 0$. In this case, we obtain

$$a = u(v\theta^{-p}), \quad b = vu, \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (1.9).

(5.7) $q_1 \neq n_1, q_1 = m_1, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = tva(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3)(m_3, tv, m_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1)(m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3).$$

(5.8) $q_1 \neq n_1, q_1 \neq m_1, q_2 = p, q_2 = 0$. In this case, we obtain

$$a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{n_1-q_1}\right)u, \quad b = st, \quad c = ts,$$

and so $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by Lemmas 1.1 and 2.1.

We observe that it is impossible that $q_1 \neq n_1, q_1 = m_1, q_2 = p, q_2 = 0$.

Case 6–(ii,ii). If $p \leq 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq m_1 \leq n_1, q_1 \geq m_1 - m_2, a = u\left(v\theta^{n_1-q_1}\right), b = \left(v\theta^{m_1-q_1}\right)u,$$
$$0 \leq q_2 \leq m_2 \leq n_2, q_2 \geq m_2 - m_3, b = s\left(t\theta^{n_2-q_2}\right), c = \left(t\theta^{m_2-q_2}\right)s.$$

In this case, $n_3 - m_3 = -p \geq 0$.

(6.1) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \left(v^{-1}\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right)c\left[\left(t\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right).$$

(6.2) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1 = n_1, p = 0, m_2 = n_2, m_3 = n_3, a = uv, b = vu, b = s\left(t\theta^{m_2-q_2}\right), c = \left(t\theta^{m_2-q_2}\right)s.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(6.3) $q_1 \neq n_1, q_1 = m_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s,$$

whence $c = (t\theta^{m_2-q_2})va[v^{-1}(t^{-1}\theta^{m_2-q_2})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, a\left[v^{-1}\left(t^{-1}\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right)\left(m_3, \left(t\theta^{m_2-q_2}\right)v, m_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left(t\theta^{m_2-q_2}\right)v, m_1\right)\left(m_1, a\left[v^{-1}\left(t^{-1}\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right).$$

(6.4) $q_2 = n_2$. In this case, we obtain

$$q_2 = m_2 = n_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{n_1-q_1}\right)u, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(6.5) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = t(v\theta^{m_1-q_1})a[(v^{-1}\theta^{m_1-q_1})t^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1}\right)t^{-1}\right]\theta^{-p}, n_3\right)\left(m_3, t\left(v\theta^{m_1-q_1}\right), m_1\right),$$
$$(m_3, c, n_3) = \left(m_3, t\left(v\theta^{m_1-q_1}\right), m_1\right)\left(m_1, a\left[\left(v^{-1}\theta^{m_1-q_1}\right)t^{-1}\right]\theta^{-p}, n_3\right).$$

(6.6) $q_1 \neq n_1, q_1 = m_1, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = tva(v^{-1}t^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3)(m_3, tv, m_1),$$
$$(m_3, c, n_3) = (m_3, tv, m_1)(m_1, a(v^{-1}t^{-1})\theta^{-p}, n_3),$$

which gives that $(m_1, a, n_1) \sim_p (m_3, c, n_3)$.

If $p \geq 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq n_1 \leq m_1, q_1 \geq m_1 - m_2, a = u\left(v\theta^{n_1-q_1}\right), b = \left(v\theta^{m_1-q_1}\right)u,$$
$$0 \leq q_2 \leq n_2 \leq m_2, q_2 \geq m_2 - m_3, b = s\left(t\theta^{n_2-q_2}\right), c = \left(t\theta^{m_2-q_2}\right)s.$$

In this case, $m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p \geq 0$.

(6.7) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(t^{-1}\theta^{n_2-q_2}\right)\right]\theta^p c\left(t\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right).$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(t^{-1}\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(t\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(t\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right), n_1\right)\left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(t^{-1}\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(6.8) $q_1 = n_1, q_1 \neq m_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s,$$

whence $a = [v^{-1}(t^{-1}\theta^{n_2-q_2})]\theta^p c(t\theta^{n_2-q_2})v$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[v^{-1}\left(t^{-1}\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(t\theta^{n_2-q_2}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(t\theta^{n_2-q_2}\right)v, n_1\right)\left(m_1, \left[v^{-1}\left(t^{-1}\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(6.9) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = s\left(t\theta^{m_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(6.10) $q_1 \neq n_1, q_1 \neq m_1, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = [(v^{-1}\theta^{n_1-q_1})t^{-1}]\theta^p ct(v\theta^{n_1-q_1})$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)t^{-1}\right]\theta^p c, n_3\right)\left(n_3, t\left(v\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, t\left(v\theta^{n_1-q_1}\right), n_1\right)\left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)t^{-1}\right]\theta^p c, n_3\right),$$

(6.11) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = u\left(v\theta^{m_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(6.12) $q_1 = n_1, q_1 \neq m_1, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = (v^{-1}t^{-1})\theta^p ctv$. This yields that

$$(m_1, a, n_1) = (m_1, (v^{-1}t^{-1})\theta^p c, n_3)(n_3, tv, n_1),$$
$$(m_3, c, n_3) = (n_3, tv, n_1)(m_1, (v^{-1}t^{-1})\theta^p c, n_3).$$

Case 7–(ii,iii). If $p \le 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \le q_1 \le m_1 \le n_1,\ q_1 \ge m_1 - m_2,\ a = u\left(v\theta^{n_1-q_1}\right),\ b = \left(v\theta^{m_1-q_1}\right)u,$$
$$0 \le q_2 \le m_2 \le n_2,\ q_2 \ge m_2 - m_3,\ b = \left(s\theta^{m_2-q_2}\right)t,\ c = t\left(s\theta^{n_2-q_2}\right).$$

In this case, $n_3 - m_3 = n_1 - m_1 = -p \ge 0$.
(7.1) $q_1 \ne n_1, q_1 \ne m_1, q_2 \ne m_2, q_2 \ne n_2$. In this case, we obtain

$$a = \left(v^{-1}\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right)c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right), m_3\right).$$

(7.2) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1 = n_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{m_2-q_2}\right).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(7.3) $q_1 \ne n_1, q_1 = m_1, q_2 \ne m_2, q_2 \ne n_2$. In this case, we obtain

$$p = m_1 - n_1 \ne 0, \quad a = u(v\theta^{-p}), \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $c = (s^{-1}\theta^{m_2-q_2})va[v^{-1}(s\theta^{m_2-q_2})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, a\left[v^{-1}\left(s\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right)\left(m_3, \left(s^{-1}\theta^{m_2-q_2}\right)v, m_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left(s^{-1}\theta^{m_2-q_2}\right)v, m_1\right)\left(m_1, a\left[v^{-1}\left(s\theta^{m_2-q_2}\right)\right]\theta^{-p}, n_3\right).$$

(7.4) $q_1 \ne n_1, q_1 \ne m_1, q_2 = m_2, q_2 \ne n_2$. In this case, we obtain

$$p = m_2 - n_2 \ne 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = (v^{-1}\theta^{m_1-q_1})sc[s^{-1}(v\theta^{m_1-q_1})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)s, m_3\right)\left(m_3, c\left[s^{-1}\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[s^{-1}\left(v\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(v^{-1}\theta^{m_1-q_1}\right)s, m_3\right).$$

(7.5) $q_2 = n_2$. In this case, we obtain

$$q_2 = m_2 = n_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{n_1-q_1}\right)u, \quad b = st, c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(7.6) $q_1 \ne n_1, q_1 = m_1, q_2 = m_2, q_2 \ne n_2$. In this case, we obtain

$$a = u(v\theta^{-p}), \quad b = vu, \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (1.9).

If $p \ge 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \le q_1 \le n_1 \le m_1,\ q_1 \ge m_1 - m_2,\ a = u\left(v\theta^{n_1-q_1}\right),\ b = \left(v\theta^{m_1-q_1}\right)u,$$
$$0 \le q_2 \le n_2 \le m_2,\ q_2 \ge m_2 - m_3,\ b = \left(s\theta^{m_2-q_2}\right)t,\ c = t\left(s\theta^{n_2-q_2}\right).$$

In this case, $m_1 - n_1 = m_2 - n_2 = m_3 - n_3 = p \geq 0$.

(7.7) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq m_2, q_2 \neq n_2$. In this case, we obtain

$$a = \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c\left(s^{-1}\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right).$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)\left(v\theta^{n_1-q_1}\right), n_1\right)\left(m_1, \left[\left(v^{-1}\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(7.8) $q_1 = n_1, q_1 \neq m_1, q_2 \neq m_2, q_2 \neq n_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $a = [v^{-1}(s\theta^{n_2-q_2})]\theta^p c(s^{-1}\theta^{n_2-q_2})v$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[v^{-1}\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)v, n_1\right)\left(m_1, \left[v^{-1}\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(7.9) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{m_2-q_2}\right).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(7.10) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2 = m_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = u\left(v\theta^{m_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(7.11) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq m_2, q_2 = n_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = u\left(v\theta^{n_1-q_1}\right), \quad b = \left(v\theta^{m_1-q_1}\right)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = [s^{-1}(v\theta^{n_1-q_1})]\theta^p a(v^{-1}\theta^{n_1-q_1})s$. This yields that

$$(m_1, a, n_1) = \left(n_1, \left(v^{-1}\theta^{n_1-q_1}\right)s, n_3\right)\left(m_3, \left[s^{-1}\left(v\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[s^{-1}\left(v\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right)\left(n_1, \left(v^{-1}\theta^{n_1-q_1}\right)s, n_3\right).$$

(7.12) $q_1 = n_1, q_1 \neq m_1, q_2 \neq m_2, q_2 = n_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence

$$c1_s = c, \quad av^{-1} = u1_v, \quad cs^{-1} = t1_s,$$
$$b1_v = (v\theta^p)u1_v = (v\theta^p)av^{-1}, \quad b1_s = (s\theta^p)t1_s = (s\theta^p)cs^{-1}.$$

Since the idempotents of T form a chain, we have $1_v1_s = 1_s$ or $1_v1_s = 1_v$. If $1_v1_s = 1_s$, then

$$(v\theta^p)av^{-1}1_s = b1_v1_s = b1_s1_v = (s\theta^p)cs^{-1}1_v,$$
$$(v\theta^p)av^{-1}1_s = (v\theta^p)av^{-1}1_s1_s = (s\theta^p)cs^{-1}1_v1_s = (s\theta^p)cs^{-1}1_s = (s\theta^p)cs^{-1}.$$

Multiplying by s on the right on both sides of the second equation above, we obtain

$$(v\theta^p)av^{-1}s = (v\theta^p)av^{-1}1_s s = (s\theta^p)cs^{-1}s = (s\theta^p)c1_s = (s\theta^p)c,$$

and hence $c = (s^{-1}v)\theta^p a(v^{-1}s)$. This yields that

$$(m_1, a, n_1) = (n_1, v^{-1}s, n_3)(m_3, (s^{-1}v)\theta^p a, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}v)\theta^p a, n_1)(n_1, v^{-1}s, n_3).$$

Similarly, we can prove the case that $1_v 1_s = 1_v$.

Case 8–(ii,iv). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq n_1 \leq m_1,\ q_1 \geq m_1 - m_2,\ a = u\left(v\theta^{n_1 - q_1}\right),\ b = \left(v\theta^{m_1 - q_1}\right)u,$$

and $0 \leq q_2 \leq p$, $b = (s\theta^{p-q_2})t$, $c = (t\theta^{q_2})s$. In this case,

$$m_1 - n_1 = m_3 - n_3 = p \geq 0, \quad m_3 - q_2 \geq m_3 - p = n_3, \quad n_1 + q_2 \geq n_1.$$

(8.1) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = \left[\left(v^{-1}\theta^{n_1 - q_1}\right)t^{-1}\right]\theta^p\left(c\theta^{p-q_2}\right)t\left(v\theta^{n_1 - q_1}\right).$$

This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, \left[\left(v^{-1}\theta^{n_1 - q_1}\right)t^{-1}\right]\theta^{q_2}c, n_3\right)\left(m_3 - q_2, t\left(v\theta^{n_1 - q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_2, t\left(v\theta^{n_1 - q_1}\right), n_1\right)\left(n_1 + q_2, \left[\left(v^{-1}\theta^{n_1 - q_1}\right)t^{-1}\right]\theta^{q_2}c, n_3\right),$$

(8.2) $q_1 = n_1, q_1 \neq m_1, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = \left(s\theta^{p-q_2}\right)t, \quad c = \left(t\theta^{q_2}\right)s,$$

whence $a = (v^{-1}t^{-1})\theta^p(c\theta^{p-q_2})tv$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, (v^{-1}t^{-1})\theta^{q_2}c, n_3\right)(m_3 - q_2, tv, n_1),$$
$$(m_3, c, n_3) = (m_3 - q_2, tv, n_1)\left(n_1 + q_2, (v^{-1}t^{-1})\theta^{q_2}c, n_3\right).$$

(8.3) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1, \quad p = 0 = q_2, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(8.4) $q_1 \neq n_1, q_1 \neq m_1, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p = q_2 \neq 0, \quad a = u\left(v\theta^{n_1 - q_1}\right), \quad b = \left(v\theta^{m_1 - q_1}\right)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = [(v^{-1}\theta^{n_1 - q_1})t^{-1}]\theta^p ct(v\theta^{n_1 - q_1})$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(v^{-1}\theta^{n_1 - q_1}\right)t^{-1}\right]\theta^p c, n_3\right)\left(n_3, t\left(v\theta^{n_1 - q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, t\left(v\theta^{n_1 - q_1}\right), n_1\right)\left(m_1, \left[\left(v^{-1}\theta^{n_1 - q_1}\right)t^{-1}\right]\theta^p c, n_3\right).$$

(8.5) $q_1 \neq n_1, q_1 \neq m_1, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = u\left(v\theta^{n_1 - q_1}\right), \quad b = \left(v\theta^{m_1 - q_1}\right)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = [s^{-1}(v\theta^{n_1 - q_1})]\theta^p a(v^{-1}\theta^{n_1 - q_1})s$. This yields that

$$(m_1, a, n_1) = \left(n_1, \left(v^{-1}\theta^{n_1 - q_1}\right)s, n_3\right)\left(m_3, \left[s^{-1}\left(v\theta^{n_1 - q_1}\right)\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[s^{-1}\left(v\theta^{n_1 - q_1}\right)\right]\theta^p a, n_1\right)\left(n_1, \left(v^{-1}\theta^{n_1 - q_1}\right)s, n_3\right).$$

(8.6) $q_1 = n_1, q_1 \neq m_1, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p = q_2 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = (v^{-1}t^{-1})\theta^p ctv$. This yields that

$$(m_1, a, n_1) = (m_1, (v^{-1}t^{-1})\theta^p c, n_3)(n_3, tv, n_1),$$
$$(m_3, c, n_3) = (n_3, tv, n_1)(m_1, (v^{-1}t^{-1})\theta^p c, n_3).$$

(8.7) $q_1 = n_1$, $q_1 \neq m_1$, $q_2 \neq p$, $q_2 = 0$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (7.12).

(8.8) $q_2 = p$, $q_2 = 0$. In this case, we obtain

$$p = 0, \quad m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = u\big(v\theta^{m_1-q_1}\big), \quad b = \big(v\theta^{m_1-q_1}\big)u, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

Case 9–(iii,i). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \le q_1 \le m_1 \le n_1, \quad q_1 \ge m_1 - m_2, \quad a = \big(u\theta^{m_1-q_1}\big)v, \quad b = v\big(u\theta^{n_1-q_1}\big),$$

and $p \le q_2 \le 0$, $b = s(t\theta^{q_2-p})$, $c = t(s\theta^{-q_2})$. In this case,

$$m_1 - n_1 = m_3 - n_3 = p \le 0, \quad n_3 + q_2 \ge n_3 + p = m_3, \quad m_1 - q_2 \ge m_1.$$

(9.1) $q_1 \neq m_1$, $q_1 \neq n_1$, $q_2 \neq p$, $q_2 \neq 0$. In this case, we obtain

$$a = \big(u\theta^{m_1-q_1}\big)s\big(c\theta^{q_2-p}\big)\big[s^{-1}\big(u^{-1}\theta^{m_1-q_1}\big)\big]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \big(m_1, \big(u\theta^{m_1-q_1}\big)s, n_3 + q_2\big)\big(m_3, c\big[s^{-1}\big(u^{-1}\theta^{m_1-q_1}\big)\big]\theta^{-q_2}, m_1 - q_2\big),$$
$$(m_3, c, n_3) = \big(m_3, c\big[s^{-1}\big(u^{-1}\theta^{m_1-q_1}\big)\big]\theta^{-q_2}, m_1 - q_2\big)\big(m_1, \big(u\theta^{m_1-q_1}\big)s, n_3 + q_2\big).$$

(9.2) $q_1 = m_1$, $q_1 \neq n_1$, $q_2 \neq p$, $q_2 \neq 0$. In this case, we obtain

$$a = uv, \quad b = v(u\theta^{-p}), \quad b = s\big(t\theta^{q_2-p}\big), \quad c = t\big(s\theta^{-q_2}\big),$$

whence $a = us(c\theta^{q_2-p})(s^{-1}u^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, us, n_3 + q_2)\big(m_3, c(s^{-1}u^{-1})\theta^{-q_2}, m_1 - q_2\big),$$
$$(m_3, c, n_3) = \big(m_3, c(s^{-1}u^{-1})\theta^{-q_2}, m_1 - q_2\big)(m_1, us, n_3 + q_2).$$

(9.3) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1 = n_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad q_2 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(9.4) $q_1 \neq m_1$, $q_1 \neq n_1$, $q_2 = p$, $q_2 \neq 0$. In this case, we obtain

$$p = q_2 \neq 0, \quad a = \big(u\theta^{m_1-q_1}\big)v, \quad b = v\big(u\theta^{n_1-q_1}\big), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = (u\theta^{m_1-q_1})sc[s^{-1}(u^{-1}\theta^{m_1-q_1})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, \big(u\theta^{m_1-q_1}\big)s, m_3\big)\big(m_3, c\big[s^{-1}\big(u^{-1}\theta^{m_1-q_1}\big)\big]\theta^{-p}, n_1\big),$$
$$(m_3, c, n_3) = \big(m_3, c\big[s^{-1}\big(u^{-1}\theta^{m_1-q_1}\big)\big]\theta^{-p}, n_1\big)\big(m_1, \big(u\theta^{m_1-q_1}\big)s, m_3\big).$$

(9.5) $q_1 \neq m_1$, $q_1 \neq n_1$, $q_2 \neq p$, $q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = \big(u\theta^{m_1-q_1}\big)v, \quad b = v\big(u\theta^{n_1-q_1}\big), \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = t(u^{-1}\theta^{m_1-q_1})a[(u\theta^{m_1-q_1})t^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, a\big[\big(u\theta^{m_1-q_1}\big)t^{-1}\big]\theta^{-p}, n_3\big)\big(m_3, t\big(u^{-1}\theta^{m_1-q_1}\big), m_1\big),$$
$$(m_3, c, n_3) = \big(m_3, t\big(u^{-1}\theta^{m_1-q_1}\big), m_1\big)\big(m_1, a\big[\big(u\theta^{m_1-q_1}\big)t^{-1}\big]\theta^{-p}, n_3\big).$$

(9.6) $q_1 = m_1, q_1 \neq n_1, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p = q_2 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = (us)c(s^{-1}u^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, us, m_3)(m_3, c[(s^{-1}u^{-1})\theta^{-p}], n_1),$$
$$(m_3, c, n_3) = (m_3, c[(s^{-1}u^{-1})\theta^{-p}], n_1)(m_1, us, m_3).$$

(9.7) $q_1 = m_1, q_1 \neq n_1, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$a = uv, \quad b = v(u\theta^{-p}), \quad b = s(t\theta^{-p}), c = ts,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (1.8).
(9.8) $q_2 = p, q_2 = 0$. In this case, we obtain

$$m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{m_1-q_1}\right), \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

Case 10–(iii,ii). If $p \leq 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq m_1 \leq n_1, q_1 \geq m_1 - m_2, a = \left(u\theta^{m_1-q_1}\right)v, b = v\left(u\theta^{n_1-q_1}\right),$$
$$0 \leq q_2 \leq m_2 \leq n_2, q_2 \geq m_2 - m_3, b = s\left(t\theta^{n_2-q_2}\right), c = \left(t\theta^{m_2-q_2}\right)s.$$

In this case, $n_3 - m_3 = n_1 - m_1 = -p \geq 0$.
(10.1) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \left(u\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right)c\left[\left(t\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left(u\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(u\theta^{m_1-q_1}\right)\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right).$$

(10.2) $q_1 = m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s,$$

whence $a = u(t^{-1}\theta^{m_2-q_2})c[(t\theta^{m_2-q_2})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, u\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(t\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right)\left(m_1, u\left(t^{-1}\theta^{m_2-q_2}\right), m_3\right).$$

(10.3) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = s\left(t\theta^{m_2-q_2}\right), \quad c = \left(t\theta^{m_2-q_2}\right)s.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(10.4) $q_2 = n_2$. In this case, we obtain

$$q_2 = m_2 = n_2, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{m_1-q_1}\right), \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(10.5) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right), \quad b = s(t\theta^{-p}), \quad c = ts,$$

whence $c = t(u^{-1}\theta^{m_1-q_1})a[(u\theta^{m_1-q_1})t^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \big(m_1, a\big[\big(u\theta^{m_1-q_1}\big)t^{-1}\big]\theta^{-p}, n_3\big)\big(m_3, t\big(u^{-1}\theta^{m_1-q_1}\big), m_1\big),$$
$$(m_3, c, n_3) = \big(m_3, t\big(u^{-1}\theta^{m_1-q_1}\big), m_1\big)\big(m_1, a\big[\big(u\theta^{m_1-q_1}\big)t^{-1}\big]\theta^{-p}, n_3\big).$$

(10.6) $q_1 = m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 = m_2$. In this case, we obtain

$$a = uv,\ b = v(u\theta^{-p}),\ b = s(t\theta^{-p}),\ c = ts,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (1.8).

If $p \geq 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq n_1 \leq m_1,\ q_1 \geq m_1 - m_2,\ a = \big(u\theta^{m_1-q_1}\big)v,\ b = v\big(u\theta^{n_1-q_1}\big),$$
$$0 \leq q_2 \leq n_2 \leq m_2,\ q_2 \geq m_2 - m_3,\ b = s\big(t\theta^{n_2-q_2}\big),\ c = \big(t\theta^{m_2-q_2}\big)s.$$

In this case, $m_1 - n_1 = m_3 - n_3 = p \geq 0$.

(10.7) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \big[\big(u\theta^{n_1-q_1}\big)\big(t^{-1}\theta^{n_2-q_2}\big)\big]\theta^p c\big(t\theta^{n_2-q_2}\big)\big(u^{-1}\theta^{n_1-q_1}\big).$$

This yields that

$$(m_1, a, n_1) = \big(m_1, \big[\big(u\theta^{n_1-q_1}\big)\big(t^{-1}\theta^{n_2-q_2}\big)\big]\theta^p c, n_3\big)\big(n_3, \big(t\theta^{n_2-q_2}\big)\big(u^{-1}\theta^{n_1-q_1}\big), n_1\big),$$
$$(m_3, c, n_3) = \big(n_3, \big(t\theta^{n_2-q_2}\big)\big(u^{-1}\theta^{n_1-q_1}\big), n_1\big)\big(m_1, \big[\big(u\theta^{n_1-q_1}\big)\big(t^{-1}\theta^{n_2-q_2}\big)\big]\theta^p c, n_3\big).$$

(10.8) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1,\quad p = 0,\quad m_2 = n_2,\quad m_3 = n_3,\quad a = uv,\quad b = vu,\quad b = s\big(t\theta^{m_2-q_2}\big),\quad c = \big(t\theta^{m_2-q_2}\big)s.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(10.9) $q_1 \neq m_1, q_1 = n_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0,\quad a = (u\theta^p)v,\quad b = vu,\quad b = s\big(t\theta^{n_2-q_2}\big),\quad c = \big(t\theta^{m_2-q_2}\big)s,$$

whence $c = [(t\theta^{n_2-q_2})u^{-1}]\theta^p au(t^{-1}\theta^{n_2-q_2})$. This yields that

$$(m_1, a, n_1) = \big(n_1, u\big(t^{-1}\theta^{n_2-q_2}\big), n_3\big)\big(m_3, \big[\big(t\theta^{n_2-q_2}\big)u^{-1}\big]\theta^p a, n_1\big),$$
$$(m_3, c, n_3) = \big(m_3, \big[\big(t\theta^{n_2-q_2}\big)u^{-1}\big]\theta^p a, n_1\big)\big(n_1, u\big(t^{-1}\theta^{n_2-q_2}\big), n_3\big).$$

(10.10) $q_1 \neq m_1, q_1 \neq n_1, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0,\quad a = \big(u\theta^{m_1-q_1}\big)v,\quad b = v\big(u\theta^{n_1-q_1}\big),\quad b = st,\quad c = (t\theta^p)s,$$

whence $a = [(u\theta^{n_1-q_1})t^{-1}]\theta^p ct(u^{-1}\theta^{n_1-q_1})$. This yields that

$$(m_1, a, n_1) = \big(m_1, \big[\big(u\theta^{n_1-q_1}\big)t^{-1}\big]\theta^p c, n_3\big)\big(n_3, t\big(u^{-1}\theta^{n_1-q_1}\big), n_1\big),$$
$$(m_3, c, n_3) = \big(n_3, t\big(u^{-1}\theta^{n_1-q_1}\big), n_1\big)\big(m_1, \big[\big(u\theta^{n_1-q_1}\big)t^{-1}\big]\theta^p c, n_3\big).$$

(10.11) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2 = m_2,\quad p = 0,\quad m_1 = n_1,\quad m_3 = n_3,\quad a = \big(u\theta^{m_1-q_1}\big)v,\quad b = v\big(u\theta^{m_1-q_1}\big),\quad b = st,\quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(10.12) $q_1 \neq m_1, q_1 = n_1, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0,\quad a = (u\theta^p)v,\quad b = vu,\quad b = st,\quad c = (t\theta^p)s,$$

whence

$$c1_s = c, \quad cs^{-1} = (t\theta^p)1_s, \quad av^{-1} = (u\theta^p)1_v,$$
$$(b\theta^p)1_v = (vu)\theta^p 1_v = (v\theta^p)(u\theta^p)1_v = (v\theta^p)av^{-1},$$
$$(b\theta^p)1_s = (st)\theta^p 1_s = (s\theta^p)(t\theta^p)1_s = (s\theta^p)cs^{-1}.$$

Since the idempotents of T form a chain, we have $1_v1_s = 1_s$ or $1_v1_s = 1_v$. If $1_s1_v = 1_s$, then

$$(v\theta^p)av^{-1}1_s = (b\theta^p)1_v1_s = (b\theta^p)1_s1_v = (s\theta^p)cs^{-1}1_v,$$
$$(v\theta^p)av^{-1}1_s = (v\theta^p)av^{-1}1_s1_s = (s\theta^p)cs^{-1}1_v1_s = (s\theta^p)cs^{-1}1_s = (s\theta^p)cs^{-1},$$

Multiplying by s from the right on both sides of the second equation above, then

$$(v\theta^p)av^{-1}s = (s\theta^p)cs^{-1}s = (s\theta^p)c1_s = (s\theta^p)c,$$

hence $c = (s^{-1}v)\theta^p a(v^{-1}s)$. This yields that

$$(m_1, a, n_1) = (n_1, v^{-1}s, n_3)(m_3, (s^{-1}v)\theta^p a, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}v)\theta^p a, n_1)(n_1, v^{-1}s, n_3).$$

Similarly, we can prove the case that $1_s1_v = 1_v$.

Case 11–(iii,iii). If $p \le 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \le q_1 \le m_1 \le n_1,\ q_1 \ge m_1 - m_2,\ a = \left(u\theta^{m_1-q_1}\right)v,\ b = v\left(u\theta^{n_1-q_1}\right),$$
$$0 \le q_2 \le m_2 \le n_2,\ q_2 \ge m_2 - m_3,\ b = \left(s\theta^{m_2-q_2}\right)t,\ c = t\left(s\theta^{n_2-q_2}\right).$$

In this case, $n_3 - m_3 = n_1 - m_1 = -p \ge 0$.

(11.1) $q_1 \ne m_1, q_1 \ne n_1, q_2 \ne n_2, q_2 \ne m_2$. In this case, we obtain

$$a = \left(u\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right)c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}.$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left(u\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(u\theta^{m_1-q_1}\right)\left(s\theta^{m_2-q_2}\right), m_3\right).$$

(11.2) $q_1 = m_1, q_1 \ne n_1, q_2 \ne m_2, q_2 \ne n_2$. In this case, we obtain

$$p = m_1 - n_1 \ne 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $a = u(s\theta^{m_2-q_2})c[(s^{-1}\theta^{m_2-q_2})u^{-1}]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, u\left(s\theta^{m_2-q_2}\right), m_3\right)\left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[\left(s^{-1}\theta^{m_2-q_2}\right)u^{-1}\right]\theta^{-p}, n_1\right)\left(m_1, u\left(s\theta^{m_2-q_2}\right), m_3\right).$$

(11.3) $q_1 = n_1$. In this case, we obtain

$$q_1 = m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = \left(s\theta^{n_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(11.4) $q_1 \ne m_1, q_1 \ne n_1, q_2 = m_2, q_2 \ne n_2$. In this case, we obtain

$$p = m_2 - n_2 \ne 0, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right), \quad b = st, c = t(s\theta^{-p}),$$

whence $a = (u\theta^{m_1-q_1})sc[s^{-1}(u^{-1}\theta^{m_1-q_1})]\theta^{-p}$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left(u\theta^{m_1-q_1}\right)s, m_3\right)\left(m_3, c\left[s^{-1}\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, c\left[s^{-1}\left(u^{-1}\theta^{m_1-q_1}\right)\right]\theta^{-p}, n_1\right)\left(m_1, \left(u\theta^{m_1-q_1}\right)s, m_3\right).$$

(11.5) $q_2 = n_2$. In this case, we obtain

$$q_2 = m_2 = n_2, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{m_1-q_1}\right), \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(11.6) $q_1 = m_1, q_1 \neq n_1, q_2 = m_2, q_2 \neq n_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = uv, \quad b = v(u\theta^{-p}), \quad b = st, \quad c = t(s\theta^{-p}),$$

whence $a = usc(s^{-1}u^{-1})\theta^{-p}$. This yields that

$$(m_1, a, n_1) = (m_1, us, m_3)(m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1),$$
$$(m_3, c, n_3) = (m_3, c(s^{-1}u^{-1})\theta^{-p}, n_1)(m_1, us, m_3).$$

If $p \geq 0$, then there exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq n_1 \leq m_1, q_1 \geq m_1 - m_2, a = \left(u\theta^{m_1-q_1}\right)v, b = v\left(u\theta^{n_1-q_1}\right),$$
$$0 \leq q_2 \leq n_2 \leq m_2, q_2 \geq m_2 - m_3, b = \left(s\theta^{m_2-q_2}\right)t, c = t\left(s\theta^{n_2-q_2}\right).$$

In this case, $m_1 - n_1 = m_3 - n_3 = p \geq 0$.
(11.7) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \left[\left(u\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c\left(s^{-1}\theta^{n_2-q_2}\right)\left(u^{-1}\theta^{n_1-q_1}\right).$$

This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(u\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right)\left(m_1, \left[\left(u\theta^{n_1-q_1}\right)\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(11.8) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{m_2-q_2}\right).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(11.9) $q_1 \neq m_1, q_1 = n_1, q_2 \neq m_2, q_2 \neq n_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $c = [(s^{-1}\theta^{n_2-q_2})u^{-1}]\theta^p au(s\theta^{n_2-q_2})$. This yields that

$$(m_1, a, n_1) = \left(n_1, u\left(s\theta^{n_2-q_2}\right), n_3\right)\left(m_3, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right)\left(n_1, u\left(s\theta^{n_2-q_2}\right), n_3\right).$$

(11.10) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2 = m_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{m_1-q_1}\right), \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(11.11) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq m_2, q_2 = n_2$. In this case, we obtain

$$p = m_2 - n_2 \neq 0, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right), \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = [s^{-1}(u^{-1}\theta^{n_1-q_1})]\theta^p a(u\theta^{n_1-q_1})s$. This yields that

$$(m_1, a, n_1) = \left(n_1, \left(u\theta^{n_1-q_1}\right)s, n_3\right)\left(m_3, \left[s^{-1}\left(u^{-1}\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[s^{-1}\left(u^{-1}\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right)\left(n_1, \left(u\theta^{n_1-q_1}\right)s, n_3\right).$$

(11.12) $q_1 \neq m_1, q_1 = n_1, q_2 \neq m_2, q_2 = n_2$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p aus$. This yields that

$$(m_1, a, n_1) = (n_1, us, n_3)(m_3, (s^{-1}u^{-1})\theta^p a, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}u^{-1})\theta^p a, n_1)(n_1, us, n_3).$$

Case 12–(iii,iv). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq n_1 \leq m_1, \quad q_1 \geq m_1 - m_2, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right)$$

and $0 \leq q_2 \leq p$, $b = (s\theta^{p-q_2})t$, $c = (t\theta^{q_2})s$. In this case,

$$m_1 - n_1 = m_3 - n_3 = p \geq 0, \quad m_3 - q_2 \geq m_3 - p = n_3, \quad n_1 + q_2 \geq n_1.$$

(12.1) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = \left[\left(u\theta^{n_1-q_1}\right)t^{-1}\right]\theta^p\left(c\theta^{p-q_2}\right)t\left(u^{-1}\theta^{n_1-q_1}\right).$$

This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, \left[\left(u\theta^{n_1-q_1}\right)t^{-1}\right]\theta^{q_2}c, n_3\right)\left(m_3 - q_2, t\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_2, t\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right)\left(n_1 + q_2, \left[\left(u\theta^{n_1-q_1}\right)t^{-1}\right]\theta^{q_2}c, n_3\right).$$

(12.2) $q_1 = m_1$. In this case, we obtain

$$q_1 = n_1 = m_1, \quad p = 0, \quad m_2 = n_2, \quad m_3 = n_3, \quad q_2 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(12.3) $q_1 \neq m_1, q_1 = n_1, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = (u\theta^p)v, \quad b = vu, \quad b = \left(s\theta^{p-q_2}\right)t, \quad c = \left(t\theta^{q_2}\right)s,$$

whence $c = [s^{-1}(u^{-1}\theta^{q_2})]\theta^p(a\theta^{q_2})(u\theta^{q_2})s$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right)\left(m_3 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_2}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_2}a, n_1\right)\left(n_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right).$$

(12.4) $q_1 \neq m_1, q_1 \neq n_1, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$p = q_2 \neq 0, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right), \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = [(u\theta^{n_1-q_1})t^{-1}]\theta^p ct(u^{-1}\theta^{n_1-q_1})$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[\left(u\theta^{n_1-q_1}\right)t^{-1}\right]\theta^p c, n_3\right)\left(n_3, t\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, t\left(u^{-1}\theta^{n_1-q_1}\right), n_1\right)\left(m_1, \left[\left(u\theta^{n_1-q_1}\right)t^{-1}\right]\theta^p c, n_3\right).$$

(12.5) $q_1 \neq m_1, q_1 \neq n_1, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = \left(u\theta^{m_1-q_1}\right)v, \quad b = v\left(u\theta^{n_1-q_1}\right), \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = [s^{-1}(u^{-1}\theta^{n_1-q_1})]\theta^p a(u\theta^{n_1-q_1})s$. This yields that

$$(m_1, a, n_1) = \left(n_1, \left(u\theta^{n_1-q_1}\right)s, n_3\right)\left(m_3, \left[s^{-1}\left(u^{-1}\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[s^{-1}\left(u^{-1}\theta^{n_1-q_1}\right)\right]\theta^p a, n_1\right)\left(n_1, \left(u\theta^{n_1-q_1}\right)s, n_3\right).$$

(12.6) $q_1 \neq m_1, q_1 = n_1, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$a = (u\theta^p)v, \quad b = vu, \quad b = st, \quad c = (t\theta^p)s,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (10.12).
(12.7) $q_1 \neq m_1, q_1 = n_1, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p = m_1 - n_1 \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p aus$. This yields that

$$(m_1, a, n_1) = (n_1, us, n_3)(m_3, (s^{-1}u^{-1})\theta^p a, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}u^{-1})\theta^p a, n_1)(n_1, us, n_3).$$

(12.8) $q_2 = p, q_2 = 0$. In this case, we obtain

$$m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad q_2 = 0, \quad a = \left(u\theta^{m_1 - q_1}\right)v, \quad b = v\left(u\theta^{m_1 - q_1}\right), \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

Case 13–(iv,i). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq p, \quad a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad p \leq q_2 \leq 0, \quad b = s\left(t\theta^{q_2 - p}\right), \quad c = t\left(s\theta^{-q_2}\right).$$

In this case,

$$p = 0, \quad q_1 = q_2 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

Case 14–(iv,ii). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq p, \quad a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u,$$
$$0 \leq q_2 \leq n_2 \leq m_2, \quad q_2 \geq m_2 - m_3, \quad b = s\left(t\theta^{n_2 - q_2}\right), \quad c = \left(t\theta^{m_2 - q_2}\right)s.$$

In this case,

$$m_1 - n_1 = m_3 - n_3 = p \geq 0, \quad m_3 - q_1 \geq m_3 - p = n_3, n_1 + q_1 \geq n_1.$$

(14.1) $q_1 \neq p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$c = \left[\left(t\theta^{n_2 - q_2}\right)u^{-1}\right]\theta^p\left(a\theta^{q_1}\right)u\left(t^{-1}\theta^{n_2 - q_2}\right).$$

This yields that

$$(m_1, a, n_1) = \left(n_1 + q_1, u\left(t^{-1}\theta^{n_2 - q_2}\right), n_3\right)\left(m_3 - q_1, \left[\left(t\theta^{n_2 - q_2}\right)u^{-1}\right]\theta^{p - q_1}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, \left[\left(t\theta^{n_2 - q_2}\right)u^{-1}\right]\theta^{p - q_1}a, n_1\right)\left(n_1 + q_1, u\left(t^{-1}\theta^{n_2 - q_2}\right), n_3\right).$$

(14.2) $q_1 = p, q_1 \neq 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = s\left(t\theta^{n_2 - q_2}\right), \quad c = \left(t\theta^{m_2 - q_2}\right)s,$$

whence $a = [v^{-1}(t^{-1}\theta^{n_2 - q_2})]\theta^p c[(t\theta^{n_2 - q_2})v]$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[v^{-1}\left(t^{-1}\theta^{n_2 - q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(t\theta^{n_2 - q_2}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(t\theta^{n_2 - q_2}\right)v, n_1\right)\left(m_1, \left[v^{-1}\left(t^{-1}\theta^{n_2 - q_2}\right)\right]\theta^p c, n_3\right).$$

(14.3) $q_1 \neq p, q_1 = 0, q_2 \neq n_2, q_2 \neq m_2$. In this case, we obtain

$$p \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = s\left(t\theta^{n_2 - q_2}\right), \quad c = \left(t\theta^{m_2 - q_2}\right)s,$$

whence $c = [(t\theta^{n_2-q_2})u^{-1}]\theta^p a[u(t^{-1}\theta^{n_2-q_2})]$. This yields that

$$(m_1, a, n_1) = \left(n_1, u\left(t^{-1}\theta^{n_2-q_2}\right), n_3\right)\left(m_3, \left[\left(t\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[\left(t\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right)\left(n_1, u\left(t^{-1}\theta^{n_2-q_2}\right), n_3\right).$$

(14.4) $q_1 \neq p, q_1 \neq 0, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = [v^{-1}(t^{-1}\theta^{p-q_1})]\theta^p(c\theta^{p-q_1})[(t\theta^{p-q_1})v]$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_1, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1}c, n_3\right)\left(m_3 - q_1, \left(t\theta^{p-q_1}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, \left(t\theta^{p-q_1}\right)v, n_1\right)\left(n_1 + q_1, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1}c, n_3\right).$$

(14.5) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2 = m_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad q_1 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(14.6) $q_1 = p, q_1 = 0$. In this case, we obtain

$$m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = s\left(t\theta^{n_2-q_2}\right), \quad c = \left(t\theta^{n_2-q_2}\right)s.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.
(14.7) $q_1 = p, q_1 \neq 0, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = (v^{-1}t^{-1})\theta^p c(tv)$. This yields that

$$(m_1, a, n_1) = (m_1, (v^{-1}t^{-1})\theta^p c, n_3)(n_3, tv, n_1),$$
$$(m_3, c, n_3) = (n_3, tv, n_1)(m_1, (v^{-1}t^{-1})\theta^p c, n_3).$$

(14.8) $q_1 \neq p, q_1 = 0, q_2 = n_2, q_2 \neq m_2$. In this case, we obtain

$$a = (u\theta^p)v, \quad b = vu, \quad b = st, \quad c = (t\theta^p)s,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (10.12).

Case 15–(iv,iii). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq p, \quad a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u,$$
$$0 \leq q_2 \leq n_2 \leq m_2, q_2 \geq m_2 - m_3, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right).$$

In this case,

$$m_1 - n_1 = m_3 - n_3 = p \geq 0, \quad m_3 - q_1 \geq m_3 - p = n_3, n_1 + q_1 \geq n_1.$$

(15.1) $q_1 \neq p, q_1 \neq 0, q_2 \neq m_2, q_2 \neq n_2$. In this case, we obtain

$$c = \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p\left(a\theta^{q_1}\right)u\left(s\theta^{n_2-q_2}\right).$$

This yields that

$$(m_1, a, n_1) = \left(n_1 + q_1, u\left(s\theta^{n_2-q_2}\right), n_3\right)\left(m_3 - q_1, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^{p-q_1}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^{p-q_1}a, n_1\right)\left(n_1 + q_1, u\left(s\theta^{n_2-q_2}\right), n_3\right).$$

(15.2) $q_1 = p, q_1 \neq 0, q_2 \neq m_2, q_2 \neq n_2$. In this case, we obtain

$$p = q_1 \neq 0, \quad a = uv, \quad b = (v\theta^p)u, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $a = [v^{-1}(s\theta^{n_2-q_2})]\theta^p c[(s^{-1}\theta^{n_2-q_2})v]$. This yields that

$$(m_1, a, n_1) = \left(m_1, \left[v^{-1}\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right)\left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, \left(s^{-1}\theta^{n_2-q_2}\right)v, n_1\right)\left(m_1, \left[v^{-1}\left(s\theta^{n_2-q_2}\right)\right]\theta^p c, n_3\right).$$

(15.3) $q_1 \neq p$, $q_1 = 0$, $q_2 \neq m_2$, $q_2 \neq n_2$. In this case, we obtain

$$p \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{n_2-q_2}\right),$$

whence $c = [(s^{-1}\theta^{n_2-q_2})u^{-1}]\theta^p au(s\theta^{n_2-q_2})$. This yields that

$$(m_1, a, n_1) = \left(n_1, u\left(s\theta^{n_2-q_2}\right), n_3\right)\left(m_3, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left[\left(s^{-1}\theta^{n_2-q_2}\right)u^{-1}\right]\theta^p a, n_1\right)\left(n_1, u\left(s\theta^{n_2-q_2}\right), n_3\right).$$

(15.4) $q_2 = m_2$. In this case, we obtain

$$q_2 = n_2 = m_2, \quad p = 0, \quad m_1 = n_1, \quad m_3 = n_3, \quad q_1 = 0, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(15.5) $q_1 \neq p$, $q_1 \neq 0$, $q_2 \neq m_2$, $q_2 = n_2$. In this case, we obtain

$$a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p(a\theta^{q_1})us$. This yields that

$$(m_1, a, n_1) = (n_1 + q_1, us, n_3)\left(m_3 - q_1, (s^{-1}u^{-1})\theta^{p-q_1}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, (s^{-1}u^{-1})\theta^{p-q_1}a, n_1\right)(n_1 + q_1, us, n_3).$$

(15.6) $q_1 = p$, $q_1 = 0$. In this case, we obtain

$$m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = \left(s\theta^{m_2-q_2}\right)t, \quad c = t\left(s\theta^{m_2-q_2}\right).$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$.

(15.7) $q_1 = p$, $q_1 \neq 0$, $q_2 \neq m_2$, $q_2 = n_2$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (7.12).

(15.8) $q_1 \neq p$, $q_1 = 0$, $q_2 \neq m_2$, $q_2 = n_2$. In this case, we obtain

$$p \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p aus$. This yields that

$$(m_1, a, n_1) = (n_1, us, n_3)(m_3, (s^{-1}u^{-1})\theta^p a, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}u^{-1})\theta^p a, n_1)(n_1, us, n_3).$$

Case 16–(iv,iv). There exist $q_1, q_2 \in \mathbb{Z}$ and $u, v, s, t \in T$ such that

$$0 \leq q_1 \leq p, \quad a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad 0 \leq q_2 \leq p, \quad b = \left(s\theta^{p-q_2}\right)t, \quad c = \left(t\theta^{q_2}\right)s.$$

In this case,

$$m_3 - q_1 \geq m_3 - p = n_3, n_1 + q_1 \geq n_1, m_3 - q_2 \geq m_3 - p = n_3, n_1 + q_2 \geq n_1.$$

(16.1) $q_1 \neq p$, $q_1 \neq 0$, $q_2 \neq p$, $q_2 \neq 0$. In this case, we have $0 < q_1 + q_2 < 2p$.

If $-p < q_1 + q_2 - p < 0$, then we have $c = [s^{-1}(u^{-1}\theta^{q_2})]\theta^p(a\theta^{q_1+q_2})(u\theta^{q_2})s$. This implies that

$$(m_1, a, n_1) = \left(n_1 + q_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right)\left(m_3 - q_1 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_1-q_2}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_1-q_2}a, n_1\right)\left(n_1 + q_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right).$$

If $0 < q_1 + q_2 - p < p$, then we have $a = [v^{-1}(t^{-1}\theta^{p-q_1})]\theta^p(c\theta^{p-(q_1+q_2-p)})(t\theta^{p-q_1})v$. This implies that

$$(m_1, a, n_1) = \left(n_1 + q_1 + q_2 - p, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1+q_2-p}c, n_3\right)\left(m_3 + p - q_1 - q_2, \left(t\theta^{p-q_1}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 + p - q_1 - q_2, \left(t\theta^{p-q_1}\right)v, n_1\right)\left(n_1 + q_1 + q_2 - p, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1+q_2-p}c, n_3\right).$$

If $q_1 + q_2 - p = 0$, then we have

$$c = \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^p(a\theta^p)\left(u\theta^{q_2}\right)s, \; a = \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^p(c\theta^p)\left(t\theta^{p-q_1}\right)v,$$

and hence $1_s(t^{-1}\theta^{p-q_1})c = (t^{-1}\theta^{p-q_1})c$, $1_v(u^{-1}\theta^{q_2})a = (u^{-1}\theta^{q_2})a$. Since the idempotents of T form a chain, we have $1_s1_v = 1_s$ or $1_s1_v = 1_v$. If $1_s1_v = 1_v$, then

$$\left(u\theta^{q_2}\right)ss^{-1}\left(u^{-1}\theta^{q_2}\right)a = \left(u\theta^{q_2}\right)1_s1_v\left(u^{-1}\theta^{q_2}\right)a = \left(u\theta^{q_2}\right)1_v\left(u^{-1}\theta^{q_2}\right)a = \left(u\theta^{q_2}\right)\left(u^{-1}\theta^{q_2}\right)a = a.$$

This implies that

$$(m_1, a, n_1) = \left(m_1, \left(u\theta^{q_2}\right)s, n_3\right)\left(n_3, s^{-1}\left(u^{-1}\theta^{q_2}\right)a, n_1\right),$$
$$(m_3, c, n_3) = \left(n_3, s^{-1}\left(u^{-1}\theta^{q_2}\right)a, n_1\right)\left(m_1, \left(u\theta^{q_2}\right)s, n_3\right).$$

If $1_s1_v = 1_s$, then

$$\left(t\theta^{p-q_1}\right)vv^{-1}\left(t^{-1}\theta^{p-q_1}\right)c = \left(t\theta^{p-q_1}\right)1_v1_s\left(t^{-1}\theta^{p-q_1}\right)c = \left(t\theta^{p-q_1}\right)1_s\left(t^{-1}\theta^{p-q_1}\right)c = \left(t\theta^{p-q_1}\right)\left(t^{-1}\theta^{p-q_1}\right)c = c.$$

This implies that

$$(m_1, a, n_1) = \left(n_1, v^{-1}\left(t^{-1}\theta^{p-q_1}\right)c, n_3\right)\left(m_3, \left(t\theta^{p-q_1}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3, \left(t\theta^{p-q_1}\right)v, n_1\right)\left(n_1, v^{-1}\left(t^{-1}\theta^{p-q_1}\right)c, n_3\right).$$

(16.2) $q_1 = p, q_1 \neq 0, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = \left(s\theta^{p-q_2}\right)t, \quad c = \left(t\theta^{q_2}\right)s,$$

whence $a = (v^{-1}t^{-1})\theta^p(c\theta^{p-q_2})tv$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, (v^{-1}t^{-1})\theta^{q_2}c, n_3\right)(m_3 - q_2, tv, n_1),$$
$$(m_3, c, n_3) = (m_3 - q_2, tv, n_1)\left(n_1 + q_2, (v^{-1}t^{-1})\theta^{q_2}c, n_3\right).$$

(16.3) $q_1 \neq p, q_1 = 0, q_2 \neq p, q_2 \neq 0$. In this case, we obtain

$$a = (u\theta^p)v, \quad b = vu, \quad b = \left(s\theta^{p-q_2}\right)t, \quad c = \left(t\theta^{q_2}\right)s,$$

whence $c = [s^{-1}(u^{-1}\theta^{q_2})]\theta^p(a\theta^{q_2})(u\theta^{q_2})s$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right)\left(m_3 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_2}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_2, \left[s^{-1}\left(u^{-1}\theta^{q_2}\right)\right]\theta^{p-q_2}a, n_1\right)\left(n_1 + q_2, \left(u\theta^{q_2}\right)s, n_3\right).$$

(16.4) $q_1 \neq p, q_1 \neq 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $a = [v^{-1}(t^{-1}\theta^{p-q_1})]\theta^p(c\theta^{p-q_1})(t\theta^{p-q_1})v$. This yields that

$$(m_1, a, n_1) = \left(n_1 + q_1, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1}c, n_3\right)\left(m_3 - q_1, \left(t\theta^{p-q_1}\right)v, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, \left(t\theta^{p-q_1}\right)v, n_1\right)\left(n_1 + q_1, \left[v^{-1}\left(t^{-1}\theta^{p-q_1}\right)\right]\theta^{q_1}c, n_3\right).$$

(16.5) $q_1 \neq p, q_1 \neq 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$a = \left(u\theta^{p-q_1}\right)v, \quad b = \left(v\theta^{q_1}\right)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p(a\theta^{q_1})(us)$. This yields that

$$(m_1, a, n_1) = (n_1 + q_1, us, n_3)\left(m_3 - q_1, (s^{-1}u^{-1})\theta^{p-q_1}a, n_1\right),$$
$$(m_3, c, n_3) = \left(m_3 - q_1, (s^{-1}u^{-1})\theta^{p-q_1}a, n_1\right)(n_1 + q_1, us, n_3).$$

(16.6) $q_1 = p, q_1 = 0$. In this case, we obtain

$$q_2 = p = 0, \quad m_1 = n_1, \quad m_2 = n_2, \quad m_3 = n_3, \quad a = uv, \quad b = vu, \quad b = st, \quad c = ts.$$

By Lemmas 1.1 and 2.1, $(m_1, a, m_1) \sim_p (m_3, c, m_3)$. It is easy to see that $q_1 = p, q_1 = 0$ if and only if $q_2 = p, q_2 = 0$. Thus, all the cases that there exist at least three items are equal to zero in $q_1 - p, q_1, q_2 - p, q_2$ reduce to this case.

(16.7) $q_1 = p, q_1 \neq 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = st, \quad c = (t\theta^p)s,$$

whence $p = q_1 \neq 0$, $a = (v^{-1}t^{-1})\theta^p ctv$. This yields that

$$(m_1, a, n_1) = (m_1, (v^{-1}t^{-1})\theta^p c, n_3)(n_3, tv, n_1),$$
$$(m_3, c, n_3) = (n_3, tv, n_1)(m_1, (v^{-1}t^{-1})\theta^p c, n_3).$$

(16.8) $q_1 = p, q_1 \neq 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$a = uv, \quad b = (v\theta^p)u, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (7.12).

(16.9) $q_1 \neq p, q_1 = 0, q_2 = p, q_2 \neq 0$. In this case, we obtain

$$a = (u\theta^p)v, \quad b = vu, \quad b = st, \quad c = (t\theta^p)s,$$

whence $(m_1, a, m_1) \sim_p (m_3, c, m_3)$ by (10.12).

(16.10) $q_1 \neq p, q_1 = 0, q_2 \neq p, q_2 = 0$. In this case, we obtain

$$p \neq 0, \quad a = (u\theta^p)v, \quad b = vu, \quad b = (s\theta^p)t, \quad c = ts,$$

whence $c = (s^{-1}u^{-1})\theta^p aus$. This yields that

$$(m_1, a, n_1) = (n_1, us, n_3)(m_3, (s^{-1}u^{-1})\theta^p, n_1),$$
$$(m_3, c, n_3) = (m_3, (s^{-1}u^{-1})\theta^p, n_1)(n_1, us, n_3).$$

We have completed the proof. □

The following example shows that $\sim_p$ is not transitive in a Bruck-Reilly extension of a general Clifford semigroup.

Example 2.4. Let $Y = \{0, 1, \alpha, \beta, \gamma, \delta\}$ be a semilattice with the greatest element 1 and the least element 0 such that the four elements $\alpha, \beta, \gamma, \delta$ are pairwise incomparable. Consider the Bruck-Reilly extension $S = \mathrm{BR}(Y, \theta)$ of Y determined by θ where $\theta : Y \to \{1\}, x \mapsto 1$. Take three elements $(0, \alpha, 1), (0, 0, 1)$ and $(0, \beta, 1)$ in S. Since

$$(0, 0, 1) = (0, \beta, 0)(0, \alpha, 1), \quad (0, \alpha, 1) = (0, \alpha, 1)(0, \beta, 0),$$
$$(0, 0, 1) = (0, \gamma, 0)(0, \delta, 1), \quad (0, \delta, 1) = (0, \delta, 1)(0, \gamma, 0),$$

we have $(0, \alpha, 1) \sim_p (0, 0, 1)$ and $(0, 0, 1) \sim_p (0, \delta, 1)$. Suppose that $(0, \alpha, 1) \sim_p (0, \delta, 1)$. In view of Lemma 2.2 and the fact that $p = 0 - 1 = -1 < 0$, we obtain that at least one of items (1), (2) and (3) in Lemma 2.2 holds. We shall show that this is impossible. In fact, if (1) holds, then there exist $q \in \mathbb{Z}, c, d \in Y$ such that

$-1 = p \leq q \leq 0$, $\alpha = c(d\theta^{q-p})$ and $\delta = d(c\theta^{-q})$ whence $q = 0$ or $q = -1$. If $q = 0$, then $\alpha = c(d\theta) = c1 = c$ and $\delta = dc$. This implies that $\delta = d\alpha$, and so $\delta \leq \alpha$. If $q = -1$, then $\alpha = cd$ and $\delta = d(c\theta) = d1 = d$. This gives that $\alpha = c\delta$, and so $\alpha \leq \delta$. If (2) holds, then there exist $q \in \mathbb{Z}$, $c, d \in Y$ such that

$$0 \leq q \leq \min\{0, 1\}, \quad q \geq 0 - 0, \quad \alpha = c(d\theta^{1-q}), \quad \delta = (d\theta^{0-q})c,$$

which shows that $q = 0$, and so $\alpha = c(d\theta) = c1 = c$ and $\delta = dc = d\alpha$. Thus, $\delta \leq \alpha$. If (3) holds, then there exist $q \in \mathbb{Z}$, $c, d \in Y$ such that

$$0 \leq q \leq \min\{0, 1\}, \quad q \geq 0 - 0, \quad \alpha = (c\theta^{0-q})d, \quad \delta = d(c\theta^{1-q}).$$

This implies that $\alpha = c\delta$, and so $\alpha \leq \delta$. We have shown that α and δ are comparable in any case. However, α and δ are incomparable by hypothesis, which is a contradiction. Thus, $\sim_p$ is not transitive.

Remark 2.5. From the Introduction part, $\sim_p$ is contained in $\sim_o = \sim_c$ in Bruck-Reilly extensions of general Clifford semigroups. We observe that the inclusion $\sim_p \subseteq \sim_o$ is proper by the following example. Let T be a Clifford monoid which has at least two $\mathcal{H}$-classes and $S = \mathrm{BR}(T, \theta)$ the Bruck-Reilly extension of T determined by θ. Assume that e and f are distinct idempotents in T. Then $(0, e, 0)$ and $(0, f, 0)$ are two idempotents in S. By the fact (1.1), we have $(0, e, 0) \sim_o (0, f, 0)$. If $(0, e, 0) \sim_p (0, f, 0)$, then by Lemma 2.1 we obtain that $e \sim_p f$ in T. Thus, there exist $c, d \in T$ such that $e = cd$ and $f = dc$, and so e, f lie in the same $\mathcal{H}$-class of T whence $e = f$, which is a contradiction.

3 The primary conjugacy is transitive in regular ω-semigroups

In this section, we shall show that the primary conjugacy $\sim_p$ is transitive in any regular ω-semigroup by using the technical results obtained in Section 2.

Theorem 3.1. *The relation $\sim_p$ in a regular ω-semigroup is transitive.*

Proof. Let S be a regular ω-semigroup. If S has no kernel, then S is a Clifford semigroup by Lemma 1.3, and so $\sim_p$ is transitive by Lemma 1.1. If S has kernel K and $S = K$, then by Lemma 1.3, S is a Bruck-Reilly extension $\mathrm{BR}(T, \theta)$ of T determined by θ in which T is a Clifford monoid whose idempotents form a chain, and so $\sim_p$ is transitive by Lemma 2.3.

Now, we consider the case that S has kernel K which is not equal to S. By Lemma 1.4, we can assume that S is the semigroup constructed in Lemma 1.4. Suppose that $a, b, c \in S$ and $a \sim_p b$, $b \sim_p c$. Then there exist $u, v, s, t \in S$ such that

$$a = u \circ v, \quad b = v \circ u, \quad b = s \circ t, \quad c = t \circ s.$$

We shall consider the following cases in the sequel.

(1) If $u, v, s, t \in K$, then $a, b, c \in K$ and $a \sim_p b$ and $b \sim_p c$ in K. Since K is a simple regular ω-semigroup, it follows that K has kernel K, and so $a \sim_p c$ in K by the above arguments. This certainly gives that $a \sim_p c$ in S.

(2) If $v, s, t \in K$ and $u \in S \backslash K$, then $u \in G_i$ for some i with $0 \leq i \leq l - 1$. By Lemma 1.4 we have

$$a = u \circ v = (u\alpha_{i,l})v, \;\; b = v \circ u = v(u\alpha_{i,l}), \;\; b = s \circ t = st, \;\; c = t \circ s = ts.$$

Observe that $u\alpha_{i,l}, v, s, t$ are all in K, it follows that $a, b, c \in K$ and $a \sim_p b$, $b \sim_p c$ in K, and so $a \sim_p c$ in K by Lemma 2.3. Hence, $a \sim_p c$ in S. Similarly, we can prove $a \sim_p c$ in S for all the cases with $|\{u, v, s, t\} \cap K| = 3$.

(3) If $u, v \in K$ and $s, t \in S \backslash K$, then $s \in G_i$ and $t \in G_j$ for some i, j with $0 \leq i, j \leq l - 1$. By Lemma 1.4 we have

$$b = v \circ u = vu \in K, \quad b = s \circ t = (s\alpha_{i,x})(t\alpha_{j,x}) \in S \backslash K,$$

where $x = \max\{i, j\}$. This leads to a contradiction. Similarly, we can prove the case that $u, v \notin K$ and $s, t \in K$ is also impossible.

(4) If $v, t \in K$ and $u, s \in S\backslash K$, then $u \in G_i$ and $s \in G_j$ for some i, j with $0 \le i, j \le l-1$. By Lemma 1.4 we have

$$a = u \circ v = (u\alpha_{i,l})v, \quad b = v \circ u = v(u\alpha_{i,l}), \quad b = s \circ t = (s\alpha_{j,l})t, \quad c = t \circ s = t(s\alpha_{j,l}).$$

By similar arguments in Case (2), we have $a \sim_p c$ in S. Similarly, we can prove $a \sim_p c$ in S for the following cases: (i) $u, s \in K, v, t \in S\backslash K$, (ii) $u, t \in K, v, s \in S\backslash K$ and (iii) $v, s \in K, u, t \in S\backslash K$.

(5) If $s \in K$ and $u, v, t \in S\backslash K$, then $u \in G_i$, $v \in G_j$ and $t \in G_r$ for some i, j, r with $0 \le i, j, r \le l-1$. By Lemma 1.4, we have

$$b = v \circ u = (v\alpha_{j,x})(u\alpha_{i,x}) \in S\backslash K, b = s \circ t = s(t\alpha_{r,l}) \in K,$$

where $x = \max\{i, j\}$. This also leads to a contradiction. Therefore, this case does not occur. Similarly, we can show that it is impossible for all the cases with $|\{u, v, s, t\} \cap K| = 1$.

(6) If $u, v, s, t \in S\backslash K$, then $u \in G_i, v \in G_j, s \in G_n$ and $t \in G_m$ for some i, j, m, n with $0 \le i, j, n, m \le l-1$. By Lemma 1.4, we have

$$a = u \circ v = (u\alpha_{i,x})(v\alpha_{j,x}), b = v \circ u = (v\alpha_{j,x})(u\alpha_{i,x}),$$
$$b = s \circ t = (s\alpha_{n,y})(t\alpha_{m,y}), c = t \circ s = (t\alpha_{m,y})(s\alpha_{n,y}),$$

where $x = \max\{i, j\}$ and $y = \max\{m, n\}$. This implies that $b \in G_x$ and $b \in G_y$ whence $x = y$. Thus, $a, b, c \in G_x$ and $a \sim_p b$, $b \sim_p c$ in the group G_x, and hence $a \sim_p c$ in G_x by Lemma 1.1. Certainly, it follows that $a \sim_p c$ in S. □

Remark 3.2. By Theorem 3.1, primary conjugacy in a regular ω-semigroup is transitive. However, the proof is too complicated. Thus, it will be a necessary work to find a simple proof of this result in future. We observe that it is impossible to give a proof by showing that $\sim_o = \sim_p$ according to Remark 2.5. Moreover, the characterizations of various kinds of conjugacy relations and the relationships among them on regular ω-semigroups are also worthy of further study.

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References

[1] G. Kudryavtseva and V. Mazorchuk, *On three approaches to conjugacy in semigroups*, Semigroup Forum **78** (2009), no. 1, 14–20, DOI: https://doi.org/10.1007/s00233-008-9047-7.

[2] G. Kudryavtseva and V. Mazorchuk, *On conjugation in some transformation and Brauer-type semigroups*, Publ. Math. Debrecen **70** (2007), no. 1–2, 19–43.

[3] F. Otto, *Conjugacy in monoids with a special Church-Rosser presentation is decidable*, Semigroup Forum **29** (1984), no. 1, 223–240, DOI: https://doi.org/10.1007/BF02573327.

[4] J. Araújo, J. Konieczny, and A. Malheiro, *Conjugation in semigroups*, J. Algebra **403** (2014), 93–134, DOI: https://doi.org/10.1016/j.jalgebra.2013.12.025.

[5] J. Araújo, M. Kinyon, J. Konieczny, and A. Malheiro, *Four notions of conjugacy for abstract semigroups*, Proc. Roy. Soc. Edinburgh Sect. A **147** (2017), no. 6, 1169–1214, DOI: https://doi.org/10.1017/S0308210517000099.

[6] J. Araújo, M. Kinyon, and J. Konieczny, *Conjugacy in inverse semigroups*, J. Algebra **533** (2019), 142–173, DOI: https://doi.org/10.1016/j.jalgebra.2019.05.022.

[7] G. Kudryavtseva, *On Conjugacy in Regular Epigroups*, 2006, DOI: https://doi.org/10.48550/arXiv.math/0605698.
[8] J. Konieczny, *A new definition of conjugacy for semigroups*, J. Algebra Appl. **17** (2018), no. 2, 1850032, DOI: https://doi.org/10.1142/S0219498818500329.
[9] J. Araújo, M. Kinyon, J. Konieczny, and A. Malheiro, *Decidability and independence of conjugacy problems in finitely presented monoids*, Theoret. Comput. Sci. **731** (2018), 88–98, DOI: https://doi.org/10.1016/j.tcs.2018.04.002.
[10] C. Choffrut, *Conjugacy in free inverse monoids*, Int. J. Algebra Comput. **3** (1993), no. 2, 169–188, DOI: https://doi.org/10.1142/S0218196793000135.
[11] A. J. Cain, A. Malheiro, and F. M. Silva, *Conjugacy in Patience Sorting Monoids*, 2018, DOI: https://doi.org/10.48550/arXiv.1803.00361.
[12] O. Ganyushkin and T. Kormysheva, *The chain decomposition of partial permutations and classes of conjugate elements of the semigroup IS_n*, Vianyk Kyiv Univ. **2** (1993), 10–18.
[13] L. Zhang, *Conjugacy in special monoids*, J. Algebra **143** (1991), no. 2, 487–497, DOI: https://doi.org/10.1016/0021-8693(91)90275-D.
[14] M. Borralho and M. Kinyon, *Variants of epigroups and primary conjugacy*, Commun. Algebra **48** (2020), no. 12, 5465–5473, DOI: https://doi.org/10.1080/00927872.2020.1791145.
[15] M. Borralho, *The transitivity of primary conjugacy in a class of semigroups*, Quasigroups Related Systems **28** (2020), no. 1, 43–46.
[16] R. C. Lyndon and M. P. Schützenberger, *The equation $a^m = b^n c^p$ in a free group*, Michigan Math. J. **9** (1962), no. 4, 289–298, DOI: https://doi.org/10.1307/mmj/1028998766.
[17] G. Lallement, Semigroups and combinatorial applications, *Pure and Applied Mathematics*, A Wiley-Interscience Publication. John Wiley & Sons, New York-Chichester-Brisbane, 1979.
[18] O. Ganyushkin and V. Mazorchuk, *Classical finite transformation semigroups*, Algebra and Applications, Vol. 9, Springer-Verlag, London, 2009, DOI: https://doi.org/10.1007/978-1-84800-281-4.
[19] J. M. Howie, *Fundamentals of semigroup theory*, London Mathematical Society Monographs, Vol. 12, Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1995.
[20] B. P. Kočin, *The structure of inverse ideal-simple ω-semigroups*, Vestnik Leningrad. Univ. Mat. Meh. Astronom. **23** (1968), no. 7, 41–50, (in Russian).
[21] W. D. Munn, *Regular ω-semigroups*, Glasgow Math. J. **9** (1968), no. 1, 46–66, DOI: https://doi.org/10.1017/S0017089500000288.
[22] N. R. Reilly, *Bisimple ω-semigroups*, Glasgow Math. J. **7** (1966), no. 3, 160–167, DOI: https://doi.org/10.1017/S2040618500035346.
[23] M. Petrich, *Inverse Semigroups*, John Wiley and Sons, New York, 1984.
[24] D. B. McAlister and P. J. Medeiros, *Compatible total orders on ω-regular semigroups*, Semigroup Forum **89** (2014), no. 1, 217–235, DOI: https://doi.org/10.1007/s00233-013-9514-7.