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Research Article

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Weak measure expansivity of C^2 dynamics

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Abstract: Let f be a C^2-diffeomorphism with Axiom A and no cycle condition on a two-dimensional smooth manifold. In this article, we prove that if f is C^2-robustly weak measure expansive, then it is Q^2-Anosov. Moreover, we expand the results of the C^2-diffeomorphism case into the C^2-vector field on a three-dimensional smooth manifold. Let X be a C^2-vector field with Axiom A and no cycle condition. We prove that if X is C^2-robustly weak measure expansive, then it is Q^2-Anosov.

Keywords: expansive, weak measure expansive, continuum-wise expansive, Q^2-Anosov, Axiom A

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1 Introduction

Under mainly C^1-topology, the main research topic of dynamical systems is the study of hyperbolicity using various properties for diffeomorphisms or flows. Franks' lemma [1] and Closing lemma [2], famous properties in dynamical system studies, only work well for the C^1-topology and play an essential role several times in the proof. In fact, we can see that Franks' lemma is used in the proof process of [3,4] and [5], and the closing lemma is used in [6] and [7]. To study more similar (closer) dynamics to a given dynamics, many mathematicians are studying C^2-topology. The papers that directly motivated our research are the following.

Theorem A. (Pujals and Sambarino [8]) *Let f be a C^2 surface diffeomorphism and $\Lambda \subset \Omega(f)$ be a compact f-invariant set admitting a dominated splitting $T_\Lambda M = E \oplus F$. Assume that all periodic points in Λ are hyperbolic saddles. Then $\Lambda = \Lambda_1 \cup \Lambda_2$, where*
– *Λ_1 is a hyperbolic set and*
– *Λ_2 is the union of finitely many pairwise disjoint normally hyperbolic circles $C_1, \ldots, C_k$*
such that $f^{m_i}(C_i) = C_i$ and $f^{m_i} : C_i \to C_i$ is an irrational rotation for some $m_i \geq 1$. Here m_i denotes the minimal number such that $f^{m_i}(C_i) = C_i$.

From this result, the following two theorems have been proved by adding a shadowing property view point.

Theorem B. (Sakai [9]) *Let f satisfy C^2-stably shadowing property on a compact surface. If periodic points of f are dense in the non-wandering set and there is a dominated splitting on the closure of periodic points of saddle type, then f satisfies both Axiom A and the strong transversality condition.*

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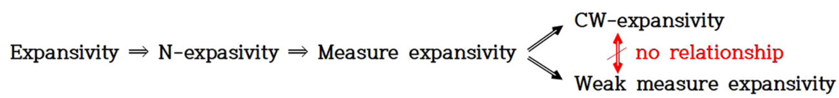
Theorem C. (Lee [10]) *Let f satisfy C^2-stably inverse shadowing property on a compact surface. If periodic points of f are dense in the non-wandering set and there is a dominated splitting on the closure of periodic points of saddle type, then f satisfies both Axiom A and the strong transversality condition.*

In addition, Artigue showed the following by adding an expansivity view point in [11].

Theorem D. (Artigue [11]) *Let f be a C^2-diffeomorphism with Axiom A and no cycle condition on a compact surface. If it is C^2-robustly CW-expansive, then f is Q^2-Anosov. Here, the meaning of a Q^2-Anosov diffeomorphism is that it is Axiom A, has no cycles, and there is no 2-tangency.*

It is well known that the relationship between the various expansivities is as follows.



In particular, we refer to [3] for an example of a homeomorphism f that satisfies weak measure expansivity but not measure expansivity. More precisely, an irrational rotation map on the unit circle is weak measure expansive but not Lebesgue measure expansive. From this example, we can see that weak measure expansivity is clearly different from other expansivities, and it can be seen that the research value is a sufficient subject.

Theorem D became the motivation of one of the main theorems in this article as follows.

Theorem 1.1. *Let f be a C^2-star diffeomorphism with Axiom A and no cycle condition on a two-dimensional smooth manifold. If f is a C^2-robustly weak measure expansive diffeomorphism, then it is Q^2-Anosov.*

This problem is worth studying because there is no relationship between CW-expansivity and weak measure expansivity. Particularly, Q^1-Anosov is quasi-Anosov and quasi-Anosov is closely related to expansivities. The fact that a C^1-robustly expansive, N-expansive, CW-expansive, measure expansive, and weak measure expansive diffeomorphism is quasi-Anosov has already been proven. However, since we do not know the relation between weak measure expansive diffeomorphism and Q^2-Anosov in C^2-dynamics, we will prove it in Theorem 1.1.

Moreover, we propose an extension of the Q^2-Anosov definition of a diffeomorphism to a flow and prove the following second main result.

Theorem 1.2. *Let X be a C^2-vector field with Axiom A and no cycle condition on a three-dimensional smooth manifold. If X is a C^2-robustly weak measure expansive vector field, then it is Q^2-Anosov.*

This result extends the result of the first main theorem, which is obtained for the case of diffeomorphisms to the case of continuous flows.

2 Basic definitions

2.1 Discrete dynamics

Let (M, d) be a compact smooth $n(\geq 1)$-dimensional manifold without boundary and let $\mathrm{Diff}^r(M)$ $(r \geq 1)$ be a set of C^r diffeomorphisms with the C^r topology. Let β be the Borel σ-algebra on M. Denote by $\mathcal{M}(M)$ the set of Borel probability measures on M endowed with the weak* topology. We say that $\mu \in \mathcal{M}(M)$ is *atomic* if there exists a point $x \in M$ such that $\mu(\{x\}) > 0$. Let $\mathcal{M}^*(M)$ be the set of nonatomic measures $\mu \in \mathcal{M}(M)$.

Recently, Morales and Sirvent [12] introduced a general notion of expansivity as follows: Let $\mu \in \mathcal{M}(M)$ be given, we say that f is *μ-expansive* if there exists a constant $\delta > 0$ such that $\mu(\Gamma_\delta^f(x)) = 0$ for all $x \in M$, where $\Gamma_\delta^f(x) = \{y \in M : d(f^i(x), f^i(y)) \le \delta, i \in \mathbb{Z}\}$. Such a δ is called an *μ-expansivity constant of f.*

From this, a concept of weak measure expansivity is introduced in [3]. It is the generalizing notion of measure expansivity and it is based on the concept of measure-sensitive partition in [13]. To do this, we say that a finite collection $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of subsets of M is a *finite δ-partition* ($\delta > 0$) of M if

(i) A_i's are disjoint, and $\cup_{i=1}^n A_i = M$;

(ii) each A_i is measurable for a Borel probability measure, $\mathrm{int}(A_i) \ne \emptyset$, and $\mathrm{diam} A_i \le \delta$ for all $i = 1, 2, \ldots, k$.

It can be easily checked that for any $\delta > 0$ there is a finite δ-partition $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of M. For convenience, we omit the "δ" and just say that $\mathcal{A}$ is a finite partition of M. However, if exact constants are needed to compare the size of elements of partitions, we will use "$\delta > 0$" and so on (for more details, see [3]). Now we introduce the notion of weak measure expansivity by using a finite partition as follows.

Definition 2.1. [3] For any $\mu \in \mathcal{M}(M)$, a homeomorphism $f : M \to M$ is said to be *weak μ-expansive* if there is a finite partition $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of M such that $\mu(\Gamma_{\mathcal{A}}^f(x)) = 0$ for all $x \in M$, where

$$\Gamma_{\mathcal{A}}^f(x) = \{y \in M : f^i(y) \in P(f^i(x)) \quad \text{for all } i \in \mathbb{Z}\}.$$

The set $\Gamma_{\mathcal{A}}^f(x)$ is called the *dynamic $\mathcal{A}$-ball of $x \in M$ with respect to f*, and $P(x)$ denotes the element of $\mathcal{A}$ containing x. Denote $\Gamma_{\mathcal{A}}^f(x)$ by $\Gamma_{\mathcal{A}}(x)$ for simplicity if there is no confusion. Note that

$$\Gamma_{\mathcal{A}}(x) = \bigcap_{i \in \mathbb{Z}} f^{-i}(P(f^i(x))).$$

A diffeomorphism $f : M \to M$ is called *weak measure expansive* if it is weak μ-expansive for all $\mu \in \mathcal{M}^*(M)$.

Note that if a diffeomorphism $f : M \to M$ is weak μ-expansive for $\mu \in \mathcal{M}(M)$, then μ is clearly nonatomic. Therefore, we can assume that μ is always an element of $\mathcal{M}^*(M)$ in this article.

Given $x \in M$, we can take $\delta_1, \delta_2 > 0$, and a C^2-coordinate chart $\varphi : U \subset M \to [-\delta_1, \delta_1] \times [-\delta_2, \delta_2]$, (here U is a neighborhood of x) such that $\varphi(x) = (0, 0)$ and two C^2 functions $g_s, g_u : [-\delta_1, \delta_1] \to [-\delta_2, \delta_2]$ such that the graph of g_s and g_u is the local expression of the local stable and the local unstable manifold of x, respectively. If the degree $r(\ge 1)$ Taylor polynomials of g_s and g_u at 0 coincide we say that there is an *r-tangency* at x.

Definition 2.2. [11] We say that C^r-diffeomorphism f is *Q^r-Anosov* if it is Axiom A, has no cycles, and there is no r-tangency.

Particularly, Q^1-Anosov is quasi-Anosov and quasi-Anosov is closely related to expansivities. It has already been demonstrated that a C^1-robustly expansive, N-expansive, CW-expansive, measure expansive, and weak measure expansive diffeomorphism is quasi-Anosov. However, since we do not know the relation between weak measure expansive diffeomorphism and Q^2-Anosov in C^2-dynamics, we will prove it in Theorem 1.1.

2.2 Continuous dynamics

Let $\mathfrak{X}^r(M)$ $(r \ge 1)$ be the set of C^r *vector fields* $X : M \to TM$ endowed with the C^r topology.

Then every $X \in \mathfrak{X}^r(M)$ generates a C^r flow $X_t : M \times \mathbb{R} \to M$, that is, a family of diffeomorphisms on M such that $X_s \circ X_t = X_{s+t}$ for all $s, t \in \mathbb{R}$, X_0 is the identity map and $\mathrm{d}X_t(x)/\mathrm{d}t\,|_{t=0} = X(x)$ for any $x \in M$. Here X_t is called the *integrated flow* of X.

For each $t \in \mathbb{R}$ the map $X_t : M \to M$ defined by $X_t(p) = X(p, t)$ is a C^r diffeomorphism. In addition, $X_{(-\varepsilon,\varepsilon)}(x) = \{X_t(x) : t \in (-\varepsilon, \varepsilon)\}$ and we denote the *orbit of x with respect to X_t* as $O_{\mathbb{R}}(x) = \{X_t(x) : t \in \mathbb{R}\}$.

For any subset $A \subset M$ we say $\mu(A) = 0$ if $\mu(B) = 0$ for every Borel set $B \subset A$. Let $\mathcal{F}(M)$ be the set of Borel probability measures on M endowed with the weak* topology and $\mathcal{F}^*_X(M) = \{\mu \in \mathcal{F}(M) : \mu(O_{\mathbb{R}}(x)) = 0$ for all $x \in M\}$.

For any $\mu \in \mathcal{F}(M)$, we say that X is *μ-expansive* if there exists a constant $\delta > 0$ such that $\mu(\Gamma^X_\delta(x)) = 0$ for all $x \in M$, where $\Gamma^X_\delta(x) = \{y \in M : d(X_t(x), X_{h(t)}(y)) \le \delta$ for some $h \in$ Rep and all $t \in \mathbb{R}\}$. Here Rep is the set of continuous maps $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ and h is said a reparametrization. Such a δ is called an *μ-expansivity constant of f.*

For $X \in \mathfrak{X}^r(M)$, a finite partition $\mathcal{A}$ of M and $x \in M$, the *dynamic $\mathcal{A}$-ball of $x \in M$ with respect to X_t*, $\Gamma^{X_t}_{\mathcal{A}}(x)$ is defined by

$$\{y \in M : X_{h(t)}(y) \in P(X_t(x)) \text{ for some } h \in \text{Rep and all } t \in \mathbb{R}\},$$

where $P(x)$ stands for the element of $\mathcal{A}$ containing x. Denote $\Gamma_{\mathcal{A}}(x)$ by $\Gamma^{X_t}_{\mathcal{A}}(x)$ for simplicity if there is no confusion.

Definition 2.3. [5] For any $\mu \in \mathcal{F}(M)$, X is said to be *weak μ-expansive* if there is a finite partition $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ of M such that $\mu(\Gamma_{\mathcal{A}}(x)) = 0$ for all $x \in M$. We say that X is *weak measure expansive* if it is weak μ-expansive for all $\mu \in \mathcal{F}^*_X(M)$.

3 Partially hyperbolic diffeomorphisms

In this section, we assume that M is a compact surface, i.e., $\dim(M) = 2$.

Lemma 3.1. [11, Lemma 4.1] *Let $p \in M$ and $U \subset M$ be a neighborhood of p. Then there are $\varepsilon > 0$ and an one-parameter family of C^∞ diffeomorphism $f_\theta : M \to M$, $|\theta - 1| < \varepsilon$, such that for all θ : $f_\theta(p) = p$, $f_\theta(x) = x$ for all $x \in M\backslash U$, $D_p f_\theta = \theta . Id$. Moreover, the function $\theta \mapsto f_\theta$ is continuous in the C^2-topology.*

Lemma 3.1 plays a very important role in the proof process of Theorem 1.1.

Recall that a point $x \in M$ is called a (*Lyapunov*) *stable point* for $f : M \to M$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f^n(x), f^n(y)) < \varepsilon$ for all $n \ge 0$. Since M is compact, we can take subsequences $\{f^{n_k}(x)\}$ and $\{f^{n_k}(y)\}$ converging to points x_1 and y_1, respectively, such that $d(x_1, y_1) > 0$ and $d(f^n(x_1), f^n(y_1)) \le \varepsilon$ for every $n \in \mathbb{Z}$. For $p \in M$, we say that p is *periodic* of period n for $f : M \to M$ if $f^n(p) = p$ for some $n \in \mathbb{N}$, but $f^m(p) \ne p$ for all $0 < m < n$. We denote by n the period $\pi(p)$ and by $\text{Per}(f)$ the set of periodic points of f. A point $x \in M$ is called *nonwandering* of f if for any neighborhood U of x in M, there is $n \ge 1$ such that $f^n(U) \cap U \ne \emptyset$. The set of nonwandering points of f is called the *nonwandering set* of f and is denoted by $\Omega(f)$. It is clear that $\overline{\text{Per}(f)} \subset \Omega(f)$.

Lemma 3.2. *If f is a weak measure expansive diffeomorphism on M then no periodic point is stable.*

Proof. Assume that f admits a stable point $q \in \text{Per}(f)$. Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a finite ε-partition of M for any $\varepsilon > 0$. Then there exists $\delta > 0$ such that

$$\Gamma_{\mathcal{A}}(q) \subset B_\delta[q] \subset A_i$$

for some $i \in \{1, 2, \dots, k\}$, where $B_\delta[q]$ is the closed δ-ball centered at q. Since q is a stable point, $f^n(B_\delta[q]) \to \{q\}$ as $n \to \infty$. For sufficiently large $N > 0$,

$$f^N(B_\delta[q]) \subset \Gamma_{\mathcal{A}}(q). \tag{3.1}$$

Put $\mathbb{B} = f^N(B_\delta[q])$ and let $\mathfrak{M}_{\mathbb{B}}$ be a normalized Lebesgue measure on $\mathbb{B}$. Define $\nu \in \mathcal{M}^*(M)$ by

$$\nu(U) = \mathfrak{M}_{\mathbb{B}}(U \cap \mathbb{B})$$

for any measurable set U of M (this is well-defined). Then we obtain $\nu(\Gamma_{\mathcal{A}}(q)) \ge \nu(\mathbb{B}) = 1$ by (3.1) and this is a contradiction. So proof is completed. □

For a diffeomorphism f, a property “P” is said to be *C^r-robust* if there is a C^r-neighborhood $\mathcal{U} \subset \mathrm{Diff}^r(M)$ of f such that for any $g \in \mathcal{U}$, g satisfies “P”.

Let f be a diffeomorphism on M and p be a periodic point of f. We say that p is *sink* (resp. *source*) if all eigenvalues of $D_p f^{\pi(p)}$ have norm less than 1 (resp. bigger than 1). In other cases, we call p is *saddle*.

Theorem 3.3. *If f is C^2-robustly weak measure expansive diffeomorphism on M, then every periodic point p of f with period $\pi(p)$ is saddle.*

Proof. Let U be a neighborhood of p such that $f^i(p) \notin \overline{U}$, for all $i = 1, \ldots, \pi(p) - 1$. Suppose that the eigenvalues of $D_p f^{\pi(p)}$ are smaller or equal than 1 in modulus. By Lemma 3.1, take a C^2-diffeomorphism f_μ of M fixing p and being the identity outside U. In particular, f_μ is the identity in a neighborhood of the points $f(p), \ldots, f^{\pi(p)-1}(p)$. Assume that $\mu \in (0, 1)$ is closer to 1. Define $g = f \circ f_\mu$. Then

- p is a periodic point of g with period $\pi(p)$,
- g is C^2-close to f, and
- eigenvalues of $D_p g^{\pi(p)}$ are $\mu\lambda_1, \mu\lambda_2$ (where λ_1, λ_2 : eigenvalues of $D_p f^{\pi(p)}$) such that modulus (strictly) smaller than 1.

Then p is a hyperbolic sink for g. Since f is a C^2-robustly weak measure expansive diffeomorphism, g is a weak measure expansive diffeomorphism. But it is contradiction by Lemma 3.2.

Finally, if the eigenvalues of $D_p f^{\pi(p)}$ are bigger, or equal to 1 in modulus, then we can obtain the contradiction through a similar method to the above process. Therefore, we can complete the proof. □

We say that f is a $C^r(r \geq 1)$*-star diffeomorphism* if there is a C^r-neighborhood $\mathcal{U}(f)$ of f such that every periodic orbit of every $g \in \mathcal{U}(f)$ is a hyperbolic set. In the case of $r = 1$, Smale et al. proved that the following three statements are equivalent.

(1) f satisfies Axiom A and no cycle condition,
(2) f is Ω-stable,
(3) f is a star diffeomorphism.

“(1) ⇒ (2)” was proved by Smale in [14], “(2) ⇒ (1)” was proved by Palis in [15], and “(3) ⇒ (1)” was proved by Hayashi in [16] and Aoki in [17], respectively.

However, we do not yet know whether an equivalence relation exists in C^2-dynamics, so we propose the following question. “If a diffeomorphism f of 2-dimensional manifold M or any dimensional manifold M is C^2-star, then does it satisfy Axiom A?” Below we give a partial answer to the question above.

Theorem 3.4. *Every C^2-robustly weak measure expansive diffeomorphism f on a compact surface M is a C^2-star diffeomorphism.*

Proof. To derive a contradiction, we suppose that there is a non-hyperbolic point $p \in \mathrm{Per}(g)$ for some $g \in \mathcal{U}(f)$. Here, $\mathcal{U}(f)$ is a C^2-neighborhood of f and $\mathrm{Per}(g)$ is the set of periodic points of g. Then $D_p g^{\pi(p)}$ has either only one eigenvalue λ with $|\lambda| = 1$, or only one pair of complex conjugated eigenvalues.

Case 1 : Let $\lambda_1 = 1$ and $\lambda_2 < 1$ (or $\lambda_2 > 1$).

Then by Lemma 3.1, we can make a C^2-diffeomorphism g_θ such that eigenvalues $\theta\lambda_1$ and $\theta\lambda_2$ of $D_p g_\theta^{\pi(p)}$ are less than 1 (or bigger than 1). This means that g_θ has a sink point p, and this is a contradiction by Lemma 3.2.

Case 2 : We can prove the second case similarly and complete the proof. □

The following proof is essentially contained in the proof of Theorem 4.4 in [11].

Proof of Theorem 1.1. Suppose that f is not Q^2-Anosov. Then there is a wandering point $p \in M$ with an 2-tangency. Take a C^2-local coordinate $\psi : U \times V \subset \mathbb{R}^2 \to M$ at p, where $U, V \subset \mathbb{R}$ are open sets.

Let $g_s, g_u : U \to V$ be C^2 functions which describe the graphs of the local stable and local unstable manifolds of p, respectively.

By the hypothesis, there exists a 2-tangency at p. So we may assume that the Taylor polynomials of order 2 of g_s and g_u vanish at 0.

Define the C^2-diffeomorphism $\xi : U \times \mathbb{R} \to U \times \mathbb{R}$ by

$$\xi(x, y) = (x, g_s(x) - g_u(x) + y).$$

For $\theta > 0$, define $\varphi_\theta : U \times V \to U \times \mathbb{R}$ by

$$\varphi_\theta(x, y) = \varsigma(\sqrt{x^2 + y^2}/\theta) \cdot \xi(x, y),$$

where $\varsigma : \mathbb{R} \to [0, 1]$ is a C^∞-function such that

$$\varsigma(a) = \begin{cases} \varsigma(a) = 1 \text{ for } a \le \frac{1}{2}, \\ \varsigma(a) = 0 \text{ for } a \ge 1. \end{cases}$$

Let $f_\theta : M \to M$ be defined by

$$f_\theta(q) = \begin{cases} f \circ \psi \circ \varphi_\theta \circ \psi^{-1}(q) & \text{for } q \in \psi(B_\theta(0, 0)), \\ f(q) & \text{otherwise.} \end{cases}$$

Let $\theta_1 < \theta$. If $\sqrt{x^2 + y^2} < \theta_1$, then $\varphi_\theta(x, y) = \xi(x, y)$. Then we have

$$\xi(x, g_u(x)) = (x, g_s(x) - g_u(x) + g_u(x)) = (x, g_s(x)),$$

that is, ξ is a map that sends the graph of g_u to the graph of g_s.

Let $\mathcal{A} = \{A_1, A_2, \dots, A_k\}$ be a finite θ-partition on M. There are $\theta_2 < \theta_1$ and an arc $\gamma = \{\psi(x, g_s(x)) : |x| < \theta_2\} \subset M$ such that
- $\mathrm{diam}(f_\theta^n(\gamma)) > 0$ if n is finite,
- $\mathrm{diam}(f_\theta^n(\gamma)) \to 0$ as $n \to \pm\infty$,
- $\gamma \subset \mathrm{int}(A_i)$ for some $A_i \in \mathcal{A}$, and
- $f_\theta^i(\gamma) \subset A_j$ for some $A_j \in \mathcal{A}$ containing $f_\theta^i(p)$.

Let $\mathfrak{M}_\gamma$ be a normalized Lebesgue measure on γ. Define $\nu \in \mathcal{M}^*_{f_\theta}(M)$ by

$$\nu(B) = \mathfrak{M}_\gamma(B \cap \gamma)$$

for any measurable set B of M (this is well-defined). For the wandering point p, the dynamic $\mathcal{A}$-ball $\Gamma^{f_\theta}_{\mathcal{A}}(p) = \{q \in M : f_\theta^i(q) \in P(f_\theta^i(p))$ of p with respect to f_θ for all $i \in \mathbb{Z}\}$ contains the arc γ. Then we obtain

$$\nu\left(\Gamma^{f_\theta}_{\mathcal{A}}(p)\right) \ge \nu(\gamma) = 1.$$

This fact shows that f_θ is not weak measure expansive, i.e., f is not C^2-robustly weak measure expansive. Therefore, this contradiction proves the theorem. □

4 Partially hyperbolic flows

Let (M, d) be as before with $\dim(M) = 3$. Let γ be a closed orbit of a vector field $X \in \mathfrak{X}^2(M)$. Through a point $x_0 \in \gamma$ we consider a section N_{x_0} transversal to the field X.

The orbit through x_0 returns to intersection N_{x_0} at time τ, where τ is the period or γ. By the continuity of the flow of X the orbit through a point $x \in N_{x_0}$ sufficiently close to x_0 also returns to intersection N_{x_0} at a time close to τ. Thus, if $V \subset N_{x_0}$ is a sufficiently small neighborhood of x_0 we can define a map $\mathbb{P} : V \to N_{x_0}$ which to each point $x \in V$ associates $\mathbb{P}(x)$, the first point where the orbit of x returns to intersection N_{x_0}. This map is called the *Poincaré map associated with the orbit* γ *(and the section* N_{x_0}*)*.

For any $X \in \mathfrak{X}^2(M)$ and $\sigma \in M$,σ is *singular* if $X(\sigma) = 0_\sigma$. Denote by Sing(X) the set of singular points of X. We say that p is *periodic* if $X_{\pi(p)}(p) = p$ for some $\pi(p) > 0$, but $X_t(p) \neq p$ for all $0 < t < \pi(p)$, and denoted by Per(X) is the set of periodic point of X. And p is *regular* if it is not singular nor periodic. A singularity or a periodic orbit of X are both called a *critical orbit* or a *critical point* of X and we denote by Crit(X) = Per$(X) \cup$ Sing(X) is the set of critical orbits.

The next lemma which is called *tubular flow theorem* describes the local behavior of the orbits in a neighborhood of a regular point.

Lemma 4.1. [18] *Let $X \in \mathfrak{X}^2(M)$ and let $p \in M$ be a regular point of X. Let $C = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : |x_i| < 1\}$ and let X_C be the vector field on C defined by $X_C(x) = (1, 0, \dots, 0)$. Then there exists a C^r diffeomorphism $g : V_p \to C$, for some neighborhood V_p of p in M, taking orbits of X to orbits of X_C.*

A *tubular flow* for $X \in \mathfrak{X}^r(M)$ is a pair (F, f) where F is an open set in M and f is a C^r diffeomorphism of F onto the cube $I^n = I \times I^{n-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : |x| < 1 \text{ and } |y_i| < 1,\ i = 1, \dots, n-1\}$, which takes the orbits of X in F to the straight lines $I \times \{y\} \subset I \times I^{n-1}$. If f_*X denotes the field in I^n induced by f and X, that is, $f_*X(x, y) = Df_{f^{-1}(x,y)} \cdot X(f^{-1}(x, y))$, then f_*X is parallel to the constant field $(x, y) \to (1, 0)$.

The open set F is called a flow box for the vector field X. By the tubular flow theorem, we know that if $p \in M$ is a regular point of X, then there is a flow box containing p.

Let X be a C^∞ vector field on M, γ a closed orbit of X, and Σ a transversal section through a point $p \in \Sigma$. Let $\mathcal{U}(X) \subset \mathfrak{X}^2(M)$ be a neighborhood of X and let $V \subset \Sigma$ be neighborhood of p such that, for all $Y \in \mathcal{U}(X)$, the Poincaré map of Y is defined on V.

Next lemma is a C^2-vector field version of [11, Lemma 4.1].

Lemma 4.2. *Let $p \in M$ and $X \in \mathfrak{X}^2(M)$. For any neighborhood U of 0_p in $\mathbb{R}^n$ and $\varepsilon' > 0$, there are $0 < \varepsilon < \varepsilon'$ and a one-parameter family of C^∞-vector field $X_\theta : \mathbb{R}^n \to \mathbb{R}^n$, $|\theta - 1| < \varepsilon$, such that*

(1) *for any θ, the flow ϕ_t^θ is generated by X_θ, the Poincaré map $\mathbb{P}_\theta$ is defined by $\mathbb{P}_\theta : \{-1\} \times I^{n-1} \to \{1\} \times I^{n-1}$, then one has $D_{(\{-1\}\times 0)}\mathbb{P}_\theta(\omega) = \theta\omega$ for all $\omega \in I^{n-1}$.*
(2) *$X_\theta = X$ in $\mathbb{R}^n \setminus U$.*
(3) *$\|X_\theta - X\|_{C^r} < \varepsilon'$.*

Proof. Let (F, f) be a tubular flow with center p such that f_*X is the constant vector field C on $[-1, 1] \times I^{n-1}$. Let $\tilde{C}$ be a C^∞ vector field on $f(F) \subset \mathbb{R}^n$ such that $\tilde{C}$ is transversal to $\{-1\} \times I^{n-1}$ and $\{1\} \times I^{n-1}$ and each orbit of X_θ through a point of $\{-1\} \times I^{n-1}$ meets $\{1\} \times I^{n-1}$. Then we can define a map $J_{\tilde{C}} : \{-1\} \times I^{n-1} \to \{1\} \times I^{n-1}$ which associates with each point of $\{-1\} \times I^{n-1}$ the intersection of its orbit with $\{1\} \times I^{n-1}$. By the Tubular Flow Theorem, $J_{\tilde{C}}$ is a diffeomorphism.

We claim that, given a neighborhood U of 0_p and $\varepsilon' > 0$, there is $\varepsilon > 0$ such that for any $0 < \theta - 1 < \varepsilon$ we can find a C^∞ vector field $\tilde{C}$ such that $\|\tilde{C} - C\|_{C^r} < \varepsilon'$ on $[-1, 1] \times I^{n-1}$, $\tilde{C} = C$ on $\mathbb{R}^n \setminus U$, and $J_{\tilde{C}}(-1, v_0) = (1, \theta v_0)$ if $v_0 \in I_{1/4}^{n-1}$.

Let $\psi : [-1, 1] \to \mathbb{R}^+$ be a C^∞ function such that

$$\begin{cases} \psi(t) = 0 & \text{for } t \in \left[-1, -\frac{1}{2}\right] \cup \left[\frac{1}{2}, 1\right], \\ \psi(t) > 0 & \text{for } t \in \left(-\frac{1}{2}, \frac{1}{2}\right). \end{cases}$$

Take $g : I^{n-1} \to \mathbb{R}^+$ such that

$$\begin{cases} g(v) = 0, & \|v\| \geq \frac{3}{4} \\ g(v) = 1, & \|v\| \leq \frac{1}{2} \\ g(v) > 0, & \frac{1}{2} < \|v\| < \frac{3}{4}. \end{cases}$$

We define $\tilde{C}(x, v) = (1, \rho g(v)\psi(x)v)$ and find $\rho \in \mathbb{R}$ where $\tilde{C}$ satisfies the required conditions. We can immediately see from the definition that $\tilde{C} = C$ is established on $\mathbb{R}^n \setminus U$. The differential equation associated with $\tilde{C}$ can be written as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = 1, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \rho g(v)\psi(x)v. \end{cases} \tag{4.1}$$

Let $v_0 \in I^{n-1}$ satisfy $\|v_0\| \le 1/4$. We have $g(v_0) = 1$ and $g(v) = 1$ in a neighborhood of v_0 (it is clear by the definition of g). The solution of (4.1) with initial conditions $x(0) = -1$ and $v(0) = v_0$ can be written as

$$x(t) = t - 1 \text{ and } v(t) = v_0 + \int_0^t \rho g(v(s))\psi(s-1)v(s)\mathrm{d}s.$$

By the continuity of $v(t)$, there is $l > 0$ such that $g(v(s)) = 1$ for all $s \in \left[0, -\frac{1}{2} + l\right]$, i.e., $v(t) = v_0 + \int_0^t \rho\psi(s-1)v(s)\mathrm{d}s$ on $\left[0, -\frac{1}{2} + l\right]$. Then we obtain

$$v(t) = v_0 \times \exp\left(\rho \int_0^t \psi(s-1)\mathrm{d}s\right)$$

on $\left[0, -\frac{1}{2} + l\right]$ by solving the differential equation. Put $\eta(t) = \exp\left(\rho \int_0^t \psi(s-1)\mathrm{d}s\right)$. Then clearly $\eta(0) = 1$ and $\rho = \frac{\ln \eta(t)}{\int_0^t \psi(s-1)\mathrm{d}s}$. Take $\varepsilon = \frac{\ln 2}{\int_0^2 \psi(s-1)\mathrm{d}s}$. Then since η is increasing, for $0 < \rho < \varepsilon$ we have $0 < \eta(2) \le 2$. Thus, $\|v(s)\| \le 2\|v_0\| \le 1/2$. Therefore, $g(v(s)) = 1$ for all $s \in [0, 2]$, so that

$$v(s) = \eta(s)v_0 \text{ and } J_{\tilde{C}}(-1, v_0) = (1, \eta v_0).$$

It is straightforward to find out that we can select ρ to form $\eta(2) = \theta$ for $0 < \theta - 1 < \varepsilon$, so that $J_{\tilde{C}}(-1, v_0) = (1, \theta v_0)$. Moreover, we explicitly obtain $\|\tilde{C} - C\|_{C^r} < \varepsilon'$ by taking ρ small.

Let X_θ be the vector field on M, which is equal to X outside $f^{-1}([-1, 1] \times I^{n-1})$ and $(f^{-1})_*\tilde{C}$ on $f^{-1}([-1, 1] \times I^{n-1})$. It is clear that X_θ is C^∞ and $\|X_\theta - X\|_{C^r} < \varepsilon'$. The expression for the Poincaré map in the local chart is $\mathbb{P}_\theta(1, v) = \theta\mathbb{P}(1, v)$ if $\|v\| \le 1/4$. Thus, $D_{(\{-1\}\times 0)}\mathbb{P}_\theta(\omega) = \theta\omega$ for all $\omega \in I^{n-1}$. □

Lemma 4.3. *Let X be a C^2 weak μ-expansive vector field for any $\mu \in \mathcal{F}^*_X(M)$. Then* $\operatorname{supp}(\mu) \cap \operatorname{Sing}(X) = \varnothing$, *where* $\operatorname{supp}(\mu) = \{x \in M : \mu(V) > 0$ *for any open neighborhood V of x*$\}$.

Proof. Since X is weak μ expansive for every $\mu \in \mathcal{F}^*_X(M)$, there is a finite partition $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ of M satisfying $\mu(\Gamma^{X_t}_{\mathcal{A}}(x)) = 0$ for all $x \in M$. Let $\tau \in M$ be a singularity of X and $V = \operatorname{int}P(\tau)$. It is clear that $\Gamma^{X_t}_{\mathcal{A}}(\tau) = \mathrm{P}(\tau)$. Thus, we obtain

$$\mu(V) \le \mu(P(\tau)) = \mu\left(\Gamma^{X_t}_{\mathcal{A}}(\tau)\right) = 0,$$

this fact means $\tau \notin \operatorname{supp}(\mu)$. □

We state that a point $x \in M$ is called a *stable point* for a vector field X if for any $\varepsilon > 0$, there is $\delta > 0$ such that if $d(x, y) < \delta$ then $d(X_t(x), X_t(y)) < \varepsilon$ for all $t \ge 0$.

Remark 4.4. If X is a weak measure expansive vector field on M, then every closed orbit is not stable. We already proved this fact holds for diffeomorphisms in Lemma 3.2. This property can naturally be extended to flows.

For a vector field X, a property "P" is said to be C^r-*robust* if there is a C^r-neighborhood $\mathcal{U} \subset \mathfrak{X}^r(M)$ of X in such that for any $Y \in \mathcal{U}$,Y satisfies "P."

A compact invariant set T is *transitive* if $T = \omega_X(q)$ for some $q \in T$ and *attracting* if $T = \cap_{t\geq 0} X_t(V)$ for some neighborhood T of C satisfying $X_t(V) \subset V$ for all $t \geq 0$. An *attractor* of X is a transitive attracting set of X and a *repeller* is an attractor for $-X$. A *sink* of X is a trivial attractor of X, namely it reduces to a single orbit of X and a *source* of X is a trivial repeller of X. Otherwise, the single orbit is called a *saddle*.

Theorem 4.5. *If X is a C^2-robustly weak measure expansive vector field on M, then every periodic orbit $\gamma \in \mathrm{Per}(X)$ is a saddle.*

Proof. To induce contradiction, assume that there exists $\gamma \in \mathrm{Per}(X)$ with period T which is not saddle. This means that for a Poincaré map $\mathbb{P}$ of X there is $q \in \gamma$ with period T such that $D_q\mathbb{P}$ has eigenvalues $|\lambda_1| < 1$ and $|\lambda_2| = 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| = 1$, respectively). Let U be a neighborhood of q. By Lemma 4.2, take a Poincaré map $\mathbb{P}_\theta : \{q\} \times N_q \to \{q\} \times N_q$ of X_θ fixing q and being identity outside U. Here $N_q = (\mathrm{Span}X(q))^\perp \subset T_qM$. Assume that $\theta \in (0, 1)$ is closer to 1. Then

- q is a periodic point of $\mathbb{P}_\theta$ with period T,
- $\mathbb{P}_\theta$ is C^2-close to $\mathbb{P}$, and
- eigenvalues of $D_q\mathbb{P}_\theta$ are $\theta\lambda_1$, $\theta\lambda_2$ such that modulus (strictly) smaller than 1.

Therefore, q is a sink for $\mathbb{P}_\theta$, that is, it is a sink for X_θ. Since X is a C^2-robustly weak measure expansive flow, X_θ is a weak measure expansive flow. But it is contradiction by Lemma 4.4. □

We say that X is a $C^r(r \geq 1)$-*star vector field* if there is a C^r-neighborhood $\mathcal{U}$ of X such that every critical orbit of every $Y \in \mathcal{U}$ is a hyperbolic set. In the case of $r = 1$, Gan and Wen proved that any nonsingular star flow satisfies Axiom A and the no cycle condition in [4]. However, as in the case of discrete dynamics, we do not yet know whether an equivalent relationship exists in C^2-dynamics in continuous dynamical systems, so we propose the following question. "If a vector field X on 3-dimensional manifold M or any dimensional manifold M is C^2-star, then does it satisfy Axiom A?" In the remainder of the article, we give a partial answer to the aforementioned question.

Lemma 4.6. *Every C^2-robustly weak measure expansive vector field X on M is a C^2-star vector field.*

Proof. To derive a contradiction, we suppose that there is a nonhyperbolic orbit $\gamma \in \mathrm{Per}(Y)$ for some $Y \in \mathcal{U}(X)$. Here, $\mathcal{U}(X)$ is a C^2-neighborhood of X. Let $x \in \gamma$ and let $\mathbb{P} : V \to \Sigma$ be a Poincaré map of Y associated with the orbit γ (and the section Σ). Then $D_x\mathbb{P}^{\pi(x)}$ has an eigenvalue λ with $\|\lambda\| = 1$, or only one pair of complex conjugated eigenvalues.

Case 1 : Let $\lambda_1 = 1$ and $\lambda_2 < 1$ (or $\lambda_2 > 1$).

Then by Lemma 4.2, we can make a C^2-vector field Y_θ and Poincaré map $\mathbb{P}_\theta$ such that eigenvalues $\theta\lambda_1$ and $\theta\lambda_2$ of $D_x\mathbb{P}_\theta^{\pi(x)}$ are less than 1 (or bigger than 1). This means that Y_θ has a sink point x, and this is a contradiction by Remark 4.4.

Case 2 : We can prove the second case similarly and complete the proof. □

Definition 4.7. We say that C^2-vector field X is a Q^2-*Anosov vector field* if it is Axiom A, has no cycles, and there is no 2-tangency.

Proof of Theorem 1.2. Suppose that X is not Q^2-Anosov. Since X is Axiom A and has no cycle condition, there exists a 2-tangency $x \in M\setminus\Omega(X)$. According to Lemmas 4.1 and 4.3, we consider a normal section N_x then there is a local chart $\phi : V_x \to \mathbb{R}^2$ and open sets $U, V \subset \mathbb{R}^2$ such that $\phi(V_x) \subseteq U \times V$. Then it is clear $V_x \subseteq \phi^{-1}(U \times V)$ and we let $W_x = [\phi^{-1}(U \times V)] \cap V_x$. We can take a C^2-local coordinate $\psi = \phi^{-1}|_{W_x} : W_x = U_x' \times V_x' \to N_x$ around x where $U_x', V_x' \subset \mathbb{R}^2$ are open sets.

Let $g_s, g_u : U'_x \to V'_x$ be C^2 functions such that their graphs describe the local stable and local unstable manifolds of x in coordinates and let $\theta > 0$. As in the proof of Theorem 1.1, we find maps ξ, φ_θ, ς and can define $f_\theta : N_x \to N_{X_{t_1}(x)}$ ($t_1 \geq 1$) by

$$f_\theta(y) = \begin{cases} \tilde{f}_X \circ \psi \circ \varphi_\theta \circ \psi^{-1}(y) & \text{for } y \in \psi(B_\theta(0,0)) \cap N_x, \\ \tilde{f}_X(y) & \text{otherwise,} \end{cases}$$

where $\tilde{f}_X : N_x \to N_{X_{t_1}(x)}$ is a time-1 map with respect to X. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ be a finite θ-partition on N_x. Then the same way of proof of Theorem 1.1, we can take an enough small constant $\tilde{\theta} < \theta$ and an arc $\gamma = \{\psi(x, g_s(x)) : |x| < \tilde{\theta}\} \subset N_x$ such that

- diam$(f_\theta^n(\gamma)) \to 0$ as $n \to \pm\infty$ and
- $\gamma \subset \operatorname{int}(A_i)$ for some $A_i \in \mathcal{A}$

and so f_θ is not weak measure expansive, i.e., $\tilde{f}_X$ is not C^2-robustly weak measure expansive. Then X is not weak measure expansive by Lee and Oh [5]. This contradicts the assumption. □

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