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Research Article

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Class-preserving Coleman automorphisms of some classes of finite groups

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Abstract: The normalizer problem of integral group rings has been studied extensively in recent years due to its connection with the longstanding isomorphism problem of integral group rings. Class-preserving Coleman automorphisms of finite groups occur naturally in the study of the normalizer problem. Let G be a finite group with a nilpotent subgroup N. Suppose that G/N acts faithfully on the center of each Sylow subgroup of N. Then it is proved that every class-preserving Coleman automorphism of G is an inner automorphism. In addition, if G is the product of a cyclic normal subgroup and an abelian subgroup, then it is also proved that every class-preserving Coleman automorphism of G is an inner automorphism. Other similar results are also obtained in this article. As direct consequence, the normalizer problem has a positive answer for such groups.

Keywords: class-preserving automorphisms, Coleman automorphisms, the normalizer property

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1 Introduction

All groups considered in this article are finite. Let M be a subgroup of G and let $\sigma \in \mathrm{Aut}(G)$. We write $\sigma|_M$ for the restriction of σ to M. Furthermore, suppose that $M \trianglelefteq G$ and σ fixes M. Then, by abuse of notation, we write $\sigma|_{G/M}$ for the automorphism of G/M induced by σ. Let g be a fixed element in G. We write $\mathrm{conj}(g)$ for the inner automorphism of G induced by g via conjugation. Denote by $\pi(G)$ the set of all primes dividing $|G|$. Other notations will be mostly standard, refer to [1,2].

Let G be a finite group and $\mathbb{Z}G$ be its integral group ring over $\mathbb{Z}$. Denote by $U(\mathbb{Z}G)$ the group of units of $\mathbb{Z}G$. The normalizer problem (see problem 43 in [2]) of integral group rings asks whether $N_{U(\mathbb{Z}G)}(G) = G \cdot Z(U(\mathbb{Z}G))$ for any finite group G, where $N_{U(\mathbb{Z}G)}(G)$ and $Z(U(\mathbb{Z}G))$ denote the normalizer of G in $U(\mathbb{Z}G)$ and the center of $U(\mathbb{Z}G)$, respectively. If the equality is valid for G, then we say that the normalizer property holds for G.

This equality was first shown to be true for finite nilpotent groups by Coleman in [3], and later this result was extended to any finite group having a normal Sylow 2-subgroup by Jackowski and Marciniak in [4]. It was Mazur who first noted that there are close connections between the normalizer problem and the isomorphism problem (see [5–7]). Based on Mazur's observations, among other things, Hertweck in [8] constructed the first counterexample to the normalizer problem and then the first counterexample to the isomorphism problem. Nevertheless, it is still of interest to determine for which groups the normalizer property holds. Recently, lots of positive results on the normalizer problem can be found in [9–13].

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For any $u \in N_{U(\mathbb{Z}G)}(G)$, we write φ_u to denote the automorphism of G induced by u via conjugation, i.e., $g^{\varphi_u} = u^{-1}gu$ for all $g \in G$. All such automorphisms of G form a subgroup of $\mathrm{Aut}(G)$, denoted by $\mathrm{Aut}_{\mathbb{Z}}(G)$. It is not hard to see that $\mathrm{Inn}(G) \leq \mathrm{Aut}_{\mathbb{Z}}(G)$. Let $\mathrm{Out}_{\mathbb{Z}}(G) := \mathrm{Aut}_{\mathbb{Z}}(G)/\mathrm{Inn}(G)$. A question (see Question 3.7 in [4]) asks whether $\mathrm{Out}_{\mathbb{Z}}(G) = 1$ for any finite group G.

It turns out that the aforementioned question is equivalent to the normalizer problem. Due to this, it is not a surprise that some classes of special automorphisms occur naturally in the study of the normalizer problem. $\mathrm{Aut}_c(G)$ denotes the class-preserving automorphism group of G, in which every automorphism sends $g \in G$ to some conjugate of g. $\mathrm{Aut}_{\mathrm{Col}}(G)$ denotes the Coleman automorphism group of G, in which the restriction of every automorphism to each Sylow subgroup of G equals the restriction of some inner automorphism of G. Set $\mathrm{Out}_c(G) = \mathrm{Aut}_c(G)/\mathrm{Inn}(G)$ and $\mathrm{Out}_{\mathrm{Col}}(G) = \mathrm{Aut}_{\mathrm{Col}}(G)/\mathrm{Inn}(G)$. It is known by Coleman's lemma (see in [3]) that $\mathrm{Out}_{\mathbb{Z}}(G) \leq \mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$. In addition, Krempa showed that $\mathrm{Out}_{\mathbb{Z}}(G)$ is an elementary abelian 2-group (proof can be found in [4]). Thus, if one can show that $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$ is of odd order, then $\mathrm{Out}_{\mathbb{Z}}(G) = 1$, namely, the normalizer property holds for such group G.

In this direction, Hertweck (see [1,12]) proved that if Sylow 2-subgroups of a finite group G are cyclic, dihedral, or generalized quaternion, then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$ is of odd order. Marciniak and Roggenkamp [9] proved that the normalizer property holds for metabelian groups with abelian Sylow 2-subgroups. For other related results, see [14–19].

The aim of this article is to investigate class-preserving Coleman automorphisms of some classes of finite groups without any restrictions on the structure of Sylow 2-subgroups. In Section 2, we present some lemmas which will be used in the sequel. In Section 3, we give some results on class-preserving Coleman automorphisms of some groups. Particularly, we prove that if G is a finite group with a nilpotent normal subgroup N and G/N acts faithfully on the center of each Sylow subgroup of N, then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$. The counterexample to the normalizer problem constructed by Hertweck (see [8]) is a metabelian group. However, we can show that if G is the product of a cyclic normal subgroup and an abelian subgroup, then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$; in particular, the normalizer property holds for G. Some other related results are also obtained in Section 3.

2 Preliminaries

In this section, some lemmas needed in the sequel are presented.

Lemma 2.1. [17] *Let G be a finite group with a nilpotent normal subgroup N. Assume that P is an arbitrary Sylow subgroup of N and G/N acts faithfully on $Z(P)$. Then $C_G(P) \leq N$. In particular, $C_G(N) \leq N$.*

Lemma 2.2. [19] *Let P be a normal p-subgroup of a finite group G. If $C_G(P) \leq P$, then G has no noninner p-central automorphisms. In particular,* $\mathrm{Out}_{\mathrm{Col}}(G) = 1$.

Lemma 2.3. *Let G be a finite group, H be a subgroup of G, and let σ be an automorphism of G of p-power order, where p is a prime. If there is $x \in G$ such that $\sigma|_H = \mathrm{conj}(x)|_H$, then there exists some $\gamma \in \mathrm{Inn}(G)$ such that $\sigma\gamma|_H = id|_H$ and $\sigma\gamma$ is still of p-power order.*

Proof. Set $o(\sigma) = p^i$, where $i \in \mathbb{N}$. Write $\beta := \mathrm{conj}(x)$. Then $\sigma|_H = \beta|_H$, i.e., $\sigma\beta^{-1}|_H = id|_H$. Let $n \in \mathbb{N}$ such that $(\sigma\beta^{-1})^n$ be the p-part of $\sigma\beta^{-1}$ with $(n, p) = 1$. Then there exists $s, t \in \mathbb{Z}$ such that $sn + tp^i = 1$. Obviously, $(\sigma\beta^{-1})^{sn}$ is of p-power order and $(\sigma\beta^{-1})^{sn}|_H = id|_H$. Note that $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$, so there exists some $\gamma \in \mathrm{Inn}(G)$ such that $(\sigma\beta^{-1})^{sn} = \sigma^{sn}\gamma = \sigma^{1-tp^i}\gamma = \sigma\gamma$. Hence, γ is the desired inner automorphism. □

Lemma 2.4. [1] *Let p be a prime, and σ an automorphism of G of p-power order. Assume further that there is $N \trianglelefteq G$ such that σ fixes all elements of N, and that σ induces the identity on G/N. Then σ induces the identity on $G/O_p(Z(N))$. If σ fixes in addition a Sylow p-subgroup of G element-wise, then σ is an inner automorphism.*

Lemma 2.5. [1] *Let $N \trianglelefteq G$ and let p be a prime which does not divide the order of G/N. Then the following hold.*

(1) *If $\sigma \in \mathrm{Aut}(G)$ is a class-preserving or Coleman automorphism of G of p-power order, then σ induces a class-preserving or a Coleman automorphism of N, respectively;*

(2) *If $\mathrm{Out}_c(N)$ or $\mathrm{Out}_{\mathrm{Col}}(N)$ is a p'-group, then so is $\mathrm{Out}_c(G)$ or $\mathrm{Out}_{\mathrm{Col}}(G)$. If $\mathrm{Out}_c(N) \cap \mathrm{Out}_{\mathrm{Col}}(N)$ is a p'-group, then so is $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$.*

Lemma 2.6. *Let G be a finite group and let N be a subgroup of G. Let σ be an automorphism of G of p-power order with p a prime. Suppose that σ fixes N and $\sigma|_N = \mathrm{conj}(x)|_N$ for some $x \in G$. Then there exists a p-element $y \in G$ such that $\sigma|_N = \mathrm{conj}(y)|_N$.*

Proof. Let $o(\sigma) = p^i$, $o(x) = p^j t$, where $i, j, t \in \mathbb{N}$ and $(p, t) = 1$. Set $k = \max\{i, j\}$. Since $(p^k, t) = 1$, it follows that there exists $u, v \in \mathbb{Z}$ such that $up^k + vt = 1$. Write $y = x^{vt}$. Then it is obvious that y is a p-element. For any $z \in N$, since $z = z^{\sigma^{up^k}} = z^{x^{up^k}}$, it follows that $z^\sigma = z^x = z^{x^{up^k+vt}} = (z^{x^{up^k}})^{x^{vt}} = z^{x^{vt}} = z^y$, namely, $\sigma|_N = \mathrm{conj}(y)|_N$. □

Lemma 2.7. [20] *Let A be an abelian p'-group and let N be a p-group on which A acts, where p is a prime. Then $C_A(N) = C_A(x) = C_A(\bar{N}) = C_A(\bar{x})$ for some $x \in N$, where $\bar{N} = N/\Phi(N)$ with $\Phi(N)$ being the Frattini subgroup of N.*

3 Proof of the theorems

Theorem 3.1. *Let G be a finite group with a nontrivial nilpotent normal subgroup N. Assume that G/N acts faithfully on the center of each Sylow subgroup of N. Then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$. In particular, the normalizer property holds for G.*

Proof. Let $q \in \pi(G)$ and let $\sigma \in \mathrm{Aut}_c(G) \cap \mathrm{Aut}_{\mathrm{Col}}(G)$ be of q-power order. We have to show that $\sigma \in \mathrm{Inn}(G)$. If $|\pi(N)| = 1$, then by Lemma 2.1, $C_G(N) \leq N$. Furthermore, by Lemma 2.2, this implies that $\mathrm{Out}_{\mathrm{Col}}(G) = 1$. In particular, $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$. Hereafter, we assume that $|\pi(N)| = r > 1$.

Claim 1. $\sigma|_N = id|_N$.

Let $\pi(N) = \{p_1, p_2, \ldots, p_r\}$ and let $P_i \in \mathrm{Syl}_{p_i}(N)$, where $i = 1, 2, \ldots, r$. Then $N = P_1 \times P_2 \times \cdots \times P_r$. Since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, there exists some $h_i \in G$ such that

$$\sigma|_{P_i} = \mathrm{conj}(h_i)|_{P_i}. \tag{1}$$

For any $z_i \in Z(P_i)$, by equation (1), we have

$$z_i^\sigma = h_i^{-1} z_i h_i. \tag{2}$$

For any $z_i \in Z(P_i)$ and $z_j \in Z(P_j)$ with $i \neq j$, by equation (2),

$$(z_i z_j)^\sigma = h_i^{-1} z_i h_i h_j^{-1} z_j h_j. \tag{3}$$

On the other hand, since $\sigma \in \mathrm{Aut}_c(G)$, there exists $h \in G$ such that

$$(z_i z_j)^\sigma = h^{-1} z_i h h^{-1} z_j h. \tag{4}$$

Combining equation (3) with (4), we obtain

$$(h_i^{-1} z_i h_i)(h_j^{-1} z_j h_j) = (h^{-1} z_i h)(h^{-1} z_j h). \tag{5}$$

Since N is nilpotent, by equation (5), we have $h_i^{-1} z_i h_i = h^{-1} z_i h$ and $h_j^{-1} z_j h_j = h^{-1} z_j h$. That is,

$$(h_i h^{-1})^{-1} z_i (h_i h^{-1}) = z_i, \tag{6}$$

$$(h_j h^{-1})^{-1} z_j (h_j h^{-1}) = z_j. \tag{7}$$

Since $\langle \bar{h}_i\bar{h}_i^{-1}\rangle$ is cyclic, there exists some element in $Z(P_i)$, say z_i, such that

$$C_{\langle \bar{h}_i\bar{h}_i^{-1}\rangle}(z_i) = \cap_{z\in Z(P_i)} C_{\langle \bar{h}_i\bar{h}_i^{-1}\rangle}(z) = C_{\langle \bar{h}_i\bar{h}_i^{-1}\rangle}(Z(P_i)) \le C_{G/N}(Z(P_i)) = 1.$$

So by equations (6) and (7), $hN = h_iN = h_jN$. As i, j are arbitrary, we have $hN = h_1N = h_2N = \cdots = h_rN$. Set $h_i = hn_i$ with $n_i \in N$, $i = 1, 2, \ldots, r$. For any $x_i \in P_i$, by equation (1),

$$x_i^{\sigma} = n_i^{-1}h^{-1}x_ihn_i. \tag{8}$$

As N is nilpotent, we may assume $n_i \in P_i$ in equation (8). Write $n := n_1n_2 \cdots n_r$. Then, by equation (8), for any $x = x_1x_2 \cdots x_r \in N$ with $x_i \in P_i$,

$$x^{\sigma} = n^{-1}h^{-1}xhn. \tag{9}$$

This shows that $\sigma|_N = \mathrm{conj}(hn)|_N$. By Lemma 2.3, we may assume that $\sigma|_N = id|_N$, as claimed.

Claim 2. $\sigma|_{G/N} = id|_{G/N}$.

For any $g \in G$ and $n \in N$, by Claim 1, $n^g = (n^g)^{\sigma} = n^{g^{\sigma}}$, implying $g^{\sigma}g^{-1} \in C_G(N)$. Recall that $C_G(N) \le N$. So the preceding equality implies that $\sigma|_{G/N} = id|_{G/N}$, as claimed.

Claim 3. $\sigma \in \mathrm{Inn}(G)$.

By Lemma 2.4, Claims 1 and 2 yield that $\sigma|_{G/O_q(Z(N))} = id|_{G/O_q(Z(N))}$. If $q \notin \pi(N)$, then the preceding equation implies that $\sigma = id$. It remains to consider the case $q \in \pi(N)$. Let Q be a Sylow q-subgroup of G fixed by σ. Then $Q_1 := Q \cap N$ is the Sylow q-subgroup of N. Since $\sigma \in \mathrm{Aut}_{\mathrm{Col}}(G)$, there exists some q-element $g \in G$ such that $\sigma|_Q = \mathrm{conj}(g)|_Q$. In particular, $\sigma|_{Z(Q_1)} = \mathrm{conj}(g)|_{Z(Q_1)}$. On the other hand, By Claim 1, $\sigma|_{Z(Q_1)} = id|_{Z(Q_1)}$. Consequently, $g \in C_G(Z(Q_1))$. It follows that $gN \in C_{G/N}(Z(Q_1))$. From this we deduce that $gN = N$ since the action of G/N on $Z(Q_1)$ is faithful. So $g \in N$ and hence $g \in Q_1$. Note that $\sigma|_{Q_1} = \mathrm{conj}(g)|_{Q_1} = id|_{Q_1}$. So $g \in Z(Q_1) \le Z(N)$. It follows that $\sigma\mathrm{conj}(g^{-1})|_N = id|_N$, $\sigma\mathrm{conj}(g^{-1})|_{G/N} = id|_{G/N}$, and $\sigma\mathrm{conj}(g^{-1})|_Q = id_Q$. So by Lemma 2.4 $\sigma \in \mathrm{Inn}(G)$. We are done. □

As immediate consequences of Theorem 3.1, we have the following results.

Corollary 3.2. *Let $G = NwrK$ be the standard wreath product of N by K, where N is a nontrivial nilpotent group and K is an arbitrary group. Then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$. In particular, the normalizer property holds for G.*

Proof. Let $|K| = r$. Then $G = NwrK = N^r \rtimes K$, where N^r is the direct product of r copies of N. Let $p \in \pi(N)$ and let $P \in \mathrm{Syl}_p(N^r)$. We will show that K acts faithfully on $Z(P)$. Since N is a nilpotent group, thus N^r is also a nilpotent group. Obviously, K acts on $Z(P)$. For any $y \in Z(P)$, if $y^h = y$, where $h \in K$. Since the intersection of $Z(P)$ with each component of N^r is nontrivial, i.e., $Z(P)$ is extensive in N^r, we deduce that $h = 1$, this shows that K acts faithfully on $Z(P)$. Thus, the assertion follows from Theorem 3.1. □

As a direct consequence of Corollary 3.2, we have the following result, which generalizes a well-known result due to Petit Lobão and Sehgal ([11], Theorem 1).

Corollary 3.3. *Let $G = NwrS_m$, where N is a finite nilpotent group and S_m is the group of all permutations on m letters. Then the normalizer property holds for G.*

Corollary 3.4. *Let G be the holomorph of an arbitrary nilpotent group N, i.e., $G = N \rtimes \mathrm{Aut}(N)$. Then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G) = 1$. In particular, the normalizer property holds for G.*

Theorem 3.5. *Let $G = NA$, where N is a nilpotent normal subgroup and A is an abelian subgroup. Then $\mathrm{Out}_c(G) \cap \mathrm{Out}_{\mathrm{Col}}(G)$ is a p'-group for each $p \in \pi(G)\backslash\pi(N) \cap \pi(A)$.*

Proof. For any $p \in \pi(G)\backslash\pi(N) \cap \pi(A)$, let $\rho \in \mathrm{Aut}_c(G) \cap \mathrm{Aut}_{\mathrm{Col}}(G)$ be of p-power order. We will show that ρ is an inner automorphism.

Case 1. $p \in \pi(N)$.

Let N_p be the Sylow p-subgroup of N. Then N_p char $N \trianglelefteq G$. It follows that N_p is a normal Sylow p-group of G. By Lemma 2.5(2) (replacing N therein with N_p), $\mathrm{Out_{Col}}(G)$ is a p'-group. In particular, $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G)$ is a p'-group.

Case 2. $p \in \pi(A)$.

Let A_p be the Sylow p-subgroup of A. Then NA_p is normal in G since G/N is abelian. Note that G/NA_p is a p'-group. By Lemma 2.5, we may assume that $G = NA_p$.

Claim 1. $\rho\mathrm{conj}(g^{-1})|_N \in \mathrm{Aut_{Col}}(N)$ for some $g \in G$.

Let $\pi(N) = \{p_1, p_2, \dots, p_r\}$ and let $P_i \in \mathrm{Syl}_{p_i}(N)$, where $i = 1, 2, \dots, r$. Then $N = P_1 \times P_2 \times \cdots \times P_r$. Since ρ is a Coleman automorphism, by Lemma 2.6, for each P_i, there exists a p-element $h_i \in G$ such that

$$\rho|_{P_i} = \mathrm{conj}(h_i)|_{P_i}. \tag{10}$$

By Lemma 2.7, $C_{A_p}(P_i) = C_{A_p}(x_i)$ for some $x_i \in P_i$, where $i = 1, 2, \dots, r$. Write $x = x_1x_2 \cdots x_r$. Then there exists $g \in G$ such that $x^\rho = x^g$, i.e., $(g^{-1}x_1g)\cdots(g^{-1}x_rg) = (h_1^{-1}x_1h_1)\cdots(h_r^{-1}x_rh_r)$. From this we obtain $g^{-1}x_ig = h_i^{-1}x_ih_i$. It follows that

$$[h_ig^{-1}, x_i] = 1. \tag{11}$$

Since $G = N \rtimes A_p$, we may set $h_ig^{-1} = na$, where $n \in N$ and $a \in A_p$. We will show $a \in C_{A_p}(P_i)$. Let $\bar{P}_i := P_i/\Phi(P_i)$, where $\Phi(P_i)$ is the Frattini subgroup of P_i. Then G acts on $\bar{P}_i$. By equation (11), we have

$$[h_ig^{-1}, \bar{x}_i] = 1. \tag{12}$$

On the other hand,

$$[h_ig^{-1}, \bar{x}_i] = [na, \bar{x}_i] = [a, \bar{x}_i]. \tag{13}$$

So equations (12) and (13) imply that $[a, \bar{x}_i] = 1$. Again by Lemma 2.7, $a \in C_{A_p}(\bar{x}_i) = C_{A_p}(P_i)$. This, together with equation (10), implies that $\rho\mathrm{conj}(g^{-1})|_{P_i} = \mathrm{conj}(n)|_{P_i}$. This shows that $\rho\mathrm{conj}(g^{-1})|_N \in \mathrm{Aut_{Col}}(N)$, as claimed.

Claim 2. $\rho \in \mathrm{Inn}(G)$.

Since N is nilpotent, it follows that $\rho\mathrm{conj}(g^{-1})|_N = \mathrm{conj}(n)|_N$ for some $n \in N$. That is, $\rho|_N = \mathrm{conj}(ng)|_N$. With this in hand, by Lemma 2.3, we may assume that

$$\rho|_N = id|_N. \tag{14}$$

Note that G/N is abelian. So we have

$$\rho|_{G/N} = id|_{G/N}. \tag{15}$$

By Lemma 2.4, equations (14) and (15) yield that $\rho|_{G/O_p(Z(N))} = id|_{G/O_p(Z(N))}$. Note that N is a p'-group. So the preceding equality is precisely $\rho = id$. We are done. □

Corollary 3.6. *Let G be an extension of a p-group by an abelian group. Then* $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G) = 1$. *In particular, the normalizer property holds for G.*

Corollary 3.7. *Let G be an extension of a nilpotent group of odd order by an abelian group. Then* $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G)$ *is of odd order. In particular, the normalizer property holds for G.*

Marciniak and Roggenkamp (see [9]) constructed a finite metabelian group $G = (C_2^4 \times C_3) \rtimes C_2^3$ for which $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G)$ is of even order. It is clear that the group G is the semidirect product of a cyclic group of 3 by a nonabelian 2-group. This shows that if G is the product of a cyclic normal subgroup and a nilpotent subgroup, then it is not necessary that $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G)$ is trivial. However, we can prove the following result.

Theorem 3.8. *Let $G = CA$, where C is a cyclic normal subgroup and A is an abelian subgroup. Then* $\mathrm{Out}_c(G) \cap \mathrm{Out_{Col}}(G) = 1$. *In particular, the normalizer property holds for G.*

Proof. Let $p \in \pi(G)$ and $\rho \in \mathrm{Aut}_c(G) \cap \mathrm{Aut}_{\mathrm{Col}}(G)$ be of p-power order. We will show that ρ is inner. If either $p \in \pi(C) \backslash \pi(A)$ or $p \in \pi(A) \backslash \pi(C)$, then by Theorem 3.5 $\rho \in \mathrm{Inn}(G)$. If $p \in \pi(C) \cap \pi(A)$, then by Lemma 2.5(2) we may assume that A itself is a p-subgroup. Since C is cyclic and $\rho \in \mathrm{Aut}_c(G)$, it follows that there exists some $g \in G$ such that $\rho|_C = \mathrm{conj}(g)|_C$. Without loss of generality, we may set $\rho|_C = id|_C$. Let P_C be the Sylow p-subgroup of C. Then $P = P_C A$ is a Sylow p-subgroup of G. Without loss of generality, we may assume that ρ fixes P and $\rho|_P = \mathrm{conj}(x)|_P$ for some p-element x. Set $x = ab$ with $a \in P_C$ and $b \in A$. Note that $id|_{P_C} = \rho|_{P_C} = \mathrm{conj}(ab)|_{P_C} = \mathrm{conj}(b)|_{P_C}$. This yields that $b \in C_A(P_C)$. Since A is abelian, it follows that $b \in Z(P)$ and thus $\rho|_P = \mathrm{conj}(x)|_P = \mathrm{conj}(ab)|_P = \mathrm{conj}(a)|_P$. From this we deduce that $\rho = \mathrm{conj}(a)$. We are done. ☐

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