DE GRUYTER Open Mathematics 2022; 20: 1470–1478

Research Article

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Existence and multiplicity of solutions for second-order Dirichlet problems with nonlinear impulses

https://doi.org/10.1515/math-2022-0519
received June 25, 2022; accepted October 7, 2022

Abstract: We are concerned with Dirichlet problems of impulsive differential equations

$$\begin{cases} -u''(x) - \lambda u(x) + g(x, u(x)) + \sum_{j=1}^{p} I_j(u(x))\delta(x - y_j) = f(x) \quad \text{for a.e. } x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

where λ is a parameter and runs near 1, $f \in L^2(0, \pi)$, $I_j \in C(\mathbb{R}, \mathbb{R})$, $j = 1, 2, \ldots, p$, $p \in \mathbb{N}$, the nonlinearity $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, $\delta = \delta(x)$ denote the Dirac delta impulses concentrated at 0, which are applied at given points $0 < y_1 < y_2 < \cdots < y_p < \pi$. We show the existence and multiplicity of solutions to the aforementioned problem for λ in a neighborhood of 1 by using degree theory and bifurcation theory.

Keywords: near resonance, impulsive differential equations, *a priori* bounds, bifurcation from infinity

MSC 2020: 34B15, 34B37, 34C23

1 Introduction and main result

In this article, we are concerned with the existence and multiplicity of solutions for Dirichlet problems of impulsive differential equations

$$\begin{cases} -u''(x) - \lambda u(x) + g(x, u(x)) + \sum_{j=1}^{p} I_j(u(x))\delta(x - y_j) = f(x) \quad \text{for a.e. } x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases} \tag{1.1}$$

where λ is a parameter and runs near 1, $f \in L^2(0, \pi)$, $I_j : \mathbb{R} \to \mathbb{R}$ is continuous, $j = 1, 2, \ldots, p$, $p \in \mathbb{N}$, the nonlinearity $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition, $\delta = \delta(x)$ denote the Dirac delta impulses concentrated at 0, i.e., $\delta(x) = 0$ for $x \neq 0$, $\delta(0) = +\infty$, and $\int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = 1$, the Dirac delta impulses δ are applied at given points $0 < y_1 < y_2 < \cdots < y_p < \pi$.

In recent years, the existence and multiplicity of solutions for problem (1.1) with $I_j \equiv 0$, $j = 1, 2, \ldots, p$, have been extensively studied by many authors, see [1–6] and references therein. In particular, Mawhin and Schmitt [1] studied the Dirichlet problem with the parameter λ near the principal eigenvalue of the form:

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$$\begin{cases} u''(x) + \lambda u(x) + g(x, u(x)) = f(x) & \text{for a.e. } x \in (0, \pi), \\ u(0) = u(\pi) = 0. \end{cases} \tag{1.2}$$

Using degree theory together with bifurcation theory, they proved that problem (1.2) had near $\lambda = 1$ at least one solution for $\lambda \geq 1$ and at least two solutions for $\lambda < 1$ provided that

$$\int_0^\pi g^-(x) f(x) \mathrm{d}x < \int_0^\pi f(x) \sin x \mathrm{d}x < \int_0^\pi g^+(x) \sin x \mathrm{d}x,$$

where

$$g^-(x) := \limsup_{s\to-\infty} g(x, s), \quad g^+(x) := \liminf_{s\to+\infty} g(x, s).$$

In addition, Chiappinell et al. [2] showed that there exists $\nu > 0$ such that problem (1.2) with λ near 1 had at least one solution for $\lambda \leq 1$ and two solutions for $1 < \lambda < 1 + \nu$ under $\lim_{s\to+\infty} g(x, s)/s = 0$ and a Landesman-Lazer-type condition. Here, we emphasize that the existence and multiplicity of solutions in works [1,2] are obtained without impulsive effects.

As far as we know, the model of the impulsive differential equation describes evolution processes in which their states change abruptly at certain moments in time, see [7,8]. There has recently been increasing interest in studying the impulsive differential equation, see, for instance, [9–26]. In [9], Drábek and Langerová studied Dirichlet problems of impulsive differential equations

$$\begin{cases} -u''(x) - \mu u(x) + \tilde{g}(u(x)) = f(x), & x \in (0, \pi)\backslash\{x_1, x_2, \ldots, x_q\}, \\ \Delta u'(x_i) := u'(x_i^+) - u'(x_i^-) = I_i(u(x_i)), & i = 1, 2, \ldots, q, \\ u(0) = u(\pi) = 0, \end{cases} \tag{1.3}$$

where $\mu \in \mathbb{R}$ is a parameter, $0 < x_1 < \cdots < x_q < \pi$, $\tilde{g} \in C(\mathbb{R}, \mathbb{R})$, and $I_i \in C(\mathbb{R}, \mathbb{R})$. Based on topological degree arguments, they showed that problem (1.3) had at least one solution under the assumption that nonlinearity $\tilde{g}$ and impulses I_i satisfy sublinear growth at $\pm\infty$. Note that the multiplicity of solutions for problem (1.3) in [9] is not studied.

Recently, Shi and Chen [10] studied Dirichlet problems of impulsive differential equations

$$\begin{cases} -u''(x) = h(x, u(x)), & x \in (0, \pi)\backslash\{x_1, x_2, \ldots, x_q\}, \\ \Delta u'(x_i) := u'(x_i^+) - u'(x_i^-) = I_i(u(x_i)), & i = 1, 2, \ldots, q, \\ u(0) = u(\pi) = 0 \end{cases} \tag{1.4}$$

with $h \in C([0, \pi] \times \mathbb{R}, \mathbb{R})$. They showed that problem (1.4) had at least three solutions provided that impulses I_i satisfy sublinear growth at $\pm\infty$. The method consists in using Morse theory in combination with the minimax arguments. In [11], Liu and Zhao considered a Dirichlet problem of impulsive differential equations. The existence of multiple solutions is obtained by using variational methods combined with a three critical point theorem.

Although papers [10,11] obtained the multiplicity results of solutions for the Dirichlet problems of impulsive differential equations, they did not consider the near resonance problem corresponding to problem (1.3) or (1.4). It is natural to ask whether it is possible to obtain some multiplicity results for Dirichlet problems of impulsive differential equations near resonance. In this article, we show that this program can be developed for problem like (1.1). Specifically, we shall employ degree theory and bifurcation theory to deal with problem (1.1) under the following assumptions:

(A1) $I_j : \mathbb{R} \to \mathbb{R}$ is continuous and

$$\lim_{|s|\to\infty} \frac{I_j(s)}{s} = 0, \quad j = 1, \ldots, p.$$

(A2) $g : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory condition and there exists $\Gamma \in L^1(0, \pi)$ with

$$g(x, s) \leq \|\Gamma\|_{L^1}.$$

(A3) There exist limits $\lim_{s\to\pm\infty} g(x,s) = g(x,\pm\infty)$ for any $x \in (0,\pi)$ and $\lim_{s\to\pm\infty} I_j(s) = I_j(\pm\infty)$ for $j = 1, 2, \dots, p$. Moreover, the following inequality holds:

$$\sum_{j=1}^{p} I_j(-\infty)\sin y_j + \int_0^\pi g(x,-\infty)\sin x\,\mathrm{d}x < \int_0^\pi f(x)\sin x\,\mathrm{d}x < \sum_{j=1}^{p} I_j(+\infty)\sin y_j + \int_0^\pi g(x,+\infty)\sin x\,\mathrm{d}x. \quad (1.5)$$

Our main result is the following:

Theorem 1.1. *Assume that* (A1)–(A3) *hold. Then for all* $\lambda \in (-\infty, 4)$, *problem* (1.1) *has at least one solution. Furthermore, if* $\lambda < 1$ *but close to* 1, *then problem* (1.1) *has at least three solutions.*

Remark 1.1. For other multiplicity results of solutions of second-order impulsive differential equations, we refer the interested readers to D'Agui et al. [19], Han et al. [20], Henderson and Luca [21], and Lee and Lee [22]. Nevertheless, the difference between these works and our work consists of the fact that we studied the near resonance problem of the impulsive differential equation. Furthermore, the method we use (bifurcation theory) is also different from the methods used in the aforementioned works.

This article is organized as follows. Section 2 is devoted to proving the regularity of a weak solution to problem (1.1). In Section 3, we introduce some preliminary results from Mawhin and Schmitt [6] which are useful for the proof of our main result. Finally, in Section 4, we give *a priori* bounds of the eventual solutions of problem (1.1) and finish the proof of Theorem 1.1.

2 Regularity of weak solutions

Denote $H := H_0^1(0,\pi)$. We say that $u \in H$ is a *weak solution* of problem (1.1) if the integral identity

$$\int_0^\pi u'(x)v'(x)\mathrm{d}x - \lambda\int_0^\pi u(x)v(x)\mathrm{d}x + \int_0^\pi g(x,u(x))v(x)\mathrm{d}x = \int_0^\pi f(x)v(x)\mathrm{d}x - \sum_{j=1}^{p} I_j(u(y_j))v(y_j) \quad (2.1)$$

for any test function $v \in H$. For simplicity, we use the following notations:

$$0 = y_0 < y_1 < y_2 < \cdots < y_p < y_{p+1} = \pi; \quad \mathcal{I}_j := (y_{j-1}, y_j),\ j = 1, 2, \dots, p+1; \quad (0,\pi)\backslash\{y_1, y_2, \dots, y_p\} = \bigcup_{j=1}^{p+1} \mathcal{I}_j.$$

Let $\mathcal{D}(\mathcal{I})$, $\mathcal{I} \subset \mathbb{R}$, be the set of all infinitely differentiable functions on $\mathcal{I}$ with compact support lying in $\mathcal{I}$.

In the sequel, we prove the regularity of a weak solution $u \in H$.

Proposition 2.1. *The impulsive problem* (1.1) *is equivalent to Dirichlet problem*

$$\begin{cases} -u''(x) - \lambda u(x) + g(x,u(x)) = f(x), & x \in \mathcal{I}_j,\ j = 1, 2, \dots, p+1, \\ u(0) = u(\pi) = 0 \end{cases} \quad (2.2)$$

with impulsive conditions

$$\Delta u'(y_j) := \lim_{x\to y_j^+} u'(x) - \lim_{x\to y_j^-} u'(x) = I_j(u(y_j)), \quad j = 1, 2, \dots, p. \quad (2.3)$$

Proof. Choose $v \in \mathcal{D}(\mathcal{I}_j)$ and extend $v = 0$ on $(0,\pi)\backslash\mathcal{I}_j$, Then $v \in H$. Integrating by parts in (2.1), we obtain

$$\int_0^\pi u'(x)v'(x)\mathrm{d}x - \lambda\int_0^\pi v(x)\mathrm{d}\int_0^x u(\tau)\mathrm{d}\tau + \int_0^\pi v(x)\mathrm{d}\int_0^x g(\tau,u(\tau))\mathrm{d}\tau = \int_0^\pi v(x)\mathrm{d}\int_0^x f(\tau)\mathrm{d}\tau - \sum_{j=1}^{p} I_j(u(y_j))v(y_j).$$

Moreover, we have

$$\int_{\mathcal{I}_j}\left[u'(x)+\lambda\int_{y_{j-1}}^{x}u(\tau)\mathrm{d}\tau+\int_{y_{j-1}}^{x}f(\tau)\mathrm{d}\tau-\int_{y_{j-1}}^{x}g(\tau,u(\tau))\mathrm{d}\tau\right]v'(x)\mathrm{d}x=0. \tag{2.4}$$

Since (2.4) holds for arbitrary $v\in\mathcal{D}(\mathcal{I}_j)$, there exists a constant k_1 such that

$$u'(x)+\lambda\int_{y_{j-1}}^{x}u(\tau)\mathrm{d}\tau+\int_{y_{j-1}}^{x}f(\tau)\mathrm{d}\tau-\int_{y_{j-1}}^{x}g(\tau,u(\tau))\mathrm{d}\tau=k_1 \tag{2.5}$$

for a.e. $x\in\mathcal{I}_j$. Therefore, $u\in C^1(\mathcal{I}_j)$. Moreover, from (2.5), we infer that

$$-u''(x)-\lambda u(x)+g(x,u(x))=f(x),\quad x\in\mathcal{I}_j.$$

In particular, the equation in (1.1) holds pointwise in $\bigcup_{j=1}^{p+1}\mathcal{I}_j$.

We have $u\in C^1[0,\pi]$ by the embedding $H\hookrightarrow C[0,\pi]$. Set

$$0\le\eta<\min_{j=1,2,\dots,p}\{y_j-y_{j-1},y_{j+1}-y_j\}.$$

Choose $v\in\mathcal{D}(y_j-\eta,y_j+\eta)$ and extend $v=0$ on $(0,\pi)\setminus(y_j-\eta,y_j+\eta)$, i.e., $v\in H$. Integration by parts in (2.1) yields

$$\int_{y_j-\eta}^{y_j+\eta}u'(x)v'(x)\mathrm{d}x-\lambda\int_{y_j-\eta}^{y_j+\eta}u(x)v(x)\mathrm{d}x+\int_{y_j-\eta}^{y_j+\eta}g(x,u(x))v(x)\mathrm{d}x=\int_{y_j-\eta}^{y_j+\eta}f(x)v(x)\mathrm{d}x-\int_{y_j-\eta}^{y_j+\eta}I_j(u(x))\delta(x-y_j)v(x)\mathrm{d}x.$$

Moreover, we have

$$\int_{y_j-\eta}^{y_j+\eta}\left[u'(x)+\lambda\int_{y_j-\eta}^{x}u(\tau)\mathrm{d}\tau-\int_{y_j-\eta}^{x}g(\tau,u(\tau))\mathrm{d}\tau+\int_{y_j-\eta}^{x}f(\tau)\mathrm{d}\tau-\int_{y_j-\eta}^{x}I_j(u(\tau))\delta(\tau-y_j)\mathrm{d}\tau\right]v'(x)\mathrm{d}x=0.$$

Since $v\in(y_j-\eta,y_j+\eta)$ is arbitrary, there are constants k_2 such that

$$u'(x)+\lambda\int_{y_j-\eta}^{x}u(\tau)\mathrm{d}\tau-\int_{y_j-\eta}^{x}g(\tau,u(\tau))\mathrm{d}\tau+\int_{y_j-\eta}^{x}f(\tau)\mathrm{d}\tau-\int_{y_j-\eta}^{x}I_j(u(\tau))\delta(\tau-y_j)\mathrm{d}\tau=k_2 \tag{2.6}$$

for a.e. $x\in(y_j-\eta,y_j+\eta)$. Moreover, from (2.6), we conclude

$$\begin{aligned}\lim_{x\to y_j^+}u'(x)&=k_2-\lambda\int_{y_j-\eta}^{y_j}u(s)\mathrm{d}s+\int_{y_j-\eta}^{y_j}g(s,u(s))\mathrm{d}s+I_j(u(y_j))-\int_{y_j-\eta}^{y_j}f(s)\mathrm{d}s,\\ \lim_{x\to y_j^-}u'(x)&=k_2-\lambda\int_{y_j-\eta}^{y_j}u(s)\mathrm{d}s+\int_{y_j-\eta}^{y_j}g(s,u(s))\mathrm{d}s+0-\int_{y_j-\eta}^{y_j}f(s)\mathrm{d}s.\end{aligned} \tag{2.7}$$

Therefore, from (2.6) and (2.7), we obtain

$$\Delta u'(y_j):=\lim_{x\to y_j^+}u'(x)-\lim_{x\to y_j^-}u'(x)=I_j(u(y_j)).$$

It follows from the aforementioned considerations that $u\in C(0,\pi)$ and that the first derivative u' is piecewise continuous with discontinuities of the first kind at the points $y_1,y_2,\dots,y_p$.

The boundary conditions $u(0)=u(\pi)=0$ are satisfied automatically due to the fact that every weak solution u belongs to H. Therefore, the impulsive problem (1.1) is equivalent to the Dirichlet problem (2.2) with impulsive conditions (2.3). □

In fact, we have just proved that every weak solution to problem (1.1) is also a classical solution. On the other hand, it is obvious that every classical solution is also a weak solution. With this result at hand, we can look for (classical) solutions as for solutions of certain operator equation induced by (2.1). Therefore, we define the scalar product and the norm of H, respectively, given by

$$\langle u, v\rangle := \int_0^\pi u'(x)v'(x)\mathrm{d}x, \quad u, v \in H \tag{2.8}$$

and

$$\|u\| := \left(\int_0^\pi (u'(x))^2\mathrm{d}x\right)^{1/2}.$$

Moreover, we define operators $J, S, G, I^* : H \to H$ and an element f^* as follows:

$$\langle Ju, v\rangle := \int_0^\pi u'(x)v'(x)\mathrm{d}x, \quad \langle Su, v\rangle := \int_0^\pi u(x)v(x)\mathrm{d}x,$$

$$\langle G(u), v\rangle := \int_0^\pi g(x, u(x))v(x)\mathrm{d}x, \quad \langle I^*(u), v\rangle := \sum_{j=1}^p I_j(u(y_j))v(y_j),$$

$$\langle f^*, v\rangle := \int_0^\pi f(x)v(x)\mathrm{d}x, \quad u, v \in H.$$

Then problem (2.2), (2.3) is then equivalent to operator equation

$$Ju - \lambda Su + Gu + I^*(u) = f^*. \tag{2.9}$$

It follows from the definitions of operators J, S, G, I^* and the compact embedding $H \hookrightarrow C[0, \pi]$ that J is just identity, S is a linear compact operator, and G is a nonlinear compact operator. In addition, the element f^* is also well defined because $f \in L^2(0, \pi)$.

3 Bifurcation and continuation

Let E be a real Banach space with the norm $\|\cdot\|_E$ and let $\mathcal{A} : E \times \mathbb{R} \to E$ be a completely continuous operator. Consider the operator equation

$$u - \mathcal{A}(u, \lambda) = 0. \tag{3.1}$$

It follows from Mawhin and Schmitt [6] that the following results.

Lemma 3.1. [6, Theorem 1] *Let there exists a bounded open set Ω in E such that*

$$\deg(Id - \mathcal{A}(\cdot,a), \Omega, 0) \neq 0. \tag{3.2}$$

Then there exist continua C^- and C^+ with

$$C^- \subset E \times (-\infty, a] \cap (Id - \mathcal{A})^{-1}(0), \quad C^+ \subset E \times [a, +\infty) \cap (Id - \mathcal{A})^{-1}(0),$$

and for both $C = C^-$ and *$C = C^+$ the following are valid*:

(*i*) *$C \cap \Omega \times \{a\} \neq \emptyset$.*

(*ii*) *Either C is unbounded or else $C \cap E\backslash\bar{\Omega} \times \{a\} \neq \emptyset$.*

Lemma 3.2. [6, Corollary 2] *Assume the conditions of* Lemma 3.1, *where* Ω *is given by*

$$\Omega := B_R(0) = \{u \in E : \|u\|_E < R\}.$$

Moreover, assume that there exists $b > a$ *such that for any* λ, $a \le \lambda \le b$, *we have that* $\|u\|_E < R$, *where* (u, λ) *is a solution of operator equation* (2.9). *Then there exists a constant* $\alpha > 0$, *such that for* $b \le \lambda \le b + \alpha$, *there exists* $(u, \lambda) \in C^+$ *with* $\|u\|_E \le 2R$.

Remark 3.1. A similar statement holds for values of λ to the left of a.

As a further tool, we need a result that guarantees bifurcation from infinity. For this, we shall assume a particular form for the completely continuous mapping $\mathcal{A}$, namely

$$\mathcal{A}(u, \lambda) := \mathcal{L}(\lambda)u + \mathcal{B}(u, \lambda), \tag{3.3}$$

where $\mathcal{L}(\lambda)$ is a family of compact linear operators and the perturbation term $\mathcal{B}$ satisfies

$$\frac{\mathcal{B}(u, \lambda)}{\|u\|_E} \to 0 \quad \text{as } \|u\|_E \to \infty. \tag{3.4}$$

Lemma 3.3. [6, Theorem 3] *Assume* (3.3) *and* (3.4) *hold and assume that for* $\lambda = \lambda_1$ *the generalized nullspace of* $Id - \mathcal{L}(\lambda_1)$ *has odd dimension. Then there exists a continuum*

$$C \subset (Id - \mathcal{A})^{-1}(0)$$

which bifurcates from infinity at $\lambda = \lambda_1$. *That is, for any* $\varepsilon > 0$, *there exists* $(u, \lambda) \in C$ *with*

$$|\lambda - \lambda_1| < \varepsilon \quad \textit{and} \quad \|u\|_E > \frac{1}{\varepsilon}.$$

4 Proof of Theorem 1.1

Theorem 4.1. *Assume that* (A1)–(A3) *hold. Then for given* γ, $1 < \gamma < 4$, *there exists a constant* $R_0 > 0$, *such that, if* $1 \le \lambda \le \gamma$, *then any solution* u *of problems* (2.2) *and* (2.3) *satisfies*

$$\|u\|_\infty := \max_{0 \le x \le \pi} |u(x)| < R_0. \tag{4.1}$$

Proof. We break the proof into two parts, according to $\lambda = 1$ or $1 < \lambda \le \gamma$.
Case 1. $1 < \lambda \le \gamma$. Assume there exists a constant $R_1 > 0$ such that

$$Ju - \lambda Su + Gu + I^*(u) - f^* \neq 0$$

for all $\|u\| = R_1$, then the properties of operators J, S, G, I^*, and f^* imply that the Leray-Schauder degree

$$\deg\big(J - \lambda S + G + I^* - f^*, B_{R_1}, 0\big)$$

is well-defined, where $B_{R_1} := \{u \in H : \|u\| < R_1\}$. If we find $R_1 > 0$ such that

$$\deg\big(J - \lambda S + G + I^* - f^*, B_{R_1}, 0\big) \neq 0, \tag{4.2}$$

then there exists $u \in B_{R_1}$ satisfying operator equation (2.9). In order to search $R_1 > 0$ such that (4.2) holds, we use the homotopy invariance property of the Leray-Schauder degree. Specifically, we introduce the homotopy

$$\mathcal{H}_\lambda(\kappa, u) := Ju - \lambda Su - (1 - \kappa)\theta Su + \kappa Gu + \kappa I^* - \kappa f^*,$$

where $\kappa \in [0, 1]$ is a homotopy parameter and $\theta > 0$ satisfies

$$\lambda + (1 - \kappa)\theta \neq 1 \quad \text{for any } \kappa \in [0, 1].$$

We prove that there exists $R_1 > 0$ such that for all $u \in H$, $\|u\| = R_1$, and all $\kappa \in [0,1]$ we have

$$\mathcal{H}_\lambda(\kappa, u) \neq 0. \tag{4.3}$$

We prove (4.3) via contradiction. Assume that there exist $u_m \in H$, $\|u_m\| \to \infty$, $\kappa_m \in [0,1]$ such that $\mathcal{H}_\lambda(\kappa_m, u_m) = 0$ for $1 < \lambda \le \gamma$. Then, setting $v_m := u_m / \|u_m\|$, this is equivalent to

$$Jv_m - \lambda S v_m - (1-\kappa_m)\theta S v_m + \kappa_m \frac{G(u_m)}{\|u_m\|} + \kappa_m \frac{I^*(u_m)}{\|u_m\|} - \kappa_m \frac{f^*}{\|u_m\|} = 0. \tag{4.4}$$

The last three terms in (4.4) tend to zero due to assumptions (A1), (A2), and the fact that f^* is a fixed element. Passing to subsequences if necessary, we may assume $v_m \rightharpoonup v$ (weakly) in H for some $v \in H$ and $\kappa_m \to \kappa \in [0,1]$. Since S is compact, $Sv_m \to Sv$ (strongly) in H. Now the strong convergence $Jv_m \to Jv$ follows from (4.4). In particular, $\|v\| = 1$ and

$$Jv - (\lambda + (1-\kappa)\theta) Sv = 0. \tag{4.5}$$

Since $\lambda + (1-\kappa)\theta \neq 1$, equation (4.5) has only trivial solution, a contradiction. Hence, there exists a positive constant R_1 which is independent of the parameters κ_m and κ such that $\|u\| \le R_1$.

Case 2. $\lambda = 1$. In this case, our goal is to prove

$$\deg\big(J - S + G + I^* - f^*; B_{R_2}, 0\big) \neq 0$$

for some $R_2 > 0$. To this end, we introduce homotopy

$$\mathcal{H}_1(\kappa, u) := Ju - Su - (1-\kappa)\theta S u + \kappa G u + \kappa I^* - \kappa f^*,$$

where $\kappa \in [0,1]$ is a homotopy parameter and $0 < \theta < 3$. We prove that there exists $R_2 > 0$ such that

$$\mathcal{H}_1(\kappa, u) \neq 0 \tag{4.6}$$

holds for all $u \in H$, $\|u\| = R_2$, and all $\kappa \in [0,1]$. Then the result follows from (4.6) and the homotopy invariance of the Leary-Schauder degree as in the proof of Case 1.

We also prove (4.6) via contradiction. Let $u_m \in H$, $\|u_m\| \to \infty$, $\kappa_m \in [0,1]$ be such that $\mathcal{H}_1(\kappa_m, u_m) = 0$. This is equivalent to

$$Jv_m - Sv_m - (1-\kappa_m)\theta S v_m + \kappa G v_m + \kappa_m \frac{I^*(u_m)}{\|u_m\|} - \kappa_m \frac{f^*}{\|u_m\|} = 0.$$

It follows from (A1) that

$$\kappa_m \frac{I^*(u_m)}{\|u_m\|} \to 0, \quad -\kappa_m \frac{f^*}{\|u_m\|} \to 0.$$

Similarly as in the proof of above, we arrive at the limit equation

$$Jv - (1 + (1-\kappa)\theta) Sv = 0. \tag{4.7}$$

Since $1 + (1-\kappa)\theta < 4$ due to the choice of θ, (4.7) may occur only if $\kappa = 1$ and $v(x) = \pm\sqrt{\frac{2}{\pi}}\sin x$. First, we assume

$$v(x) = \sqrt{\frac{2}{\pi}}\sin x.$$

Taking the inner product of $\mathcal{H}_1(\kappa_m, u_m) = 0$ with $\sin x$, we obtain

$$(1-\kappa_m)\theta \int_0^\pi u_m(x)\sin x \, dx + \kappa_m \sum_{j=1}^p I_j(u_m(y_j))\sin y_j - \kappa_m \int_0^\pi (f(x) - g(x, u_m))\sin x \, dx = 0. \tag{4.8}$$

It follows from the embedding $H \hookrightarrow C[0,\pi]$ that $v_m(x) \rightrightarrows \sqrt{\frac{2}{\pi}}\sin x$ (uniformly) on $[0,\pi]$. Therefore, we have

$$\int_0^\pi u_m(x)\sin x \, dx > 0, \quad m \gg 1. \tag{4.9}$$

Combining (4.8) and (4.9), we have

$$\sum_{j=1}^{p} I_j(u_m(y_j))\sin y_j + \int_0^{\pi} g(x, u_m(x))\sin x\mathrm{d}x \le \int_0^{\pi} f(x)\sin x\mathrm{d}x. \tag{4.10}$$

Passing to the limit for $m \to \infty$ in (4.10), we obtain

$$\sum_{j=1}^{p} I_j(+\infty)\sin y_j + \int_0^{\pi} g(x, +\infty)\sin x\mathrm{d}x \le \int_0^{\pi} f(x)\sin x\mathrm{d}x.$$

However, this contradicts the second inequality in (1.5). If

$$v(x) = -\sqrt{\frac{2}{\pi}}\sin x,$$

we derive similarly a contradiction with the first inequality in (1.5). Hence, there exists a positive constant R_2, which is independent of the parameters κ_m and κ such that $\|u\| \le R_2$.

Consequently, set $R_3 := \max\{R_1, R_2\}$, we obtain

$$\|u\| \le R_3. \tag{4.11}$$

Moreover, (4.11) together with the compact embedding $H \hookrightarrow\hookrightarrow C[0, \pi]$ imply that

$$\|u\|_\infty \le R_0,$$

where R_0 is a positive constant. This completes the proof. □

Proof of Theorem 1.1. From assumption (A2), g is bounded in $\mathbb{R}$. Assume that $\lambda \in (-\infty, 1) \cup (1, \gamma)$. Then we may apply the Schauder fixed point theorem. To prove the other parts of the result, we shall employ Lemmas 3.2 and 3.3.

The necessary $L^1(0, \pi)$ bound of course follows immediately. Moreover, since the *a priori* bounds (4.1) of Theorem 4.1 are independent of the parameter κ, we may, for $1 < \lambda \le \gamma$, compute the Leray-Schauder degree (3.2) as that of a linear homeomorphism, which is nonzero (in our case -1). Therefore, we may apply Lemma 3.2 to deduce the existence of solutions for $1 \le \lambda < 4$.

On the other hand, $\lambda = 1$ is the principle eigenvalue of the linear eigenvalue problem

$$\begin{cases} -u''(x) = \lambda u(x), & x \in (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

and which is of multiplicity one. Hence, Lemma 3.3 also is applied and we may conclude that there is bifurcation from infinity at $\lambda = 1$. However, we have established *a priori* bounds for the eventual solutions of problem (1.1) with $1 \le \lambda \le \gamma < 4$, the continua bifurcating from infinity, must do so for $\lambda < 1$ but close to 1, there must exist large positive and a large negative solution. This coupled with Lemma 3.2 and what has been proved above yields the existence of three solutions for $\lambda < 1$ close to 1. □

Acknowledgements: The authors are very grateful to the anonymous referees for their valuable suggestions.

Funding information: This research was supported by the NSFC (No. 12061064).

Author contributions: The authors claim that the research was realized in collaboration with the same responsibility. All authors read and approved the last of the manuscript.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: Data sharing not applicable to this article as no datasets were generated.

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