Research Article

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Extension of isometries in real Hilbert spaces

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Abstract: The main purpose of this article is to develop a theory that extends the domain of any local isometry to the whole space containing the domain, where a local isometry is an isometry between two proper subsets. In fact, the main purpose of this article consists of the following three detailed objectives: The first objective is to extend the bounded domain of any local isometry to the first-order generalized linear span. The second one is to extend the bounded domain of any local isometry to the second-order generalized linear span. The third objective of this article is to extend the bounded domain of any local isometry to the whole Hilbert space.

Keywords: isometry, extension of isometry, generalized linear span

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1 Introduction

In the course of the development of mathematics in the last century, the problem of extending the domain of a function while keeping/preserving the characteristic properties of a function defined in a local domain has had a great influence on the development of functional analysis.

For example, in topology, the Tietze extension theorem states that all continuous functions defined on a closed subset of a normal topological space can be extended to the entire space.

Theorem 1.1. (Tietze) *Let X be a normal space, E be a nonempty closed subset of X, and let $[-L, L]$ be a closed real interval. If $f : E \to [-L, L]$ is a continuous function, then there exists a continuous extension of f to X, i.e., there exists a continuous function $F : X \to [-L, L]$ such that $F(x) = f(x)$ for all $x \in E$.*

The Tietze extension theorem has a wide range of applications and is an interesting theorem, so there are many variations in this theorem.

In 1972, Mankiewicz published his article [1] determining whether an isometry $f : E \to Y$ from a subset E of a real normed space X into a real normed space Y admits an extension to an isometry from X onto Y. Indeed, he proved that every isometry $f : E \to Y$ can be uniquely extended to an affine isometry between the whole spaces when either E and $f(E)$ are both convex bodies or E is nonempty open connected and $f(E)$ is open, where a convex body is a convex set with a nonempty interior.

Theorem 1.2. (Mankiewicz) *Let X and Y be real normed spaces, E be a nonempty subset of X, and let $f : E \to f(E)$ be a surjective isometry, where $f(E)$ is a subset of Y. If either both E and $f(E)$ are convex bodies, or E is open and connected and $f(E)$ is open, then f can be uniquely extended to an affine isometry $F : X \to Y$.*

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This conclusion particularly holds for the closed unit balls. Based on this fact, with the same research direction, Tingley [2] intuitively paid attention to the unit spheres and posed the following problem, which is now known as Tingley's problem.

Problem 1.1. (Tingley) Is every surjective isometry between the unit spheres of two Banach spaces a restriction to the unit sphere of a surjective real-linear isometry between the whole spaces?

Recently, many articles have been devoted to the study of the extension of isometries and Tingley's problem. Among them, a result of Ding [3, Theorem 2.2], which is related to Problem 1.1, will be introduced.

Theorem 1.3. (Ding) *Let X and Y be real Hilbert spaces and let $f : S_1(X) \to S_1(Y)$ be a function between unit spheres. If $-f(S_1(X)) \subset f(S_1(X))$ and $\|f(x_1) - f(x_2)\| \le \|x_1 - x_2\|$ for all $x_1, x_2 \in S_1(X)$, then f can be extended to a real-linear isometry from X into Y (see also* [4–13]).

The research in this article is strongly motivated by Theorems 1.2 and 1.3, and [14, Theorem 2.5], among others (refer to [15–17] also).

The main purpose of this article is to develop a theory that extends the (bounded) domain of any local isometry to the real Hilbert space M_a containing the domain, where a local isometry is an isometry between two proper subsets of the Hilbert space M_a, which is defined in Section 2 of this article. In Section 3, we introduce some concepts such as first-order generalized linear span and index set, which are essential to prove the final result of this article. Section 4 is devoted to the problem of extending the domain of a local isometry to the first-order generalized linear span. Solving this problem is the first objective of this article. We introduce the concept of a second-order generalized linear span in Section 5 and develop the theory of extension of the domain of a local isometry to the second-order generalized linear span in Section 7, which is the second objective of this article. Finally, we prove in Theorem 8.1 that the domain of a local isometry can be extended to the real Hilbert space M_a including that domain, which is the third objective of this article.

We observe that the domain of a local isometry is assumed to be bounded and contains at least two elements, but it need not be a convex body nor an open set. This indicates that the main results of this article are more general than those previously published.

2 Preliminaries

Throughout this article, the symbol $\mathbb{R}^\omega$ will denote the space of all real sequences. From now on, we denote by $(\mathbb{R}^\omega, \mathcal{T})$ the product space $\prod_{i=1}^{\infty}\mathbb{R}$, where $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is the usual topological space. Then, since $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ is a Hausdorff space, $(\mathbb{R}^\omega, \mathcal{T})$ is a Hausdorff space.

Let $a = \{a_i\}$ be a sequence of positive real numbers satisfying the following condition:

$$\sum_{i=1}^{\infty} a_i^2 < \infty. \tag{2.1}$$

With this sequence $a = \{a_i\}$, we define

$$M_a = \left\{ (x_1, x_2, \ldots) \in \mathbb{R}^\omega : \sum_{i=1}^{\infty} a_i^2 x_i^2 < \infty \right\}.$$

Then, M_a is a vector space over $\mathbb{R}$, and we can define an inner product $\langle \cdot, \cdot \rangle_a$ on M_a by

$$\langle x, y \rangle_a = \sum_{i=1}^{\infty} a_i^2 x_i y_i$$

for all $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ of M_a, with which $(M_a, \langle\cdot,\cdot\rangle_a)$ becomes a real inner product space. This inner product induces the norm in the natural way

$$\|x\|_a = \sqrt{\langle x, x\rangle_a}$$

for all $x \in M_a$, so that $(M_a, \|\cdot\|_a)$ becomes a real normed space.

Remark 2.1. M_a is the set of all elements $x \in \mathbb{R}^\omega$ satisfying $\|x\|_a^2 < \infty$, i.e.,

$$M_a = \{(x_1, x_2, \ldots) \in \mathbb{R}^\omega : \|x\|_a^2 < \infty\}.$$

We define the metric d_a on M_a by

$$d_a(x, y) = \|x - y\|_a = \sqrt{\langle x - y, x - y\rangle_a}$$

for all $x, y \in M_a$. Thus, (M_a, d_a) is a real metric space. Let $(M_a, \mathcal{T}_a)$ be the topological space generated by the metric d_a.

Similar to [18, Theorem 70.4], we can prove Remark 2.2(i).

Remark 2.2. We note that
(i) $(M_a, \langle\cdot,\cdot\rangle_a)$ is a Hilbert space over $\mathbb{R}$;
(ii) $(M_a, \mathcal{T}_a)$ is a Hausdorff space as a subspace of the Hausdorff space $(\mathbb{R}^\omega, \mathcal{T})$.

Definition 2.1. Given c in M_a, the *translation* by c is the mapping $T_c : M_a \to M_a$ defined by $T_c(x) = x + c$ for all $x \in M_a$.

3 First-order generalized linear span

In [14, Theorem 2.5], we were able to extend the domain of a d_a-isometry f to the entire space when the domain of f is a *nondegenerate basic cylinder* (see Definition 6.1 for the exact definition of nondegenerate basic cylinders). However, we shall see in Definition 4.1 and Theorem 4.2 that the domain of a d_a-isometry f can be extended to its first-order generalized linear span whenever f is defined on a bounded set that contains more than one element.

From now on, it is assumed that E, E_1, and E_2 are subsets of M_a, and each of them contains more than one element, unless specifically stated for their cardinalities. If the set has only one element or no element, this case will not be covered here because the results derived from this case are trivial and uninteresting.

Definition 3.1. Assume that E is a nonempty bounded subset of M_a and p is a fixed element of E. We define the *first-order generalized linear span* of E with respect to p as

$$\mathrm{GS}(E, p) = \left\{ p + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(x_{ij} - p) \in M_a : m \in \mathbb{N};\ x_{ij} \in E \quad \text{and} \quad \alpha_{ij} \in \mathbb{R} \text{ for all } i \text{ and } j \right\}.$$

We remark that if a bounded subset E of M_a contains more than one element, then E is a proper subset of its first-order generalized linear span $\mathrm{GS}(E, p)$, because $x = p + (x - p) \in \mathrm{GS}(E, p)$ for any $x \in E$ and $p + \alpha(x - p) \in \mathrm{GS}(E, p)$ for any $\alpha \in \mathbb{R}$, which implies that $\mathrm{GS}(E, p)$ is unbounded. Moreover, we note that $\alpha x + \beta y \in M_a$ for all $x, y \in M_a$ and $\alpha, \beta \in \mathbb{R}$, because $\|\alpha x + \beta y\|_a \le |\alpha|\|x\|_a + |\beta|\|y\|_a < \infty$. Therefore, $\mathrm{GS}(E, p) - p$ is a real vector space, because the double sum in the definition of $\mathrm{GS}(E, p)$ guarantees $\alpha x + \beta y \in \mathrm{GS}(E, p) - p$ for all $x, y \in \mathrm{GS}(E, p) - p$ and $\alpha, \beta \in \mathbb{R}$ and because $\mathrm{GS}(E, p) - p$ is a subspace of a real vector space M_a (*cf.* Lemma 5.3(i)).

For each $i \in \mathbb{N}$, we set $e_i = (0, \ldots, 0, 1, 0, \ldots)$, where 1 is in the ith position. Then, $\left\{\frac{1}{a_i} e_i\right\}$ is a complete orthonormal sequence in M_a.

Definition 3.2. Let E be a nonempty subset of M_a.

(i) We define the *index set* of E by

$$\Lambda(E) = \{i \in \mathbb{N} : \text{ there are an } x \in E \text{ and an } \alpha \in \mathbb{R} \backslash \{0\} \text{ satisfying } x + \alpha e_i \in E\}.$$

Each $i \in \Lambda(E)$ is called an *index* of E. If $\Lambda(E) \neq \mathbb{N}$, then the set E is called *degenerate*. Otherwise, E is called *nondegenerate*.

(ii) Let $\beta = \{\beta_i\}_{i \in \mathbb{N}}$ be another complete orthonormal sequence in M_a. We define the β*-index set* of E by

$$\Lambda_\beta(E) = \{i \in \mathbb{N} : \text{ there are an } x \in E \text{ and an } \alpha \in \mathbb{R} \backslash \{0\} \text{ satisfying } x + \alpha\beta_i \in E\}.$$

Each $i \in \Lambda_\beta(E)$ is called a β*-index* of E.

We will find that the concept of an index set in Hilbert space sometimes takes over the role that the concept of dimension plays in vector space. According to the definition above, if i is a β-index of E, i.e., $i \in \Lambda_\beta(E)$, then there are $x \in E$ and $x + \alpha\beta_i \in E$ for some $\alpha \neq 0$. Since $x \neq x + \alpha\beta_i$, we remark that if $\Lambda_\beta(E) \neq \varnothing$, then the set E contains at least two elements.

In the following lemma, we prove that if i is an index of E and $p \in E$, then the first-order generalized linear span $\mathrm{GS}(E, p)$ contains the line through p in the direction e_i.

Lemma 3.1. *Assume that E is a bounded subset of M_a, and $\mathrm{GS}(E, p)$ is the first-order generalized linear span of E with respect to a fixed element $p \in E$. If $i \in \Lambda(E)$, then $p + \alpha e_i \in \mathrm{GS}(E, p)$ for all $\alpha \in \mathbb{R}$.*

Proof. By Definition 3.2, if $i \in \Lambda(E)$, then there exist an $x \in E$ and an $\alpha_0 \neq 0$, which satisfy $x + \alpha_0 e_i \in E$. Since $x \in E$ and $x + \alpha_0 e_i \in E$, by Definition 3.1, we obtain

$$p + \alpha_0\beta e_i = p + \beta(x + \alpha_0 e_i - p) - \beta(x - p) \in \mathrm{GS}(E, p)$$

for all $\beta \in \mathbb{R}$. Setting $\alpha = \alpha_0\beta$ in the above relation, we obtain $p + \alpha e_i \in \mathrm{GS}(E, p)$ for any $\alpha \in \mathbb{R}$. □

We now introduce a lemma, which is a generalized version of [14, Lemma 2.3] and whose proof runs in the same way. We prove that the function $T_{-q} \circ f \circ T_p : E_1 - p \to E_2 - q$ preserves the inner product. This property is important in proving the following theorems as a necessary condition for f to be a d_a-isometry.

Lemma 3.2. *Assume that E_1 and E_2 are bounded subsets of M_a that are d_a-isometric to each other via a surjective d_a-isometry $f : E_1 \to E_2$. Assume that p is an element of E_1 and q is an element of E_2 with $q = f(p)$. Then, the function $T_{-q} \circ f \circ T_p : E_1 - p \to E_2 - q$ preserves the inner product, i.e.,*

$$\langle (T_{-q} \circ f \circ T_p)(x - p), (T_{-q} \circ f \circ T_p)(y - p)\rangle_a = \langle x - p, y - p\rangle_a$$

for all $x, y \in E_1$.

Proof. Since $T_{-q} \circ f \circ T_p : E_1 - p \to E_2 - q$ is a d_a-isometry, we have

$$\|(T_{-q} \circ f \circ T_p)(x - p) - (T_{-q} \circ f \circ T_p)(y - p)\|_a = \|(x - p) - (y - p)\|_a$$

for any $x, y \in E_1$. If we put $y = p$ in the previous equality, then we obtain

$$\|(T_{-q} \circ f \circ T_p)(x - p)\|_a = \|x - p\|_a$$

for each $x \in E_1$. Moreover, it follows from the previous equality that

$$\begin{aligned}\|(T_{-q} \circ f \circ T_p)(x - p) - (T_{-q} \circ f \circ T_p)(y - p)\|_a^2 &= \langle (T_{-q} \circ f \circ T_p)(x - p) - (T_{-q} \circ f \circ T_p)(y - p), \\ (T_{-q} \circ f \circ T_p)(x - p) - (T_{-q} \circ f \circ T_p)(y - p)\rangle_a &= \|x - p\|_a^2 - 2\langle (T_{-q} \circ f \circ T_p)(x - p), (T_{-q} \circ f \circ T_p)(y - p)\rangle_a + \|y - p\|_a^2\end{aligned}$$

and

$$\|(x-p)-(y-p)\|_a^2 = \langle (x-p)-(y-p), (x-p)-(y-p)\rangle_a = \|x-p\|_a^2 - 2\langle x-p, y-p\rangle_a + \|y-p\|_a^2.$$

Finally, comparing the last two equalities yields the validity of our assertion. □

4 First-order extension of isometries

In the previous section, we made all the necessary preparations to extend the domain E_1 of the surjective d_a-isometry $f : E_1 \to E_2$ to its first-order generalized linear span $\mathrm{GS}(E_1, p)$.

Although E_1 is a bounded set, $\mathrm{GS}(E_1, p) - p$ is a real vector space. Now we will extend the d_a-isometry $T_{-q} \circ f \circ T_p$ defined on the bounded set $E_1 - p$ to the d_a-isometry $T_{-q} \circ F \circ T_p$ defined on the vector space $\mathrm{GS}(E_1, p) - p$.

Definition 4.1. Assume that E_1 and E_2 are nonempty bounded subsets of M_a that are d_a-isometric to each other via a surjective d_a-isometry $f : E_1 \to E_2$. Let p be a fixed element of E_1 and let q be an element of E_2 that satisfies $q = f(p)$. We define a function $F : \mathrm{GS}(E_1, p) \to M_a$ as

$$(T_{-q} \circ F \circ T_p)\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p)\right) = \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij}-p)$$

for any $m \in \mathbb{N}$, $x_{ij} \in E_1$, and for all $\alpha_{ij} \in \mathbb{R}$ satisfying $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) \in M_a$.

We note that in the definition above, it is important for the argument of $T_{-q} \circ F \circ T_p$ to belong to M_a. Now, we show that the function $F : \mathrm{GS}(E_1, p) \to M_a$ is well defined.

Lemma 4.1. *Assume that E_1 and E_2 are bounded subsets of M_a that are d_a-isometric to each other via a surjective d_a-isometry $f : E_1 \to E_2$. Let p be an element of E_1 and let q be an element of E_2 that satisfy $q = f(p)$. The function $F : \mathrm{GS}(E_1, p) \to M_a$ given in* Definition 4.1 *is well defined.*

Proof. First, we will check that the range of F is a subset of M_a. For any $m, n_1, n_2 \in \mathbb{N}$ with $n_2 > n_1$, $x_{ij} \in E_1$, and for all $\alpha_{ij} \in \mathbb{R}$, it follows from Lemma 3.2 that

$$\begin{aligned}
&\left\|\sum_{i=1}^{m}\sum_{j=1}^{n_2}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij}-p) - \sum_{i=1}^{m}\sum_{j=1}^{n_1}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij}-p)\right\|_a^2 \\
&= \left\langle \sum_{i=1}^{m}\sum_{j=n_1+1}^{n_2}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij}-p), \sum_{k=1}^{m}\sum_{\ell=n_1+1}^{n_2}\alpha_{k\ell}(T_{-q} \circ f \circ T_p)(x_{k\ell}-p)\right\rangle_a \\
&= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=n_1+1}^{n_2}\alpha_{ij}\sum_{\ell=n_1+1}^{n_2}\alpha_{k\ell}\langle (T_{-q} \circ f \circ T_p)(x_{ij}-p), (T_{-q} \circ f \circ T_p)(x_{k\ell}-p)\rangle_a \\
&= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=n_1+1}^{n_2}\alpha_{ij}\sum_{\ell=n_1+1}^{n_2}\alpha_{k\ell}\langle x_{ij}-p, x_{k\ell}-p\rangle_a \\
&= \left\langle \sum_{i=1}^{m}\sum_{j=n_1+1}^{n_2}\alpha_{ij}(x_{ij}-p), \sum_{k=1}^{m}\sum_{\ell=n_1+1}^{n_2}\alpha_{k\ell}(x_{k\ell}-p)\right\rangle_a \\
&= \left\|\sum_{i=1}^{m}\sum_{j=n_1+1}^{n_2}\alpha_{ij}(x_{ij}-p)\right\|_a^2 = \left\|\sum_{i=1}^{m}\sum_{j=1}^{n_2}\alpha_{ij}(x_{ij}-p) - \sum_{i=1}^{m}\sum_{j=1}^{n_1}\alpha_{ij}(x_{ij}-p)\right\|_a^2.
\end{aligned} \tag{4.1}$$

Indeed, equality (4.1) holds for all $m, n_1, n_2 \in \mathbb{N}$.

We now assume that $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) \in M_a$ for some $x_{ij} \in E_1$ and $\alpha_{ij} \in \mathbb{R}$, where m is a fixed positive integer. Then, since $(M_a, \mathcal{T}_a)$ is a Hausdorff space on account of Remark 2.2(ii) and the topology $\mathcal{T}_a$ is consistent with the metric d_a and with the norm $\|\cdot\|_a$, the sequence $\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}(x_{ij}-p)\right\}_n$ converges to $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p)$ (in M_a), and hence, the sequence $\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}(x_{ij}-p)\right\}_n$ is a Cauchy sequence in M_a.

We know by (4.1) and the definition of Cauchy sequences that for each $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that

$$\left\|\sum_{i=1}^{m}\sum_{j=1}^{n_2}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p) - \sum_{i=1}^{m}\sum_{j=1}^{n_1}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p)\right\|_a = \left\|\sum_{i=1}^{m}\sum_{j=1}^{n_2}\alpha_{ij}(x_{ij}-p) - \sum_{i=1}^{m}\sum_{j=1}^{n_1}\alpha_{ij}(x_{ij}-p)\right\|_a < \varepsilon$$

for all integers $n_1, n_2 > N_\varepsilon$, which implies that $\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p)\right\}_n$ is also a Cauchy sequence in M_a. By Remark 2.2(i), we observe that $(M_a, \langle\cdot,\cdot\rangle_a)$ is a real Hilbert space. Thus, M_a is not only complete, but also a Hausdorff space, so the Cauchy sequence $\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p)\right\}_n$ converges in M_a, i.e., by Definition 4.1, we have

$$\begin{aligned}(T_{-q}\circ F\circ T_p)\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p)\right) &= \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p)\\ &= \lim_{n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p) \in M_a,\end{aligned}$$

which implies

$$F\left(p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p)\right) \in M_a + q = M_a$$

for all $x_{ij} \in E_1$ and $\alpha_{ij} \in \mathbb{R}$ with $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) \in M_a$, i.e., the image of each element of $\mathrm{GS}(E_1, p)$ under F belongs to M_a.

We now assume that $\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) = \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij}-p) \in M_a$ for some $m_1, m_2 \in \mathbb{N}$, $x_{ij}, y_{ij} \in E_1$, and $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$. It then follows from Definition 4.1 and Lemma 3.2 that

$$\begin{aligned}&\left\|(T_{-q}\circ F\circ T_p)\left(\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p)\right) - (T_{-q}\circ F\circ T_p)\left(\sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij}-p)\right)\right\|_a^2\\
&= \left\|\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p) - \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(T_{-q}\circ f\circ T_p)(y_{ij}-p)\right\|_a^2\\
&= \Bigg\langle \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(x_{ij}-p) - \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(T_{-q}\circ f\circ T_p)(y_{ij}-p),\\
&\qquad \sum_{k=1}^{m_1}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(T_{-q}\circ f\circ T_p)(x_{k\ell}-p) - \sum_{k=1}^{m_2}\sum_{\ell=1}^{\infty}\beta_{k\ell}(T_{-q}\circ f\circ T_p)(y_{k\ell}-p)\Bigg\rangle_a\\
&= \cdots = \left\langle \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) - \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij}-p), \sum_{k=1}^{m_1}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(x_{k\ell}-p) - \sum_{k=1}^{m_2}\sum_{\ell=1}^{\infty}\beta_{k\ell}(y_{k\ell}-p)\right\rangle_a\\
&= \left\|\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p) - \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij}-p)\right\|_a^2 = 0,\end{aligned}$$

which implies that

$$(T_{-q} \circ F \circ T_p)\left(\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p)\right) = (T_{-q} \circ F \circ T_p)\left(\sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij} - p)\right)$$

for all $m_1, m_2 \in \mathbb{N}$, $x_{ij}, y_{ij} \in E_1$, and $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$, satisfying $\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p) = \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij} - p) \in M_a$. □

We prove in the following theorem that the domain of a d_a-isometry $f : E_1 \to E_2$ can be extended to the first-order generalized linear span $\mathrm{GS}(E_1, p)$ whenever E_1 is a nonempty bounded subset of M_a. Therefore, Theorem 4.2 is a generalization of [19, Theorem 2.2] for M_a.

Theorem 4.2. *Assume that E_1 and E_2 are bounded subsets of M_a that are d_a-isometric to each other via a surjective d_a-isometry $f : E_1 \to E_2$. Assume that p is an element of E_1 and q is an element of E_2 with $q = f(p)$. The function $F : \mathrm{GS}(E_1, p) \to M_a$ defined in Definition 4.1 is a d_a-isometry and the function $T_{-q} \circ F \circ T_p$: $\mathrm{GS}(E_1, p) - p \to M_a$ is a linear d_a-isometry. In particular, F is an extension of f.*

Proof. (*a*) Let u and v be arbitrary elements of the first-order generalized linear span $\mathrm{GS}(E_1, p)$ of E_1 with respect to p. Then,

$$u - p = \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p) \in M_a \quad \text{and} \quad v - p = \sum_{i=1}^{n}\sum_{j=1}^{\infty}\beta_{ij}(y_{ij} - p) \in M_a \tag{4.2}$$

for some $m, n \in \mathbb{N}$, $x_{ij}, y_{ij} \in E_1$, and $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$. Then, according to Definition 4.1, we have

$$\begin{aligned}(T_{-q} \circ F \circ T_p)(u - p) &= \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij} - p),\\ (T_{-q} \circ F \circ T_p)(v - p) &= \sum_{i=1}^{n}\sum_{j=1}^{\infty}\beta_{ij}(T_{-q} \circ f \circ T_p)(y_{ij} - p).\end{aligned} \tag{4.3}$$

(*b*) By Lemma 3.2, (4.2), and (4.3), we obtain

$$\begin{aligned}&\langle (T_{-q} \circ F \circ T_p)(u - p), (T_{-q} \circ F \circ T_p)(v - p)\rangle_a\\ &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ f \circ T_p)(x_{ij} - p), \sum_{k=1}^{n}\sum_{\ell=1}^{\infty}\beta_{k\ell}(T_{-q} \circ f \circ T_p)(y_{k\ell} - p)\right\rangle_a\\ &= \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\beta_{k\ell}\langle (T_{-q} \circ f \circ T_p)(x_{ij} - p), (T_{-q} \circ f \circ T_p)(y_{k\ell} - p)\rangle_a\\ &= \sum_{i=1}^{m}\sum_{k=1}^{n}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\beta_{k\ell}\langle x_{ij} - p, y_{k\ell} - p\rangle_a\\ &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p), \sum_{k=1}^{n}\sum_{\ell=1}^{\infty}\beta_{k\ell}(y_{k\ell} - p)\right\rangle_a\\ &= \langle u - p, v - p\rangle_a\end{aligned} \tag{4.4}$$

for all $u, v \in \mathrm{GS}(E_1, p)$. That is, $T_{-q} \circ F \circ T_p$ preserves the inner product. Indeed, equality (4.4) is an extended version of Lemma 3.2.

(*c*) By using equality (4.4), we further obtain

$$\begin{aligned}d_a(F(u), F(v))^2 &= \|F(u) - F(v)\|_a^2\\ &= \|(T_{-q} \circ F \circ T_p)(u - p) - (T_{-q} \circ F \circ T_p)(v - p)\|_a^2\\ &= \langle (T_{-q} \circ F \circ T_p)(u - p) - (T_{-q} \circ F \circ T_p)(v - p),\end{aligned}$$

$$(T_{-q} \circ F \circ T_p)(u-p) - (T_{-q} \circ F \circ T_p)(v-p)\rangle_a = \langle u-p, u-p\rangle_a - \langle u-p, v-p\rangle_a - \langle v-p, u-p\rangle_a + \langle v-p, v-p\rangle_a$$
$$= \langle (u-p)-(v-p), (u-p)-(v-p)\rangle_a$$
$$= \|(u-p)-(v-p)\|_a^2$$
$$= \|u-v\|_a^2$$
$$= d_a(u,v)^2$$

for all $u, v \in \mathrm{GS}(E_1, p)$, i.e., F is a d_a-isometry.

(d) Now, let u and v be arbitrary elements of $\mathrm{GS}(E_1, p)$. Then, it holds that $u - p \in \mathrm{GS}(E_1, p) - p$, $v - p \in \mathrm{GS}(E_1, p) - p$, and $\alpha(u-p) + \beta(v-p) \in \mathrm{GS}(E_1, p) - p$ for any $\alpha, \beta \in \mathbb{R}$, because $\mathrm{GS}(E_1, p) - p$ is a real vector space.

We obtain

$$\begin{aligned}&\|(T_{-q} \circ F \circ T_p)(\alpha(u-p) + \beta(v-p)) - \alpha(T_{-q} \circ F \circ T_p)(u-p) - \beta(T_{-q} \circ F \circ T_p)(v-p)\|_a^2 \\ &= \langle (T_{-q} \circ F \circ T_p)(\alpha(u-p) + \beta(v-p)) - \alpha(T_{-q} \circ F \circ T_p)(u-p) \\ &\quad - \beta(T_{-q} \circ F \circ T_p)(v-p), (T_{-q} \circ F \circ T_p)(\alpha(u-p) + \beta(v-p)) - \alpha(T_{-q} \circ F \circ T_p)(u-p) \\ &\quad - \beta(T_{-q} \circ F \circ T_p)(v-p)\rangle_a.\end{aligned}$$

Since $\alpha(u-p) + \beta(v-p) = w - p$ for some $w \in \mathrm{GS}(E_1, p)$, we further use (4.4) to obtain

$$\begin{aligned}&\|(T_{-q} \circ F \circ T_p)(\alpha(u-p) + \beta(v-p)) - \alpha(T_{-q} \circ F \circ T_p)(u-p) - \beta(T_{-q} \circ F \circ T_p)(v-p)\|_a^2 \\ &= \langle w-p, w-p\rangle_a - \alpha\langle w-p, u-p\rangle_a - \beta\langle w-p, v-p\rangle_a - \alpha\langle u-p, w-p\rangle_a + \alpha^2\langle u-p, u-p\rangle_a \\ &\quad + \alpha\beta\langle u-p, v-p\rangle_a - \beta\langle v-p, w-p\rangle_a + \alpha\beta\langle v-p, u-p\rangle_a + \beta^2\langle v-p, v-p\rangle_a = 0,\end{aligned}$$

which implies that the function $T_{-q} \circ F \circ T_p : \mathrm{GS}(E_1, p) - p \to M_a$ is linear.

(e) Finally, we set $\alpha_{11} = 1$ and $\alpha_{ij} = 0$ for any $(i, j) \neq (1, 1)$, and $x_{11} = x$ in (4.2) and (4.3) to see

$$(T_{-q} \circ F \circ T_p)(x-p) = (T_{-q} \circ f \circ T_p)(x-p)$$

for every $x \in E_1$, which implies that $F(x) = f(x)$ for every $x \in E_1$, i.e., F is an extension of f. □

5 Second-order generalized linear span

For any element x of M_a and $r > 0$, we denote by $B_r(x)$ the open ball defined by $B_r(x) = \{y \in M_a: \|y - x\|_a < r\}$.

Definitions 3.1 and 4.1 will be generalized to the cases of $n \geq 2$ in the following definition. We introduce the concept of nth-order generalized linear span $\mathrm{GS}^n(E_1, p)$, which generalizes the concept of first-order generalized linear span $\mathrm{GS}(E, p)$. Moreover, we define the d_a-isometry F_n, which extends the domain of a d_a-isometry f to $\mathrm{GS}^n(E_1, p)$.

It is surprising, however, that this process of generalization does not go far. Indeed, we will find in Proposition 5.4 and Theorem 7.2 that $\mathrm{GS}^2(E_1, p)$ and F_2 are their limits.

Definition 5.1. Let E_1 be a nonempty bounded subset of M_a that is d_a-isometric to a subset E_2 of M_a via a surjective d_a-isometry $f : E_1 \to E_2$. Let p be an element of E_1 and q an element of E_2 with $q = f(p)$. Assume that r is a positive real number satisfying $E_1 \subset B_r(p)$.

(i) We define $\mathrm{GS}^0(E_1, p) = E_1$ and $\mathrm{GS}^1(E_1, p) = \mathrm{GS}(E_1, p)$. In general, we define the *nth-order generalized linear span* of E_1 with respect to p as $\mathrm{GS}^n(E_1, p) = \mathrm{GS}(\mathrm{GS}^{n-1}(E_1, p) \cap B_r(p), p)$ for all $n \in \mathbb{N}$.

(ii) We define $F_0 = f$ and $F_1 = F$, where F is defined in Definition 4.1. Moreover, for any $n \in \mathbb{N}$, we define the function $F_n : \mathrm{GS}^n(E_1, p) \to M_a$ by

$$(T_{-q} \circ F_n \circ T_p)\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p)\right) = \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ F_{n-1} \circ T_p)(x_{ij} - p)$$

for all $m \in \mathbb{N}$, $x_{ij} \in \mathrm{GS}^{n-1}(E_1, p) \cap B_r(p)$, and $\alpha_{ij} \in \mathbb{R}$ with $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p) \in M_a$.

Proposition 5.1. *Let E be a nonempty bounded subset of M_a. If s and t are positive real numbers that satisfy $E \subset B_s(p) \cap B_t(p)$, then*

$$\mathrm{GS}(\mathrm{GS}(E, p) \cap B_s(p), p) = \mathrm{GS}(\mathrm{GS}(E, p) \cap B_t(p), p).$$

Proof. Assume that $0 < s < t$. Then, there exists a real number $c > 1$ with $s > \frac{t}{c}$, and it is obvious that $B_{t/c}(p) \subset B_s(p)$. Assume that x is an arbitrary element of $\mathrm{GS}(\mathrm{GS}(E, p) \cap B_t(p), p)$. Then, there exist some $m \in \mathbb{N}$, $u_{ij} \in \mathrm{GS}(E, p) \cap B_t(p)$, and $\alpha_{ij} \in \mathbb{R}$ such that $x = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) \in M_a$. We note that

$$(\mathrm{GS}(E, p) - p) \cap (B_t(p) - p) = \{u - p \in M_a : u \in \mathrm{GS}(E, p) \cap B_t(p)\}.$$

Since $\mathrm{GS}(E, p) - p$ is a real vector space, $\frac{t}{c} < s$, and since $u_{ij} - p \in (\mathrm{GS}(E, p) - p) \cap (B_t(p) - p)$ for any i and j, we have

$$\frac{1}{c}(u_{ij} - p) \in (\mathrm{GS}(E, p) - p) \cap (B_s(p) - p).$$

Hence, we can choose a $v_{ij} \in \mathrm{GS}(E, p) \cap B_s(p)$ such that $\frac{1}{c}(u_{ij} - p) = v_{ij} - p$. Thus, we obtain

$$x = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}c\alpha_{ij}(v_{ij} - p) \in \mathrm{GS}(\mathrm{GS}(E, p) \cap B_s(p), p),$$

which implies that $\mathrm{GS}(\mathrm{GS}(E, p) \cap B_t(p), p) \subset \mathrm{GS}(\mathrm{GS}(E, p) \cap B_s(p), p)$.

The reverse inclusion is obvious, since $B_s(p) \subset B_t(p)$. □

We generalize Lemma 3.2 and formula (4.4) in the following lemma. Indeed, we prove that the function $T_{-q} \circ F_n \circ T_p : \mathrm{GS}^n(E_1, p) - p \to M_a$ preserves the inner product. This property is important in proving the following theorems as a necessary condition for F_n to be a d_a-isometry.

Lemma 5.2. *Let E_1 be a bounded subset of M_a that is d_a-isometric to a subset E_2 of M_a via a surjective d_a-isometry $f : E_1 \to E_2$. Assume that p and q are elements of E_1 and E_2, which satisfy $q = f(p)$. If $n \in \mathbb{N}$, then*

$$\langle (T_{-q} \circ F_n \circ T_p)(u - p), (T_{-q} \circ F_n \circ T_p)(v - p)\rangle_a = \langle u - p, v - p\rangle_a$$

for all $u, v \in \mathrm{GS}^n(E_1, p)$.

Proof. Our assertion for $n = 1$ was already proved in (4.4). Considering Proposition 5.1, assume that r is a positive real number satisfying $E_1 \subset B_r(p)$. Now we assume that the assertion is true for some $n \in \mathbb{N}$. Let u, v be arbitrary elements of $\mathrm{GS}^{n+1}(E_1, p)$. Then, there exist some $m_1, m_2 \in \mathbb{N}$, $x_{ij}, y_{k\ell} \in \mathrm{GS}^n(E_1, p) \cap B_r(p)$, and $\alpha_{ij}, \beta_{k\ell} \in \mathbb{R}$ such that

$$u - p = \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij} - p) \in M_a \quad \text{and} \quad v - p = \sum_{k=1}^{m_2}\sum_{\ell=1}^{\infty}\beta_{k\ell}(y_{k\ell} - p) \in M_a.$$

Using Definition 5.1(*ii*) and our assumption, we obtain

$$\begin{aligned}&\langle (T_{-q} \circ F_{n+1} \circ T_p)(u - p), (T_{-q} \circ F_{n+1} \circ T_p)(v - p)\rangle_a \\ &\quad = \left\langle \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ F_n \circ T_p)(x_{ij} - p), \sum_{k=1}^{m_2}\sum_{\ell=1}^{\infty}\beta_{k\ell}(T_{-q} \circ F_n \circ T_p)(y_{k\ell} - p)\right\rangle_a\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m_1}\sum_{k=1}^{m_2}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\beta_{k\ell}\langle (T_{-q}\circ F_n\circ T_p)(x_{ij}-p), (T_{-q}\circ F_n\circ T_p)(y_{k\ell}-p)\rangle_a \\
&= \sum_{i=1}^{m_1}\sum_{k=1}^{m_2}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\beta_{k\ell}\langle x_{ij}-p, y_{k\ell}-p\rangle_a \\
&= \left\langle \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(x_{ij}-p), \sum_{k=1}^{m_2}\sum_{\ell=1}^{\infty}\beta_{k\ell}(y_{k\ell}-p)\right\rangle_a \\
&= \langle u-p, v-p\rangle_a
\end{aligned}
$$

for all $u, v \in \mathrm{GS}^{n+1}(E_1, p)$. By mathematical induction, we may then conclude that our assertion is true for all $n \in \mathbb{N}$. □

When $n = 1$ and $p = p'$, the first assertion in (i) of the following lemma is obvious, so we have used that fact several times before, omitting the proof. The assertion (iv) in the following lemma seems to be related in some way to Proposition 5.1.

Lemma 5.3. *Let E be a bounded subset of M_a and $p, p' \in E$. Assume that r is a positive real number satisfying $E \subset B_r(p)$.*

(i) $\mathrm{GS}^n(E, p) - p'$ *is a vector space over* $\mathbb{R}$ *for each* $n \in \mathbb{N}$.

(ii) $\mathrm{GS}^n(E, p) \subset \mathrm{GS}^{n+1}(E, p)$ *for each* $n \in \mathbb{N}$.

(iii) $\mathrm{GS}^2(E, p) = \overline{\mathrm{GS}(E, p)}$, *where* $\overline{\mathrm{GS}(E, p)}$ *is the closure of* $\mathrm{GS}(E, p)$ *in* M_a.

(iv) $\Lambda(\mathrm{GS}^n(E, p)) = \Lambda(\mathrm{GS}^n(E, p) \cap B_r(p))$ *for all* $n \in \mathbb{N}$.

Proof. (i) By using Definitions 3.1 and 5.1, we prove that $\mathrm{GS}(E, p) - p'$ is a real vector space. (We can prove similarly for the case of $n > 1$.) Given $x, y \in \mathrm{GS}(E, p) - p'$, we may choose some $m_1, m_2 \in \mathbb{N}$, $u_{ij}, v_{ij} \in E$, and $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ such that $x = (p - p') + \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) \in M_a$ and $y = (p - p') + \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(v_{ij} - p) \in M_a$. Since M_a is a real vector space, $\alpha\sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) + \beta\sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta_{ij}(v_{ij} - p) \in M_a$ for all $\alpha, \beta \in \mathbb{R}$.

Moreover, we see that

$$\alpha x + \beta y = \left(p + (1-\alpha-\beta)(p'-p) + \sum_{i=1}^{m_1}\sum_{j=1}^{\infty}\alpha\alpha_{ij}(u_{ij}-p) + \sum_{i=1}^{m_2}\sum_{j=1}^{\infty}\beta\beta_{ij}(v_{ij}-p)\right) - p' \in \mathrm{GS}(E, p) - p'$$

for all $\alpha, \beta \in \mathbb{R}$. Hence, $\mathrm{GS}(E, p) - p'$ is a real vector space as a subspace of real vector space M_a.

(ii) Let r be a positive real number with $E \subset B_r(p)$. If $x \in \mathrm{GS}^n(E, p)$ for some $n \in \mathbb{N}$, then $x - p \in \mathrm{GS}^n(E, p) - p$. Since $\mathrm{GS}^n(E, p) - p$ is a real vector space by (i) and $B_r(p) - p = B_r(0)$, we can choose a (positive or negative but sufficiently small) real number $\mu \neq 0$ such that $\mu(x - p) \in (\mathrm{GS}^n(E, p) - p) \cap (B_r(p) - p)$. We note that

$$(\mathrm{GS}^n(E, p) - p) \cap (B_r(p) - p) = \{v - p \in M_a : v \in \mathrm{GS}^n(E, p) \cap B_r(p)\}. \tag{5.1}$$

Thus, we see that $\mu(x - p) = v - p$ for some $v \in \mathrm{GS}^n(E, p) \cap B_r(p)$. Since $x = p + \frac{1}{\mu}(v - p)$, it holds that $x \in \mathrm{GS}^{n+1}(E, p)$. Therefore, we conclude that $\mathrm{GS}^n(E, p) \subset \mathrm{GS}^{n+1}(E, p)$ for every $n \in \mathbb{N}$.

(iii) Let x be an arbitrary element of $\overline{\mathrm{GS}(E, p)}$. Then, there exists some sequence $\{x_n\}$ in $\mathrm{GS}(E, p)$ that converges to x, where $x_n \neq x$ for all $n \in \mathbb{N}$. We now set $y_1 = x_1$ and $y_i = x_i - x_{i-1}$ for each integer $i \geq 2$. Then, we have

$$x_n = \sum_{i=1}^{n} y_i,$$

where $y_i = (x_i - p) - (x_{i-1} - p) \in \mathrm{GS}(E, p) - p$ for $i \geq 2$. Since $\mathrm{GS}(E, p) - p$ is a real vector space and $B_r(p) - p = B_r(0)$, we can select a real number $\mu_i \neq 0$ such that

$$\mu_i y_i \in \mathrm{GS}(E, p) - p \quad \text{and} \quad \mu_i y_i \in B_r(p) - p$$

for every integer $i \geq 2$. Thus, it follows from (5.1) that

$$x_n = \sum_{i=1}^{n} y_i = y_1 + \sum_{i=2}^{n} \frac{1}{\mu_i}(\mu_i y_i) = x_1 + \sum_{i=2}^{n} \frac{1}{\mu_i}(v_i - p),$$

where $v_i \in \mathrm{GS}(E,p) \cap B_r(p)$ for $i \geq 2$. Since the sequence $\{x_n\}$ is assumed to converge to x, the sequence $\left\{x_1 + \sum_{i=2}^{n} \frac{1}{\mu_i}(v_i - p)\right\}_n$ converges to x. Hence, we have

$$x_1 + \sum_{i=2}^{\infty} \frac{1}{\mu_i}(v_i - p) = \lim_{n\to\infty} x_n = x \in M_a. \tag{5.2}$$

(Since M_a is a Hausdorff space, x is the unique limit point of the sequence $\{x_n\}$.)

Furthermore, there exists a real number $\mu_1 \neq 0$ that satisfies $\mu_1(x_1 - p) \in \mathrm{GS}(E,p) - p$ and $\mu_1(x_1 - p) \in B_r(p) - p$, i.e., $\mu_1(x_1 - p) \in (\mathrm{GS}(E,p) - p) \cap (B_r(p) - p)$. Thus, there exists a $v_1 \in \mathrm{GS}(E,p) \cap B_r(p)$ such that $\mu_1(x_1 - p) = v_1 - p$ or $x_1 - p = \frac{1}{\mu_1}(v_1 - p)$. Therefore,

$$x = p + (x_1 - p) + \sum_{i=2}^{\infty} \frac{1}{\mu_i}(v_i - p) = p + \sum_{i=1}^{\infty} \frac{1}{\mu_i}(v_i - p), \tag{5.3}$$

where $v_i \in \mathrm{GS}(E,p) \cap B_r(p)$ for each $i \in \mathbb{N}$. It follows from (5.2) that $\sum_{i=1}^{\infty} \frac{1}{\mu_i}(v_i - p) \in M_a$. Thus, by (5.3), we see that $x \in \mathrm{GS}^2(E,p)$, which implies that $\overline{\mathrm{GS}(E,p)} \subset \mathrm{GS}^2(E,p)$.

On the other hand, let $y \in \mathrm{GS}^2(E,p)$. Then, there are some $m \in \mathbb{N}$, $v_{ij} \in \mathrm{GS}(E,p) \cap B_r(p)$, and $\alpha_{ij} \in \mathbb{R}$ such that $y = p + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(v_{ij} - p) \in M_a$. Let us define $y_n = p + \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}(v_{ij} - p)$ for every $n \in \mathbb{N}$. Since $v_{ij} - p \in \mathrm{GS}(E,p) - p$ for all i and j and $\mathrm{GS}(E,p) - p$ is a real vector space, we know that $y_n - p = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_{ij}(v_{ij} - p) \in \mathrm{GS}(E,p) - p$, and hence, $y_n \in \mathrm{GS}(E,p)$ for all $n \in \mathbb{N}$. Since $\mathrm{GS}(E,p)$ is a Hausdorff space, y is the unique limit point of the sequence $\{y_n\}$. Thus, we see that

$$y = p + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(v_{ij} - p) = \lim_{n\to\infty} y_n \in \overline{\mathrm{GS}(E,p)},$$

which implies that $\mathrm{GS}^2(E,p) \subset \overline{\mathrm{GS}(E,p)}$.

(*iv*) Let $i \in \Lambda(\mathrm{GS}^n(E,p))$. In view of Definition 3.2, there exist $x \in \mathrm{GS}^n(E,p)$ and $\alpha \neq 0$ with $x + \alpha e_i \in \mathrm{GS}^n(E,p)$. Furthermore, $x = p + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p)$ for some $m \in \mathbb{N}$, $u_{ij} \in \mathrm{GS}^{n-1}(E,p) \cap B_r(p)$, and $\alpha_{ij} \in \mathbb{R}$. Since $\sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p) + \alpha e_i = x - p + \alpha e_i \in \mathrm{GS}^n(E,p) - p$, it holds that $\mu(\sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p) + \alpha e_i) \in (\mathrm{GS}^n(E,p) - p) \cap (B_r(p) - p)$ for any sufficiently small $\mu \neq 0$, or equivalently, it follows from (5.1) that

$$\left(p + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \mu\alpha_{ij}(u_{ij} - p)\right) + \mu\alpha e_i \in \mathrm{GS}^n(E,p) \cap B_r(p). \tag{5.4}$$

On the other hand, since $\sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p) = x - p \in \mathrm{GS}^n(E,p) - p$, it holds that $\sum_{i=1}^{m} \sum_{j=1}^{\infty} \mu\alpha_{ij}(u_{ij} - p) \in (\mathrm{GS}^n(E,p) - p) \cap (B_r(p) - p)$ for any sufficiently small $\mu \neq 0$. Hence, it follows from (5.1) that $p + \sum_{i=1}^{m} \sum_{j=1}^{\infty} \mu\alpha_{ij}(u_{ij} - p) \in \mathrm{GS}^n(E,p) \cap B_r(p)$ for any sufficiently small $\mu \neq 0$. Thus, by Definition 3.2 and (5.4), it holds that $i \in \Lambda(\mathrm{GS}^n(E,p) \cap B_r(p))$, which implies that $\Lambda(\mathrm{GS}^n(E,p)) \subset \Lambda(\mathrm{GS}^n(E,p) \cap B_r(p))$. Obviously, the inverse inclusion is true. □

As we mentioned earlier, we will see that the second-order generalized linear span is the last step in this kind of domain extension.

Proposition 5.4. *If E is a bounded subset of M_a and $p \in E$, then*

$$E \subset \mathrm{GS}(E,p) \subset \overline{\mathrm{GS}(E,p)} = \mathrm{GS}^2(E,p) = \mathrm{GS}^n(E,p)$$

for any integer $n \geq 3$. Indeed, $\mathrm{GS}^n(E,p) - p$ is a real Hilbert space for $n \geq 2$.

Proof. (*a*) Considering Proposition 5.1, we can choose a real number $r > 0$ that satisfies $E \subset B_r(p)$. Assume that $x \in \mathrm{GS}^3(E, p)$. Then, there exist some $m_0 \in \mathbb{N}$, $u_{ij} \in \mathrm{GS}^2(E, p) \cap B_r(p)$, and $\alpha_{ij} \in \mathbb{R}$ such that $x = p + \sum_{i=1}^{m_0}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) \in M_a$.

We define $x_m = p + \sum_{i=1}^{m_0}\sum_{j=1}^{m}\alpha_{ij}(u_{ij} - p)$ for each $m \in \mathbb{N}$. Since $u_{ij} \in \mathrm{GS}^2(E, p)$, there exist some $m_{ij} \in \mathbb{N}$, $v_{ijk\ell} \in \mathrm{GS}(E, p) \cap B_r(p)$, and $\beta_{ijk\ell} \in \mathbb{R}$ such that $u_{ij} = p + \sum_{k=1}^{m_{ij}}\sum_{\ell=1}^{\infty}\beta_{ijk\ell}(v_{ijk\ell} - p) \in M_a$. Hence, it holds that

$$x_m = p + \sum_{i=1}^{m_0}\sum_{j=1}^{m}\sum_{k=1}^{m_{ij}}\sum_{\ell=1}^{\infty}\alpha_{ij}\beta_{ijk\ell}(v_{ijk\ell} - p) \in M_a,$$

which implies that $x_m \in \mathrm{GS}^2(E, p)$ for all $m \in \mathbb{N}$. Thus, $\{x_m\}$ is a sequence in $\mathrm{GS}^2(E, p)$ that converges to x. Therefore, $x \in \mathrm{GS}^2(E, p)$ because $\mathrm{GS}^2(E, p)$ is closed. Thus, $\mathrm{GS}^3(E, p) \subset \mathrm{GS}^2(E, p)$. The inverse inclusion is of course true due to Lemma 5.3(*ii*). We have proved that $\mathrm{GS}^2(E, p) = \mathrm{GS}^3(E, p)$.

(*b*) Assume that $\mathrm{GS}^2(E, p) = \cdots = \mathrm{GS}^n(E, p) = \mathrm{GS}^{n+1}(E, p)$ for some integer $n \geq 2$.

(*c*) If we replace $\mathrm{GS}(E, p)$, $\mathrm{GS}^2(E, p)$, and $\mathrm{GS}^3(E, p)$ in the previous part (*a*) with $\mathrm{GS}^n(E, p)$, $\mathrm{GS}^{n+1}(E, p)$, and $\mathrm{GS}^{n+2}(E, p)$, respectively, and if we consider the fact that $\mathrm{GS}^{n+1}(E, p) = \mathrm{GS}^2(E, p)$ is closed in M_a by Lemma 5.3(*iii*) and our assumption (*b*), then we arrive at the conclusion that $\mathrm{GS}^{n+1}(E, p) = \mathrm{GS}^{n+2}(E, p)$.

(*d*) With the conclusion of mathematical induction we prove that $\mathrm{GS}^n(E, p) = \mathrm{GS}^2(E, p)$ for every integer $n \geq 3$. Moreover, when $n \geq 2$, $\mathrm{GS}^n(E, p)$ is complete as a closed subset of a real Hilbert space M_a (ref. Remark 2.2). Therefore, $\mathrm{GS}^n(E, p) - p$ is a real Hilbert space for $n \geq 2$. □

The following lemma is an extension of Lemma 3.1 for the second-order generalized linear span $\mathrm{GS}^2(E, p)$. Indeed, we prove that if $i \in \Lambda(\mathrm{GS}^2(E, p))$, then the second-order generalized linear span of E contains all the lines through $\mathrm{GS}(E, p)$ in the direction e_i.

Lemma 5.5. *Assume that a bounded subset E of M_a contains at least two elements and $p \in E$. If $i \in \Lambda(\mathrm{GS}^2(E, p))$ and $p' \in \mathrm{GS}(E, p)$, then $p' + \alpha_i e_i \in \mathrm{GS}^2(E, p)$ for any $\alpha_i \in \mathbb{R}$.*

Proof. Let r be a positive real number with $E \subset B_r(p)$. Assume that $i \in \Lambda(\mathrm{GS}^2(E, p))$. Considering Lemma 5.3(*iv*) and Proposition 5.4, if we substitute $\mathrm{GS}^2(E, p) \cap B_r(p)$ for E in Lemma 3.1, then $p + \alpha_i e_i \in \mathrm{GS}^3(E, p) = \mathrm{GS}^2(E, p)$ for all $\alpha_i \in \mathbb{R}$. Thus, there are some $m \in \mathbb{N}$, $w_{ij} \in \mathrm{GS}(E, p) \cap B_r(p)$, and $\beta_{ij} \in \mathbb{R}$ with $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\beta_{ij}(w_{ij} - p) \in M_a$ such that $p + \alpha_i e_i = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\beta_{ij}(w_{ij} - p)$, and hence, we have

$$p' + \alpha_i e_i = p + \alpha_i e_i + (p' - p) = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\beta_{ij}(w_{ij} - p) + (p' - p). \tag{5.5}$$

Because $p' - p$ belongs to $\mathrm{GS}(E, p) - p$, which is a real vector space by Lemma 5.3(*i*), and $B_r(p) - p = B_r(0)$, we can choose some sufficiently small real number $\mu \neq 0$ such that

$$\mu(p' - p) \in \mathrm{GS}(E, p) - p \quad \text{and} \quad \mu(p' - p) \in B_r(p) - p. \tag{5.6}$$

Considering (5.1), (5.5), and (5.6), if we put $\mu(p' - p) = w - p$ with a $w \in \mathrm{GS}(E, p) \cap B_r(p)$, then we have

$$p' + \alpha_i e_i = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\beta_{ij}(w_{ij} - p) + \frac{1}{\mu}(w - p) \in \mathrm{GS}^2(E, p)$$

for all $\alpha_i \in \mathbb{R}$. □

6 Basic cylinders and basic intervals

First, we will define the infinite dimensional intervals, which were simply defined in [14], more precisely divided into nondegenerate basic cylinders, degenerate basic cylinders, and basic intervals.

Definition 6.1. For any positive integer n, we define the infinite dimensional interval by

$$J = \prod_{i=1}^{\infty} J_i, \quad \text{where } J_i = \begin{cases} [0, p_{2i}] & (\text{for } i \in \Lambda_1), \\ [p_{1i}, p_{2i}] & (\text{for } i \in \Lambda_2), \\ [p_{1i}, 1] & (\text{for } i \in \Lambda_3), \\ \{p_{1i}\} & (\text{for } i \in \Lambda_4), \\ [0, 1] & (\text{otherwise}) \end{cases}$$

for some disjoint finite subsets Λ_1, Λ_2, and Λ_3 of $\{1, 2, \ldots, n\}$ and $0 < p_{1i} < p_{2i} < 1$ for $i \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ and $0 \le p_{1i} \le 1$ for $i \in \Lambda_4$. If $\Lambda_4 = \varnothing$, then J is called a *nondegenerate basic cylinder*. When Λ_4 is a nonempty finite set, J is called a *degenerate basic cylinder*. If Λ_4 is an infinite set, then J will be called a *basic interval*.

Remark 6.1.
(i) In order for an infinite dimensional interval J to become a basic cylinder, Λ_4 must be a finite set.
(ii) We remark that $\Lambda_4 = \mathbb{N} \setminus \Lambda(J)$ and $\Lambda(J) = \mathbb{N} \setminus \Lambda_4$. That is, $\mathbb{N}$ is the disjoint union of $\Lambda(J)$ and Λ_4.
(iii) If $p = (p_1, p_2, \ldots, p_i, \ldots)$ is an element of J, then $J_i = \{p_i\}$ for each $i \notin \Lambda(J)$.

We note that the basic cylinder or the basic interval J defined in Definition 6.1 can be expressed as

$$J = \left\{ \sum_{i=1}^{\infty} \alpha_i \left(\frac{1}{a_i} e_i \right) : \alpha_i \in a_i J_i \quad \text{for all } i \in \mathbb{N} \right\},$$

where J_i is the interval defined in Definition 6.1.

Definition 6.2. Let $\beta = \{\beta_i\}_{i\in\mathbb{N}}$ be a complete orthonormal sequence in M_a, J_i the interval given in Definition 6.1, and let n be a positive integer. We define

$$J_\beta = \left\{ \sum_{i=1}^{\infty} \alpha_i \beta_i : \alpha_i \in a_i J_i \quad \text{for all } i \in \mathbb{N} \right\}$$

for some disjoint finite subsets Λ_1, Λ_2, and Λ_3 of $\{1, 2, \ldots, n\}$; $0 < p_{1i} < p_{2i} < 1$ for $i \in \Lambda_1 \cup \Lambda_2 \cup \Lambda_3$; and $0 \le p_{1i} \le 1$ for $i \in \Lambda_4$. If $\Lambda_4 = \varnothing$, then J_β is called a *nondegenerate β-basic cylinder*. When Λ_4 is a nonempty finite set, J_β is called a *degenerate β-basic cylinder*. If Λ_4 is an infinite set, then J_β will be called a *β-basic interval*.

Using Definitions 6.1 and 6.2, Remark 6.1(*ii*) is generalized as follows:

Remark 6.2. Let $\beta = \{\beta_i\}_{i\in\mathbb{N}}$ be a complete orthonormal sequence in M_a and let J_β be a β-basic cylinder or a β-basic interval. It holds that $\Lambda_\beta(J_\beta) = \mathbb{N} \setminus \Lambda_4$, where Λ_4 is given in Definitions 6.1 and 6.2.

Proof. In general, if $i \in \Lambda_4$, then it follows from Definition 6.2 that

$$\langle x, \beta_i \rangle_a = \left\langle \sum_{j=1}^{\infty} \alpha_j \beta_j, \beta_i \right\rangle_a = \alpha_i \in a_i J_i = \{a_i p_{1i}\}$$

for all $x \in J_\beta$. That is, $\langle x, \beta_i \rangle_a = \alpha_i = a_i p_{1i}$ for all $x \in J_\beta$ and $i \in \Lambda_4$. If $i \in \Lambda_4$, then $\langle x + \alpha\beta_i, \beta_i \rangle_a = \langle x, \beta_i \rangle_a + \alpha = a_i p_{1i} + \alpha \neq a_i p_{1i}$ for all $x \in J_\beta$ and $\alpha \neq 0$, which implies that $x + \alpha\beta_i \notin J_\beta$. That is, in view of Definition 3.2(*ii*), we conclude that $i \notin \Lambda_\beta(J_\beta)$.

We now assume that $i \notin \Lambda_\beta(J_\beta)$. Then, by Definition 3.2(*ii*), it holds that

$$x + \alpha\beta_i \notin J_\beta \tag{6.1}$$

for any $x \in J_\beta$ and $\alpha \neq 0$. Using Definition 6.2 again, we have

$$x + \alpha\beta_i = \sum_{j \notin \Lambda_4} \alpha_j \beta_j + \sum_{j \in \Lambda_4} a_j p_{1j} \beta_j + \alpha\beta_i \tag{6.2}$$

for all $x \in J_\beta$ and $\alpha \neq 0$. We assume on the contrary that $i \notin \Lambda_4$. In view of (6.2) and by the structure of J_i ($a_i J_i$ is indeed a nondegenerate interval for $i \notin \Lambda_4$), it holds that

$$x + \alpha\beta_i = \sum_{j \notin \Lambda_4 \cup \{i\}} \alpha_j \beta_j + (\alpha_i + \alpha)\beta_i + \sum_{j \in \Lambda_4} a_j p_{1j} \beta_j \in J_\beta$$

for some $x \in J_\beta$ and $\alpha \neq 0$, which is contrary to (6.1). (We note that, for each $i \notin \Lambda_4$, $\alpha_i \in a_i J_i$ and there exists an $\alpha \neq 0$ satisfying $\alpha_i + \alpha \in a_i J_i$.) Therefore, we conclude that if $i \notin \Lambda_\beta(J_\beta)$, then $i \in \Lambda_4$. □

Theorem 6.1. *Let $\beta = \{\beta_i\}_{i \in \mathbb{N}}$ be a complete orthonormal sequence in M_a and let J_β be either a translation of a β-basic cylinder or a translation of a β-basic interval and $p \in J_\beta$. Then,*

$$\mathrm{GS}(J_\beta, p) = \left\{ p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i \in M_a : \alpha_i \in \mathbb{R} \quad \textit{for all } i \in \Lambda_\beta(J_\beta) \right\}.$$

Proof. Assume that x is an arbitrary element of $\mathrm{GS}(J_\beta, p)$. By Definition 3.1, we have

$$x - p = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \varepsilon_{ij}(x_{ij} - p) \in M_a$$

for some $m \in \mathbb{N}$, $\varepsilon_{ij} \in \mathbb{R}$, and $x_{ij} \in J_\beta$. Furthermore, since $x_{ij}, p \in J_\beta$, by Definition 6.2, we obtain

$$x_{ij} = \sum_{k=1}^{\infty} \gamma_k \beta_k = \sum_{k \in \mathbb{N} \setminus \Lambda_4} \gamma_k \beta_k + \sum_{k \in \Lambda_4} a_k p_{1k} \beta_k$$

and

$$p = \sum_{k=1}^{\infty} \delta_k \beta_k = \sum_{k \in \mathbb{N} \setminus \Lambda_4} \delta_k \beta_k + \sum_{k \in \Lambda_4} a_k p_{1k} \beta_k$$

for some $\gamma_k, \delta_k \in a_k J_k$.

Since $\{\beta_i\}_{i \in \mathbb{N}}$ is a complete orthonormal sequence in M_a, it follows from Definition 6.2 and Remark 6.2 that

$$x - p = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \varepsilon_{ij}(x_{ij} - p) = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \varepsilon_{ij} \sum_{k \in \mathbb{N} \setminus \Lambda_4} (\gamma_k - \delta_k)\beta_k = \sum_{k \in \mathbb{N} \setminus \Lambda_4} \omega_k \beta_k = \sum_{i \in \Lambda_\beta(J_\beta)} \omega_i \beta_i$$

for some real numbers ω_i. Since $x \in \mathrm{GS}(J_\beta, p) \subset M_a$, it holds that

$$x = p + \sum_{i \in \Lambda_\beta(J_\beta)} \omega_i \beta_i (\in M_a) \in \left\{ p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i \in M_a : \alpha_i \in \mathbb{R} \quad \text{for all } i \in \Lambda_\beta(J_\beta) \right\},$$

which implies that

$$\mathrm{GS}(J_\beta, p) \subset \left\{ p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i \in M_a : \alpha_i \in \mathbb{R} \quad \text{for all } i \in \Lambda_\beta(J_\beta) \right\}.$$

It remains to prove the reverse inclusion. According to the structure of J_β given in Definition 6.2, for each $i \in \Lambda_\beta(J_\beta)$, there exists a real number $\gamma_i \neq 0$ such that $p + \gamma_i \beta_i \in J_\beta$. In other words, for each $i \in \Lambda_\beta(J_\beta)$, there exists a $u_i \in J_\beta$ such that $\gamma_i \beta_i = u_i - p$. Thus, if we assume that

$$p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i \in M_a$$

for some $\alpha_i \in \mathbb{R}$, then

$$p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i = p + \sum_{i \in \Lambda_\beta(J_\beta)} \frac{\alpha_i}{\gamma_i}(\gamma_i \beta_i) = p + \sum_{i \in \Lambda_\beta(J_\beta)} \frac{\alpha_i}{\gamma_i}(u_i - p) \in \mathrm{GS}(J_\beta, p),$$

since $u_i \in J_\beta$ for all $i \in \Lambda_\beta(J_\beta)$, which implies that

$$\mathrm{GS}(J_\beta, p) \supset \left\{ p + \sum_{i \in \Lambda_\beta(J_\beta)} \alpha_i \beta_i \in M_a : \alpha_i \in \mathbb{R} \quad \text{for all } i \in \Lambda_\beta(J_\beta) \right\}.$$

We end the proof in this way. □

Since in some ways, index sets have some properties of dimensions in vector space, the following theorem may seem to be obvious.

Theorem 6.2. *Assume that a bounded subset E of M_a contains at least two elements and $p \in E$. Then,* $\Lambda(\mathrm{GS}^2(E, p)) = \mathbb{N}$ *if and only if* $\mathrm{GS}^2(E, p) = M_a$.

Proof. Let x be an arbitrary element of M_a. There exist some real numbers α_i such that

$$x = \sum_{i=1}^{\infty} \alpha_i e_i \in M_a. \tag{6.3}$$

If $\Lambda(\mathrm{GS}^2(E, p)) = \mathbb{N}$, then it follows from Lemma 5.5 that

$$p + \alpha_i e_i \in \mathrm{GS}^2(E, p)$$

for all $i \in \mathbb{N}$. In other words,

$$\alpha_i e_i \in \mathrm{GS}^2(E, p) - p$$

for all $i \in \mathbb{N}$.

By Lemma 5.3(i), we obtain

$$x_n := \sum_{i=1}^{n} \alpha_i e_i \in \mathrm{GS}^2(E, p) - p$$

for any $n \in \mathbb{N}$. Due to Lemma 5.3(iii) and (6.3), we further obtain

$$x = \sum_{i=1}^{\infty} \alpha_i e_i = \lim_{n \to \infty} x_n \in \mathrm{GS}^2(E, p) - p,$$

which implies that $M_a \subset \mathrm{GS}^2(E, p) - p$, or equivalently, $M_a \subset \mathrm{GS}^2(E, p)$.

The reverse inclusion is trivial. □

7 Second-order extension of isometries

It was proved in Theorem 4.2 that the domain of a d_a-isometry $f : E_1 \to E_2$ can be extended to the first-order generalized linear span $\mathrm{GS}(E_1, p)$ whenever E_1 is a nonempty bounded subset of M_a, whether degenerate or nondegenerate.

Now we generalize Theorem 4.2 into the following theorem. More precisely, we prove that the domain of f can be extended to its second-order generalized linear span $\mathrm{GS}^2(E_1, p)$. It follows from Lemma 5.3(iii) that $\mathrm{GS}^2(E_1, p) = \overline{\mathrm{GS}(E_1, p)}$. Therefore, the following theorem is a further generalization of [19, Theorem 2.2].

Although the closure $\overline{\mathrm{span}}\, E_1$ of the linear span of E_1 is a real Hilbert space, it seems difficult to extend the domain E_1 of a local isometry to the Hilbert space $\overline{\mathrm{span}}\, E_1$. However, we can extend the domain E_1 of a local isometry to the second-order generalized linear span, as we see in the following theorem. For this reason, we use the second-order generalized linear span instead of the closure of linear span of E_1.

In the proof, we use the fact that $\mathrm{GS}^n(E_1, p) - p$ is a real vector space.

Theorem 7.1. *Let E_1 be a bounded subset of M_a that is d_a-isometric to a subset E_2 of M_a via a surjective d_a-isometry $f : E_1 \to E_2$. Assume that p and q are elements of E_1 and E_2, which satisfy $q = f(p)$. The function $F_2 : \mathrm{GS}^2(E_1, p) \to M_a$ is a d_a-isometry and the function $T_{-q} \circ F_2 \circ T_p : \mathrm{GS}^2(E_1, p) - p \to M_a$ is linear. In particular, F_2 is an extension of F.*

Proof.

(a) Suppose r is a positive real number satisfying $E_1 \subset B_r(p)$. Referring to the changes presented in the table below and following the first part of proof of Theorem 4.2, we can easily prove that F_2 is a d_a-isometry.

Theorem 4.2:	E_1	$\mathrm{GS}(E_1, p)$	f	F	Definition 4.1	Lemma 3.2
Here:	$\mathrm{GS}(E_1, p) \cap B_r(p)$	$\mathrm{GS}^2(E_1, p)$	F	F_2	Definition 5.1	Lemma 5.2

(b) We prove the linearity of $T_{-q} \circ F_n \circ T_p : \mathrm{GS}^n(E_1, p) - p \to M_a$ in a more general setting for $n \geq 2$. Referring to the changes presented in the table below and following (d) of the proof of Theorem 4.2, we can easily prove that $T_{-q} \circ F_n \circ T_p$ is linear.

Theorem 4.2:	$\mathrm{GS}(E_1, p)$	F	(4.4)
Here:	$\mathrm{GS}^n(E_1, p)$	F_n	Lemma 5.2

(c) According to Definition 5.1(i), for any $m \in \mathbb{N}$, $x_{ij} \in \mathrm{GS}(E_1, p) \cap B_r(p)$, and any $\alpha_{ij} \in \mathbb{R}$ with $\sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(x_{ij} - p) \in M_a$, there exists a $u \in \mathrm{GS}^2(E_1, p)$ satisfying

$$u - p = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(x_{ij} - p) \in M_a. \tag{7.1}$$

Due to Definition 5.1(ii), we further have

$$(T_{-q} \circ F_2 \circ T_p)(u - p) = \sum_{i=1}^{m} \sum_{j=1}^{\infty} \alpha_{ij}(T_{-q} \circ F \circ T_p)(x_{ij} - p). \tag{7.2}$$

If we set $\alpha_{11} = 1$, $\alpha_{ij} = 0$ for each $(i, j) \neq (1, 1)$, and $x_{11} = x$ in (7.1) and (7.2), we see that

$$(T_{-q} \circ F_2 \circ T_p)(x - p) = (T_{-q} \circ F \circ T_p)(x - p) \tag{7.3}$$

for all $x \in \mathrm{GS}(E_1, p) \cap B_r(p)$.

Let w be an arbitrary element of $\mathrm{GS}(E_1, p)$. Then, $w - p \in \mathrm{GS}(E_1, p) - p$. Since $\mathrm{GS}(E_1, p) - p$ is a real vector space and $B_r(p) - p = B_r(0)$, there exists a (sufficiently small) real number $\mu \neq 0$ such that

$$\mu(w - p) \in (\mathrm{GS}(E_1, p) - p) \cap (B_r(p) - p).$$

Hence, by (5.1), we can choose a $v \in \mathrm{GS}(E_1, p) \cap B_r(p)$ such that $\mu(w - p) = v - p$. Since both $T_{-q} \circ F_2 \circ T_p$ and $T_{-q} \circ F \circ T_p$ are linear and $\mathrm{GS}(E_1, p) \subset \mathrm{GS}^2(E_1, p)$, it follows from (7.3) that

$$\begin{aligned}
\mu(T_{-q} \circ F_2 \circ T_p)(w - p) &= (T_{-q} \circ F_2 \circ T_p)(\mu(w - p)) \\
&= (T_{-q} \circ F_2 \circ T_p)(v - p) \\
&= (T_{-q} \circ F \circ T_p)(v - p) \\
&= (T_{-q} \circ F \circ T_p)(\mu(w - p)) \\
&= \mu(T_{-q} \circ F \circ T_p)(w - p).
\end{aligned}$$

Therefore, it follows that $(T_{-q} \circ F_2 \circ T_p)(w - p) = (T_{-q} \circ F \circ T_p)(w - p)$ for all $w \in \mathrm{GS}(E_1, p)$, i.e., $F_2(w) = F(w)$ for all $w \in \mathrm{GS}(E_1, p)$. In other words, F_2 is an extension of F. Also, because of Theorem 4.2, we see that F_2 is obviously an extension of f. □

On account of Proposition 5.4, it holds that

$$\mathrm{GS}^2(E_1, p) = \cdots = \mathrm{GS}^{n-1}(E_1, p) = \mathrm{GS}^n(E_1, p)$$

for every integer $n \geq 3$.

Theorem 7.2. *Let E_1 be a bounded subset of M_a that is d_a-isometric to a subset E_2 of M_a via a surjective d_a-isometry $f : E_1 \to E_2$. Assume that p and q are elements of E_1 and E_2, which satisfy $q = f(p)$. Then, F_n is identically the same as F_2 for any integer $n \geq 3$, where F_2 and F_n are defined in Definition* 5.1.

Proof. Let r be a fixed positive real number satisfying $E_1 \subset B_r(p)$. We assume that $F_2 \equiv F_3 \equiv \cdots \equiv F_{n-1}$ on $\mathrm{GS}^2(E_1, p)$. Let x be an arbitrary element of $\mathrm{GS}^n(E_1, p)$. Then, in view of (5.1), there exist a real number $\mu \neq 0$ and an element u of $\mathrm{GS}^n(E_1, p) \cap B_r(p)$ such that

$$u - p = \mu(x - p) \in (\mathrm{GS}^n(E_1, p) - p) \cap (B_r(p) - p).$$

If we put $\alpha_{11} = 1$, $\alpha_{ij} = 0$ for all $(i, j) \neq (1, 1)$, and $x_{11} = v$ in Definition 5.1(*ii*), then we obtain

$$(T_{-q} \circ F_n \circ T_p)(v - p) = (T_{-q} \circ F_{n-1} \circ T_p)(v - p) \tag{7.4}$$

for all $v \in \mathrm{GS}^{n-1}(E_1, p) \cap B_r(p) = \mathrm{GS}^n(E_1, p) \cap B_r(p)$ by Proposition 5.4.

Since $T_{-q} \circ F_n \circ T_p$ is linear by (b) in the proof of Theorem 7.1, it follows from (7.4) and our assumption that

$$\begin{aligned}\mu(T_{-q} \circ F_n \circ T_p)(x - p) &= (T_{-q} \circ F_n \circ T_p)(u - p)\\ &= (T_{-q} \circ F_{n-1} \circ T_p)(u - p)\\ &= (T_{-q} \circ F_2 \circ T_p)(u - p)\\ &= \mu(T_{-q} \circ F_2 \circ T_p)(x - p),\end{aligned}$$

i.e., $F_n(x) = F_2(x)$ for every $x \in \mathrm{GS}^n(E_1, p) = \mathrm{GS}^2(E_1, p)$. By mathematical induction, we conclude that F_n is identically the same as F_2 for every integer $n \geq 3$. □

Assume that J is either a translation of a basic cylinder or a translation of a basic interval, and p is an element of J. Due to Definition 6.1, Remark 6.1, and Theorem 6.1, $\mathrm{GS}(J, p)$ is a closed subset of M_a.

Remark 7.1. $\mathrm{GS}(J, p)$ is a closed subset of M_a.

Proof. Assume that $p = (p_1, p_2, \ldots, p_i, \ldots)$ is a fixed element of J, where J is a translation of a basic cylinder or a translation of a basic interval. In view of Definition 3.1 and Remark 6.1(*iii*), we note that $x_i = p_i$ for each $x = (x_1, x_2, \ldots, x_i, \ldots) \in \mathrm{GS}(J, p)$ and each $i \notin \Lambda(J)$.

Assume that $\{z_n\}_{n\in\mathbb{N}}$ is a sequence of elements in $\mathrm{GS}(J, p)$, which converges to an element $z = (z_1, z_2, \ldots, z_i, \ldots)$ of M_a. Let us denote by z_{ni} the ith component of z_n for any $i, n \in \mathbb{N}$. Since $z_n \in \mathrm{GS}(J, p)$ for every $n \in \mathbb{N}$, the previous argument implies that $z_{ni} = p_i$ for each $i \notin \Lambda(J)$. Thus, we conclude that $z_i = p_i$ for each $i \notin \Lambda(J)$. This fact and Theorem 6.1 with $\beta = \left\{\frac{1}{a_i} e_i\right\}_{i\in\mathbb{N}}$ imply that $z \in \mathrm{GS}(J, p)$. Therefore, we conclude that $\mathrm{GS}(J, p)$ is a closed subset of M_a. □

We note that $\left\{\frac{1}{a_i} e_i\right\}_{i\in\mathbb{N}}$ is a complete orthonormal sequence in M_a. On account of Theorem 6.1 with $\beta = \left\{\frac{1}{a_i} e_i\right\}_{i\in\mathbb{N}}$, we note that $\Lambda(J) = \Lambda(\mathrm{GS}(J, p))$.

Remark 7.2. $\mathrm{GS}^2(J,p) = \mathrm{GS}(J,p)$.

Proof. Referring to the changes presented in the table below

Proposition 5.4:	$\mathrm{GS}(E,p) \cap B_r(p)$	$\mathrm{GS}^2(E,p)$	$\mathrm{GS}^3(E,p)$	x	x_m
Here:	J	$\mathrm{GS}(J,p)$	$\mathrm{GS}^2(J,p)$	u	u_m

and following the part (a) in the proof of Proposition 5.4, we can easily show that $\mathrm{GS}^2(J,p) = \mathrm{GS}(J,p)$. □

Hence, by Theorem 6.1 with $\beta = \left\{\frac{1}{a_i}e_i\right\}_{i\in\mathbb{N}}$ and Remark 7.2, we have

$$\begin{aligned} u - p &= \sum_{i=1}^{\infty}\left\langle u-p, \frac{1}{a_i}e_i\right\rangle_a \frac{1}{a_i}e_i \\ &= \sum_{i=1}^{\infty} a_i(u_i - p_i)\frac{1}{a_i}e_i \\ &= \sum_{i\in\Lambda(J)} a_i(u_i - p_i)\frac{1}{a_i}e_i \\ &= \sum_{i\in\Lambda(J)}\left\langle u-p, \frac{1}{a_i}e_i\right\rangle_a \frac{1}{a_i}e_i \end{aligned} \tag{7.5}$$

for all $u \in \mathrm{GS}^2(J,p) = \mathrm{GS}^n(J,p)$, where $n \in \mathbb{N}$.

Using a similar approach to the proof of [14, Theorem 2.4], we can apply Lemma 5.2 to prove the following theorem.

Theorem 7.3. *Assume that J is either a translation of a basic cylinder or a translation of a basic interval, K is a subset of M_a, and that there exists a surjective d_a-isometry $f : J \to K$. Suppose p is an element of J and q is an element of K with $q = f(p)$. For any $n \in \mathbb{N}$, the d_a-isometry $F_n : \mathrm{GS}^n(J,p) \to M_a$ given in Definition 5.1 satisfies*

$$(T_{-q}\circ F_n \circ T_p)(u-p) = \sum_{i\in\Lambda(J)}\left\langle u-p, \frac{1}{a_i}e_i\right\rangle_a \frac{1}{a_i}(T_{-q}\circ F_n\circ T_p)(e_i)$$

for all $u \in \mathrm{GS}^n(J,p)$.

Proof. First, we have

$$\begin{aligned} &\Bigg\langle (T_{-q}\circ F_n\circ T_p)(u-p) - \sum_{i\in\Lambda(J)}\left\langle u-p,\frac{1}{a_i}e_i\right\rangle_a\frac{1}{a_i}(T_{-q}\circ F_n\circ T_p)(e_i), (T_{-q}\circ F_n\circ T_p)(u-p) \\ &\qquad - \sum_{j\in\Lambda(J)}\left\langle u-p,\frac{1}{a_j}e_j\right\rangle_a\frac{1}{a_j}(T_{-q}\circ F_n\circ T_p)(e_j)\Bigg\rangle_a \\ &= \langle (T_{-q}\circ F_n\circ T_p)(u-p), (T_{-q}\circ F_n\circ T_p)(u-p)\rangle_a \\ &\qquad - \sum_{j\in\Lambda(J)}\left\langle u-p,\frac{1}{a_j}e_j\right\rangle_a\frac{1}{a_j}\langle (T_{-q}\circ F_n\circ T_p)(u-p), (T_{-q}\circ F_n\circ T_p)(e_j)\rangle_a \\ &\qquad - \sum_{i\in\Lambda(J)}\left\langle u-p,\frac{1}{a_i}e_i\right\rangle_a\frac{1}{a_i}\langle (T_{-q}\circ F_n\circ T_p)(e_i), (T_{-q}\circ F_n\circ T_p)(u-p)\rangle_a \\ &\qquad + \sum_{i\in\Lambda(J)}\sum_{j\in\Lambda(J)}\left\langle u-p,\frac{1}{a_i}e_i\right\rangle_a\left\langle u-p,\frac{1}{a_j}e_j\right\rangle_a\frac{1}{a_ia_j}\langle (T_{-q}\circ F_n\circ T_p)(e_i), (T_{-q}\circ F_n\circ T_p)(e_j)\rangle_a \end{aligned}$$

for all $u \in \mathrm{GS}^n(J,p)$.

Since $p + e_i \in \mathrm{GS}^n(J, p)$ for each $i \in \Lambda(J)$, it follows from Lemma 5.2 that

$$\begin{aligned}
&\Big\langle (T_{-q} \circ F_n \circ T_p)(u-p) - \sum_{i\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_i}e_i \Big\rangle_a \frac{1}{a_i}(T_{-q} \circ F_n \circ T_p)(e_i), (T_{-q} \circ F_n \circ T_p)(u-p) \\
&\quad - \sum_{j\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a \frac{1}{a_j}(T_{-q} \circ F_n \circ T_p)(e_j) \Big\rangle_a \\
&= \langle u-p, u-p\rangle_a - \sum_{j\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a - \sum_{i\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_i}e_i \Big\rangle_a \Big\langle \frac{1}{a_i}e_i, u-p \Big\rangle_a \qquad (7.6)\\
&\quad + \sum_{i\in\Lambda(J)}\sum_{j\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_i}e_i \Big\rangle_a \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a \Big\langle \frac{1}{a_i}e_i, \frac{1}{a_j}e_j \Big\rangle_a \\
&= \langle u-p, u-p\rangle_a - \sum_{j\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a \Big\langle u-p, \frac{1}{a_j}e_j \Big\rangle_a
\end{aligned}$$

for all $u \in \mathrm{GS}^n(J, p)$, since $\left\{\frac{1}{a_i}e_i\right\}$ is an orthonormal sequence in M_a.

Furthermore, we note that each $u \in \mathrm{GS}^n(J, p)$ has the expression given in (7.5). Hence, if we replace $u - p$ in (7.6) with the expression (7.5), then we have

$$\Big\| (T_{-q} \circ F_n \circ T_p)(u-p) - \sum_{i\in\Lambda(J)} \Big\langle u-p, \frac{1}{a_i}e_i \Big\rangle_a \frac{1}{a_i}(T_{-q} \circ F_n \circ T_p)(e_i) \Big\|_a^2 = 0$$

for all $u \in \mathrm{GS}^n(J, p)$, which implies the validity of our assertion. □

According to the following theorem, the image of the first-order generalized linear span of E_1 with respect to p under the d_a-isometry F is just the first-order generalized linear span of $F(E_1)$ with respect to $F(p)$. This assertion holds also for the second-order generalized linear span and F_2.

Theorem 7.4. *Assume that E_1 and E_2 are bounded subsets of M_a that are d_a-isometric to each other via a surjective d_a-isometry $f : E_1 \to E_2$. Suppose p is an element of E_1 and q is an element of E_2 with $q = f(p)$. If $F_n : \mathrm{GS}^n(E_1, p) \to M_a$ is the extension of f defined in* Definition 5.1, *then $\mathrm{GS}^n(E_2, q) = F_n(\mathrm{GS}^n(E_1, p))$ for every $n \in \mathbb{N}$.*

Proof.

(*a*) First, we prove that our assertion is true for $n = 1$, i.e., we prove that $\mathrm{GS}(E_2, q) = F(\mathrm{GS}(E_1, p))$. Let r be a fixed positive real number satisfying $E_1 \subset B_r(p)$.

(*b*) Due to Definition 3.1, for any $y \in F(\mathrm{GS}(E_1, p))$, there exists an element $x \in \mathrm{GS}(E_1, p)$ with

$$y = F(x) = F\Big(p + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p)\Big)$$

for some $m \in \mathbb{N}$, $u_{ij} \in E_1 \cap B_r(p)$, and $\alpha_{ij} \in \mathbb{R}$ with $x = p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) \in M_a$.

On the other hand, by Definition 4.1, we have

$$(T_{-q} \circ F \circ T_p)\Big(\sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p)\Big) = \sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(T_{-q} \circ f \circ T_p)(u_{ij} - p),$$

which is equivalent to

$$F(x) - q = F\Big(p + \sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(u_{ij} - p)\Big) - q = \sum_{i=1}^{m}\sum_{j=1}^{\infty} \alpha_{ij}(f(u_{ij}) - q).$$

Since $u_{ij} \in E_1 = E_1 \cap B_r(p)$ for all i and j, it holds that $f(u_{ij}) \in f(E_1) = E_2$ for each i and j. Moreover, since $u_{ij} \in E_1 \cap B_r(p)$ for all i and j, it follows from Lemma 3.2 that

$$\begin{aligned}\|f(u_{ij}) - q\|_a^2 &= \|(T_{-q} \circ f \circ T_p)(u_{ij} - p)\|_a^2 \\ &= \langle (T_{-q} \circ f \circ T_p)(u_{ij} - p), (T_{-q} \circ f \circ T_p)(u_{ij} - p)\rangle_a \\ &= \langle u_{ij} - p, u_{ij} - p\rangle_a \\ &= \|u_{ij} - p\|_a^2 \\ &< r^2\end{aligned}$$

for all i and j. Hence, $f(u_{ij}) \in E_2 \cap B_r(q)$ for all i and j.

Furthermore, it follows from Lemma 3.2 that

$$\begin{aligned}\left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(f(u_{ij}) - q)\right\|_a^2 &= \left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ f \circ T_p)(u_{ij} - p)\right\|_a^2 \\ &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q} \circ f \circ T_p)(u_{ij} - p), \sum_{k=1}^{m}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(T_{-q} \circ f \circ T_p)(u_{k\ell} - p)\right\rangle_a \\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\alpha_{k\ell}\langle (T_{-q} \circ f \circ T_p)(u_{ij} - p), (T_{-q} \circ f \circ T_p)(u_{k\ell} - p)\rangle_a \\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\alpha_{k\ell}\langle u_{ij} - p, u_{k\ell} - p\rangle_a \\ &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p), \sum_{k=1}^{m}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(u_{k\ell} - p)\right\rangle_a \\ &= \left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p)\right\|_a^2 < \infty,\end{aligned}$$

since $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij} - p) = x - p \in M_a$.

Thus, on account of Remark 2.1, we see that $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(f(u_{ij}) - q) \in M_a$.

Therefore, in view of Definition 3.1, we obtain

$$y = F(x) = q + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(f(u_{ij}) - q) \in \mathrm{GS}(E_2, q)$$

and we conclude that $F(\mathrm{GS}(E_1, p)) \subset \mathrm{GS}(E_2, q)$.

(c) Now we assume that $y \in \mathrm{GS}(E_2, q)$. By Definition 3.1, there exist some $m \in \mathbb{N}$, $v_{ij} \in E_2 \cap B_r(q)$, and $\alpha_{ij} \in \mathbb{R}$ such that $y - q = \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(v_{ij} - q) \in M_a$. Since $f : E_1 \to E_2$ is surjective, there exists a $u_{ij} \in E_1$ satisfying $v_{ij} = f(u_{ij})$ for any i and j. Moreover, by Lemma 3.2, we have

$$\begin{aligned}\|u_{ij} - p\|_a^2 &= \langle u_{ij} - p, u_{ij} - p\rangle_a \\ &= \langle (T_{-q} \circ f \circ T_p)(u_{ij} - p), (T_{-q} \circ f \circ T_p)(u_{ij} - p)\rangle_a \\ &= \langle f(u_{ij}) - q, f(u_{ij}) - q\rangle_a \\ &= \langle v_{ij} - q, v_{ij} - q\rangle_a \\ &= \|v_{ij} - q\|_a^2 < r^2\end{aligned}$$

for any i and j. So we conclude that $u_{ij} \in E_1 \cap B_r(p)$ and $v_{ij} = f(u_{ij})$ for all i and j.

On the other hand, using Lemma 3.2, we have

$$\begin{aligned}\left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij}-p)\right\|_a^2 &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij}-p), \sum_{k=1}^{m}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(u_{k\ell}-p)\right\rangle_a \\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\alpha_{k\ell}\langle u_{ij}-p, u_{k\ell}-p\rangle_a \\ &= \sum_{i=1}^{m}\sum_{k=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}\sum_{\ell=1}^{\infty}\alpha_{k\ell}\langle (T_{-q}\circ f\circ T_p)(u_{ij}-p), (T_{-q}\circ f\circ T_p)(u_{k\ell}-p)\rangle_a \\ &= \left\langle \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(u_{ij}-p), \sum_{k=1}^{m}\sum_{\ell=1}^{\infty}\alpha_{k\ell}(T_{-q}\circ f\circ T_p)(u_{k\ell}-p)\right\rangle_a \\ &= \left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(u_{ij}-p)\right\|_a^2 \\ &= \left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(f(u_{ij})-q)\right\|_a^2 \\ &= \left\| \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(v_{ij}-q)\right\|_a^2 < \infty\end{aligned}$$

since $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(v_{ij}-q) = y-q \in M_a$. Thus, Remark 2.1 implies that $\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij}-p) \in M_a$.

Hence, it follows from Definition 4.1 that

$$\begin{aligned} y &= q + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(f(u_{ij})-q) \\ &= q + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(T_{-q}\circ f\circ T_p)(u_{ij}-p) \\ &= q + (T_{-q}\circ F\circ T_p)\left(\sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij}-p)\right) \\ &= F\left(p + \sum_{i=1}^{m}\sum_{j=1}^{\infty}\alpha_{ij}(u_{ij}-p)\right) \\ &\in F(\mathrm{GS}(E_1,p)).\end{aligned}$$

Thus, we conclude that $\mathrm{GS}(E_2,q) \subset F(\mathrm{GS}(E_1,p))$.

(d) Similarly, referring to the changes presented in the tables below and following the previous parts (b) and (c) in this proof, we can prove that $\mathrm{GS}^2(E_2,q) = F_2(\mathrm{GS}^2(E_1,p))$.

The case $n = 1$:	E_1	E_2	$\mathrm{GS}(E_1,p)$	$\mathrm{GS}(E_2,q)$	f	F
The case $n = 2$:	$\mathrm{GS}(E_1,p)$	$\mathrm{GS}(E_2,q)$	$\mathrm{GS}^2(E_1,p)$	$\mathrm{GS}^2(E_2,q)$	F	F_2

The case $n = 1$:	Definition 3.1	Definition 4.1	Lemma 3.2
The case $n = 2$:	Definition 5.1(i)	Definition 5.1(ii)	(4.4)

(e) Finally, according to Proposition 5.4, Theorem 7.2, and (d), we further have

$$\mathrm{GS}^n(E_2,q) = \mathrm{GS}^2(E_2,q) = F_2(\mathrm{GS}^2(E_1,p)) = F_n(\mathrm{GS}^n(E_1,p))$$

for any integer $n \geq 3$. ◻

8 Extension of isometries to the entire space

Let $I^\omega = \prod_{i=1}^{\infty} I$ be the *Hilbert cube*, where $I = [0, 1]$ is the unit closed interval. From now on, we assume that E_1 and E_2 are nonempty subsets of I^ω. They are bounded, of course.

In our main theorem (Theorem 8.1), we will prove that the domain of a local d_a-isometry $f : E_1 \to E_2$ can be extended to any real Hilbert space including the domain E_1.

Definition 8.1. Let E_1 be a nonempty subset of I^ω that is d_a-isometric to a subset E_2 of I^ω via a surjective d_a-isometry $f : E_1 \to E_2$. Let p be an element of E_1 and q be an element of E_2 with $q = f(p)$. Assume that $\left\{\frac{1}{a_i}e_i\right\}_{i\in\Lambda_\alpha}$ is a complete orthonormal sequence in the Hilbert space $\mathrm{GS}^2(E_1, p) - p$, where Λ_α is a nonempty proper subset of $\mathbb{N}$. Moreover, assume that $\{\beta_i\}_{i\in\mathbb{N}}$ is a complete orthonormal sequence in the Hilbert space M_a such that $\beta_i = \frac{1}{a_i}(T_{-q} \circ F_2 \circ T_p)(e_i)$ for each $i \in \Lambda_\alpha$, where $F_2 : \mathrm{GS}^2(E_1, p) \to M_a$ is defined in Definition 5.1. Let p_i be the ith component of p, i.e., $p = \sum_{i=1}^{\infty} p_i e_i$. For any set Λ satisfying $\Lambda_\alpha \subset \Lambda \subset \mathbb{N}$, we define a basic cylinder or a basic interval $\tilde{J}$ by

$$\tilde{J} = \prod_{i=1}^{\infty} \tilde{J}_i, \quad \text{where } \tilde{J}_i = \begin{cases} [0,1] & (\text{for } i \in \Lambda), \\ \{p_i\} & (\text{for } i \notin \Lambda). \end{cases}$$

Moreover, referring to Theorem 7.3, we define the function $G_2 : \mathrm{GS}^2(\tilde{J}, p) \to M_a$ by

$$(T_{-q} \circ G_2 \circ T_p)(u - p) = \sum_{i\in\Lambda(\tilde{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \beta_i \tag{8.1}$$

for all $u \in \mathrm{GS}^2(\tilde{J}, p)$.

The following theorem states that the domain of a local d_a-isometry can be extended to any real Hilbert space including the domain of the local d_a-isometry.

Theorem 8.1. *Let E_1 be a bounded subset of I^ω that contains at least two elements. Suppose E_1 is d_a-isometric to a subset E_2 of I^ω via a surjective d_a-isometry $f : E_1 \to E_2$. Let p and q be elements of E_1 and E_2 satisfying $q = f(p)$. Assume that $\left\{\frac{1}{a_i}e_i\right\}_{i\in\Lambda_\alpha}$ is a complete orthonormal sequence in the Hilbert space* $\mathrm{GS}^2(E_1, p) - p$, *where Λ_α is a nonempty proper subset of $\mathbb{N}$. Moreover, assume that $\{\beta_i\}_{i\in\mathbb{N}}$ is a complete orthonormal sequence in the Hilbert space M_a such that $\beta_i = \frac{1}{a_i}(T_{-q} \circ F_2 \circ T_p)(e_i)$ for each $i \in \Lambda_\alpha$. Let Λ be a set satisfying $\Lambda_\alpha \subset \Lambda \subset \mathbb{N}$ and let $\tilde{J}$ be defined as in* Definition 8.1. *Then, the function $G_2 : \mathrm{GS}^2(\tilde{J}, p) \to M_a$ is a d_a-isometry and the function $T_{-q} \circ G_2 \circ T_p : \mathrm{GS}^2(\tilde{J}, p) - p \to M_a$ is linear. In particular, G_2 is an extension of F_2.*

Proof.

(a) First, we assert that the function $T_{-q} \circ G_2 \circ T_p : \mathrm{GS}^2(\tilde{J}, p) - p \to M_a$ preserves the inner product. Assume that u and v are arbitrary elements of $\mathrm{GS}^2(\tilde{J}, p)$. Since $\Lambda = \Lambda(\tilde{J})$, it follows from (7.5), (8.1), and the orthonormality of $\left\{\frac{1}{a_i}e_i\right\}_{i\in\mathbb{N}}$ and $\{\beta_i\}_{i\in\mathbb{N}}$ that

$$\begin{aligned}\langle (T_{-q} \circ G_2 \circ T_p)(u - p), (T_{-q} \circ G_2 \circ T_p)(v - p)\rangle_a &= \left\langle \sum_{i\in\Lambda} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \beta_i, \sum_{j\in\Lambda} \left\langle v - p, \frac{1}{a_j}e_j \right\rangle_a \beta_j \right\rangle_a \\ &= \sum_{i\in\Lambda} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \sum_{j\in\Lambda} \left\langle v - p, \frac{1}{a_j}e_j \right\rangle_a \langle \beta_i, \beta_j\rangle_a \\ &= \sum_{i\in\Lambda} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \sum_{j\in\Lambda} \left\langle v - p, \frac{1}{a_j}e_j \right\rangle_a \left\langle \frac{1}{a_i}e_i, \frac{1}{a_j}e_j \right\rangle_a\end{aligned}$$

$$= \left\langle \sum_{i\in\Lambda} \left\langle u - p, \frac{1}{a_i} e_i \right\rangle_a \frac{1}{a_i} e_i, \ \sum_{j\in\Lambda} \left\langle v - p, \frac{1}{a_j} e_j \right\rangle_a \frac{1}{a_j} e_j \right\rangle_a$$
$$= \langle u - p, v - p \rangle_a$$

for all $u, v \in \mathrm{GS}^2(\tilde{J}, p)$, i.e., $T_{-q} \circ G_2 \circ T_p$ preserves the inner product.

(b) We assert that G_2 is a d_a-isometry. Let u and v be arbitrary elements of $\mathrm{GS}^2(\tilde{J}, p)$. Since $T_{-q} \circ G_2 \circ T_p$ preserves the inner product by (a), we have

$$d_a(G_2(u), G_2(v))^2 = \|(T_{-q} \circ G_2 \circ T_p)(u - p) - (T_{-q} \circ G_2 \circ T_p)(v - p)\|_a^2$$
$$= \langle (T_{-q} \circ G_2 \circ T_p)(u - p) - (T_{-q} \circ G_2 \circ T_p)(v - p),$$

$$\begin{aligned} &(T_{-q} \circ G_2 \circ T_p)(u - p) - (T_{-q} \circ G_2 \circ T_p)(v - p) \rangle_a \\ &= \langle u - p, u - p \rangle_a - \langle u - p, v - p \rangle_a - \langle v - p, u - p \rangle_a + \langle v - p, v - p \rangle_a \\ &= \langle (u - p) - (v - p), (u - p) - (v - p) \rangle_a \\ &= \|(u - p) - (v - p)\|_a^2 \\ &= \|u - v\|_a^2 \\ &= d_a(u, v)^2 \end{aligned}$$

for all $u, v \in \mathrm{GS}^2(\tilde{J}, p)$, i.e., $G_2 : \mathrm{GS}^2(\tilde{J}, p) \to M_a$ is a d_a-isometry.

(c) Now, we assert that the function $T_{-q} \circ G_2 \circ T_p : \mathrm{GS}^2(\tilde{J}, p) - p \to M_a$ is linear. Assume that u and v are arbitrary elements of $\mathrm{GS}^2(\tilde{J}, p)$ and α and β are real numbers. Since $\mathrm{GS}^2(\tilde{J}, p) - p$ is a real vector space, it holds that $\alpha(u - p) + \beta(v - p) \in \mathrm{GS}^2(\tilde{J}, p) - p$. Thus, $\alpha(u - p) + \beta(v - p) = w - p$ for some $w \in \mathrm{GS}^2(\tilde{J}, p)$. Hence, referring to the changes presented in the table below and following (d) of the proof of Theorem 4.2, we can easily prove that $T_{-q} \circ G_2 \circ T_p$ is linear.

Theorem 4.2:	$\mathrm{GS}(E_1, p)$	F	(4.4)
Here:	$\mathrm{GS}^2(\tilde{J}, p)$	G_2	(a)

(d) Finally, we assert that G_2 is an extension of F_2. Let $\hat{J}$ be either a basic cylinder or a basic interval defined by

$$\hat{J} = \prod_{i=1}^{\infty} \hat{J}_i, \quad \text{where } \hat{J}_i = \begin{cases} [0, 1] & (\text{for } i \in \Lambda_\alpha), \\ \{p_i\} & (\text{for } i \notin \Lambda_\alpha). \end{cases}$$

We see that $p = (p_1, p_2, \ldots) \in \hat{J} \cap E_1$ and $\Lambda(\hat{J}) = \Lambda_\alpha = \Lambda(\mathrm{GS}^2(E_1, p))$.

According to Lemma 5.5, if $i \in \Lambda(\mathrm{GS}^2(E_1, p))$, then $\alpha_i e_i \in \mathrm{GS}^2(E_1, p) - p$ for all $\alpha_i \in \mathbb{R}$. Since $\mathrm{GS}^2(E_1, p) - p$ is a real vector space, if we set $\Lambda_n = \{i \in \Lambda(\mathrm{GS}^2(E_1, p)) : i < n\}$, then we have

$$\sum_{i\in\Lambda_n} \alpha_i e_i \in \mathrm{GS}^2(E_1, p) - p$$

for all $n \in \mathbb{N}$ and all $\alpha_i \in \mathbb{R}$. For now, with all α_i fixed, we define $x_n = p + \sum_{i\in\Lambda_n} \alpha_i e_i$ for any $n \in \mathbb{N}$. Then, $\{x_n\}$ is a sequence in $\mathrm{GS}^2(E_1, p)$. When $\{x_n\}$ converges in M_a, it holds that

$$p + \sum_{i\in\Lambda(\mathrm{GS}^2(E_1,p))} \alpha_i e_i = \lim_{n\to\infty} x_n \in \mathrm{GS}^2(E_1, p),$$

because $\mathrm{GS}^2(E_1, p)$ is closed by Lemma 5.3(iii). That is,

$$\left\{ p + \sum_{i\in\Lambda(\mathrm{GS}^2(E_1,p))} \alpha_i e_i \in M_a : \alpha_i \in \mathbb{R} \quad \text{for all } i \in \Lambda(\mathrm{GS}^2(E_1, p)) \right\} \subset \mathrm{GS}^2(E_1, p).$$

Hence, by the previous inclusion and Theorem 6.1 with $\beta = \left\{\frac{1}{a_i} e_i\right\}_{i\in\mathbb{N}}$ and $J_\beta = \hat{J}$, we obtain

$$\begin{aligned}\mathrm{GS}(\hat{J}, p) - p &= \left\{ \sum_{i\in\Lambda(\hat{J})} \alpha_i e_i \in M_a : \alpha_i \in \mathbb{R} \text{ for all } i \in \Lambda(\hat{J}) \right\} \\ &= \left\{ \sum_{i\in\Lambda(\mathrm{GS}^2(E_1,p))} \alpha_i e_i \in M_a : \alpha_i \in \mathbb{R} \text{ for all } i \in \Lambda(\mathrm{GS}^2(E_1, p)) \right\} \\ &\subset \mathrm{GS}^2(E_1, p) - p.\end{aligned}$$

So, we have

$$\hat{J} \cap B_r(p) \subset \mathrm{GS}(\hat{J}, p) \cap B_r(p) \subset \mathrm{GS}^2(E_1, p) \cap B_r(p)$$

for some real number $r > 0$, and hence, we further have

$$\mathrm{GS}(\hat{J}, p) \subset \mathrm{GS}^2(\hat{J}, p) \subset \mathrm{GS}^3(E_1, p) = \mathrm{GS}^2(E_1, p).$$

Moreover, by Remark 7.2, we know that $\mathrm{GS}^2(\hat{J}, p) = \mathrm{GS}(\hat{J}, p)$. Hence, we have

$$\mathrm{GS}(\hat{J}, p) = \mathrm{GS}^2(\hat{J}, p) \subset \mathrm{GS}^2(E_1, p).$$

On the other hand, since $\left\{\frac{1}{a_i}e_i\right\}_{i\in\Lambda_\alpha}$ is a complete orthonormal sequence in $\mathrm{GS}^2(E_1, p) - p$, it follows from Theorem 6.1 with $\beta = \left\{\frac{1}{a_i}e_i\right\}_{i\in\mathbb{N}}$ that

$$x = \sum_{i\in\Lambda_\alpha} \left\langle x, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}e_i \in \mathrm{GS}(\hat{J}, p) - p = \mathrm{GS}^2(\hat{J}, p) - p$$

for all $x \in \mathrm{GS}^2(E_1, p) - p$, which implies that $\mathrm{GS}^2(E_1, p) = \mathrm{GS}^2(\hat{J}, p) = \mathrm{GS}(\hat{J}, p)$.

Let u be an arbitrary element of $\mathrm{GS}^2(E_1, p)$. Then, by (7.5) with $\hat{J}$ instead of J, we have

$$u - p = \sum_{i\in\Lambda(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}e_i, \tag{8.2}$$

and since $T_{-q} \circ F_2 \circ T_p$ is linear and continuous, we use (8.1), (8.2), and the facts $\mathrm{GS}^2(E_1, p) = \mathrm{GS}^2(\hat{J}, p) = \mathrm{GS}(\hat{J}, p)$, and $\Lambda(\hat{J}) = \Lambda_\alpha = \Lambda(\mathrm{GS}^2(E_1, p))$ to have

$$\begin{aligned}(T_{-q} \circ G_2 \circ T_p)(u - p) &= \sum_{i\in\Lambda(\tilde{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \beta_i \\ &= \sum_{i\in\Lambda(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \beta_i \\ &= \sum_{i\in\Lambda(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}(T_{-q} \circ F_2 \circ T_p)(e_i) \\ &= \lim_{n\to\infty} \sum_{i\in\Lambda_n(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}(T_{-q} \circ F_2 \circ T_p)(e_i) \\ &= \lim_{n\to\infty} (T_{-q} \circ F_2 \circ T_p)\left(\sum_{i\in\Lambda_n(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}e_i \right) \\ &= (T_{-q} \circ F_2 \circ T_p)\left(\sum_{i\in\Lambda(\hat{J})} \left\langle u - p, \frac{1}{a_i}e_i \right\rangle_a \frac{1}{a_i}e_i \right) \\ &= (T_{-q} \circ F_2 \circ T_p)(u - p),\end{aligned}$$

where we set $\Lambda_n(\hat{J}) = \{i \in \Lambda(\hat{J}) : i < n\}$ for every $n \in \mathbb{N}$.

Therefore, it follows that $(T_{-q} \circ G_2 \circ T_p)(u-p) = (T_{-q} \circ F_2 \circ T_p)(u-p)$ for all $u \in \mathrm{GS}^2(E_1, p)$, i.e., $G_2(u) = F_2(u)$ for all $u \in \mathrm{GS}^2(E_1, p)$. In other words, G_2 is an extension of F_2. □

9 Applications

Given an integer $n > 0$, let $a = \{a_i\}_{i\in\mathbb{N}}$ be a sequence of positive real numbers that satisfies

$$a_1 = \cdots = a_n = 1 \quad \text{and} \quad \sum_{i=n+1}^{\infty} a_i^2 < \infty.$$

We know that the n dimensional real vector space $\mathbb{R}^n$ can be identified with a subspace of M_a. More precisely, it holds that $\mathbb{R}^n \simeq \{(x_1, \dots, x_n, 0, 0, \dots) : x_i \in \mathbb{R} \text{ for } 1 \le i \le n\}$.

We can define an inner product $\langle\cdot,\cdot\rangle_a$ for $\mathbb{R}^n$ by

$$\langle x, y\rangle_a = \sum_{i=1}^{\infty} a_i^2 x_i y_i = \sum_{i=1}^{n} x_i y_i$$

for all $x, y \in \mathbb{R}^n$, with which $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_a)$ becomes a real Hilbert space. This inner product induces the Euclidean norm in the natural way as follows:

$$\|x\|_a = \sqrt{\langle x, x\rangle_a} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

for all $x \in \mathbb{R}^n$, and $(\mathbb{R}^n, \|\cdot\|_a)$ becomes a real Banach space.

If we replace M_a, $\mathbb{N}$, and I^ω with $\mathbb{R}^n$, $\{1, 2, \dots, n\}$, and $[0,1]^n$, respectively, in the definitions and theorems in the previous sections, it is not difficult to see that they also hold for the n dimensional Euclidean space $\mathbb{R}^n$.

The following theorem is the finite dimensional real Hilbert space version of Theorem 8.1, the main theorem of this article. More specifically, Theorem 8.1, which applies to infinite dimensional real Hilbert spaces, is also applicable to finite dimensional real Hilbert spaces.

Theorem 9.1. *Let E_1 be a bounded subset of $[0,1]^n$ that contains at least two elements. Suppose E_1 is d_a-isometric to a subset E_2 of $[0,1]^n$ via a surjective d_a-isometry $f : E_1 \to E_2$. Let p and q be elements of E_1 and E_2 satisfying $q = f(p)$. Assume that $\left\{\frac{1}{a_i}e_i\right\}_{i\in\Lambda_\alpha}$ is a complete orthonormal sequence in the Hilbert space $\mathrm{GS}^2(E_1, p) - p$, where Λ_α is a nonempty proper subset of $\{1, 2, \dots, n\}$. Moreover, assume that $\{\beta_i\}_{i\in\{1,2,\dots,n\}}$ is a complete orthonormal sequence in the Hilbert space $\mathbb{R}^n$ such that $\beta_i = \frac{1}{a_i}(T_{-q} \circ F_2 \circ T_p)(e_i)$ for each $i \in \Lambda_\alpha$. Let Λ be a set satisfying $\Lambda_\alpha \subset \Lambda \subset \{1, 2, \dots, n\}$ and let $\tilde{J}$ be defined as in* Definition 8.1. *Then, the function $G_2 : \mathrm{GS}^2(\tilde{J}, p) \to \mathbb{R}^n$ is a d_a-isometry and the function $T_{-q} \circ G_2 \circ T_p : \mathrm{GS}^2(\tilde{J}, p) - p \to \mathbb{R}^n$ is linear. In particular, G_2 is an extension of F_2.*

10 Discussion

The pair (X, Y) of Hilbert spaces is said to have the *isometric extension property* if for every isometry f from an arbitrary subset S of X into Y, there exists an isometry F of X into Y such that the restriction of F to S coincides with f.

The following theorem is a well-known result due to [20, Theorem 11.4].

Theorem 10.1. (Wells and Williams) *If H is a Hilbert space, then (H, H) has the isometric extension property if and only if H is finite dimensional. In general, if $S \subset H$ and $f : S \to H$ is an isometry, then f can be extended as an isometry to the closed linear span of S.*

We note that Theorem 10.1 does not imply Theorem 8.1. For example, let E_1 and E_2 be subsets of the Hilbert cube I^ω. Then, $\mathrm{GS}^2(\tilde{J}, p) - p$ is a proper subspace of the real Hilbert space M_a, and $\mathrm{GS}^2(E_1, p) - p$ is a proper subspace of $\mathrm{GS}^2(\tilde{J}, p) - p$. Nevertheless, it follows from Theorem 8.1 that every surjective isometry $f : E_1 \to E_2$ can be extended to an isometry $G_2 : \mathrm{GS}^2(\tilde{J}, p) \to M_a$. On the other hand, we cannot expect to obtain this result using Theorem 10.1, since the closed linear span of E_1 is a proper subset of $\mathrm{GS}^2(\tilde{J}, p) - p$, which implies that Theorem 8.1 is not only different from Theorem 10.1 but also has a number of advantages.

Moreover, for any bounded subset S of I^ω, it is clear that $\overline{\mathrm{span}}\, S \subset \mathrm{GS}^2(S, p)$. But, it is not yet clear whether $\overline{\mathrm{span}}\, S = \mathrm{GS}^2(S, p)$, where $\overline{\mathrm{span}}\, S$ denotes the closed linear span of S. If $\overline{\mathrm{span}}\, S \neq \mathrm{GS}^2(S, p)$ is correct, Theorem 8.1 has more advantages than Theorem 10.1.

According to Theorem 8.1, the domain of a local d_a-isometry can be extended to any real Hilbert space containing that domain.

11 Conclusion

In view of Theorem 8.1, the domain of a local d_a-isometry can be extended to any real Hilbert space containing that domain. The domain of a local d_a-isometry does not need to be a convex body or an open set required by [1], it just needs to be bounded and contain at least two elements. Therefore, the coverage of our result is wider than that of previous results. This is the biggest advantage of this article compared to the previous results.

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