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Research Article

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Some estimates for commutators of bilinear pseudo-differential operators

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Abstract: We obtain a class of commutators of bilinear pseudo-differential operators on products of Hardy spaces by applying the accurate estimates of the Hörmander class. And we also prove another version of these types of commutators on Herz-type spaces.

Keywords: bilinear pseudo-differential operator, commutator, Lipschitz function, Hardy space, Herz-type space

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1 Introduction and main results

Let T be a linear operator. Given a function b, the commutator $[T, b]$ is defined by

$$[T, b](f) := T(bf) - bT(f).$$

It is very interesting that T is a pseudo-differential operator because its theory plays an important role in many aspects of harmonic analysis and it has had quite a success in a linear setting. As one of the most meaningful branches, the study of bilinear pseudo-differential operators was motivated not only as generalizations of the theory of linear ones but also its natural appearance in Harmonic. This topic is continuous to attract many researchers.

Let b be a Lipschitz function and $1 < p < \infty$. The estimates of the form

$$\|[T, b](f)\|_{L^p} \lesssim \|b\|_{\mathrm{Lip}^1}\|f\|_{L^p}, \quad \text{for all } f \in L^p(\mathbb{R}^n) \tag{1.1}$$

have been studied extensively. In particular, Calderón proved that (1.1) holds when T is a pseudo-differential operator whose kernel is homogeneous of degree of $-n-1$ in [1]. Coifman and Meyer showed (1.1) when $T = T_\sigma$ and σ is a symbol in the Hörmander class $S^1_{1,0}$ in [2,3], and this result was later extended by Auscher and Taylor in [4] to $\sigma \in \mathcal{B}S^1_{1,1}$, where the class $\mathcal{B}S^1_{1,1}$, which contains $S^1_{1,0}$ modulo symbols associated with smoothing operators, consists of symbols whose Fourier transforms in the first n-dimensional variable are appropriately compactly supported. The method in the proofs of [2,3] mainly showed that, for each Lipschitz continuous function b on $\mathbb{R}^n$, $[T, b]$ is a Calderón-Zygmund singular integral whose kernel constants are controlled by $\|b\|_{\mathrm{Lip}^1}$. For another thing, Auscher and Taylor proved (1.1) in two different ways: one method is based on the paraproducts while another is based on the Calderón-Zygmund singular integral operator approach that relies on the $T(1)$ theorem. For a more systematic study of these (and even more general) spaces, we refer the readers to [5,6].

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Given a bilinear operator T and a function b, the following two kinds of commutators are, respectively, defined by

$$[T,b]_1(f,g)=T(bf,g)-bT(f,g)$$

and

$$[T,b]_2(f,g)=T(f,bg)-bT(f,g).$$

In 2014, Bényi and Oh proved that (1.1) is also valid for this bilinear setting in [7]. More precisely, given a bilinear pseudo-differential operator T_σ with σ in the bilinear Hörmander class $\mathrm{BS}^1_{1,0}$ and a Lipschitz function b on $\mathbb{R}^n$, it was proved in [7, Theorem 1] that $[T,b]_1$ and $[T,b]_2$ are bilinear Calderón-Zygmund operators. The main aim of this article is to study (1.1) of $[T_\sigma,b]_j (j=1,2)$ on the products of Hardy spaces and Herz-type spaces with $\sigma\in\mathcal{B}\mathrm{BS}^1_{1,1}$. Before stating our main results, we need to recall some definitions and notations.

We say that a function b defined on $\mathbb{R}^n$ is Lipschitz continuous if

$$\|b\|_{\mathrm{Lip}^1}:=\sup_{x,y\in\mathbb{R}^n}\frac{|b(x)-b(y)|}{|x-y|}<\infty.$$

Let $\delta\geq 0$, $\rho>0$ and $m\in\mathbb{R}$. An infinitely differentiable function $\sigma:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ belongs to the bilinear Hörmander class $\mathrm{BS}^m_{\rho,\delta}$ if for all multi-indices $\alpha,\beta,\gamma\in\mathbb{N}^n_0$ there exists a positive constant $C_{\alpha,\beta,\gamma}$ such that

$$|\partial^\alpha_x\partial^\beta_\xi\partial^\gamma_\eta\sigma(x,\xi,\eta)|\leq C(1+|\xi|+|\eta|)^{m+\delta|\alpha|-\rho(|\beta|+|\gamma|)}.$$

Given a $\sigma(x,\xi,\eta)\in\mathrm{BS}^m_{\rho,\delta}$, the bilinear pseudo-differential operator associated with σ is defined by

$$T_\sigma(f,g)(x)=\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\sigma(x,\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)e^{2\pi ix\cdot(\xi+\eta)}\mathrm{d}\xi\mathrm{d}\eta,\quad\text{for all } x\in\mathbb{R}^n, f,g\in\mathcal{S}(\mathbb{R}^n).$$

In 1980, Meyer [8] first introduced the linear $\mathrm{BS}^m_{1,1}$, and corresponding boundedness of $[T_\sigma,a]_j(j=1,2)$ is obtained by Bényi and Oh in [7], that is given $m\in\mathbb{R}$ and $r>0$, an infinitely differentiable function $\sigma:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ belongs to $\mathcal{B}_r\mathrm{BS}^m_{1,1}$ if

$$\sigma\in BS^m_{1,1},\ \mathrm{supp}(\hat{\sigma}^1)\subset\{(\tau,\xi,\eta)\in\mathbb{R}^{3n}:|\tau|\leq r(|\xi|+|\eta|)\},$$

where $\hat{\sigma}^1$ denotes the Fourier transform of σ with respect to its first variable in $\mathbb{R}^n$, that is, $\hat{\sigma}^1(\tau,\xi,\eta)=\widehat{\sigma(\cdot,\xi,\eta)}(\tau)$. for all $\tau,\xi,\eta\in\mathbb{R}^n$. The class $\mathcal{B}\mathrm{BS}^m_{1,1}$ is defined as

$$\mathcal{B}\mathrm{BS}^m_{1,1}=\bigcup_{r\in(0,\frac{1}{7})}\mathcal{B}_r\mathrm{BS}^m_{1,1}.$$

Recently, many authors are interested in bilinear operators, which is a natural generalization of linear case. With further research, Bényi and Naibo proved the boundedness for the commutators of bilinear pseudo-differential operators and Lipschitz functions with $\sigma\in\mathcal{B}\mathrm{BS}^1_{1,1}$ on the Lebesgue spaces in [9]. In 2018, Tao and Li proved that the boundedness of the commutators of bilinear pseudo-differential operators was also true on the classical and generalized Morrey spaces in [10]. Motivated by the results mentioned above, a natural and interesting problem is to consider whether or not (1.1) is true on the products of Hardy spaces and Herz-type spaces with $\sigma\in\mathcal{B}\mathrm{BS}^1_{1,1}$. The purpose of this article is to give a surely an answer. Our proofs are based on the pointwise estimates of the sharp maximal function proved in the next section.

Suppose that $\sigma\in\mathcal{B}\mathrm{BS}^1_{1,1}$. Let K and K_j denote the kernel of T_σ and $[T_\sigma,b]_j$ $(j=1,2)$, respectively. We have

$$\begin{aligned}K(x,y,z)&=\int\int e^{i\xi\cdot(x-y)}e^{i\eta\cdot(x-z)}\sigma(x,\xi,\eta)\mathrm{d}\xi\mathrm{d}\eta,\\K_1(x,y,z)&=(b(y)-b(x))K(x,y,z),\\K_2(x,y,z)&=(b(z)-b(x))K(x,y,z).\end{aligned}$$

Then the following consequences are true.

Theorem A. [7, Lemma 3] *If $x \neq y$ or $x \neq z$, then we have*
(1) $|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K(x, y, z)| \leq C_{\alpha,\beta,\gamma}(|x-y| + |x-z|)^{-2n-1-|\alpha|-|\beta|-|\gamma|}$,
(2) $|K_j(x, y, z)| \lesssim \|b\|_{\mathrm{Lip}^1}(|x-y| + |x-z| + |y-z|)^{-2n}$.

Let $m \geq 1$ be a positive integer and $K(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ and C be a positive constant. We say that K is a multilinear Calderón-Zygmund kernel if it satisfies the size condition that for all $(x, y_1, \ldots, y_m) \in (\mathbb{R}^m)^{m+1}$ with $x \neq y_s$ for some $1 \leq s \leq m$,

$$|K(x, y_1, \ldots, y_m)| \leq C(|x-y_1| + \cdots + |x-y_m|)^{-mn}$$

and satisfies the regularity condition that

$$|K(x, y_1, \ldots, y_m) - K(x', y_1, \ldots, y_m)| \leq C|x-x'|(|x-y_1| + \cdots + |x-y_m|)^{-mn-1}$$

whenever $\max_{1\leq k\leq m}|x-y_k| \geq 2|x-x'|$, and also that for each fixed k with $1 \leq k \leq m$,

$$\begin{aligned} &|K(x, y_1, \ldots, y_{k-1}, y_k, y_{k+1}, \ldots, y_m) - K(x, y_1, \ldots, y_{k-1}, y_k', y_{k+1}, \ldots, y_m)| \\ &\leq C|x-x'|(|x-y_1| + \cdots + |x-y_m|)^{-mn-1} \end{aligned} \tag{1.2}$$

whenever $\max_{1\leq s\leq m}|x-y_s| \geq 2|y_k - y_k'|$.

The statements of our main theorems are presented as follows.

Theorem 1.1. *Let $\sigma \in \mathcal{B}\mathrm{BS}^1_{1,1}$ and b be a Lipschitz function on $\mathbb{R}^n$. Suppose that for all bounded functions a_i supported on some cubes in $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} a_i(x)\mathrm{d}x = 0$ for $i = 1, 2$,*

$$\int_{\mathbb{R}^m} [T_\sigma, b]_j(a_1, a_2)(x)x^\alpha \mathrm{d}x = 0$$

for every multi-index α with $|\alpha| \leq n$, where $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. Then $[T_\sigma, b]_j$ $(j = 1, 2)$ extends boundedly from $H^{p_1}(\mathbb{R}^n) \times H^{p_2}(\mathbb{R}^n)$ into $H^p(\mathbb{R}^n)$ for $p_1, p_2 \in n/(n+1), 1$ and p with $1/p = 1/p_1 + 1/p_2$ and $\lfloor n(1/p - 1) = m\rfloor$, where $\lfloor s\rfloor$ for any $s \in \mathbb{R}$ denotes the integer not greater than s.

Theorem 1.2. *Let $\sigma \in \mathcal{B}\mathrm{BS}^1_{1,1}$ and b be a Lipschitz function on $\mathbb{R}^n$. Suppose $0 \leq p_1, p_2 \leq 1$, $1 < q_1, q_2 < \infty$, $1/p = 1/p_1 + 1/p_2$, $1/q = 1/q_1 + 1/q_2$. If $[T_\sigma, b]_j$ $(j = 1, 2)$ is bounded from $L^{q_1} \times L^{q_2}$ into $L^{q,\infty}$ with controlled by $\|a\|_{\mathrm{Lip}^1}$, then $[T_\sigma, b]_j$ $(j = 1, 2)$ is bounded from $\dot{K}_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n) \times \dot{K}_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)$ into $W\dot{K}_{q_2}^{n(2-1/q),p}(\mathbb{R}^n)$.*

Throughout this article, for $1 \leq p \leq \infty$, p' is the conjugate index of p, that is, $1/p + 1/p' = 1$. $B(x, R)$ denotes the ball centered at x with radius $R > 0$ and $f_B = \frac{1}{|B(x,R)|}\int_{B(x,R)} f(y)\mathrm{d}y$. The boundedness of commutators on product of Hardy spaces is presented in Section 2. The boundedness of commutators on product of Herz-type spaces is given in Section 3.

2 Boundedness on product of Hardy spaces

Definition 2.1. [11] Let $p \in (0, 1]$. The Hardy space $H^p(\mathbb{R}^n)$ is defined by

$$H^p(\mathbb{R}^n) \equiv \{f \in \mathcal{S}(\mathbb{R}^n) : \varphi^+(f) \equiv \sup_{t>0}|\varphi_t * f| \in L^p(\mathbb{R}^n)\},$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n}\varphi(x)\mathrm{d}x = 1$, and for any $y \in \mathbb{R}^n$ and $t \in (0,\infty)$, $\varphi_t(y) = t^{-n}\varphi(y/t)$. Moreover, define

$$\|f\|_{H^p(\mathbb{R}^n)} \equiv \|\varphi^+(f)\|_{L^p(\mathbb{R}^n)}.$$

It is known that the definition of Hardy space $H^p(\mathbb{R}^n)$ does not depend on the choice of φ (see [11]).

Definition 2.2. [12] Let $p \in (0,1]$ and $q \in [1,\infty]$ with $p \neq q$. For $s \in \mathbb{Z}$ satisfying $s \geq \lfloor n(1/p-1)\rfloor$, a real-valued function $a(x)$ is called a (p,q,s)-atom centered at x_0 if
- (i) $a \in L^q(\mathbb{R}^n)$ and is supported in a cube Q centered at x_0;
- (ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |Q|^{1/q-1/p}$;
- (iii) $\int_{\mathbb{R}^n} a(x)x^\alpha \mathrm{d}x = 0$ for every multi-index α with $|\alpha| \leq s$.

Let $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ be the set of all finite linear combinations of (p,q,s)-atoms. For any $f \in H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$, define

$$\|f\|_{H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)} \equiv \left\{\left(\sum_{i=1}^{k}|\lambda_i|^p\right)^{1/p} : f = \sum_{i=1}^{k}\lambda_i a_i,\ k \in \mathbb{N},\ \{a_i\}_{i=1}^k \text{ are } (p,q,s)\text{-atoms}\right\}.$$

Denote by $C(\mathbb{R}^n)$ the set of all continuous. Meda et al. proved the following result in [13], which ensures that a bounded linear operator on $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ with $q < \infty$ or $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ can be extended to be a bounded operator on $H^p(\mathbb{R}^n)$.

Lemma 2.1. [13] *Let $p \in (0,1]$, $q \in [1,\infty]$ with $p \neq q$ and $s \in \mathbb{Z}$ satisfying $s \geq \lfloor n(1/p-1)\rfloor$. The quasi-norms $\|\cdot\|_{H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^p(\mathbb{R}^n)}$ are equivalent on $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n)$ where $q < \infty$ and on $H^{p,q,s}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ when $q = \infty$.*

To prove Theorem 1.2, we need the boundedness of $[T_\sigma, a]_j (j = 1,2)$ on products of Lebesgue spaces.

Lemma 2.2. [9] *If $\sigma \in \mathcal{B}\mathrm{BS}^m_{1,1}$ and b is a Lipschitz function on $\mathbb{R}^n$, then the commutators $[T_\sigma, b]_j$ $(j = 1,2)$ are bilinear Calderón-Zygmund operators. In particular, $[T_\sigma, b]_j$, $j = 1,2$ are bounded from $L^{p_1} \times L^{p_2}$ into L^p for $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $1 < p_1, p_2 < \infty$ and verify appropriate end-point boundedness properties. Moreover, the corresponding norms of the operators are controlled by $\|b\|_{\mathrm{Lip}^1}$.*

Lemma 2.3. *Let $\sigma \in \mathcal{B}\mathrm{BS}^1_{1,1}$ and b be a Lipschitz function on $\mathbb{R}^n$. Suppose that a_1 is a $(p_1,\infty,0)$-atom supported on Q_1 and a_2 be a $(p_2,\infty,0)$-atom supported on Q_2, with $p_1, p_2 \in (n/n+1, 1]$. Then*
- *(i) for any $y \in (2Q_1)^c$,*

$$|[T_\sigma, b]_j(a_1,a_2)(y)| \lesssim \|b\|_{\mathrm{Lip}^1}|Q_2|^{-\frac{1}{p_2}}\frac{|Q_1|^{1+\frac{1}{n}-\frac{1}{p_1}}}{|y-x_{Q_1}|^{n+1}},$$

while for any $y \in (2Q_2)^c$,

$$|[T_\sigma, b]_j(a_1,a_2)(y)| \lesssim \|b\|_{\mathrm{Lip}^1}|Q_1|^{-\frac{1}{p_2}}\frac{|Q_2|^{1+\frac{1}{n}-\frac{1}{p_1}}}{|y-x_{Q_2}|^{n+1}};$$

- *(ii) for any $y \in (2Q_1)^c \cap y \in (2Q_2)^c$,*

$$|[T_\sigma, b]_j(a_1,a_2)(y)| \lesssim \|b\|_{\mathrm{Lip}^1}\min\left\{\frac{|Q_1|^{1+\frac{1}{n}-\frac{1}{p_1}}|Q_2|^{1-\frac{1}{p_2}}}{(|y-x_{Q_1}|+|y-x_{Q_2}|)^{2n+1}}, \frac{|Q_2|^{1+\frac{1}{n}-\frac{1}{p_2}}|Q_1|^{1-\frac{1}{p_1}}}{(|y-x_{Q_1}|+|y-x_{Q_2}|)^{2n+1}}\right\};$$

- *(iii) for any cube $R \in \mathbb{R}^n$*

$$\int_{\mathbb{R}}\varphi^+[T_\sigma, b]_j(a_1,a_2)(x)\mathrm{d}x \lesssim \|b\|_{\mathrm{Lip}^1}|R|^{\frac{1}{2}}\min\left\{|Q_1|^{\frac{1}{2}-\frac{1}{p_1}}|Q_2|^{-\frac{1}{p_2}}, |Q_1|^{-\frac{1}{p_2}}|Q_2|^{\frac{1}{2}-\frac{1}{p_1}}\right\}.$$

Proof of Lemma 2.3. To obtain the conclusion of Lemma 2.3, we only prove (ii) and (iii). Together with (1.2), Theorem A and the vanishing moment of a_1, it follows that for any $y \in (2Q_1)^c \cap y \in (2Q_2)^c$,

$$\begin{aligned}|[T_\sigma, b]_j(a_1, a_2)(y)| &\lesssim \int\limits_{(\mathbb{R}^n)^2} |K_j(y, z_1, z_2) - K_j(y, x_{Q_1}, z_2)\big)||a_1(z_1)||a_2(z_2)|\mathrm{d}z_1\mathrm{d}z_2 \\ &\lesssim \|b\|_{\mathrm{Lip}^1} \int\limits_{(\mathbb{R}^n)^2} \frac{|z_1 - x_{Q_1}|}{(|y - z_1| + |y - z_2|)^{2n+1}} |a_1(z_1)||a_2(z_2)|\mathrm{d}z_1\mathrm{d}z_2 \\ &\lesssim \|b\|_{\mathrm{Lip}^1} \frac{l(Q_1)}{(|y - x_{Q_1}| + |y - x_{Q_2}|)^{2n+1}} \prod_{k=1}^{2} \|a_k\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathrm{Lip}^1} \frac{|Q_1|^{1+\frac{1}{n}-\frac{1}{p_1}} |Q_2|^{1-\frac{1}{p_2}}}{(|y - x_{Q_1}| + |y - x_{Q_2}|)^{2n+1}}.\end{aligned}$$

Similarly, applying the vanishing moment of a_2, we obtain

$$|[T_\sigma, b]_j(a_1, a_2)(y)| \lesssim |b|_{\mathrm{Lip}^1} \frac{|Q_2|^{1+\frac{1}{n}-\frac{1}{p_2}} |Q_1|^{1-\frac{1}{p_1}}}{(|y - x_{Q_1}| + |y - x_{Q_2}|)^{2n+1}},$$

and conclusion (ii) follows directly.

To prove conclusion (iii), we first observe that for any $g \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\varphi^+(g)(x) \lesssim M(g)(x),$$

where M is the Hardy-Littlewood maximal operator. By Hölder's inequality, the fact φ^+ is bounded on $L^2(\mathbb{R}^n)$, Lemma 2.1 and the size condition of a_1 and a_2, we have

$$\begin{aligned}\int\limits_R \varphi^+[T_\sigma, b]_j(a_1, a_2)(x)\mathrm{d}x &\lesssim |R|^{\frac{1}{2}} \|\varphi^+[T_\sigma, b]_j(a_1, a_2)\|_{L^2(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathrm{Lip}^1} |R|^{\frac{1}{2}} \|a_1\|_{L^2(\mathbb{R}^n)} \|a_1\|_{L^\infty(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathrm{Lip}^1} |R|^{\frac{1}{2}} |Q_1|^{\frac{1}{2}-\frac{1}{p_1}} |Q_2|^{-\frac{1}{p_2}},\end{aligned}$$

and similarly,

$$\int\limits_R \varphi^+[T_\sigma, b]_j(a_1, a_2)(x)d \lesssim \|b\|_{\mathrm{Lip}^1} |R|^{\frac{1}{2}} |Q_1|^{-\frac{1}{p_1}} |Q_2|^{\frac{1}{2}-\frac{1}{p_2}}.$$

This leads conclusion (iii) and completes the proof of Lemma 2.3. □

Lemma 2.4. [14] *Let $p \in (0, 1]$. Then there exists a positive constant C_p such that for all finite collections of cubes $\{Q_k\}_{k=1}^K$ in $\mathbb{R}^n$ and all nonnegative integrable functions g_k with* $\mathrm{supp}(g_k) \in Q_k$,

$$\left\| \sum_{k=1}^{K} g_k \right\|_{L^p(\mathbb{R}^n)} \leq C_p \left\| \sum_{k=1}^{K} \left[\frac{1}{|Q_k|} \int\limits_{Q_k} g_k(x)\mathrm{d}x \right] \chi_{Q_k^*} \right\|_{L^p(\mathbb{R}^n)},$$

where Q_k^* denotes the cube with the same center as Q_k and $2\sqrt{n}$ its side-length.

Proof of Theorem 1.1. By Lemma 2.1 and a density argument, it suffices to prove that $[T_\sigma, b]_j$ $(j = 1, 2)$ is bounded from $(H^{p_1,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n)) \times (H^{p_2,\infty,0}_{\mathrm{fin}}(\mathbb{R}^n) \cap C(\mathbb{R}^n))$ into $H^p(\mathbb{R}^n)$, $i = 1, 2$, we decompose f_i as

$$f_i(x) = \sum_{k_i} \lambda_{i,k_i} a_{i,k_i}(x),$$

where a_{i,k_i} are $(p_i, \infty, 0)$-atoms in Definition 2.2. This means that a_{i,k_i} are functions supported in cubes Q_{i,k_i} and satisfy the properties $\|a_{i,k_i}\|_{L^\infty}(\mathbb{R}^n) \leq |Q_{i,k_i}|^{-1/p_i}$ and $\int_{Q_{i,k_i}} a_{i,k_i}\mathrm{d}x = 0$. Without loss of generality, we may assume that for fixed Q_{1,k_1} and Q_{2,k_2}, $l(Q_{1,k_1}) \leq l(Q_{2,k_2})$. It is easy to see that there exists a cube R_{k_1,k_2} such that

$$\left(Q_{1,k_1}^* \cap Q_{2,k_2}^*\right) \subset R_{k_1,k_2} \subset R_{k_1,k_2}^* \subset \left(Q_{1,k_1}^{**} \cap Q_{2,k_2}^{**}\right)$$

and $|R_{k_1,k_2}| \geq C_1|Q_{1,k_1}|$, where $C_1 \in (0,1)$ is a constant independent of R_{k_1,k_2} and Q_{i,k_i} with $i = 1, 2$. Our purpose is to show

$$\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\right\|_{L^p(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^2\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{1/p_i}. \tag{2.1}$$

For this goal, we write

$$\begin{aligned}
&\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\right\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\chi_{Q_{1,k_1}^*\cap Q_{2,k_2}^*}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad + \left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\chi_{\left(Q_{1,k_1}^*\right)^c\cap\left(Q_{2,k_2}^*\right)^c}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad + \left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\chi_{\left(Q_{1,k_1}^*\right)^c\cap Q_{2,k_2}^*}\right\|_{L^p(\mathbb{R}^n)}\\
&\quad + \left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)]\chi_{Q_{1,k_1}^*\cap\left(Q_{2,k_2}^*\right)^c}\right\|_{L^p(\mathbb{R}^n)}\\
&:= \sum_{i=1}^4 E_i.
\end{aligned}$$

It follows from Lemma 2.4, (iii) of Lemma 2.3 and Hölder's inequality that

$$\begin{aligned}
E_1 &\lesssim \left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{1}{|R_{k_1,k_2}|}\int_{R_{k_1,k_2}}\varphi^+[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)](x)\mathrm{d}x\chi_{R_{k_1,k_2}^*}\right\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{1}{|R_{k_1,k_2}|^{1/2}}|Q_{1,k_1}|^{\frac{1}{2}-\frac{1}{p_1}}|Q_{2,k_2}|^{-\frac{1}{p_2}}\chi_{R_{k_1,k_2}^*}\right\|_{L^p(\mathbb{R}^n)}\\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^2\left\|\sum_{k_i}|\lambda_{i,k_i}||Q_{i,k_i}|^{-i/p_i}\chi_{Q_{i,k_i}^{**}}\right\|\\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^2\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{1/p_i}.
\end{aligned}$$

We now turn to estimate E_2. Since $[T_\sigma, b]_j(a_{1,k_1}, a_{2,k_2})$ has vanishing moment up to order n, we can subtract the Taylor polynomial P^n of the function $\varphi(x-y)$ at the point $(x - x_{Q_{1,k_1}})$ with $x \in (Q_{1,k_1}^*)^c \cap (Q_{2,k_2}^*)^c$ and $y \in \mathbb{R}^n$. Set

$$\Delta(t, x, y) := \varphi_t(x, y) - P_t^n\left(x - x_{Q_{1,k_1}}\right),$$

and for all $x \in (Q_{1,k_1}^*)^c \cap (Q_{2,k_2}^*)^c$, write

$$\varphi^+\left[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)\right](x) \lesssim \sup_{t>0}\left|\int_{2Q_{1,k_1}\cap 2Q_{2,k_2}}\Delta(t, x, y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y\right|$$

$$
\begin{aligned}
&+\sup_{t>0}\left|\int\limits_{(2Q_{1,k_1})^c\cap(2Q_{2,k_2})^c}\Delta(t,x,y)[T_\sigma,b]_j\big(a_{1,k_1},a_{2,k_2}\big)(y)\mathrm{d}y\right|\\
&+\sup_{t>0}\left|\int\limits_{(2Q_{1,k_1})^c\cap 2Q_{2,k_2}}\Delta(t,x,y)[T_\sigma,b]_j\big(a_{1,k_1},a_{2,k_2}\big)(y)\mathrm{d}y\right|\\
&+\sup_{t>0}\left|\int\limits_{2Q_{1,k_1}\cap(2Q_{2,k_2})^c}\Delta(t,x,y)[T_\sigma,b]_j\big(a_{1,k_1},a_{2,k_2}\big)(y)\mathrm{d}y\right|\\
&:=\sum_{i=1}^{4}\Phi_i\big(a_{1,k_1},a_{2,k_2}\big)(x).
\end{aligned}
$$

Thus,

$$
E_2\lesssim\sum_{i=1}^{4}\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\Phi_i\big(a_{1,k_1},a_{2,k_2}\big)\chi_{(Q_{1,k_1}^*)^c\cap(Q_{2,k_2}^*)^c}\right\|_{L^p(\mathbb{R}^n)}:=E_{2i}.
$$

Our assumption $p_1,p_2\in(n/(n+1),1]$ and $1/p=1/p_1+1/p_2$ with $\lfloor n(1/p-1)\rfloor=n$ guarantee that we can choose $\theta\in(0,1)$ such that $p_1>n(n+\theta)$ and $p_2>n(n+1-\theta)$. On the other hand, we can verify that for all $x\in(Q_{1,k_1}^*)^c\cap(Q_{2,k_2}^*)^c$ and $y\in 2Q_{1,k_1}\cap 2Q_{1,k_1}$, $|x-x_{Q_{1,k_1}}|\sim|x-y|\sim|x-x_{Q_{2,k_2}}|$, and then for all $t\in(0,\infty)$,

$$
|\Delta(t,x,y)|\lesssim\frac{|y-x_{Q_{1,k_1}}|^{n+1}}{|x-x_{Q_{1,k_1}}|^{2n+1}}\lesssim\frac{|Q_{1,k_1}|^{\frac12+\frac{\theta}{n}}}{|x-x_{Q_{1,k_1}}|^{n+\theta}}\frac{|Q_{1,k_1}|^{\frac12+\frac1n-\frac{\theta}{n}}}{|x-x_{Q_{1,k_1}}|^{n+1-\theta}},
$$

since we assume that $l(Q_{1,k_1})\le l(Q_{2,k_2})$. This, together with Lemma 2.2 and Hölder's inequality, tells us that

$$
\begin{aligned}
E_{21}\lesssim&\left\{\int\limits_{\mathbb{R}^n}\left[\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{|Q_{1,k_1}|^{\frac12}+\frac{\theta}{n}}{|x-x_{Q_{1,k_1}}|^{n+\theta}}\frac{|Q_{2,k_2}|^{\frac12}+\frac1n-\frac{\theta}{n}}{|x-x_{Q_{2,k_2}}|^{n+1-\theta}}\right.\right.\\
&\left.\left.\times\left[\int\limits_{2Q_{1,k_1}\cap 2Q_{2,k_2}}|[T_\sigma,b]_j\big(a_{1,k_1},a_{2,k_2}\big)(y)|\mathrm{d}y\right]\chi_{(Q_{1,k_1}^*)^c\cap(Q_{2,k_2}^*)^c}(x)\right]^p\mathrm{d}x\right\}^{\frac1p}\\
\lesssim&\left\{\int\limits_{\mathbb{R}^n}\left[\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{|Q_{1,k_1}|^{\frac12}+\frac{\theta}{n}}{|x-x_{Q_{1,k_1}}|^{n+\theta}}\frac{|Q_{2,k_2}|^{\frac12}+\frac1n-\frac{\theta}{n}}{|x-x_{Q_{2,k_2}}|^{n+1-\theta}}\right.\right.\\
&\times\|b\|_{\mathrm{Lip}^1}\|a_{1,k_1}\|_{L^2(\mathbb{R}^n)}\|a_{2,k_2}\|_{L^2(\mathbb{R}^n)}\chi_{(Q_{1,k_1}^*)^c\cap(Q_{2,k_2}^*)^c}(x)]^p\mathrm{d}x\}^{\frac1p}\\
\lesssim&\|a_{1,k_1}\|_{L^2(\mathbb{R}^n)}\left\{\sum_{k_1}|\lambda_{1,k_1}|^{p_1}\int\limits_{(Q_{1,k_1}^*)^c}\frac{|Q_{1,k_1}|^{(1+\frac{\theta}{n}-\frac{1}{p_1})p_1}}{|x-x_{Q_{1,k_1}}|^{(n+\theta)^{p_1}}}\mathrm{d}x\right\}^{1/p_1}\\
&\times\|b\|_{\mathrm{Lip}^1}\|a_{1,k_1}\|_{L^2(\mathbb{R}^n)}\left\{\sum_{k_2}|\lambda_{2,k_2}|^{p_2}\int\limits_{(Q_{2,k_2}^*)^c}\frac{|Q_{2,k_2}|^{(1+\frac1n-\frac{\theta}{n}-\frac{1}{p_2})p_1}}{|x-x_{Q_{2,k_2}}|^{(n+1-\theta)^{p_2}}}\mathrm{d}x\right\}^{1/p_2}\\
\lesssim&\|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{1/p_i}.
\end{aligned}
$$

The estimate for E_{22} is cumbersome but straightforward. For $i, s = 0, 1, \dots$, let

$$I_{is} := \left(2^{i+1}Q_{1,k_1}^* \backslash 2^i Q_{1,k_1}^*\right) \cap \left(2^{s+1}Q_{2,k_2}^* \backslash 2^s Q_{2,k_2}^*\right).$$

For simplicity, we may assume that $2Q_{1,k_1} \subset \frac{1}{2}Q_{1,k_1}^*$ and $2Q_{2,k_2} \subset \frac{1}{2}Q_{2,k_2}^*$. Otherwise, for any $i, s = 0, 1, \dots$, replace $2^{i-1}Q_{1,k_1}^*$ and $2^{i-1}Q_{2,k_2}^*$ by $\max\{2^{i-1}Q_{1,k_1}^*, 2Q_{1,k_1}\}$ and $\max\{2^{i-1}Q_{2,k_2}^*, 2Q_{2,k_2}\}$, respectively, in the remaining part of the proof. For all $x \in I_{is}$ with $i, s = 0, 1, \dots$, write

$$\begin{aligned}\Phi_2\left(a_{1,k_1}, a_{2,k_2}\right)(x) \lesssim \sup_{t>0} &\left| \int\limits_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c} \Delta(t, x, y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y \right| \\ &+ \sup_{t>0} \int\limits_{\left(2^{i-1}Q_{1,k_1}^* \backslash 2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c} |\Delta(t, x, y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y| \\ &+ \sup_{t>0} \int\limits_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^* \backslash 2Q_{2,k_2}\right)} 2|\Delta(t, x, y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y| \\ &+ \sup_{t>0} \int\limits_{\left(2^{i-1}Q_{1,k_1}^* \backslash 2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2}^* \backslash 2Q_{2,k_2}\right)} 2|\Delta(t, x, y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y| \\ &:= \sum_{k=1}^{4} \Phi_{2k}\left(a_{1,k_1}, a_{2,k_2}\right)(x).\end{aligned}$$

Consequently,

$$E_{22} \lesssim \sum_{k=1}^{4} \left\{ \sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \int\limits_{\mathbb{R}^n} [|\lambda_{1,k_1}||\lambda_{2,k_2}|\Phi_{2k}\left(a_{1,k_1}, a_{2,k_2}\right)(x)\chi_{I_{is}}(x)]^p \mathrm{d}x \right\}^{1/p} = \sum_{k=1}^{4} E_{22}^k.$$

To estimate E_{11}^1, we further decompose that for all $x \in I_{is}$ with $i, s = 0, 1, \dots$,

$$\begin{aligned}\Phi_{21}\left(a_{1,k_1}, a_{2,k_2}\right)(x) \lesssim \sup_{t>0} &\left| \int\limits_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c} \varphi_t(x-y)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y \right| \\ &+ \sup_{t>0} \int\limits_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c} |P_t^n\left(x - x_{1,Q_1}\right)[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y| \\ &\lesssim \varphi^+\left[[T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)\chi_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c}\right](x) \\ &+ \frac{1}{|x - x_{Q_{1,k_1}}|^n} \int\limits_{\left(2^{i-1}Q_{1,k_1}^*\right)^c \cap \left(2^{s-1}Q_{2,k_2}^*\right)^c} [T_\sigma, b]_j\left(a_{1,k_1}, a_{2,k_2}\right)(y)\mathrm{d}y \\ &= \Lambda_1\left(a_{1,k_1}, a_{2,k_2}\right)(x) + \Lambda_2\left(a_{1,k_1}, a_{2,k_2}\right)(x),\end{aligned}$$

and also

$$E_{22}^1 \lesssim \sum_{k=1}^{2} \left\{ \sum_{i=0}^{\infty}\sum_{s=0}^{\infty} \int\limits_{\mathbb{R}^n} \left[\sum_{k_1}\sum_{k_2} \Lambda_k\left(a_{1,k_1}, a_{2,k_2}\right)(x)\chi_{I_{is}}(x) \right]^p \mathrm{d}x \right\}^{1/p} = \sum_{k=1}^{2} F_k.$$

Note that there exists some cube R_{is} such that

$$\left(2^{i+1}Q_{1,k_1}^*\right) \cap \left(2^{s+1}Q_{2,k_2}^*\right) \subset R_{is} \subset R_{is}^* \subset \left[\left(2^{i+1}Q_{1,k_1}^*\right)^* \cap \left(2^{s+1}Q_{2,k_2}^*\right)^*\right],$$

and $|R_{is}| \geq C_1|2^{i+1}Q_{1,k_1}^*|$. Thus, by Hölder's inequality, the fact that φ^+ is bounded on $L^2(\mathbb{R}^n)$, (ii) of Lemma 2.3, and the assumption $l(Q_{1,k_1}) \leq l(Q_{2,k_2})$, then we have that

$$\begin{aligned}
&\int\limits_{R_{is}} \varphi^{+}\Big[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{\left(2^{i-1}Q_{1,k_1}^{*}\right)^c \cap \left(2^{s-1}Q_{2,k_2}^{*}\right)^c}\Big](x)\mathrm{d}x \\
&\quad \lesssim |R_{is}|^{1/2}\left\{\int\limits_{\left(2^{i-1}Q_{1,k_1}^{*}\right)^c \cap \left(2^{s-1}Q_{2,k_2}^{*}\right)^c} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(x)|^2 \mathrm{d}x\right\}^{1/2} \\
&\quad \lesssim \|b\|_{\mathrm{Lip}^1}\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}|R_{is}|,
\end{aligned}$$

which, along with Lemma 2.4, implies that

$$\begin{aligned}
F_1 &\lesssim \left\{\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\left\|\left[|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{1}{|R_{is}|}\left(\int\limits_{\mathbb{R}_s}\varphi^{+}[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(x)\chi_{\left(2^{i-1}Q_{1,k_1}^{*}\right)^c \cap \left(2^{s-1}Q_{2,k_2}^{*}\right)^c}](x)\right)\chi_{R_{is}^{*}}\right\|_{L^p(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\left\|\left[|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}\lambda_{R_{is}^{*}}\right]\right\|_{L^p(\mathbb{R}^n)}^p\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.
\end{aligned}$$

On the other hand, a trivial computation states that

$$\int\limits_{\left(2^{i-1}Q_{1,k_1}^{*}\right)^c \cap \left(2^{s-1}Q_{2,k_2}^{*}\right)^c}\frac{1}{(|y-x_{Q_{1,k_1}}|+|y-x_{Q_{2,k_2}}|)^{2n+1}}\mathrm{d}y \lesssim \frac{1}{|2^iQ_{1,k_1}|^{\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}.$$

Thus, by (ii) of Lemma 2.3 and the assumption $l(Q_{1,k_1}) \le l(Q_{2,k_2})$, we obtain that for all $x \in I_{is}$ with i, $s = 0, 1, \dots,$

$$\begin{aligned}
\Lambda_2\big(a_{1,k_1}, a_{2,k_2}\big)(x) &\lesssim \|b\|_{\mathrm{Lip}^1}\frac{1}{|x-x_{Q_{1,k_1}}|^n}\int\limits_{\left(2^{i-1}Q_{1,k_1}^{*}\right)^c \cap \left(2^{s-1}Q_{2,k_2}^{*}\right)^c}\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1-\frac{1}{p_2}}}{(|y-x_{Q_{1,k_1}}|+|y-x_{Q_{2,k_2}}|)^{2n+1}}\mathrm{d}y \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
F_2 &\lesssim \left\{\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\left\|\left[|\lambda_{1,k_1}||\lambda_{2,k_2}|\|b\|_{\mathrm{Lip}^1}\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}\right]^p|I_{is}|\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\left(|\lambda_{1,k_1}|^{p_1}\left(\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|}\right)^p|2^iQ_{1,k_1}^{*}|\right)^{p/p_1}\sum_{i=0}^{\infty}\left(|\lambda_{2,k_2}|^{p_2}\left(\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|}\right)^p|2^iQ_{1,k_2}^{*}|\right)^{2/p_2}\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.
\end{aligned}$$

Combining the statements for F_1 and F_2 yields that

$$E_{22}^1 \lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.$$

To estimate methods for E_{22}^2 and E_{22}^3 are similar. We only deal with E_{22}^2. Observe that for all $x \in I_{is}$ with i, $s = 0, 1, \dots$, and $y \in (2^{i-1}Q_{1,k_1^*} \backslash 2Q_{1,k_1}) \cap (2^{g+1}Q_{2,k_2^*} \backslash 2^g Q_{2,k_2}^*)$ with $g = s-1, s, \dots$,

$$2^i l\big(Q_{1,k_1}\big) \lesssim |x-y| \lesssim 2^g l\big(Q_{2,k_2}\big).$$

(ii) of Lemma 2.3 and the assumption $l(Q_{1,k_1}) \lesssim l(Q_{2,k_2})$ then tell us that for all $x \in I_{is}$ with $i, s = 0, 1, \dots$,

$$\begin{aligned}
\Phi_{22}\big(a_{1,k_1}, a_{2,k_2}\big)(x) &\lesssim \int\limits_{\left(2^{i-1}Q_{1,k_1^*}\backslash 2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2^*}\right)^c} \left(\frac{1}{|x-y|^n} + \frac{1}{|x-x_{Q_{1,k_1}}|^n}\right) |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)| dy \\
&\lesssim \|b\|_{\mathrm{Lip}^1} dy \\
&\lesssim \|b\|_{\mathrm{Lip}^1} \frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}} |Q_{2,k_2}|^{1-\frac{1}{p_2}}}{|2^i Q_{1,k_1}|} \sum_{g=s-1}^{\infty} \int\limits_{2^{g+1}Q_{2,k_2}^* \backslash 2^g Q_{2,k_2}^*} \frac{1}{|y-x_{Q_{2,k_2}}|^{2n+1}} dy \\
&\lesssim \|b\|_{\mathrm{Lip}^1} \frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}} |Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^i Q_{1,k_1}|^{1+\frac{\theta}{n}} |2^s Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}},
\end{aligned}$$

which in turn leads to that

$$\begin{aligned}
E_{22}^2 &\lesssim \left\{ \sum_{i=0}^{\infty} \sum_{s=0}^{\infty} \left[\sum_{k_1} \sum_{k_2} |\lambda_{1,k_1}| |\lambda_{2,k_2}| \|b\|_{\mathrm{Lip}^1} \frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}} |Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^i Q_{1,k_1}|^{1+\frac{\theta}{n}} |2^s Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}} \right]^p |I_{is}| \right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1} \prod_{i=1}^{2} \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}}.
\end{aligned}$$

Similarly, we have

$$E_{22}^3 \lesssim \|b\|_{\mathrm{Lip}^1} \prod_{i=1}^{2} \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}}.$$

We now turn to estimate E_{22}^4. For this purpose, we first note that for all $x \in I_{is}$ with $i, s = 0, 1, \dots$,

$$\begin{aligned}
&\int\limits_{\left(2^{i-1}Q_{1,k_1^*}\right) \backslash \left(2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2^*}\right) \backslash \left(2Q_{2,k_2}\right)} \frac{|y-x_{Q_{1,k_1}}|^{n+1}}{|x-y|^{2n+1} (|y-x_{Q_{1,k_1}}| + |y-x_{Q_{2,k_2}}|)^{2n+1}} dy \\
&\lesssim \frac{1}{|2^i Q_{1,k_1}|^{1+\frac{\theta}{n}} |2^s Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}} \int\limits_{\left(2^{i-1}Q_{2,k_2^*} \backslash 2Q_{1,k_1}\right)} \frac{1}{|y-x_{Q_{1,k_1}}|^n} dy \\
&\lesssim \frac{i}{|2^i Q_{1,k_1}|^{1+\frac{\theta}{n}} |2^s Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}},
\end{aligned}$$

which, along with (ii) of Lemma 2.3 and the assumption $l(Q_{1,k_1}) \lesssim l(Q_{2,k_2})$, gives us that for all $x \in I_{is}$ with $i, s = 0, 1, \dots$,

$$\begin{aligned}
\Phi_{24}\big(a_{1,k_1}, a_{2,k_2}\big)(x) &\lesssim \int\limits_{\left(2^{i-1}Q_{1,k_1^*}\right) \backslash \left(2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2^*}\right) \backslash \left(2Q_{2,k_2}\right)} \frac{|y-x_{Q_{1,k_1}}|^{n+1}}{|x-y|^{2n}} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)| dy \\
&\lesssim \|b\|_{\mathrm{Lip}^1} \int\limits_{\left(2^{i-1}Q_{1,k_1^*}\right) \backslash \left(2Q_{1,k_1}\right) \cap \left(2^{s-1}Q_{2,k_2^*}\right) \backslash \left(2Q_{2,k_2}\right)} \frac{|y-x_{Q_{1,k_1}}|^{n+1} |Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}} |Q_{2,k_2}|^{1-\frac{1}{p_2}}}{|x-y|^{2n} (|y-x_{Q_{1,k_1}}| + |y-x_{Q_{2,k_2}}|)^{2n+1}} dy \\
&\lesssim i \|b\|_{\mathrm{Lip}^1} \frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}} |Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^i Q_{1,k_1}|^{1+\frac{\theta}{n}} |2^s Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}.
\end{aligned}$$

Thus,

$$E_{22}^4 \lesssim \left\{\sum_{i=0}^{\infty}\sum_{s=0}^{\infty}\int_{\mathbb{R}^n}\left[\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|i\|b\|_{\mathrm{Lip}^1}\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}}}|I_{is}|\right]^p\mathrm{d}x\right\}^{1/p}$$

$$\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}i^p\left[\sum_{k_1}|\lambda_{1,k_1}|^{p_1}\int_{2^{i+1}Q_{1,k_1}^*\backslash 2^iQ_{1,k_1}^*}\left(\frac{|Q_{1,k_1}|^{1+\frac{\theta}{n}-\frac{1}{p_1}}}{|2^iQ_{1,k_1}|^{1+\frac{\theta}{n}}}\right)^{p_1}\mathrm{d}x\right]^{p/p_1}\right.$$

$$\left.\times\sum_{s=0}^{\infty}\left[\sum_{k_2}|\lambda_{2,k_2}|^{p_2}\int_{2^{s+1}Q_{2,k_2}^*\backslash 2^sQ_{2,k_2}^*}\left(\frac{|Q_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}-\frac{1}{p_2}|}}{|2^sQ_{2,k_2}|^{1+\frac{1}{n}-\frac{\theta}{n}|}}\right)^{p_2}\mathrm{d}x\right]^{p/p_2}\right\}^{1/p}$$

$$\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.$$

Combining with the estimates for E_{22}^k with $k = 1, 2, 3, 4$ yields that

$$E_{22} \lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.$$

Some computations similar to those used in the estimates for E_{22} give us that

$$E_{23} + E_{24} \lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.$$

Our desired estimate for E_2 now follows directly.

It remains to consider the terms E_3 and E_4. Since the estimates for these two terms are similar, we only deal with E_3. Using the vanishing moment of $[T_\sigma, b]_j(a_{1,k_1}, a_{2,k_2})(j = 1, 2)$, we have that for all $x \in (Q_{1,k_1}^*)^c \cap Q_{2,k_2}^*$,

$$\varphi^+\Big[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\Big](x) \lesssim \sup_{t>0}\left|\int_{2Q_{1,k_1}}\Big[\varphi_t(x-y) - \varphi_t\big(x - x_{Q_{1,k_1}}\big)\Big][T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)\mathrm{d}y\right|$$

$$+ \sup_{t>0}\left|\int_{(2Q_{1,k_1})^c}\Big[\varphi_t(x-y) - \varphi_t\big(x - x_{Q_{1,k_1}}\big)\Big][T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)\mathrm{d}y\right|$$

$$:= \Psi_1\big(a_{1,k_1}, a_{2,k_2}\big)(x) + \Psi_2\big(a_{1,k_1}, a_{2,k_2}\big)(x).$$

Then,

$$E_3 \lesssim \sum_{i=1}^{2}\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}\lambda_{2,k_2}|\Psi_i\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{(Q_{1,k_1}^*)^c\cap Q_{2,k_2}^*}\right\|_{L^p}(\mathbb{R}^n) := E_{31} + E_{32}.$$

A trivial computation gives us that for all $x \in (Q_{1,k_1}^*)^c \cap Q_{2,k_2}^*$,

$$\Psi_1\big(a_{1,k_1}, a_{2,k_2}\big)(x) \lesssim \frac{|Q_{1,k_1}|^{1/n}}{|x - x_{Q_{1,k_1}}|^{n+1}}\int_{2Q_{1,k_1}}|[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)|\mathrm{d}y,$$

which, via Hölder's inequality and Lemma 2.2, leads to that

$$E_{31} \lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}| \frac{|Q_{1,k_1}|^{1/n}}{|x - x_{Q_{1,k_1}}|^{n+1}} \|[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\|_{L^2(\mathbb{R}^n)} |Q_{1,k_1}|^{1/2} \right]^p \chi_{\left(Q^*_{1,k_1}\right)^c \cap Q^*_{2,k_2}}(x)\mathrm{d}x \right\}^{1/p}$$

$$\lesssim \|b\|_{\mathrm{Lip}^1} \left\{ \sum_{k_1} |\lambda_{1,k_1}|^{p_1} \int_{\left(Q^*_{1,k_1}\right)^c} \frac{|Q_{1,k_1}|^{(1+\frac{1}{n}-\frac{1}{p_1})p_1}}{|x - x_{Q_{1,k_1}}|^{(n+1)p_1}} \mathrm{d}x \right\}^{p_1} \left\{ \sum_{k_2} |\lambda_{2,k_2}|^{p_2} |Q_{2,k_2}|^{-1} \int_{Q^*_{2,k_2}} \chi_{Q^*_{2,k_2}}(x)\mathrm{d}x \right\}^{1/p_2}$$

$$\lesssim \|b\|_{\mathrm{Lip}^1} \prod_{i=1}^{2} \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}}.$$

Let us estimate E_{32}. Observe that for all $x \in 2^{i+1}Q^*_{1,k_1} \backslash 2^i Q^*_{1,k_1}$ with $i = 0, 1, \dots,$

$$\begin{aligned}
\Psi_2\big(a_{1,k_1}, a_{2,k_2}\big)(x) &\le \sup_{t>0} \left| \int_{\left(2^{i-1}Q^*_{1,k_1}\right)^c} \varphi_t(x-y)[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)\mathrm{d}y \right| \\
&\quad + \sup_{t>0} \int_{\left(2^{i-1}Q^*_{1,k_1}\right)^c} |\varphi_t\big(x - x_{1,Q_1}\big)||[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)\mathrm{d}y|\mathrm{d}y \\
&\quad + \sup_{t>0} \int_{2^{i-1}Q^*_{1,k_1} \backslash 2Q_{1,k_1}} |\varphi_t(x-y) - \varphi_t\big(x - x_{1,Q_1}\big)||[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)\mathrm{d}y|\mathrm{d}y \\
&\lesssim \varphi^+\Big[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{\left(2^{i-1}Q^*_{1,k_1}\right)^c}\Big](x) + \frac{1}{|x - x_{Q_{1,k_1}}|^n} \int_{\left(2^{i-1}Q^*_{1,k_1}\right)^c} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)|\mathrm{d}y \\
&\quad + \int_{2^{i-1}Q^*_{1,k_1}} \frac{|y - x_{Q_{1,k_1}}|}{|x-y|^{n+1}} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)|\mathrm{d}y.
\end{aligned}$$

We then decompose

$$\begin{aligned}
E_{32} &\lesssim \left\{ \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} \left[\sum_{k_1}\sum_{k_2} |\lambda_{1,k_1}||\lambda_{2,k_2}| \varphi^+[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{\left(2^{i-1}Q^*_{1,k_1}\right)^c}](x)\chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^i Q^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x) \right] \mathrm{d}x \right\}^{1/p} \\
&\quad + \left\{ \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} \left[\sum_{k_1}\sum_{k_2} \frac{|\lambda_{1,k_1}||\lambda_{2,k_2}|}{|x - x_{Q_{1,k_1}}|^n} \int_{\left(2^{i-1}Q^*_{1,k_1}\right)} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)|\mathrm{d}y \right]^p \chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^i Q^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x)\mathrm{d}x \right\}^{1/p} \\
&\quad + \left\{ \sum_{i=0}^{\infty} \int_{\mathbb{R}^n} \left[\sum_{k_1}\sum_{k_2} \int_{2^{i-1}Q^*_{1,k_1}\backslash 2Q_{1,k_1}} \frac{|y - x_{Q_{1,k_1}}|}{|x-y|^{n+1}} |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(y)|\mathrm{d}y \right]^p \chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^i Q^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x)\mathrm{d}x \right\}^{1/p} \\
&:= \sum_{k=1}^{3} E^k_{32}.
\end{aligned}$$

The estimate for E^1_{32} is similar to that for F_1. For $2^{i+1}Q^*_{1,k_1}$ with $i = 0, 1, \dots,$ and Q^*_{2,k_2}, there exists a cube R_i such that

$$(2^{i+1}Q^*_{1,k_1}) \cap \big(Q^*_{2,k_2}\big) \subset R_i \subset R^*_i \subset \Big[\big(2^{i+1}Q^*_{1,k_1}\big)^* \cap \big(Q^*_{2,k_2}\big)^*\Big],$$

and $|R_i| \ge C_1|2^{i+1}Q^*_{1,k_1}|$. A familiar argument involving Hölder's inequality, the fact that φ^+ is bounded on $L^1(\mathbb{R}^n)$ and (i) of Lemma 2.3 leads to that

$$
\begin{aligned}
&\int_{R_i} \varphi^+\Big[[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{\left(2^{i-1}Q^*_{1,k_1}\right)^c}\Big](x)\mathrm{d}x \\
&\quad \lesssim |R|^{1/2}\left\{\int \left(2^{i-1}Q^*_{1,k_1}\right)^c |[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)(x)|^2 \mathrm{d}x\right\}^{1/2} \\
&\quad \lesssim \|b\|_{\mathrm{Lip}^1}|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}\frac{|Q_{2,k_2}|^{-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{1}{n}}}|R_i|.
\end{aligned}
$$

This now states that

$$
\begin{aligned}
E^1_{32} &\lesssim \left\{\sum_{i=0}^{\infty}\left\| \sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{1}{|R_i|}\left(\int_{\mathbb{R}^n}\varphi^+[T_\sigma, b]_j\big(a_{1,k_1}, a_{2,k_2}\big)\chi_{\left(2^{i-1}Q^*_{1,k_1}\right)^c}(x)\mathrm{d}x\right)\chi_{R_i^*}\right\|^p_{L^p(\mathbb{R}^n)}\right\}^{1/p} \\
&\lesssim \left\{\sum_{i=0}^{\infty}\left\|\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}|\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}}{|2^iQ_{1,k_1}|^{1+\frac{1}{n}}}|Q_{2,k_2}|^{-\frac{1}{p_2}}\chi_{\left(2^{i+1}Q^*_{1,k_1}\right)^*\cap\left(Q^*_{2,k_2}\right)^*}\right\|^p_{L^p(\mathbb{R}^n)}\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.
\end{aligned}
$$

The estimate for E^2_{32} is straightforward. In fact, by (i) of Lemma 2.3 and the fact $p_1 > n/(n+1)$, we obtain that

$$
\begin{aligned}
E^2_{32} &\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}\left[\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{-\frac{1}{p_2}}}{|x-x_{Q_{1,k_1}}|^n}\times\int_{\left(2^{i-1}Q_{1,k_1}\right)^c}\frac{\mathrm{d}y}{|y-x_{Q_{1,k_1}}|^{n+1}}\right]^p\chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^iQ^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x)\mathrm{d}x\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}\left[\sum_{k_1}|\lambda_{1,k_1}|^p\left(\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}}{|2^iQ_{1,k_1}|^{1+\frac{1}{n}}}\right)^p|2^{i+1}Q^*_{1,k_1}\backslash 2^iQ^*_{1,k_1}|\right)^{\frac{p}{p_1}}\times\left[\sum_{k_2}|\lambda_{2,k_2}|^{p_2}|Q_{1,k_2}|^{-1}\int_{Q^*_{2,k_2}}\chi_{Q^*_{2,k_2}}(x)\mathrm{d}x\right]^{\frac{p}{p_2}}\right\}^{\frac{1}{p}} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.
\end{aligned}
$$

Another application of (i) of Lemma 2.3 gives us that

$$
\begin{aligned}
E^3_{32} &\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}\left[\sum_{k_1}\sum_{k_2}|\lambda_{1,k_1}||\lambda_{2,k_2}\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}|Q_{2,k_2}|^{-\frac{1}{p_2}}}{|2^{iQ_{1,k_1}}|^{1+\frac{1}{n}}}\right.\right. \\
&\qquad \left.\left.\times\int_{2^{i-1}Q_{1,k_1}\backslash 2Q_{1,k_1}}\frac{\mathrm{d}y}{|y-x_{Q_{1,k_1}}|^n}\chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^iQ^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x)\right]^p\mathrm{d}x\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}\int_{\mathbb{R}^n}\left[\sum_{k_1}|\lambda_{1,k_1}||\lambda_{2,k_2}|i\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}|Q_{1,k_1}|^{-\frac{1}{p_2}}}{|2^iQ_{1,k_1}|^{1+\frac{1}{n}}}\chi_{\left(2^{i+1}Q^*_{1,k_1}\backslash 2^iQ^*_{1,k_1}\right)\cap Q^*_{2,k_2}}(x)\right]^p\mathrm{d}x\right\}^{1/p} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\left\{\sum_{i=0}^{\infty}i^p\left[\sum_{k_1}|\lambda_{1,k_1}|^{p_1}\int_{2^{i+1}Q^*_{1,k_1}\backslash 2^iQ^*_{1,k_1}}\left(\frac{|Q_{1,k_1}|^{1+\frac{1}{n}-\frac{1}{p_1}}}{|2^iQ_{1,k_1}|^{1+\frac{1}{n}}}\right)^{p_1}\mathrm{d}x\right]^{\frac{p}{p_1}}\left[\sum_{k_2}|\lambda_{2,k_2}|^{p_2}|Q_{1,k_2}|^{-1}\int_{Q^*_{2,k_2}}\mathrm{d}x\right]^{\frac{p}{p_2}}\right\}^{\frac{1}{p}} \\
&\lesssim \|b\|_{\mathrm{Lip}^1}\prod_{i=1}^{2}\left(\sum_{k_i}|\lambda_{i,k_i}|^{p_i}\right)^{\frac{1}{p_i}}.
\end{aligned}
$$

This estimate for E_{32}^k with $k = 1, 2, 3$ gives us that

$$E_{32} \lesssim \|b\|_{\mathrm{Lip}^1} \prod_{i=1}^{2} \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}},$$

which, together with the estimates for E_{31}, proves that

$$E_3 \lesssim \|b\|_{\mathrm{Lip}^1} \prod_{i=1}^{2} \left(\sum_{k_i} |\lambda_{i,k_i}|^{p_i} \right)^{\frac{1}{p_i}}.$$

The estimates for E_i with $i = 1, 2, 3, 4$ show that inequality (2.1) holds. This completes the proof of Theorem 1.1. □

3 Boundedness on product of Herz-type spaces

For $k \in \mathbb{Z}$ and measurable function $f(x)$ on $\mathbb{R}^n$, let $m_k(\sigma, f) = |\{x \in E_k : |f(x) > \sigma|\}|$. For $k \in \mathbb{Z}$, let $\tilde{m}_k(\sigma, f) = m_k(\sigma, f)$ and $\tilde{m}_{k_0}(\sigma, f) = |\{x \in B(0, 1) : |f(x) > \sigma|\}|$. In this section, we shall prove Theorem 1.2. In order to do this, let us recall some definitions.

Definition 3.1. [15] Let $\alpha \in \mathbb{R}$, $0 < q < \infty$ and $0 < p \leq \infty$.

A measurable function $f(x)$ on $\mathbb{R}^n$ is said to belong to the homogeneous weak Herz space $W\dot{H}_q^{\alpha,p}$, if

$$\|f\|_{W\dot{H}_q^{\alpha,p}(\mathbb{R}^n)} = \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha p} m_k(\lambda, f)^{p/q} \right\}^{1/p} < \infty,$$

where the usual modification is made when $p = \infty$.

Proof of Theorem 1.2. Let f_1, f_2 be functions, respectively, in $\dot{K}_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)$ and $\dot{K}_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)$. Since $f_i(x) = \sum_{l_i=-\infty}^{\infty} f_i(x)\chi_{l_i}(x)$, $i = 1, 2$, and $[T_\sigma, b]_j$, $j = 1, 2$ are bilinear Calderón-Zygmund operators, together with Theorem A, we can deduce the weaker condition (see [16]), namely, for any integrable function f_1, f_2 with $\mathrm{supp} f_i \in B(0, r_i)$, $r_i > 0$, $i = 1, 2$ and $x \notin \cap_{i=1}^2 B(0, 2r_i)$, there is

$$|[T_\sigma, b]_j(f_1, f_2)(x)| \lesssim \|b\|_{\mathrm{Lip}^1} |x|^{-mn} \prod_{i=1}^{2} \|f_i\|_{L^1(\mathbb{R}^n)}. \tag{3.1}$$

We have

$$\begin{aligned}
&\|[T_\sigma, b]_j(f_1, f_2)\|_{W\dot{K}_q^{n(2-1/q),p}(\mathbb{R}^n)} \\
&\lesssim \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \left| \left\{ x \in E_k : \left| [T_\sigma, b]_j \left(\sum_{i_1=k-1}^{\infty} f_1\chi_{I_1}, \sum_{i_2=k-1}^{\infty} f_2\chi_{I_2} \right)(x) \right| > \frac{\lambda}{4} \right\} \right|^{p/q} \right\}^{1/p} \\
&\quad + \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \left| \left\{ x \in E_k : \sum_{i_1=-\infty}^{k-3} \sum_{i_2=k-2}^{\infty} |[T_\sigma, b]_j(f_1\chi_{I_1}, f_2\chi_{I_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \right\}^{1/p} \\
&\quad + \sup_{\lambda>0} \lambda \left\{ \sum_{k=-\infty}^{\infty} 2^{kn(2-1/q)p} \left| \left\{ x \in E_k : \sum_{i_1=-\infty}^{k-3} \sum_{i_2=-\infty}^{k-3} |[T_\sigma, b]_j(f_1\chi_{I_1}, f_2\chi_{I_2})(x)| > \frac{\lambda}{4} \right\} \right|^{p/q} \right\}^{1/p}
\end{aligned}$$

$$+\sup_{\lambda>0}\lambda\left\{\sum_{k=-\infty}^{\infty}2^{kn(2-1/q)p}\left|\left\{x\in E_k:\sum_{i_1=k-2}^{\infty}\sum_{i_2=-\infty}^{k-3}|[T_\sigma,b]_j(f_1\chi_{I_1},f_2\chi_{I_2})(x)|>\frac{\lambda}{4}\right\}\right|^{p/q}\right\}^{1/p}$$
$$:=I_1+I_2+I_3+I_4.$$

Using the fact that $[T_\sigma,b]_j$ $(j=1,2)$ is bounded from $L^{q_1}\times L^{q_2}$ into $L^{q,\infty}$ with controlled by $\|a\|_{\mathrm{Lip}^1}$ and $1<q_1,q_2<\infty$, $1/p=1/p_1+1/p_2$, by Hölder's inequality, we see that

$$\begin{aligned}
I_1&\lesssim\left\{\sum_{k=-\infty}^{\infty}2^{kn(2-1/q)p}\left\|\sum_{i_1=k-2}^{\infty}f_1\chi_{i_1}\right\|_{L^{q_1}}^{p}\left\|\sum_{i_2=k-2}^{\infty}f_2\chi_{i_2}\right\|_{L^{q_2}}^{p}\right\}^{1/p}\\
&\lesssim\|b\|_{\mathrm{Lip}^1}\left\{\sum_{k=-\infty}^{\infty}2^{kn(2-1/q_1)p_1}\left\|\sum_{i_1=k-2}^{\infty}f_1\chi_{i_1}\right\|_{L^{q_1}}^{p_1}\right\}^{1/p_1}\left\{\sum_{k=-\infty}^{\infty}2^{kn(2-1/q_2)p_2}\left\|\sum_{i_2=k-2}^{\infty}f_1\chi_{i_2}\right\|_{L^{q_2}}^{p_2}\right\}^{1/p_2}\\
&\lesssim\|b\|_{\mathrm{Lip}^1}\left\{\sum_{i_1=-\infty}^{\infty}\|f_1\chi_{i_1}\|_{L^{q_1}}^{p_1}\sum_{k=-\infty}^{i_1+2}2^{kn(1-1/q_1)p_1}\right\}^{1/p_1}\left\{\sum_{i_2=-\infty}^{\infty}\|f_2\chi_{i_2}\|_{L^{q_2}}^{p_2}\sum_{k=-\infty}^{i_2+2}2^{kn(1-1/q_2)p_2}\right\}^{1/p_2}\\
&=\|b\|_{\mathrm{Lip}^1}\left\{\sum_{i_1=-\infty}^{\infty}2^{i_1n(1-1/q_1)p_1}\|f_1\chi_{i_1}\|_{L^{q_1}}^{p_1}\right\}^{1/p_1}\left\{\sum_{i_2=-\infty}^{\infty}2^{i_2n(1-1/q_2)p_2}\|f_2\chi_{i_2}\|_{L^{q_2}}^{p_2}\right\}^{1/p_2}\\
&=\|b\|_{\mathrm{Lip}^1}\|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)}\|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)}.
\end{aligned}$$

For I_2, noting that $x\in E_k$, $\operatorname{supp}f_i\chi_{l_i}\in\{x\in\mathbb{R}^n:|x|\le 2^{l_i}\}$, $i=1,2$, and $l_1\le k-3$, by (3.1) we have

$$\begin{aligned}
\sum_{l_1=-\infty}^{k-3}\sum_{l_2=k-2}^{\infty}|[T_\sigma,b]_j(f_1\chi_{I_1},f_2\chi_{I_2})(x)|&\lesssim\|b\|_{\mathrm{Lip}^1}|x|^{-2n}\left(\sum_{l_1=-\infty}^{k-3}\|f_1\chi_{l_1}\|_{L^1}\right)\left(\sum_{l_2=k-2}^{\infty}\|f_2\chi_{l_2}\|_{L^1}\right)\\
&\lesssim\|b\|_{\mathrm{Lip}^1}2^{-2kn}\|f_1\chi_{\{|x|\le2^{k-3}\}}\|_{L^1}2^{-2kn}\|f_2\chi_{\{|x|\le2^{k-3}\}}\|_{L^1}\\
&\lesssim\|b\|_{\mathrm{Lip}^1}2^{-2kn}\|f_1\|_{L^1}\|f_2\|_{L^1}.
\end{aligned}\tag{3.2}$$

It is easy to see from the definition of Herz space that $\dot K_{q_i}^{n(1-1/q_i),p_i}(\mathbb{R}^n)\in L^1(\mathbb{R}^n)$. Given any fixed $\lambda>0$, if

$$\left|\left\{x\in E_k:\sum_{l_1=-\infty}^{k-3}\sum_{l_2=k-2}^{\infty}|[T_\sigma,b]_j(f_1\chi_{I_1},f_2\chi_{I_2})(x)|>\frac{\lambda}{4}\right\}\right|\ne0,$$

then by (3.2),

$$\frac{\lambda}{4}\lesssim\|b\|_{\mathrm{Lip}^1}2^{-2kn}\|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)}\|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)}.$$

That is,

$$k\lesssim2^{-1}n^{-1}\log_2\left(4\|b\|_{\mathrm{Lip}^1}2^{-2kn}\|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)}\|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)}\right):=N_\lambda.$$

From this, we can obtain

$$\begin{aligned}
I_2&\lesssim\sup_{\lambda>0}\lambda\left(\sum_{k=-\infty}^{\lfloor N_\lambda\rfloor}2^{kn(2-1/q)p}2^{knp/q}\right)^{1/p}\\
&=\sup_{\lambda>0}\lambda2^{2n\lfloor N_\lambda\rfloor}\\
&\lesssim\sup_{\lambda>0}(\lambda\cdot4\|b\|_{\mathrm{Lip}^1}2^{-2kn}\|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)}\|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)})\\
&=\|b\|_{\mathrm{Lip}^1}\|f_1\|_{\dot K_{q_1}^{n(1-1/q_1),p_1}(\mathbb{R}^n)}\|f_2\|_{\dot K_{q_2}^{n(1-1/q_2),p_2}(\mathbb{R}^n)}.
\end{aligned}$$

From the similar way of I_2, the proof of I_3 and I_4 can be deduced. Combining all estimates on I_1, I_2, I_3 and I_4, the desired result can be established. This completes the proof of Theorem 1.2. □

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