Research Article

Kai Yu, Benxue Gong, and Zhenyu Zhao*

A modified Tikhonov regularization for unknown source in space fractional diffusion equation

https://doi.org/10.1515/math-2022-0513
received May 4, 2022; accepted September 21, 2022

Abstract: In this article, we consider the identification of an unknown steady source in a class of fractional diffusion equations. A modified Tikhonov regularization method based on Hermite expansion is presented to deal with the ill-posedness of the problem. By using the properties of Hermitian functions, we construct a modified penalty term for the Tikhonov functional. It can be proved that the method can adaptively achieve the order optimal results when we choose the regularization parameter by the discrepancy principle. Some examples are also provided to verify the effectiveness of the method.

Keywords: ill-posed problem, super order regularization, unknown source, discrepancy principle, hermite approximation

MSC 2020: 35R30, 47A52, 65M30, 65M32

1 Introduction

Due to its wide application in many fields, including physical, and mechanical engineering, signal processing and systems identification, control theory, and finance, fractional differential equations have received extensive attention during the past decades. One of the most important applications of fractional differential equations is that it can characterize the abnormal diffusion effectively. For a review, the reader can refer to [1–3] and the references therein.

In this article, we consider a space fractional diffusion equation that can describe the probability distribution of particles with superdiffusion [4]. It has the following forms:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -\dfrac{\partial J^{\alpha}_{m,n}(x,t)}{\partial x} + f(x), & -\infty < x < \infty, 0 < t < T, \\ u(x,T) = g(x), & u(x,0) = 0, \end{cases} \tag{1}$$

where

$$J^{\alpha}_{m,n}(x,t) = -\kappa_{\alpha}\{m_{-\infty}D^{\alpha-1}_{x} - n_{x}D^{\alpha-1}_{\infty}\}u$$

with $m + n = 1$, and $\kappa_{\alpha} > 0$. For $m, n \le 0$. We define the right Riemann–Liouville fractional derivatives ${}_{x}D^{\alpha}_{\infty}$ and the left Riemann–Liouville fractional derivatives ${}_{-\infty}D^{\alpha}_{x}(1 < \alpha < 2)$ as follows:

*** Corresponding author: Zhenyu Zhao,** School of Mathematics and Statistics, Shandong University of Technology, Zibo, 255049, China, e-mail: wozitianshanglai@163.com
Kai Yu, Benxue Gong: School of Mathematics and Statistics, Shandong University of Technology, Zibo, 255049, China

$$ {}_xD_{\infty}^{\alpha}u(x,t) = \frac{(-1)^{\lceil\alpha\rceil}}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_x^{\infty} u(\zeta,t)(\zeta-x)^{1-\alpha}\mathrm{d}\zeta $$

and

$$ {}_{-\infty}D_x^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}\int_{-\infty}^{x} u(\zeta,t)(x-\zeta)^{1-\alpha}\mathrm{d}\zeta, $$

where $\lceil\alpha\rceil$ is the smallest integer no less than α. Our goal is to identify the source function $f(x)$ in (1) from the additional observed data $g(x)$. Generally, the data $g(x)$ is measured, and we only have the noisy data $g^{\delta} \in L^2(\mathbb{R})$ that satisfies

$$ \|g - g^{\delta}\| \le \delta. \tag{2} $$

This problem is referred as an inverse source problem.

The inverse problems occur in many fields: for example, crack identification, heat conduction problems, pollutant detection, target tracking, and antenna synthesis [5–13]. The main difficulty of the inverse source problem is usually ill-posed. Regularization technique has to be introduced to obtain stable numerical solution. A few progress has been developed for solving the inverse source problems of space-fractional diffusion equation [4,14,15–17]. In [16], a truncated Hermite expansion method has been proposed to deal with problem (1). The method is effective, but the source condition for deriving convergence result is not natural, and the convergence rate of the method is not ordered optimal. To overcome the aforementioned shortcomings, we proposed a modified Tikhonov regularization based on Hermite expansion for problem (1). This method has been successfully applied to solving the numerical differentiation problem [18] and the Cauchy problem of the Laplace equation [19]. For the new method, we can obtain the convergence results under a weaker condition, and the convergence rates is the order optimal when the regularization parameter is chosen by the discrepancy principle.

This article has the following structure. In Section 2, we describe the process of this method. Section 3 is about the error estimate of the method. To verify the effectiveness of the method, we also give some numerical experiments in Section 4.

2 Basic description of the method

2.1 Problem of the solution

Let $\mathbb{L}^2(\mathbb{R})$ and $\mathbb{H}^p(\mathbb{R})(p > 0)$ are the usual Lebesgue and Sobolev spaces on $\mathbb{R}$, and $\|\cdot\|_s$ denote the norms in $\mathbb{H}^p(\mathbb{R})$. We define the Fourier transform of the function $f \in \mathbb{L}^2(\mathbb{R})$ as follows:

$$ \hat{f}(\xi) = \mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}\xi x}\mathrm{d}x. $$

It is well known that the norm of Sobolev space $\mathbb{H}^s(\mathbb{R})$ can be defined as follows:

$$ \|f\|_s = \left(\int_{\mathbb{R}} |\hat{f}|^2(1+|\xi|^2)^s\mathrm{d}\xi\right)^{1/2}. $$

The Fourier transform of $\mathcal{F}\{J_{m,n}^{\alpha}(x,t)\}$ can be given as follows [20]:

$$ \mathcal{F}\{J_{m,n}^{\alpha}(x,t)\} = [m(\mathrm{i}\xi)^{\alpha} + n(-\mathrm{i}\xi)^{\alpha}]\hat{u}(\xi,t), \quad \xi \in \mathbb{R}. \tag{3} $$

From [4],

$$m(\mathrm{i}\xi)^\alpha + n(-\mathrm{i}\xi)^\alpha = |\xi|^\alpha\left[(m-n)\,\mathrm{sgn}(\xi)\sin\left(\frac{\pi\alpha}{2}\right)\mathrm{i} + \cos\left(\frac{\pi\alpha}{2}\right)\right],$$

where $\mathrm{sgn}(x)$ represents the signum function.

We can deduce the following equation in Fourier space by using the the Fourier transform to problem (1):

$$\begin{cases} \dfrac{\partial\hat{u}(\xi,t)}{\partial t} = \hat{f}(\xi) + \lambda_\alpha(\xi)\hat{u}(\xi,t), & \xi\in\mathbb{R},\ 0<t<T, \\ \hat{u}(\xi,T) = \hat{g}(\xi), \quad \hat{u}(\xi,0)=0, & \xi\in\mathbb{R}, \end{cases} \tag{4}$$

where

$$\lambda_\alpha(\xi) = \kappa_\alpha\cos\left(\frac{\pi\alpha}{2}\right)|\xi|^\alpha + \left[\kappa_\alpha(m-n)\sin\left(\pm\frac{\pi\alpha}{2}\right)|\xi|^\alpha\right]\mathrm{i}.$$

We have a solution to problem (4) in the following form:

$$\hat{u}(\xi,t) = \frac{e^{\lambda_\alpha(\xi)t}-1}{\lambda_\alpha(\xi)}\hat{f}(\xi).$$

We define

$$A_\alpha(\xi) := \frac{\lambda_\alpha(\xi)}{e^{\lambda_\alpha(\xi)T}-1},$$

and then it is easy to see that

$$f = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} A_\alpha(\xi)e^{\mathrm{i}\xi x}\hat{g}(\xi)\,\mathrm{d}\xi =: \mathcal{K}g. \tag{5}$$

$A_\alpha(\xi)$ is unbounded as $|\xi|$ tends to infinity, so the problem is difficult to solve. Small errors can have a huge impact on the results. Therefore, special regularization technique is required to deal with it.

2.2 The modified Tikhonov regularization method

Let $H_\ell(x)$ be the normalized Hermite function of degree ℓ. According to [21], the Hermite functions $\{H_\ell(x)\}_{\ell=0}^\infty$ have the following orthogonality relations:

$$\int_{\mathbb{R}} H_\ell H_k(x)(x)\,\mathrm{d}x = \delta_{\ell,k}.$$

The Hermite expansion of a function $f\in\mathbb{L}^2(\mathbb{R})$ can be given as follows:

$$f(x) = \sum_{\ell=0}^{\infty}\mathbf{f}_\ell H_\ell(x),$$

where

$$\mathbf{f}_\ell = \int_{\mathbb{R}} f(x)H_\ell\,\mathrm{d}x.$$

Let $\vec{\mathbf{f}} = (\mathbf{f}_0, \mathbf{f}_1, \dots \mathbf{f}_n, \dots)^{\mathrm{T}} \in \ell^2$, we define operators

$$(\mathcal{H}\vec{\mathbf{f}})(x) = \sum_{\ell=0}^{\infty}\mathbf{f}_\ell H_\ell(x), \quad \mathcal{P}_N\vec{\mathbf{f}} = \mathcal{H}^{-1}\mathcal{F}^{-1}[\widehat{\mathcal{H}\vec{\mathbf{f}}}(\xi)\chi_N(\xi)], \quad \mathcal{R}\vec{\mathbf{f}} = \mathcal{H}^{-1}\mathcal{F}^{-1}[\widehat{\mathcal{H}\vec{\mathbf{f}}}(\xi)\cosh(\xi)], \tag{6}$$

where χ_N is the characteristic function of the interval $[-N,N]$.

Suppose that f, g satisfy equation (1), i.e., $f = \mathcal{K}g$ and the condition (2) holds. Moreover, we define an *a priori* bound on unknown source,

$$\|f\|_s \leq E, \quad s > 0, \tag{7}$$

where $E > 0$ is a constant. Now we devote to develop a method to obtain a stable approximation of f from the noisy data g^δ. Let $\alpha > 0$ and $\mathcal{T} = \mathcal{K}^{-1}\mathcal{H}$. We propose a modified Tikhonov functional of the following form:

$$\Phi(\overrightarrow{\mathbf{f}}) = \|\mathcal{T}\overrightarrow{\mathbf{f}} - g^\delta\|^2 + \beta\|\mathcal{R}\overrightarrow{\mathbf{f}}\|^2_{\ell^2}. \tag{8}$$

If $\overrightarrow{\mathbf{f}}^\delta_\beta$ is the minimizer of above functional, then

$$f^\delta_\beta = \mathcal{H}\overrightarrow{\mathbf{f}}^\delta_\beta \tag{9}$$

is chosen as the approximation of f. It can be deduced that $\overrightarrow{\mathbf{f}}^\delta_\beta$ can be obtained by solving the following equation [22]:

$$(\mathcal{T}^*\mathcal{T} + \beta\mathcal{R}^2)\overrightarrow{\mathbf{f}} = \mathcal{T}^*g^\delta. \tag{10}$$

Lemma 1. *If we let $\mathcal{E} = \mathcal{T}\mathcal{R}^{-1}$, then*

$$\overrightarrow{\mathbf{f}}^\delta_\beta = \mathcal{R}^{-1}l_\beta(\mathcal{E}^*\mathcal{E})\mathcal{E}^*g^\delta \quad \text{with } l_\beta(\theta) = \frac{1}{\beta + \theta}.$$

The function $l_\beta(\theta)$ has the following properties [23]:

$$\sup_{\theta>0} \theta^{1/2}|l_\beta(\theta)| \leq \frac{1}{2\sqrt{\beta}}, \quad \sup_{\theta>0} \theta|l_\beta(\theta)| \leq 1, \tag{11}$$

and

$$\sup_{\theta>0} \beta^{1/2}|1 - \theta l_\beta(\theta)| \leq \frac{\sqrt{\beta}}{2}, \quad \sup_{\theta>0}|1 - \theta l_\beta(\theta)| \leq 1. \tag{12}$$

3 Error estimate of the method

In this section, we deduce the error estimate of the new method. The following auxiliary results are needed.

Lemma 2. [4] *For* $1 < \alpha < 2$, *we have*

$$c_\alpha(1 + |\xi|^2)^{\frac{\alpha}{2}} \leq |A_\alpha(\xi)| \leq C_\alpha(1 + |\xi|^2)^{\frac{\alpha}{2}}, \quad \xi \in \mathbb{R}, \tag{13}$$

where c_α and C_α are two constants.

Lemma 3. [16] *For any $r > 0$, if $h \in \mathbb{H}^r(\mathbb{R})$, then*

$$\|h\| \leq c_\alpha\|\mathcal{K}^{-1}h\|^{\frac{r}{r+\alpha}}\|h\|_r^{\frac{\alpha}{r+\alpha}}. \tag{14}$$

Suppose that the Fourier Hermite coefficients vector of $h \in \mathbb{L}^2(\mathbb{R})$ is $\overrightarrow{\mathbf{h}} = (\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_n, \ldots)^T$, i.e.,

$$h(x) = (\mathcal{H}\overrightarrow{\mathbf{h}})(x),$$

then we let

$$\overrightarrow{\mathbf{h}}_N = \mathcal{P}_N\overrightarrow{\mathbf{h}}, \quad h_N = \mathcal{H}\overrightarrow{\mathbf{h}}_N, \tag{15}$$

where $N < \infty$ is a positive integer that has to be chosen properly. (It should be noted that N is only used in theoretical analysis.)

Lemma 4. *If $h \in \mathbb{H}^r(\mathbb{R})$, then we have*

$$\|\mathcal{T}\overrightarrow{\mathbf{h}} - \mathcal{T}\overrightarrow{\mathbf{h}}_N\| \le c_N N^{-r-\alpha}E \quad \text{and} \quad \|\mathcal{R}\overrightarrow{\mathbf{h}}_N\|_{\ell^2} \le C_N E, \tag{16}$$

where

$$c_N = \frac{1}{c_\alpha} \quad \text{and} \quad C_N = \max\left(1, \frac{e^N}{2N^r}\right).$$

Proof. According to Parseval's formula and Lemma 2, we obtain

$$\begin{aligned}
\|\mathcal{T}(\overrightarrow{\mathbf{h}} - \overrightarrow{\mathbf{h}}_N)\|^2 &= \int\limits_{|\xi|>N} A_\alpha^{-2}(\xi)|\hat{h}(\xi)|^2 \mathrm{d}\xi \\
&\le \frac{1}{c_\alpha^2} \int\limits_{|\xi|>N} (1 + |\xi|^2)^{-\alpha}|\hat{h}(\xi)|^2 \mathrm{d}\xi \\
&= \frac{1}{c_\alpha^2} \int\limits_{|\xi|>N} (1 + |\xi|^2)^{-(r+\alpha)}|\hat{h}(\xi)|^2(1+\xi^2)^r \mathrm{d}\xi \\
&\le \frac{1}{c_\alpha^2} N^{-2(r+\alpha)} \int\limits_{|\xi|>N} |\hat{h}(\xi)|^2(1+\xi^2)^r \mathrm{d}\xi \\
&= \frac{1}{c_\alpha^2} N^{-2(r+\alpha)}\|h\|_r^2,
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{R}\overrightarrow{\mathbf{h}}_N\|_{\ell^2}^2 &= \int\limits_{|\xi|\le N} \cosh^2(\xi)|\hat{h}(\xi)|^2 \mathrm{d}\xi \\
&= \int\limits_{|\xi|\le N} \frac{\cosh^2(\xi)}{(1+\xi^2)^r}|\hat{h}(\xi)|^2(1+\xi^2)^r \mathrm{d}\xi \\
&\le \max\left(1, \frac{e^{2N}}{4N^{2r}}\right)\|h\|_r^2.
\end{aligned}$$

□

Lemma 5. *Let k_i $(i = 1, 2, 3)$ is some fixed constants, if the vector $\overrightarrow{\mathbf{h}}^\delta = (\mathbf{h}_0^\delta, \mathbf{h}_1^\delta, \dots, \mathbf{h}_n^\delta, \dots)^{\mathrm{T}}$ sequence satisfies*

$$\|\mathcal{T}\overrightarrow{\mathbf{h}}^\delta\| \le k_1\delta, \quad \|\mathcal{R}\overrightarrow{\mathbf{h}}^\delta\|_{\ell^2} \le k_2 e^{k_3\delta^{-1/s}}\delta, \quad \delta \to 0. \tag{17}$$

Then there exists a constant $M > 0$ such that

$$\|\mathcal{H}\overrightarrow{\mathbf{h}}^\delta\|_s \le M. \tag{18}$$

Proof. Let $N_0 = k_3\delta^{-1/(s+\alpha)}$, then by using the triangle inequality, we have

$$\|\mathcal{H}\overrightarrow{\mathbf{h}}^\delta\|_s \le \|\mathcal{H}\left(I - \mathcal{P}_{N_0}\right)\overrightarrow{\mathbf{h}}^\delta\|_s + \|\mathcal{H}\mathcal{P}_{N_0}\overrightarrow{\mathbf{h}}^\delta\|_s.$$

According to Parseval's formula, we have the following results:

$$\begin{aligned}
\|\mathcal{H}\left(I - \mathcal{P}_{N_0}\right)\overrightarrow{\mathbf{h}}^\delta\|_s^2 &= \int\limits_{|\xi|>N_0} (1 + |\xi|^2)^s|\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^\delta}(\xi)|^2 \mathrm{d}\xi \\
&= \int\limits_{|\xi|>N_0} \frac{(1 + |\xi|^2)^s}{\cosh^2(\xi)} \left|\cosh(\xi)\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^\delta}(\xi)\right|^2 \mathrm{d}\xi
\end{aligned}$$

$$\leq \frac{(1+N_0)^{2s}}{\cosh^2(N_0)} \int\limits_{|\xi|>N_0} |\cosh(\xi)\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^{\delta}}(\xi)|^2 \, d\xi$$
$$\leq \frac{4N_0^2}{e^{2(N_0-1)}} \|\mathcal{R}\overrightarrow{\mathbf{h}}^{\delta}\|_{\ell^2}^2 \leq 4e^{-2}k_2^2k_3^{2s}$$

and

$$\|\mathcal{H}\mathcal{P}_{N_0}\overrightarrow{\mathbf{h}}^{\delta}\|_s^2 = \int\limits_{|\xi|\leq N_0} (1+|\xi|^2)^s|\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^{\delta}}(\xi)|^2 d\xi$$
$$= \int\limits_{|\xi|\leq N_0} (1+|\xi|^2)^s A_\alpha^2(\xi)|A_\alpha^{-1}(\xi)\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^{\delta}}(\xi)|^2 d\xi$$
$$\leq C_\alpha N_0^{2(s+\alpha)} \int\limits_{|\xi|\leq N_0} |A_\alpha^{-1}(\xi)\widehat{\mathcal{H}\overrightarrow{\mathbf{h}}^{\delta}}(\xi)|^2 d\xi$$
$$= C_\alpha N_0^{2(s+\alpha)}\|\mathcal{T}\overrightarrow{\mathbf{h}}^{\delta}\|^2 = C_\alpha k_1^2 k_3^{2(s+\alpha)}.$$

This completes the proof. □

Theorem 6. *If f_β^δ is defined by* (9) *and the conditions* (2) *and* (7) *hold. In addition, the regularization parameter β is determined by the following equation*:

$$\|g^\delta - \mathcal{T}\overrightarrow{\mathbf{f}}^{\delta}_{\beta}\| = C\delta, \quad C > 1. \tag{19}$$

and then we have

$$\|f_\beta^\delta - f\| = O(\delta^{s/s+\alpha}). \tag{20}$$

Proof. Due to (2), (16), and (19), we can obtain the following result by using the triangle inequality:

$$\|\mathcal{T}(\overrightarrow{\mathbf{f}}^{\delta}_{\beta} - \overrightarrow{\mathbf{f}}_N)\| \leq \|\mathcal{T}\overrightarrow{\mathbf{f}}^{\delta}_{\beta} - g^\delta\| + \|g^\delta - g\| + \|\mathcal{T}(\overrightarrow{\mathbf{f}} - \overrightarrow{\mathbf{f}}_N)\| \leq (C+1)\delta + c_N N^{-s-\alpha}E. \tag{21}$$

If we define $\overrightarrow{\mathbf{f}}_{\beta,N} = \mathcal{R}^{-1}l_\beta(\mathcal{E}^*\mathcal{E})\mathcal{E}^*f_N$, then we have

$$\begin{aligned}\mathcal{R}(\overrightarrow{\mathbf{f}}^{\delta}_{\beta} - \overrightarrow{\mathbf{f}}_{\beta,N}) &= l_\beta(\mathcal{E}^*\mathcal{E})\mathcal{E}^*(g^\delta - \mathcal{T}\overrightarrow{\mathbf{f}}_N),\\ \mathcal{R}(\overrightarrow{\mathbf{f}}_N - \overrightarrow{\mathbf{f}}_{\beta,N}) &= [I - l_\beta(\mathcal{E}^*\mathcal{E})\mathcal{E}^*\mathcal{E}]\mathcal{R}\overrightarrow{\mathbf{f}}_N.\end{aligned} \tag{22}$$

Hence, by using the triangle inequality, (6), (16), (22), and Lemma 1,

$$\|\mathcal{R}(\overrightarrow{\mathbf{f}}^{\delta}_{\beta} - \overrightarrow{\mathbf{f}}_N)\|_{\ell^2} \leq \|\mathcal{R}(\overrightarrow{\mathbf{f}}^{\delta}_{\beta} - \overrightarrow{\mathbf{f}}_{\beta,N})\|_{\ell^2} + \|\mathcal{R}(\overrightarrow{\mathbf{f}}_N - \overrightarrow{\mathbf{f}}_{\beta,N})\|_{\ell^2}$$
$$\leq \frac{1}{2\sqrt{\beta}}\|g^\delta - \mathcal{T}\overrightarrow{\mathbf{f}}_N\| + \|\mathcal{R}\overrightarrow{\mathbf{f}}_N\|_{\ell^2}$$
$$\leq \frac{1}{2\sqrt{\beta}}(\delta + c_N N^{-s-\alpha}E) + C_N E.$$

Let $\mathcal{G}_\beta = I - l_\beta(\mathcal{E}\mathcal{E}^*)\mathcal{E}\mathcal{E}^*$, note that $g^\delta - \mathcal{T}\overrightarrow{\mathbf{f}}^{\delta}_{\beta} = \mathcal{G}_\beta g^\delta$. According to the triangle inequality, (16), (22), and Lemma 1, we obtain

$$\|g^\delta - \mathcal{T}\overrightarrow{\mathbf{f}}^{\delta}_{\beta}\| \leq \|\mathcal{G}_\beta\mathcal{T}\overrightarrow{\mathbf{f}}_N\| + \|\mathcal{G}_\beta(g - \mathcal{T}\overrightarrow{\mathbf{f}}_N)\| + \|\mathcal{G}_\beta(g^\delta - g)\|$$
$$\leq \|\mathcal{G}_\beta\mathcal{S}\|\cdot\|\mathcal{R}\overrightarrow{\mathbf{f}}_N\| + \|g - \mathcal{T}\overrightarrow{\mathbf{f}}_N\| + \delta$$
$$\leq \frac{\sqrt{\beta}}{2}C_N E + c_N N^{-s-\alpha}E + \delta.$$

Suppose that N satisfies

$$N^{-s-\alpha}E = \frac{C-1}{2}\delta, \tag{23}$$

then

$$\begin{aligned}
&\|\mathcal{T}(\vec{\mathbf{f}}^{\delta}_{\beta} - \vec{\mathbf{f}}_N)\|_s \le k_1\delta,\\
&\|\mathcal{R}(\vec{\mathbf{f}}^{\delta}_{\beta} - \vec{\mathbf{f}}_N)\|_{\ell^2} \le k_2 e^{k_3\delta^{-1/s}}\delta
\end{aligned}$$

holds with constants $k_i (i = 1, 2, 3)$. Hence, according to Lemma 5, there exists a constant M,

$$\|\mathcal{H}(\vec{\mathbf{f}}^{\delta}_{\beta} - \vec{\mathbf{f}}_N)\|_s \le M.$$

So we can obtain

$$\begin{aligned}
\|\mathcal{H}\vec{\mathbf{f}}^{\delta}_{\beta} - f\|_s &= \|\mathcal{H}(\vec{\mathbf{f}}^{\delta}_{\beta} - \vec{\mathbf{f}}_N)\|_s + \|f - f_N\|_s\\
&\le \|\mathcal{H}(\vec{\mathbf{f}}^{\delta}_{\beta} - \vec{\mathbf{f}}_N)\|_s + \|f\|_s\\
&\le M + E.
\end{aligned} \tag{24}$$

Due to (2) and (19), we can obtain the following result by using triangle inequality:

$$\|g - \mathcal{T}\vec{\mathbf{f}}^{\delta}_{\beta}\| \le \|g - g^{\delta}\| + \|g^{\delta} - \mathcal{T}\vec{\mathbf{f}}^{\delta}_{\beta}\| \le (C+1)\delta. \tag{25}$$

The assertion can be obtained by using (24), (25), and Lemma 3. □

4 Numerical experiments

To verify the effectiveness of the proposed method, we present some numerical tests in this section. For simplicity, let $m = n = 0.5$, $N = 256$, $T = 1$, and $\kappa_\alpha = 1$. We also test the effect of the method when the parameters are different, and the result is similar.

We perform the numerical tests in a finite interval $[-B, B]$, and $f(x)$ approaches zero as $|x| > B$. Let the knots $x_i = -B + ih$, $i = 0, 1, \ldots, m$ with $m = 256$. The datum $g = \{g(x_i)\}_{i=1}^{m}$ represents values of $g(x)$ on the grid. Then the perturbation data g^{δ} is obtained by adding random uniformly distributed perturbation to g, i.e.,

$$g^{\delta} = g + \delta_1 \operatorname{rand}(\operatorname{size}(g)), \tag{26}$$

where δ_1 is the noise level and the noise δ is measured by

$$\delta = \|g^{\delta} - g\|_{\ell^2} = \sqrt{\frac{1}{m}\sum_{i=1}^{m}(g^{\delta}(x_i) - g(x_i))^2}. \tag{27}$$

The analytical solution of equation (1) is usually difficult to obtain, so we have to use the numerical method to obtain the datum g for a given $f(x)$. This step is similar to the method presented in [16]. The following relative error of ℓ^2 norm is used to measure the accuracy of the numerical approximation:

$$E_r = \left(\frac{\sum_{i=0}^{m}(f^{\delta}_{\beta}(x_i) - f(x_i))^2}{\sum_{i=0}^{m} f(x_i)^2}\right)^{1/2}. \tag{28}$$

Moreover, we would like to compare the relative errors obtained by the method in this article (M1) with that in [16] (M2). All the results are obtained by using Matlab2017b in the case of $C = 1.05$ in (19).

Example 1. We take the function f as follows:

$$f(x) = \left(\frac{x^3}{2} - \frac{3x}{4}\right)e^{-\frac{x^2}{2}}, \tag{29}$$

and the numerical text is implemented in the interval $[-10, 10]$.

The comparisons of the exact function and its approximations for various α with $\delta_1 = 0.1$ are given in Figure 1. The relative errors of M1 and M2 for various α and δ_1 are presented in Table 1. It can be seen that when alpha is close to 1, the results of the two methods are close. With the increase of α, method 1 performs better than method 2, and the numerical results are more stable and the relative errors are smaller. Figures 2 and 3 exhibit the variation of E_r with the changes of α and δ_1, respectively. It can be seen that the method is stable for various α and δ_1.

Example 2. Consider a nonsmooth function:

$$f(x) = \begin{cases} 0, & -10 \le x \le -4, \\ x+4, & -4 < x \le 0, \\ x-4, & 0 < x \le 4, \\ 0, & 4 < x \le 10. \end{cases} \tag{30}$$

The errors are given in Table 2, and the comparisons of the exact function and its approximations for various α with $\delta_1 = 0.01$ are shown in Figure 4. It can be seen that the method is still stable, and the results of M1 are also better than M2 in this case.

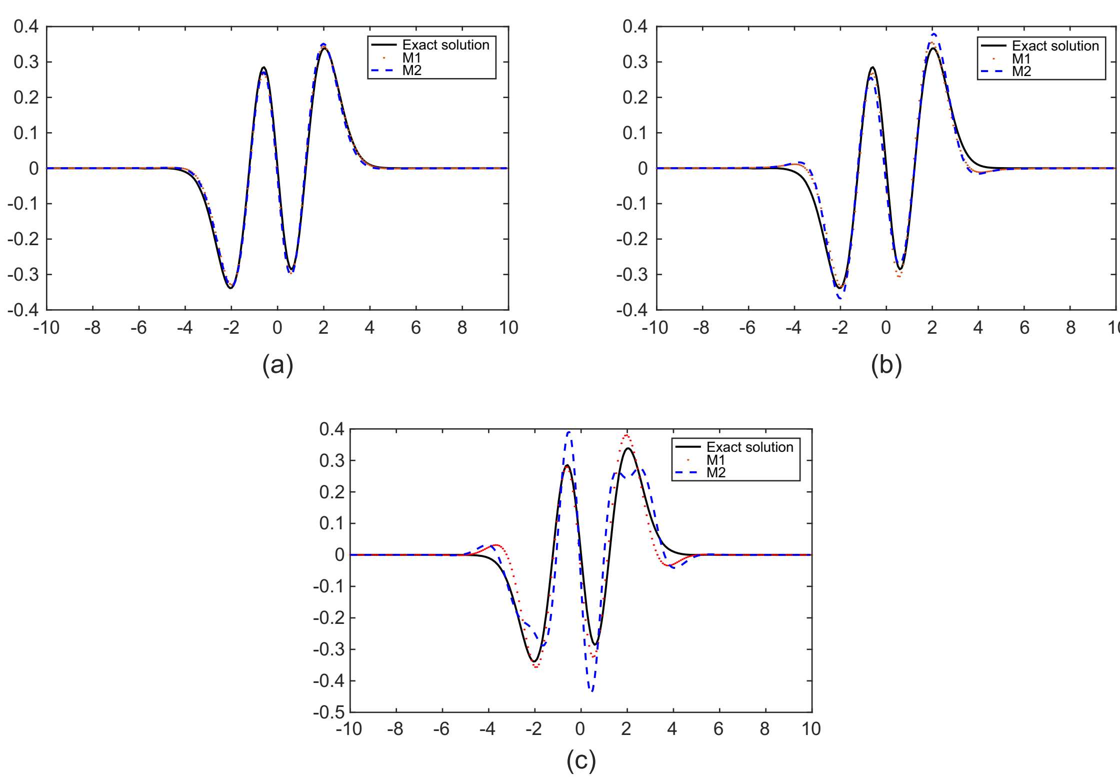

Figure 1: Comparison of exact functions and their approximation (Example 1). (a) $\alpha = 1.1$. (b) $\alpha = 1.5$. (c) $\alpha = 1.9$.

Table 1: The relative errors of Example 1

δ_1	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
	M1	**M2**	**M1**	**M2**	**M1**	**M2**
1×10^{-1}	0.0729	0.0684	0.1588	0.1934	0.2102	0.4258
1×10^{-2}	0.0093	0.0087	0.0376	0.0581	0.0554	0.1584
1×10^{-3}	0.0018	0.0020	0.0057	0.0095	0.0117	0.0222

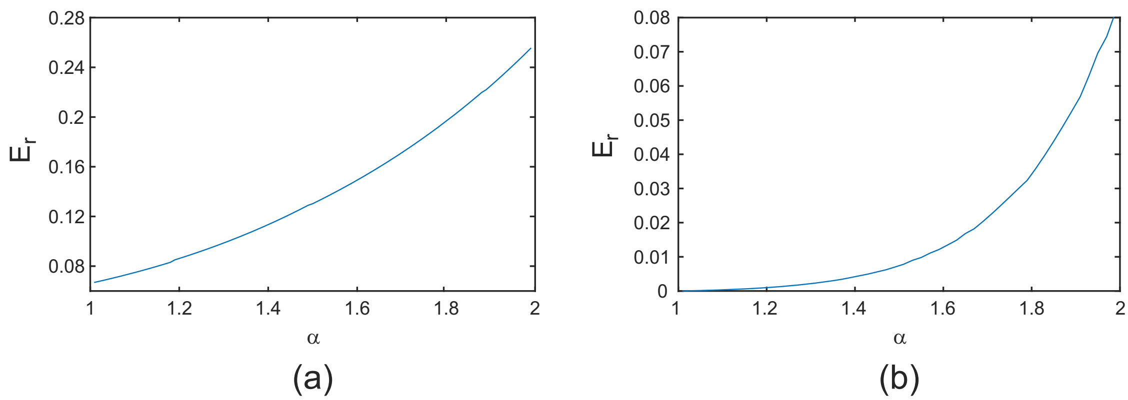

Figure 2: The variation of E_r with α (Example 1). (a) $\delta_1 = 0.1$. (b) $\delta_1 = 0.01$.

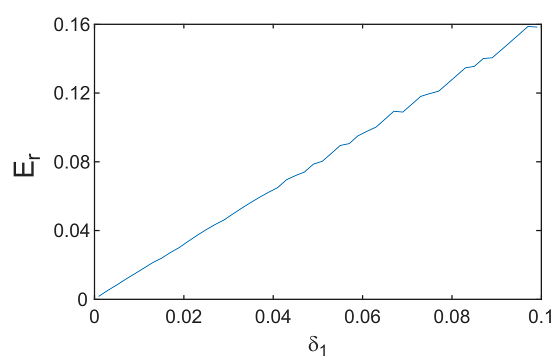

Figure 3: The variation of E_r with δ_1 (Example 1). $\alpha = 1.5$.

Table 2: The relative errors of Example 2

δ_1	$\alpha = 1.1$		$\alpha = 1.5$		$\alpha = 1.9$	
	M1	**M2**	**M1**	**M2**	**M1**	**M2**
1×10^{-1}	0.1046	0.1065	0.1732	0.2256	0.2509	0.4522
1×10^{-2}	0.0128	0.0135	0.0548	0.0846	0.0782	0.2126
1×10^{-3}	0.0082	0.0094	0.0157	0.0213	0.0468	0.0952

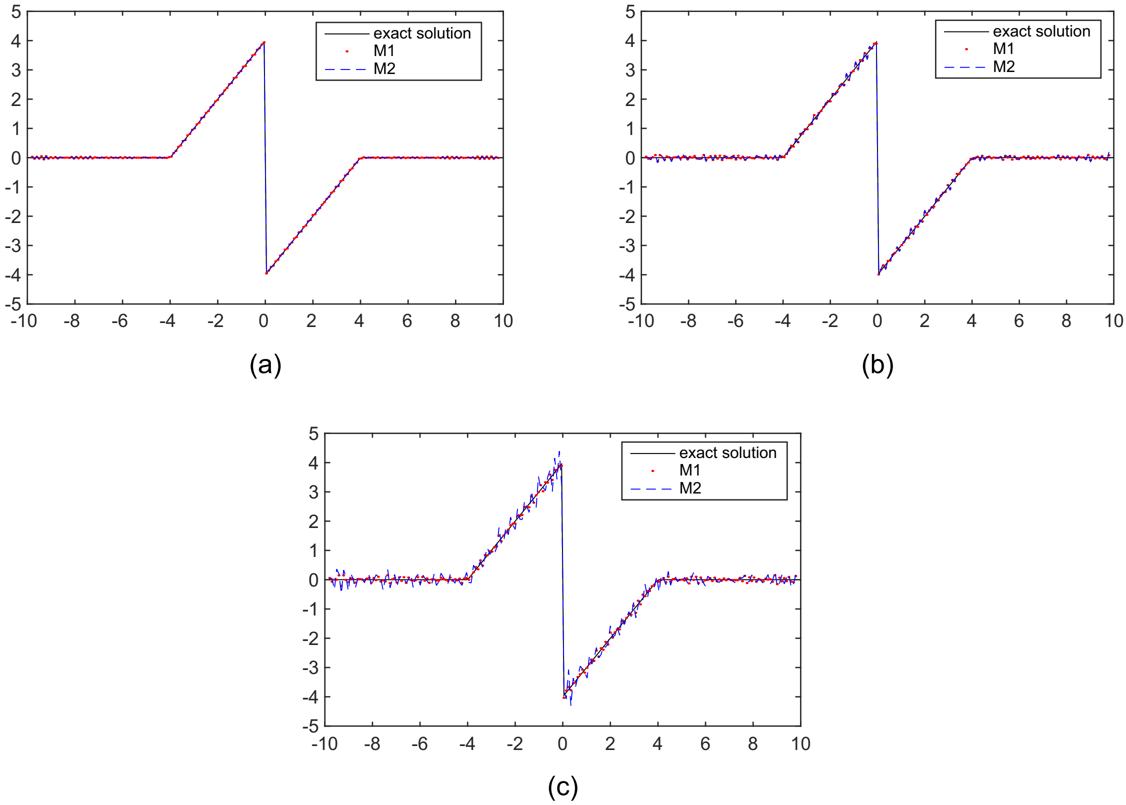

Figure 4: Comparison of exact functions and their approximation (Example 2). (a) $\alpha = 1.1$. (b) $\alpha = 1.5$. (c) $\alpha = 1.9$.

5 Conclusion

On the basis of the Hermite extension method, we propose a modified Tikhonov regularization to solve an unknown source problem in the space fractional diffusion equation. In addition, the framework of this approach can be used to deal with other ill-posed problems.

Acknowledgements: The authors thank the referees for valuable comments and suggestions, which improved the presentation of this manuscript.

Funding information: The project is supported by the project of enhancing school with innovation of Guangdong ocean university (Q18306).

Conflict of interest: The authors state no conflict of interest.

References

[1] I. Podlubny, *Geometric and physical interpretation of fractional integration and fractional differentiation*, Fract. Calc. Appl. Anal. **5** (2001), no. 4, 230–237.

[2] R. Metzler and J. Klafter, *The random walkas guide to anomalous diffusion: a fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 1–77.

[3] R. Metzler and J. Klafter, *The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics*, J. Phys. A **37** (2004), no. 31, R161.

[4] W. Y. Tian, C. Li, W. H. Deng, and Y. J. Wu, *Regularization methods for unknown source in space fractional diffusion equation*, Math. Comput. Simulation **85** (2012), 45–56.

[5] J. R. Cannon and P. Duchateau, *Structural identification of an unknown source term in a heat equation*, Inverse Problems **14** (1999), no. 3, 535–551.

[6] S. Andrieux and A. B. Abda, *Identification of planar cracks by complete overdetermined data: inversion formulae*, Inverse Problems **12** (1999), no. 5, 553.

[7] J. I. Frankel, *Residual-minimization least-squares method for inverse heat conduction*, Comput. Math. Appl. **32** (1996), no. 4, 117–130.

[8] A. J. Devaney and E. A. Marengo, *Inverse source problem in nonhomogeneous background media*, SIAM J. Appl. Math. **67** (2007), no. 5, 1353–1378.

[9] R. A. Albanese and P. B. Monk, *The inverse source problem for Maxwell's equations*, Inverse Problems **22** (2006), no. 3, 1023–1035.

[10] H. Liu and G. Uhlmann, *Determining both sound speed and internal source in thermo- and photo-acoustic tomography*, Inverse Problems **31** (2015), no. 10, 105005.

[11] M. Eller and N. P. Valdivia, *Acoustic source identification using multiple frequency information*, Inverse Problems **28** (2009), no. 11, 1183–1223.

[12] G. Wang, F. M. Ma, Y. K. Guo, and J. Z. Li, *Solving the multi-frequency electromagnetic inverse source problem by the Fourier method*, J. Differential Equations **265** (2018), 417–443.

[13] N. Magnoli and G. A. Viano, *The source identification problem in electromagnetic theory*, J. Math. Phys. **38** (1997), no. 5, 2366–2388.

[14] H. Wei, W. Chen, H. Sun, and X. Li, *A coupled method for inverse source problem of spatial fractional anomalous diffusion equations*, Inverse Probl. Sci. Eng. **18** (2010), no. 7, 945–956.

[15] F. Yang, C. L. Fu, and X. X. Li, *Identifying an unknown source in space-fractional diffusion equation*, Acta Math. Sci. **34B** (2014), no. 4, 1012–1024.

[16] Z. Y. Zhao, O. Xie, L. You, and Z. H. Meng, *A truncation method based on Hermite expansion for unknown source in space fractional diffusion equation*, Math. Model. Anal. **19** (2014), no. 3, 430–442.

[17] D. D. Trong, D. Hai, and N. D. Minh, *Optimal regularization for an unknown source of space-fractional diffusion equation*, Appl. Math. Comput. **349** (2019), 184–206.

[18] Z. Y. Zhao, *A Hermite extension method for numerical differentiation*, Appl. Numer. Math. **159** (2021), 46–60.

[19] Z. Y. Zhao, L. You, and Z. H. Meng, *A modified Tikhonov regularization method based on Hermite expansion for solving the Cauchy problem of the Laplace equation*, Open Math. **18** (2020), no. 1, 1685–1697.

[20] I. Podlubny, *Fractional Differential Equations*, Elsevier, Amsterdam, 1998.

[21] S. Thangavelu, *Lectures on Hermite and Laguerre Expansions*, Princeton University Press, Princeton, 1993.

[22] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of Inverse Problems*, Springer Science & Business Media, Berlin, 2000.

[23] M. T. Nair, S. V. Pereverzev, and U. Tautenhahn, *Regularization in Hilbert scales under general smoothing conditions*, Inverse Problems **21** (2005), no. 6, 1851.