Research Article

Dohoon Choi and Youngmin Lee*

Modular forms of half-integral weight on $\Gamma_0(4)$ with few nonvanishing coefficients modulo ℓ

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Abstract: Let k be a nonnegative integer. Let K be a number field and O_K be the ring of integers of K. Let $\ell \geq 5$ be a prime and v be a prime ideal of O_K over ℓ. Let f be a modular form of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ such that its Fourier coefficients are in O_K. In this article, we study sufficient conditions that if f has the form

$$f(z) \equiv \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v}$$

with square-free integers s_i, then f is congruent to a linear combination of iterated derivatives of a single theta function modulo v.

Keywords: Fourier coefficients of modular forms, Galois representations, modular forms of half-integral weight, theta functions

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1 Introduction

The Fourier coefficients of modular forms of half-integral weight are related to various objects in number theory and combinatorics such as the algebraic parts of the central critical values of modular L-functions, orders of Tate-Shafarevich groups of elliptic curves, the number of partitions of a positive integer, and so on. With a lot of application to these objects, Bruinier [1], Bruinier and Ono [2], Ono and Skinner [3], Ahlgren and Boylan [4,5], and the others studied congruence properties modulo a power of a prime for Fourier coefficients of modular forms of half-integral weight. Many of them considered modular forms of half-integral weight whose the Fourier coefficients are supported on only finitely many square classes modulo a prime ℓ.

Let f be a modular form of half-integral weight on $\Gamma_1(4N)$. Vignéras [6] proved that if the q-expansion of f has the form

$$f(z) = a_f(0) + \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2}, \quad q := e^{2\pi i z}$$

* **Corresponding author: Youngmin Lee,** School of Mathematics, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea, e-mail: youngminlee@kias.re.kr
Dohoon Choi: Department of Mathematics, Korea University, 145 Anam-ro, Seongbuk-gu, Seoul 02841, Republic of Korea, e-mail: dohoonchoi@korea.ac.kr

with a positive integer t and square-free integers s_i, then f is a linear combination of single variable theta functions (a different proof of this result was given by Bruinier [1]). Many of the aforementioned results can be considered as positive characteristic extensions of Vignéras' result on classification of modular forms of half-integral weight such that their nonvanishing Fourier coefficients lie in only finitely many square classes. Especially, Ahlgren et al. [7] obtained an explicit mod ℓ analog of the result of Vignéras for modular forms of half-integral weight on $\Gamma_0(4)$ satisfying the Kohnen-plus condition.

Let K be a number field and O_K be the ring of integers of K. Let $M_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ (resp. $S_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$) be the space of modular forms (resp. cusp forms) of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$ such that their Fourier coefficients are in O_K and $S^+_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ be the subspace of $S_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ consisting of $f \in S_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ satisfying the Kohnen-plus condition.

Let $\ell \geq 5$ be a prime and v be a prime ideal of O_K over ℓ. For $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$, Ahlgren et al. [7] proved that if

$$k+\frac{1}{2} < \ell\left(\ell+\frac{3}{2}\right) \tag{1.1}$$

and

$$f(z) \equiv \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v} \tag{1.2}$$

with square-free integers s_i, then k is even and

$$f(z) \equiv a_f(1)\sum_{n=1}^{\infty} n^k q^{n^2} \pmod{v}.$$

In this article, we study sufficient conditions that if f has the form (1.2), then f is congruent to a linear combination of iterated derivatives of a single theta function modulo v.

For a positive number ε, let P_ε be the set of primes ℓ such that for every $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ with $k+\frac{1}{2} < \ell^2(\log\ell)^{2-\varepsilon}$, if

$$f(z) \equiv \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v}$$

with square-free integers s_i, then

$$f(z) \equiv a_f(1)\left(\sum_{n=1}^{\infty} n^k q^{n^2}\right) + a_f(\ell)\left(\sum_{n=1}^{\infty} n^{k+\frac{\ell-1}{2}} q^{\ell n^2}\right) \pmod{v}.$$

The following theorem proves that the portion of P_ε in the set of primes is one.

Theorem 1.1. *For a positive integer X, there is an absolute constant C such that*

$$\#\{\ell : \ell \notin P_\varepsilon \text{ and } \ell \leq X\} \leq C_0 \frac{X}{(\log X)^{1+\frac{\varepsilon}{2}}}\left(1 + C\frac{\log\log X}{\log X}\right),$$

where $C_0 := \frac{2\sqrt{2}\pi^2}{3}\prod_{p>2}\frac{p^2}{p^2-1}$.

For a nonnegative real number r, we define an operator Θ^r on $\mathbb{C}[[q]]$ by

$$\Theta^r\left(\sum_{n=0}^{\infty} a(n)q^n\right) := \begin{cases} \sum_{n=0}^{\infty} n^r a(n) q^n & \text{if } r \in \mathbb{Z}_{>0}, \\ 0 & \text{elsewhere.} \end{cases}$$

For convenience, we let $\Theta := \Theta^1$. As in Theorem 1.1, the previous results on modular forms of half-integral weight having the form (1.2) such as [1,2,4,5,7] and so on imply that in many cases, if f has the form (1.2), then $\Theta(f)$ is congruent to a linear combination of iterated derivatives of a single theta function modulo v. These lead us to the following conjecture on modular forms f of half-integral weight having the form (1.2).

Conjecture 1.2. *Let K be a number field and O_K be the ring of integers of K. Let $\ell \geq 5$ be a prime and v be a prime ideal of O_K over ℓ. Assume that $f \in S_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form*

$$\Theta(f)(z) \equiv \sum_{n=1}^{\infty} sn^2 a_f(sn^2) q^{sn^2} \pmod{v}$$

with a square-free integer s, then

$$\Theta(f)(z) \equiv \frac{1}{2} a_f(1) \left(\sum_{\substack{n\in\mathbb{Z} \\ \ell \nmid n}} n^{k+2} q^{n^2} \right) \pmod{v}.$$

Assume that ℓ is a prime and m is a nonnegative integer. Let $r_\ell(m)$ be the least positive integer such that

$$r_\ell(m) \equiv m \pmod{\ell - 1}.$$

Let $\alpha(\ell, m)$ be the smallest nonnegative integer i such that

$$m + \frac{1}{2} < \ell^{2i} \left(r_\ell(m) \frac{\ell + 1}{2} + \frac{1}{2} \right),$$

and $\beta(\ell, m)$ be the smallest nonnegative integer i such that

$$m + \frac{1}{2} < \ell^{2i+1} \left(r_\ell \left(m + \frac{\ell - 1}{2} \right) \frac{\ell + 1}{2} + \frac{1}{2} \right).$$

Let

$$T(z) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2}.$$

For convenience, let

$$\sum_{n=a}^{b}{}' a_n := \begin{cases} \sum_{n=a}^{b} a_n & \text{if } a \leq b, \\ 0 & \text{if } a > b. \end{cases}$$

By using Conjecture 1.2, we have an explicit formula for modular forms of half-integral weight having the form (1.2).

Theorem 1.3. *Let K, O_K, ℓ, and v be as in* Conjecture 1.2. *Assume that $f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$.* Conjecture 1.2 *implies that if f has the form*

$$f(z) \equiv a_f(0) + \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v} \tag{1.3}$$

with square-free integers s_i, then the following statements are true.

(1) *If $r_\ell(k) \neq \ell - 1$ and $r_\ell(k) \neq \frac{\ell-1}{2}$, then*

$$f(z) \equiv \frac{1}{2} \sum_{i=0}^{\alpha(\ell,k)-1}{}' a_f(\ell^{2i}) \Theta^{k/2}(T)(\ell^{2i} z) + \frac{1}{2} \sum_{i=0}^{\beta(\ell,k)-1}{}' a_f(\ell^{2i+1}) \Theta^{(2k+\ell-1)/4}(T)(\ell^{2i+1} z) \pmod{v}.$$

(2) *If $r_\ell(k) = \ell - 1$, then*

$$f(z) \equiv a_f(0)T(z) + \frac{1}{2} \sum_{i=0}^{\alpha(\ell,k)-1}{}' (a_f(\ell^{2i}) - 2a_f(0)) \Theta^{k/2}(T)(\ell^{2i} z) + \frac{1}{2} \sum_{i=0}^{\beta(\ell,k)-1}{}' a_f(\ell^{2i+1}) \Theta^{(2k+\ell-1)/4}(T)(\ell^{2i+1} z) \pmod{v}.$$

(3) *If* $r_\ell(k) = \frac{\ell-1}{2}$, *then*

$$f(z) \equiv a_f(0)T(\ell z) + \frac{1}{2}\sum_{i=0}^{\alpha(\ell,k)-1'} a_f(\ell^{2i})\Theta^{k/2}(T)(\ell^{2i}z) + \frac{1}{2}\sum_{i=0}^{\beta(\ell,k)-1'} (a_f(\ell^{2i+1}) - 2a_f(0))\Theta^{(2k+\ell-1)/4}(T)(\ell^{2i+1}z) \pmod{v}.$$

To give numerical evidence for Conjecture 1.2, we consider a basis of the space of modular forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$. Let $F_2(z) = \sum_{n=0}^{\infty}\sigma(2n+1)q^{2n+1}$ be the modular form of weight 2 on $\Gamma_0(4)$, where $\sigma(n)$ is the sum of positive divisors of n. Then

$$\{F_2^jT^{2k+1-4j}\}_{0\le j\le\lfloor\frac{k}{2}\rfloor}$$

is a $\mathbb{C}$-basis of the space of modular forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$. Let $A_{k,m}$ be an $m\times\left(\left\lfloor\frac{k}{2}\right\rfloor+1\right)$ matrix such that the (i,j)-entry of $A_{k,m}$ is the $(i-1)$th Fourier coefficient of $F_2^{j-1}T^{2k+5-4j}$ modulo ℓ. Let $B_{k,m}$ be a submatrix of $A_{k,m}$ obtained by removing n^2+1th rows for all nonnegative integers n with $(\ell,n)=1$. Let $\mathrm{Null}(B_{k,m})$ be the null space of $B_{k,m}$. With this notation, we give the following conjecture.

Conjecture 1.4. *Let $\ell \ge 5$ be a prime. Let $\mathbb{1}_+$ be the characteristic function of the set of positive real numbers. Then, for a positive even integer k, we have*

$$\lim_{m\to\infty}\dim\mathrm{Null}(B_{k,m}) = \mathbb{1}_+(\alpha(\ell,k)).$$

By comparing the intersection of the null spaces of $B_{k,m}$ and the space of mod v modular forms of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$ having the form

$$f(z) \equiv \sum_{\ell\nmid n} a(n^2)q^{n^2} \pmod{v},$$

we have the following theorem.

Theorem 1.5. *Conjecture* 1.2 *is equivalent to Conjecture* 1.4.

Let us note that $\mathrm{Null}(B_{k,m})$ is stable for sufficiently large m. In the proof of Theorem 1.5, we prove that $\dim\mathrm{Null}(B_{k,m})$ is larger than or equal to $\mathbb{1}_+(\alpha(\ell,k))$ for all positive integers m. Hence, if there is a positive integer m such that $\dim\mathrm{Null}(B_{k,m}) = \mathbb{1}_+(\alpha(\ell,k))$, then Conjecture 1.2 is true. To compute $\dim\mathrm{Null}(B_{k,m})$, we consider the row echelon form of $B_{k,m}$. We use C++ in this process. Then we have the following theorem.

Theorem 1.6. *Assume that $k \le 1{,}000$, or that $\ell \in \{5, 7, 11, 13, 17, 19\}$ and $k \le 10{,}000$. Then, Conjecture* 1.2 *is true.*

The remainder of this article is organized as follows. In Section 2, we review some properties of f having the form (1.3) and the filtration for modular forms. In Section 3, we prove Theorems 1.1, 1.3, 1.5, and 1.6.

2 Preliminaries

In this section, we review some notions and properties of the filtration for modular forms, and then we introduce some properties about modular forms of half-integral weight on $\Gamma_0(4)$ such that their Fourier coefficients are supported on finitely many square classes modulo a prime ℓ. For further details, see [8].

Throughout the rest of this article, we fix the following notation. For a congruence subgroup Γ and $w\in\frac{1}{2}\mathbb{Z}$, let $M_w(\Gamma)$ (resp. $S_w(\Gamma)$) be the space of modular forms (resp. cusp forms) of weight w on Γ.

For a Dirichlet character χ modulo N, let $M_w(\Gamma_0(N),\chi)$ (resp. $S_w(\Gamma_0(N),\chi)$) be the space of modular forms (resp. cusp forms) of weight w on $\Gamma_0(N)$ with character χ.

Let k be a nonnegative integer and $\ell \geq 5$ be a prime. Let K be a number field and O_K be the ring of integers of K. Let v be a prime ideal of O_K over ℓ. Let $M_{k+\frac{1}{2}}(\Gamma_0(4N);\ O_K)$ (resp. $S_{k+\frac{1}{2}}(\Gamma_0(4N);\ O_K)$) be the space of modular forms (resp. cusp forms) of weight $k+\frac{1}{2}$ on $\Gamma_0(4N)$ such that their Fourier coefficients are in O_K and $S^+_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ be the subspace of $S_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ consisting of $f \in S_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$ satisfying the Kohnen-plus condition.

Now, we review the basic notions and properties about the Shimura correspondence. Assume that f is a cusp form of weight $k+\frac{1}{2}$ on $\Gamma_0(4)$. For a square-free integer t, we define $A_t(n)$ by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := \sum_{n=1}^{\infty} \left(\frac{(-1)^k t}{n}\right) \frac{1}{n^{s-k+1}} \sum_{n=1}^{\infty} \frac{a_{tn^2}(f)}{n^s}.$$

Then, the Shimura lift $\mathrm{Sh}_t(f)$ of f is defined by

$$\mathrm{Sh}_t(f)(z) := \sum_{n=1}^{\infty} A_t(n) q^n.$$

Note that $\mathrm{Sh}_t(f) \in S_{2k}(\Gamma_0(2))$. In particular, if $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4))$, then $\mathrm{Sh}_t(f) \in S_{2k}(\Gamma_0(1))$. For each odd prime p with $p \nmid t$, we have

$$\mathrm{Sh}_t\left(f | T_{p^2,k+\frac{1}{2}}\right) = \mathrm{Sh}_t(f) | T_{p,2k},$$

where $T_{n,w}$ denotes the nth Hecke operator on the space of modular forms of weight w. For each prime ℓ, operators U_ℓ and V_ℓ on formal power series are defined by

$$\left(\sum_{n=0}^{\infty} a(n) q^n\right) \Big| U_\ell := \sum_{n=0}^{\infty} a(\ell n) q^n$$

and

$$\left(\sum_{n=0}^{\infty} a(n) q^n\right) \Big| V_\ell := \sum_{n=0}^{\infty} a(n) q^{\ell n}.$$

2.1 Filtration for modular forms of half integral weight modulo a prime ℓ

The theory of filtration for modular forms of integral weight was developed by Serre [9], Swinnerton-Dyer [10], Katz [11], and Gross [12]. From this, the theory of filtration for modular forms of half-integral weight on $\Gamma_0(4)$ was studied. In this section, we review some properties of filtration for modular forms of half-integral weight on $\Gamma_0(4)$. For the details, we refer to [13, Section 2].

We say that $\sum_{n=0}^{\infty} a(n)q^n$ is congruent to $\sum_{n=0}^{\infty} b(n)q^n$ modulo v, i.e.,

$$\sum_{n=0}^{\infty} a(n) q^n \equiv \sum_{n=0}^{\infty} b(n) q^n \pmod{v},$$

if $a(n) \equiv b(n) \pmod{v}$ for all nonnegative integers n. For $f \in M_{k+\frac{1}{2}}(\Gamma_0(4);\ O_K)$, we define a filtration $\omega(f)$ of f modulo v by

$$\omega(f) := \inf\left\{k' + \frac{1}{2} : \quad \text{there is } f' \in M_{k'+\frac{1}{2}}(\Gamma_0(4);\ O_K) \text{ such that } f' \equiv f \pmod{v}\right\}.$$

For convenience, if $f \equiv 0 \pmod{v}$, then let $\omega(f) = -\infty$. We summarize the properties of $\omega(f)$ in the following lemma.

Lemma 2.1. *Let $f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$. Then, the following statements are true.*

(1) $k \equiv \omega(f) - \frac{1}{2} \pmod{\ell - 1}$.

(2) $\omega(f^\ell) = \ell \cdot \omega(f)$.

(3) *There is a nonnegative integer k' such that*

$$k' \equiv k + \frac{\ell - 1}{2} \pmod{\ell - 1},$$

and there is $g \in M_{k'+\frac{1}{2}}(\Gamma_0(4); O_K)$ such that $g \equiv f|U_\ell \pmod{v}$. Moreover, if $f(z) \equiv \sum_{n=0}^{\infty} a_f(\ell n)q^{\ell n} \pmod{v}$, then there is a nonnegative integer k' such that

$$k' \equiv k + \frac{\ell - 1}{2} \pmod{\ell - 1} \text{ and } k' + \frac{1}{2} \leq \frac{1}{\ell}\left(k + \frac{1}{2}\right),$$

and there is $g \in M_{k'+\frac{1}{2}}(\Gamma_0(4); O_K)$ such that $g \equiv f|U_\ell \pmod{v}$.

(4) *There is $h \in S_{k+\ell+\frac{3}{2}}(\Gamma_0(4))$ such that $h \equiv \Theta(f) \pmod{v}$. In particular, if $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4))$, then $h \in S^+_{k+\ell+\frac{3}{2}}(\Gamma_0(4))$.*

Proof. The proofs of (1) and (2) are in [13, Proposition 2.2]. The proof of (3) is obtained by combining [7, Lemma 4.2] and [13, Proposition 2.2]. To prove (4), let

$$h := \left(k + \frac{1}{2}\right)\Theta(E_{\ell-1})f - (\ell - 1)E_{\ell-1}\Theta(f),$$

where $E_{\ell-1}$ denotes the Eisenstein series of weight $\ell - 1$. Since $E_{\ell-1} \equiv 1 \pmod{v}$, we have $h \equiv \Theta(f) \pmod{v}$. By [14, Corollary 7.2], we obtain $h \in S_{k+\ell+\frac{3}{2}}(\Gamma_0(4))$. When f satisfies the Kohnen-plus condition, the proof of (4) is in [7, Lemma 4.1]. □

2.2 Modular forms of half-integral weight such that their Fourier coefficients are supported on finitely many square classes modulo ℓ

In this section, we introduce some properties of modular forms of half-integral weight on $\Gamma_0(4)$ such that their Fourier coefficients are supported on finitely many square classes modulo v.

Ahlgren and Boylan [4] obtained the necessary conditions for the weight of $f \in M_{k+\frac{1}{2}}(\Gamma_0(4))$ such that their Fourier coefficients are supported on finitely many square classes modulo v by using the theory of Galois representations. This was reproved in [15] by using only the theory of filtration for modular forms of integral weight. The Choi and Kilbourn [16] improved the necessary conditions for the weight by using only the theory of filtration for modular forms of integral weight. We review the results [4,16] in the following theorem.

Theorem 2.2. *Let N be a positive integer and $\ell \geq 5$ be a prime with $(\ell, N) = 1$. Assume that $f(z) \in M_{k+\frac{1}{2}}(\Gamma_1(4N)) \cap O_K[[q]]$ has the form*

$$f(z) \equiv a_f(0) + \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2)q^{s_i n^2} \pmod{v}$$

with square-free integers s_i. Let $\overline{k}$ and i_k be nonnegative integers, which satisfy $k = (\ell - 1)i_k + \overline{k}$ and $\overline{k} < \ell - 1$. Then, the following statements are true.

(1) *If $\ell \nmid n_i$ for some i, then*

$$\overline{k} \leq 2i_k + 1.$$

(2) *If* $\ell | n_i$ *for all* i *and* $\overline{k} \le \frac{\ell-3}{2}$, *then*

$$\overline{k} \le i_k - \frac{\ell+1}{2}.$$

(3) *If* $\ell | n_i$ *for all* i *and* $\overline{k} > \frac{\ell-3}{2}$, *then*

$$\overline{k} \le i_k + \frac{\ell-1}{2}.$$

Bruinier and Ono [2, Theorem 3.1] proved the following theorem by using an argument in [1].

Theorem 2.3. *Let* N *be a positive integer and* $\ell \ge 5$ *be a prime with* $(\ell, N) = 1$. *Let* χ *be a real Dirichlet character modulo* $4N$ *and* $f(z) \in S_{k+\frac{1}{2}}(\Gamma_0(4N), \chi) \cap O_K[[q]]$. *For each prime* p *with* $(p, 4N\ell) = 1$, *if there exists* $\varepsilon_p \in \{\pm 1\}$ *such that*

$$f(z) \equiv \sum_{\left(\frac{n}{p}\right) \in \{0, \varepsilon_p\}} a_f(n) q^n \pmod{v},$$

then

$$(p-1) f | T_{p^2, k+\frac{1}{2}} \equiv \varepsilon_p \left(\frac{(-1)^k}{p} \right) \chi(p)(p^k + p^{k-1})(p-1) f \pmod{v}.$$

Ahlgren et al. [7] proved that if $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ and the Fourier coefficients of f are supported on finitely many square classes modulo v, then f has the form

$$f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2) q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2) q^{\ell n^2} \pmod{v}.$$

By using the theory of Galois representations, we extend the result [7] to cusp forms of half-integral weight on $\Gamma_0(4)$ without the Kohnen-plus condition.

Proposition 2.4. *Assume that* $f \in S_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ *has the form*

$$f(z) \equiv \sum_{n=1}^{\infty} \sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v} \tag{2.1}$$

with square-free integers s_i. *Then, the following statements are true.*

(1) *If* $2 | k$ *and* $\ell \equiv 1 \pmod 4$, *then*

$$f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2) q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2) q^{\ell n^2} \pmod{v}.$$

(2) *If* $2 | k$ *and* $\ell \equiv 3 \pmod 4$, *then*

$$f(z) \equiv \sum_{n=1}^{\infty} a_f(n^2) q^{n^2} \pmod{v}.$$

(3) *If* $2 \nmid k$ *and* $\ell \equiv 3 \pmod 4$, *then*

$$f(z) \equiv \sum_{n=1}^{\infty} a_f(\ell n^2) q^{\ell n^2} \pmod{v}.$$

(4) *If* $2 \nmid k$ *and* $\ell \equiv 1 \pmod 4$, *then*

$$f(z) \equiv 0 \pmod{v}.$$

Proof. Assume that for each $i \in \{1,\dots,t\}$, there is a positive integer n_i such that $a_f(s_i n_i^2) \not\equiv 0 \pmod{v}$. Following the proof of Lemma 4.1 in [4], there exist distinct odd primes $p_{i,1},\dots,p_{i,r_i}$, each relatively to $n_i s_i \ell$, and a modular form $f_i \in S_{k+\frac{1}{2}}(\Gamma_0(4\prod_{j=1}^{r_i} p_{i,j}^2); O_K)$ such that

$$f_i(z) \equiv \sum_{\substack{n=1 \\ \gcd\left(n, \prod_{j=1}^{r_i} p_{i,j}\right)=1}}^{\infty} a_{f_i}(s_i n^2) q^{s_i n^2} \not\equiv 0 \pmod{v}.$$

By Theorem 2.3, for each prime p with $p \nmid 2 s_i \ell \prod_{j=1}^{r_i} p_{i,j}$ and $p \not\equiv 1 \pmod{\ell}$, we have

$$f_i | T_{p^2, k+\frac{1}{2}} \equiv \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1}) f_i \pmod{v}.$$

Since $S_{\frac{1}{2}}(\Gamma_0(4)) = S_{\frac{3}{2}}(\Gamma_0(4)) = \{0\}$, we may assume that $k \ge 2$. Let $F_i := \mathrm{Sh}_{s_i}(f_i) \in S_{2k}(\Gamma_0(2\prod_{j=1}^{r_i} p_{i,j}^2))$ be the Shimura lift of f_i. Since the Shimura correspondence commutes with the Hecke operators, for each prime p with $p \nmid 2 s_i \ell \prod_{j=1}^{r_i} p_{i,j}$ and $p \not\equiv 1 \pmod{\ell}$, we obtain

$$F_i | T_{p,2k} \equiv \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1}) F_i \pmod{v}.$$

Then, there is an integer N_i such that $N_i | 2\prod_{j=1}^{r_i} p_{i,j}^2$, and there is a newform $G_i \in S_{2k}(\Gamma_0(N_i))$ such that for each prime p with $p \nmid 2 s_i \ell \prod_{j=1}^{r_i} p_{i,j}$ and $p \not\equiv 1 \pmod{\ell}$,

$$\lambda_i(p) \equiv \left(\frac{(-1)^k s_i}{p}\right)(p^k + p^{k-1}) \pmod{v}.$$

Here, $\lambda_i(p)$ denotes the pth Hecke eigenvalue of G_i. Let $\mathbb{F}_v := O_K/v$. Note that there is a semi-simple Galois representation

$$\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_v),$$

such that for each prime p with $p \nmid N_i \ell$

$$\mathrm{tr}(\rho_i(\mathrm{Frob}_p)) \equiv \lambda_i(p) \pmod{v} \text{ and } \det(\rho_i(\mathrm{Frob}_p)) \equiv p^{2k-1} \pmod{v},$$

where Frob_p denotes any Frobenius element at p. Let $\chi_\ell : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_\ell^*$ be the mod-ℓ cyclotomic character. Following the argument of the proof of [5, Proposition 4.3], we have

$$\rho_i \cong \begin{cases} \begin{pmatrix} \left(\frac{(-1)^k s_i}{\cdot}\right)\chi_\ell^k & 0 \\ 0 & \left(\frac{(-1)^k s_i}{\cdot}\right)\chi_\ell^{k-1} \end{pmatrix} & \text{if } \ell \nmid s_i, \\ \begin{pmatrix} \left(\frac{(-1)^{k+\frac{\ell-1}{2}} s_i'}{\cdot}\right)\chi_\ell^{k+\frac{\ell-1}{2}} & 0 \\ 0 & \left(\frac{(-1)^{k+\frac{\ell-1}{2}} s_i'}{\cdot}\right)\chi_\ell^{k+\frac{\ell-3}{2}} \end{pmatrix} & \text{if } \ell | s_i, \end{cases} \tag{2.2}$$

where $\ell s_i' = s_i$.

By the result of Carayol [17], the conductor of ρ_i divides N_i. By (2.2), we obtain that if $\ell \nmid s_i$, then s_i^2 divides the conductor of ρ_i, and if $\ell | s_i$, then $(s_i')^2$ divides the conductor of ρ_i. Since $N_i | 2\prod_{j=1}^{r_i} p_{i,j}^2$ and $\gcd(s_i, \prod_{j=1}^{r_i} p_{i,j}) = 1$, we have $s_i \in \{1, \ell\}$. Moreover, the conductor of ρ_i is not divided by 4. Therefore, we conclude that if k is odd, then $s_i \neq 1$ and if $k + \frac{\ell-1}{2}$ is odd, then $s_i \neq \ell$. □

We extend Proposition 2.4 to general modular forms of half-integral weight including noncusp forms in the following proposition.

Proposition 2.5. *Assume that $f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form*

$$f(z) \equiv a_f(0) + \sum_{n=1}^{\infty}\sum_{i=1}^{t} a_f(s_i n^2) q^{s_i n^2} \pmod{v} \tag{2.3}$$

with square-free integers s_i. Then,

$$f(z) \equiv a_f(0) + \sum_{n=1}^{\infty} a_f(n^2) q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2) q^{\ell n^2} \pmod{v}.$$

Proof. Without loss of generality, we assume that there is a positive integer n_1 such that $a_f(s_1 n_1^2) \not\equiv 0 \pmod{v}$. Let a be the exponent of ℓ in $s_1 n_1^2$. Then, there is a unique square-free integer s_1' such that $s_1 n_1^2 = \ell^a s_1' m_1^2$ for some positive integer m_1. By Lemma 2.1 (3), there is an integer k' and a modular form $g \in M_{k'+\frac{1}{2}}(\Gamma_0(4))$ such that $g \equiv f|U_{\ell^a} \pmod{v}$. By Lemma 2.1 (4), there is $h \in S_{k'+\ell+\frac{3}{2}}(\Gamma_0(4))$ such that $h \equiv \Theta(g) \pmod{v}$. Since $a_f(s_1 n_1^2) \not\equiv 0 \pmod{v}$, we have $a_h(s_1' m_1^2) \not\equiv 0 \pmod{v}$ and then h has the form (2.1). Then, $s_1' = 1$ by Proposition 2.4. This implies that $s_1 \in \{1, \ell\}$. Therefore, Proposition 2.5 is proved. □

Combining Theorem 2.2 and Proposition 2.5, we obtain an explicit formula of $f \in M_{k+\frac{1}{2}}(\Gamma_0(4))$ having the form (2.3) when $k < \ell - 1$.

Lemma 2.6. *Assume that $f \in M_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form* (2.3) *and $f \not\equiv 0 \pmod{v}$. If $k < \ell - 1$, then $k \in \{0, \frac{\ell-1}{2}\}$. Moreover,*

$$f(z) \equiv a_f(0)\left(1 + 2\sum_{n=1}^{\infty} q^{n^2}\right) \pmod{v} \quad \text{if } k = 0$$

and

$$f(z) \equiv a_f(0)\left(1 + 2\sum_{n=1}^{\infty} q^{\ell n^2}\right) \pmod{v} \quad \text{if } k = \frac{\ell-1}{2}.$$

Proof. We assume that $k < \ell - 1$. By Theorem 2.2, we have $k \in \{0, 1, \frac{\ell-1}{2}\}$. Note that $M_{\frac{1}{2}}(\Gamma_0(4))$ is generated by T. Thus, when $k = 0$, we obtain that f is a constant multiple of T. If f has the form (2.3), then $a_f(2) \equiv 0 \pmod{v}$ by Proposition 2.5. Note that $M_{\frac{3}{2}}(\Gamma_0(4))$ is generated by T^3 and $a_{T^3}(2) = 3$. Thus, when $k = 1$, we have $f \equiv 0 \pmod{v}$. When $k = \frac{\ell-1}{2}$, we have by Theorem 2.2

$$f(z) \equiv \sum_{n=0}^{\infty} a_f(\ell n) q^{\ell n} \pmod{v}.$$

By Lemma 2.1 (3), there is $g \in M_{\frac{1}{2}}(\Gamma_0(4))$ such that $g \equiv f|U_\ell \pmod{v}$. Since g is a constant multiple of T, f is congruent to a constant multiple of $T|V_\ell$ modulo v. □

3 Proof of Theorems

In this section, we prove Theorems 1.1, 1.3, 1.5, and 1.6. First, we prove Theorem 1.3.

Proof of Theorem 1.3. We fix a prime $\ell \geq 5$. We prove Theorem 1.3 by induction on k. When $k < \ell - 1$, Theorem 1.3 is true by Lemma 2.6. Thus, we assume that Theorem 1.3 is true when $k < k_0$ with a fixed positive integer k_0, where k_0 is a positive integer larger than $\ell - 1$.

To prove Theorem 1.3, it is enough to show that Theorem 1.3 is true when $k = k_0$ by induction on k. Assume that $f \in M_{k_0+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form (1.3). Then by Lemma 2.5, f has the form

$$f(z) \equiv a_f(0) + \sum_{n=1}^{\infty} a_f(n^2)q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n^2)q^{\ell n^2} \pmod{v},$$

and

$$\Theta^{(\ell-1)/2}(f)(z) \equiv \frac{1}{2}\left(\sum_{\substack{n\in\mathbb{Z}\\ \ell\nmid n}} a_f(n^2)q^{n^2}\right) \pmod{v}.$$

By Lemma 2.1 (4), there is $g_0 \in S_{k_0+\frac{\ell^2}{2}}(\Gamma_0(4))$ such that

$$g_0 \equiv \Theta^{(\ell-1)/2}(f) \pmod{v}.$$

Let $k_1 := \max\left(k_0 + \frac{1}{2}, \omega(g_0)\right) - \frac{1}{2}$. Then, there is $g_1 \in M_{k_1+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K)$ such that

$$g_1(z) \equiv (f - \Theta^{(\ell-1)/2}(f))(z) \equiv a_f(0) + \sum_{n=1}^{\infty} a_f(\ell n^2)q^{\ell n^2} + \sum_{n=1}^{\infty} a_f(\ell^2 n^2)q^{\ell^2 n^2} \pmod{v}.$$

Let k_2 be the largest integer satisfying

$$k_2 + \frac{1}{2} \le \frac{1}{\ell}\left(k_1 + \frac{1}{2}\right) \text{ and } k_2 \equiv \frac{\ell-1}{2} + k_1 \equiv \frac{\ell-1}{2} + k_0 \pmod{\ell - 1}. \tag{3.1}$$

By Lemma 2.1 (3), there is $g_2 \in M_{k_2+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K)$ such that

$$g_2(z) \equiv g_1|U_\ell(z) \equiv a_f(0) + \sum_{n=1}^{\infty} a_f(\ell n^2)q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell^2 n^2)q^{\ell n^2} \pmod{v}.$$

Since $k_0 > \frac{\ell}{2}$, we have

$$k_2 + \frac{1}{2} \le \frac{1}{\ell}\left(k_1 + \frac{1}{2}\right) \le \frac{1}{\ell}\left(k_0 + \frac{\ell^2}{2}\right) < k_0 + \frac{1}{2}.$$

For a nonnegative integer k, we define a subset $\mathcal{B}_k$ of $M_{k+\frac{1}{2}}(\Gamma_0(4))$ by

$$\mathcal{B}_k := \begin{cases} \{\Theta^{k/2}(T)|V_{\ell^{2i}}\}_{0\le i<\alpha(\ell,k)} \cup \{\Theta^{(2k+\ell-1)/4}(T)|V_{\ell^{2i+1}}\}_{0\le i<\beta(\ell,k)} \cup \{T\} & \text{if } r_\ell(k) = \ell - 1,\\ \{\Theta^{k/2}(T)|V_{\ell^{2i}}\}_{0\le i<\alpha(\ell,k)} \cup \{\Theta^{(2k+\ell-1)/4}(T)|V_{\ell^{2i+1}}\}_{0\le i<\beta(\ell,k)} \cup \{T|V_\ell\} & \text{if } r_\ell(k) = \frac{\ell-1}{2},\\ \{\Theta^{k/2}(T)|V_{\ell^{2i}}\}_{0\le i<\alpha(\ell,k)} \cup \{\Theta^{(2k+\ell-1)/4}(T)|V_{\ell^{2i+1}}\}_{0\le i<\beta(\ell,k)} & \text{otherwise.} \end{cases}$$

To prove Theorem 1.3, it is enough to show that if $f \in M_{k_0+\frac{1}{2}}(\Gamma_0(4); \mathcal{O}_K)$ has the form (1.3), then f is congruent to a linear combination of $\mathcal{B}_{k_0}$ modulo v.

By Proposition 2.4, if k_0 is odd, then $g_0 \equiv 0 \pmod{v}$. Combining the assumption that Conjecture 1.2 is true, we have

$$g_0 \equiv \frac{a_f(1)}{2}\Theta^{k_0/2}(T) \pmod{v}.$$

Since $k_2 \equiv k_0 + \frac{\ell-1}{2} \pmod{\ell - 1}$, it follows that $\Theta^{k_0/2}(T) \equiv \Theta^{(2k_2+\ell-1)/4}(T) \pmod{v}$. By the induction hypothesis, g_2 is congruent to a linear combination of $\mathcal{B}_{k_2}$. Since

$$f \equiv \left(f - \Theta^{\frac{\ell-1}{2}}(f)\right) + \Theta^{\frac{\ell-1}{2}}(f) \equiv g_2|V_\ell + g_0 \pmod{v},$$

we deduce that f is congruent to a linear combination of

$$\begin{cases} \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} \cup \{T|V_\ell\} & \text{if } r_\ell(k_2) = \ell - 1,\\ \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} \cup \left\{T|V_{\ell^2}\right\} & \text{if } r_\ell(k_2) = \frac{\ell-1}{2},\\ \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} & \text{otherwise.} \end{cases}$$

If $r_\ell(k_2) = \frac{\ell-1}{2}$, then

$$T|V_{\ell^2} \equiv T - \Theta^{(\ell-1)/2}(T) \equiv T - \Theta^{(2k_2+\ell-1)/4}(T) \pmod{v}.$$

Thus, f is congruent to a linear combination of

$$\begin{cases} \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} \cup \{T|V_\ell\} & \text{if } r_\ell(k_2) = \ell-1, \\ \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} \cup \{T\} & \text{if } r_\ell(k_2) = \dfrac{\ell-1}{2}, \\ \{\Theta^{k_2/2}(T)|V_{\ell^{2i+1}}\}_{0\le i<\alpha(\ell,k_2)} \cup \{\Theta^{(2k_2+\ell-1)/4}(T)|V_{\ell^{2i}}\}_{0\le i<\beta(\ell,k_2)+1} & \text{otherwise.} \end{cases}$$

To complete the proof, it is sufficient to show that

$$\alpha(\ell, k_2) \le \beta(\ell, k_0) \text{ and } \beta(\ell, k_2) + 1 \le \alpha(\ell, k_0). \tag{3.2}$$

First, we assume that $k_0 + \frac{1}{2} \ge \frac{\ell^2}{2}$. Since $\Theta^m(T) \equiv \Theta^{(2m+\ell-1)/2}(T)$ for any positive integer m, we have $\omega(g_0) \le \omega(\Theta^{k_0/2}(T)) \le \frac{\ell^2}{2}$. This implies that

$$k_1 = \max\left(k_0, \omega(g_0) - \frac{1}{2}\right) = k_0.$$

Then by (3.1), we obtain (3.2).

Now, we assume that $k_0 + \frac{1}{2} < \frac{\ell^2}{2}$. In this case, we have

$$k_2 + \frac{1}{2} \le \frac{1}{\ell}\left(k_1 + \frac{1}{2}\right) \le \frac{1}{\ell} \cdot \max\left(k_0 + \frac{1}{2}, \omega(g_0)\right) \le \frac{\ell}{2}.$$

Further, assume that $k_2 \ne 0$ and $k_2 \ne \frac{\ell-1}{2}$. Then $\alpha(\ell, k_2) = \beta(\ell, k_2) = \beta(\ell, k_0) = 0$. By Lemma 2.6, we have $g_2 \equiv 0 \pmod{v}$, and then

$$f \equiv \Theta^{\frac{\ell-1}{2}}(f) \equiv \frac{a_f(1)}{2}\Theta^{k_0/2}(T) \pmod{v}.$$

Note that $\Theta^{(\ell-1)/2}(T) \equiv T - T^{\ell^2} \pmod{v}$, we have $\omega(\Theta^{(\ell-1)/2}(T)) = \frac{\ell^2}{2}$. Then, for a positive integer m with $m \le \frac{\ell-1}{2}$, we have

$$\omega(\Theta^m(T)) = (\ell+1)m + \frac{1}{2}. \tag{3.3}$$

By (3.3), we have

$$\omega(\Theta^{k_0/2}(T)) = r_\ell(k_0)\cdot \frac{\ell+1}{2} + \frac{1}{2} \le k_0 + \frac{1}{2}.$$

It implies that $\alpha(\ell, k_0) = 1$. Hence, $\alpha(\ell, k_2) = \beta(\ell, k_0)$ and $\beta(\ell, k_2) + 1 = \alpha(\ell, k_0)$. For the cases when $k_0 = 0$ and $k_0 = \frac{\ell-1}{2}$, we obtain (3.2) by direct computation. Thus, we conclude that if $f \in M_{k_0+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form (1.3), then f is congruent to a linear combination of $\mathcal{B}_{k_0}$ modulo v. Therefore, Theorem 1.3 is proved by induction on k. □

To prove Theorem 1.1, we use the following theorem which gives a sufficient condition for the weight $k + \frac{1}{2}$ that Conjecture 1.2 holds for $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$. It was proved in the proof of [7, Theorem 5.2].

Theorem 3.1. *Assume that $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form*

$$f(z) \equiv \frac{1}{2}\sum_{\substack{n\in\mathbb{Z} \\ \ell\nmid n}} a_f(n^2)q^{n^2} \pmod{v} \tag{3.4}$$

and $f \not\equiv 0 \pmod{v}$. Let p_ℓ be the smallest positive prime p such that $p \equiv 1 \pmod{\ell}$. If $2k+1 < p_\ell^2$, then k is even and

$$f \equiv \frac{1}{2}a_f(1)\Theta^{k/2}(T) \pmod{v}.$$

Proof. We follow the proof of [7, Theorem 5.2]. By Proposition 2.4, we obtain that k is even. By Theorem 2.3, for each odd prime p with $p \not\equiv 0, 1 \pmod{\ell}$, we have

$$f|T_{p^2,k+\frac{1}{2}} \equiv (p^k + p^{k-1})f \pmod{v}.$$

Hence, for any positive odd integer m which is not divisible by any prime p with $p \equiv 0, 1 \pmod{v}$, we have

$$a_f(m^2) \equiv a_f(1)m^k \pmod{v}.$$

Let $k_1 := \max\left(k, \frac{r_\ell(k)}{2}(\ell + 1)\right)$. Then, there is $g_1 \in S^+_{k_1+\frac{1}{2}}(\Gamma_0(4); O_K)$ such that

$$g_1 \equiv f - \frac{1}{2}a_f(1)\Theta^{r_\ell(k)/2}(T) \pmod{v}.$$

Let $h := g_1 - g_1|U_4|V_4 \in S^+_{k_1+\frac{1}{2}}(\Gamma_0(16))$. Then, $a_h(n) \equiv 0 \pmod{v}$ for $n < p_\ell^2$. Since

$$\frac{1}{12}\left(k_1 + \frac{1}{2}\right)\cdot[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(16)] = 2k_1 + 1 < p_\ell^2,$$

we have $h \equiv 0 \pmod{v}$ by the result of Sturm [18] called the Sturm bound. Then,

$$g_1(z) \equiv g_1|U_4|V_4(z) \equiv \sum_{m=1}^{\infty} a_{g_1}(4m^2)q^{4m^2} \pmod{v}.$$

From the proof of [7, Theorem 5.2], we have $g_1 \equiv 0 \pmod{v}$. Then,

$$f(z) \equiv \frac{1}{2}a_f(1)\Theta^{k/2}(T)(z) \equiv \frac{1}{2}a_f(1)\left(\sum_{\substack{n\in\mathbb{Z}\\ \ell\nmid n}} n^k q^{n^2}\right) \pmod{v}.$$ □

The following proposition is a refinement of Theorem 1.1.

Proposition 3.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $\sqrt{g(x)}\log x$ is an increasing function and $\lim_{x\to\infty} g(x) = 0$. Let P be a set of primes ℓ such that for every $f \in S^+_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ with $k + \frac{1}{2} < g(\ell)\ell^2(\log\ell)^2$, if f has the form* (1.2), *then*

$$f(z) \equiv \frac{1}{2}\sum_{i=0}^{\alpha(\ell,k)-1'} a_f(\ell^{2i})\Theta^{k/2}(T)(\ell^{2i}z) + \frac{1}{2}\sum_{i=0}^{\beta(\ell,k)-1'} a_f(\ell^{2i+1})\Theta^{(2k+\ell-1)/4}(T)(\ell^{2i+1}z) \pmod{v}.$$

Then, there is an absolute constant C such that

$$\#\{\ell : \ell \notin P \text{ and } \ell \le X\} \le C_0\sqrt{g(X)}\frac{X}{\log X}\left(1 + C\frac{\log\log X}{\log X}\right),$$

where $C_0 := \frac{2\sqrt{2}\pi^2}{3}\prod_{p>2}\frac{p^2}{p^2-1}$.

Proof. Let p_ℓ be the smallest positive prime p with $p \equiv 1 \pmod{\ell}$. By using Theorem 3.1 to follow the proof of Theorem 1.3, we deduce that if $p_\ell^2 > 2g(\ell)\ell^2(\log\ell)^2$, then $\ell \in P$. From this, for a positive number X, we have

$$\#\{\ell : \ell \notin P \text{ and } \ell \le X\} \le \#\{\ell : p_\ell^2 \le 2g(\ell)\ell^2(\log\ell)^2 \text{ and } \ell \le X\}.$$

For convenience, let $h(x) := \sqrt{\frac{g(x)}{2}}$. Then, we have

$$
\begin{aligned}
\#\{\ell : p_\ell^2 \le 2g(\ell)\ell^2(\log\ell)^2 \text{ and } \ell \le X\} &= \#\{\ell : p_\ell \le 2h(\ell)\ell\log\ell \text{ and } \ell \le X\} \\
&\le \sum_{n=1}^{\infty} \#\{\ell : p_\ell = 2n\ell + 1,\, n < h(\ell)\log\ell \text{ and } \ell \le X\} \\
&\le \sum_{n=1}^{\infty} \#\{\ell : p_\ell = 2n\ell + 1,\, n < h(X)\log X \text{ and } \ell \le X\} \\
&\le \sum_{n=1}^{\lfloor h(X)\log X\rfloor} \#\{\ell : p_\ell = 2n\ell + 1 \text{ and } \ell \le X\} \\
&\le \sum_{n=1}^{\lfloor h(X)\log X\rfloor} \#\{\ell : 2n\ell + 1 \text{ is a prime and } \ell \le X\}.
\end{aligned} \tag{3.5}
$$

By [19, Theorem 3.12], for any positive integer n, there is an absolute constant C such that

$$
\#\{\ell : 2n\ell + 1 \text{ is a prime and } \ell \le X\} \le A\left(\prod_{2<p|n} \frac{p-1}{p-2}\right)\frac{X}{(\log X)^2}\left(1 + C\frac{\log\log X}{\log X}\right),
$$

where

$$
A := 8\prod_{2<p}\left(1 - \frac{1}{(p-1)^2}\right).
$$

Note that for any positive integer n, we have

$$
\prod_{2<p|n} \frac{p-1}{p-2} \le \prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)}\prod_{p|n}\frac{p+1}{p} \le \left(\prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)}\right)\frac{\sigma(n)}{n}.
$$

From this, we have

$$
\begin{aligned}
\sum_{n=1}^{\lfloor h(X)\log X\rfloor} \prod_{2<p|n}\frac{p-1}{p-2} &\le \prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)} \sum_{n=1}^{\lfloor h(X)\log X\rfloor} \frac{\sigma(n)}{n} \\
&= \prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)} \sum_{n=1}^{\lfloor h(X)\log X\rfloor} \sum_{d|n}\frac{1}{d} \\
&\le \prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)} \sum_{d=1}^{\lfloor h(X)\log X\rfloor} \frac{1}{d}\cdot\frac{h(X)\log X}{d} \\
&\le \frac{\pi^2}{6}\prod_{2<p}\frac{p(p-1)}{(p+1)(p-2)} h(X)\log X.
\end{aligned}
$$

Thus, (3.5) becomes

$$
\#\{\ell : p_\ell \le 2h(\ell)\ell\log\ell \text{ and } \ell \le X\} \le \frac{4\pi^2}{3}\prod_{2<p}\frac{p^2}{p^2-1}\cdot h(X)\frac{X}{\log X}\left(1 + C\frac{\log\log X}{\log X}\right).
$$

Therefore, we conclude that

$$
\#\{\ell : \ell \notin P \text{ and } \ell \le X\} \le \left(\frac{2\sqrt{2}\pi^2}{3}\prod_{2<p}\frac{p^2}{p^2-1}\right)\cdot\sqrt{g(X)}\frac{X}{\log X}\left(1 + C\frac{\log\log X}{\log X}\right). \qquad \square
$$

By using Proposition 3.2, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $g(x) = (\log x)^{-\varepsilon}$. When $0 \le \varepsilon \le 2$, we obtain Theorem 1.1 by Proposition 3.2. If $\varepsilon > 2$, then there is no prime ℓ satisfying $p_\ell^2 \le 2g(\ell)\ell^2(\log\ell)^2$. Therefore, Theorem 1.1 is proved. $\square$

Now, we prove Theorem 1.5.

Proof of Theorem 1.5. To prove Theorem 1.5, first, we prove that if $\alpha(\ell, k) \ge 1$ and k is even, then $\dim \mathrm{Null}(B_{k,m}) \ge 1$ for any positive integer m. Since $\alpha(\ell, k) \ge 1$, we have

$$\omega(\Theta^{r_\ell(k)/2}(T)) = \frac{r_\ell(k)}{2} \cdot (\ell + 1) + \frac{1}{2} \le k + \frac{1}{2}.$$

Then, there is $h \in M_{k+\frac{1}{2}}(\Gamma_0(4); \mathbb{Z})$ such that $h \equiv \Theta^{r_\ell(k)/2}(T) \pmod{v}$. Let $(c(0), \ldots, c(k/2)) \in \mathbb{Z}^{(k/2)+1}$ such that

$$h = \sum_{j=0}^{k/2} c(j) F_2^j T^{2k+1-4j}.$$

Then, $(\overline{c(0)}, \ldots, \overline{c(k/2)}) \in \mathrm{Null}(B_{k,m})$ for any positive integer m since h has the form

$$h(z) \equiv \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ \ell \nmid n}} a_h(n) q^{n^2} \pmod{v}.$$

Here, $\overline{c(j)}$ is the reduction of $c(j)$ modulo ℓ. Thus, we conclude that $\dim \mathrm{Null}(B_{k,m}) \ge 1$ for any positive integer m, when $\alpha(\ell, k) \ge 1$ and k is even.

Now, we assume that Conjecture 1.2 is true. Let $v := (\overline{v(0)}, \ldots, \overline{v(\lfloor \frac{k}{2} \rfloor)}) \in \mathrm{Null}(B_{k,m})$ for all positive integers m, and let $v(j)$ be an integer such that the reduction of $v(j)$ modulo ℓ is equal to $\overline{v(j)}$. Let

$$f_v := \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} v(j) F_2^j T^{2k+1-4j} \in M_{k+\frac{1}{2}}(\Gamma_0(4)).$$

Then f_v has the form

$$f_v(z) \equiv \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ \ell \nmid n}} a_{f_v}(n^2) q^{n^2} \pmod{v}.$$

Note that $f_v \equiv \Theta^{(\ell-1)/2}(f_v) \pmod{v}$. We assume that k is even. By the assumption that Conjecture 1.2 is true, we have

$$f_v \equiv \frac{a_{f_v}(1)}{2} \Theta^{r_\ell(k)/2}(T) \pmod{v}.$$

Thus, $\lim_{m\to\infty} \dim \mathrm{Null}(B_{k,m})$ is less than or equal to 1. If $\lim_{m\to\infty} \dim \mathrm{Null}(B_{k,m}) = 1$, then there is $f \in S_{k+\frac{1}{2}}(\Gamma_0(4); \mathbb{Z})$ such that

$$f \equiv \Theta^{r_\ell(k)/2}(T) \pmod{v}.$$

This implies that

$$r_\ell(k) \cdot \frac{\ell + 1}{2} + \frac{1}{2} = \omega(\Theta^{r_\ell(k)/2}(T)) \le k + \frac{1}{2}.$$

By the definition of $\alpha(\ell, k)$, we have $\alpha(\ell, k) \ge 1$. Hence, we conclude that Conjecture 1.4 is true.

To complete the proof of Theorem 1.5, we assume that Conjecture 1.4 is true. Further, assume that $f \in S_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ has the form

$$\Theta(f) \equiv \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ \ell \nmid n}} sn^2 a_f(sn^2) q^{sn^2} \pmod{v}$$

with a square-free integer s and $\Theta(f) \not\equiv 0 \pmod{v}$. Then, k is even and $s = 1$ by Proposition 2.4. By Lemma 2.1, there is $f_0 \in S_{k+\ell+\frac{3}{2}}(\Gamma_0(4))$ such that $f_0 \equiv \Theta(f) \pmod{v}$. Let $(d(0), \ldots, d((k + \ell + 1)/2)) \in O_K^{(k+\ell+3)/2}$ satisfying

$$f_0 = \sum_{j=0}^{(k+\ell+1)/2} d(j) F_2^j T^{2k+2\ell+3-4j}.$$

Let $\mathbb{F}_v := O_K/v$. Then, for any positive integer m, we have

$$(\overline{d(0)}, \dots, \overline{d((k+\ell+1)/2)}) \in \mathrm{Null}(B_{k+\ell+1,m}) \otimes_{\mathbb{F}_\ell} \mathbb{F}_v,$$

where $\overline{d(j)}$ is the reduction of $d(j)$ modulo v. By the assumption that Conjecture 1.4 is true, the dimension of $\mathrm{Null}(B_{k+\ell+1,m})$ is 1 for a sufficiently large m. Hence, f_0 is congruent to a constant multiple of $\Theta^{r_\ell(k+\ell+1)/2}(T)$ modulo v. Since $r_\ell(k+\ell+1) = r_\ell(k+2)$, we conclude that $\Theta(f)$ is congruent to a constant multiple of $\Theta^{r_\ell(k+2)/2}(T)$ modulo v. □

We confirm Conjecture 1.2 under the assumption that $k \le 1{,}000$, or that $\ell \in \{5, 7, 11, 13, 17, 19\}$ and $k \le 10{,}000$.

Proof of Theorem 1.6. Note that if $\Theta(f) \equiv 0 \pmod{v}$, then Conjecture 1.2 is true since $a_f(1) \equiv 0 \pmod{v}$. Thus, we may assume that $\Theta(f) \not\equiv 0 \pmod{v}$. By Proposition 2.4, $s = 1$ and k is even. Then, f has the form

$$f(z) \equiv \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ \ell \nmid n}} a_f(n^2) q^{n^2} + \sum_{n=1}^{\infty} a_f(\ell n) q^{\ell n} \pmod{v}.$$

From this, we have

$$(f - \Theta^{(\ell-1)/2}(f))(z) \equiv \sum_{n=1}^{\infty} a_f(\ell n) q^{\ell n} \pmod{v}.$$

Assume that $k < \frac{\ell-1}{2}$. By Lemma 2.1 (3), if $f \not\equiv \Theta^{(\ell-1)/2}(f) \pmod{v}$, then there is a nonnegative integer k_0 such that

$$k_0 \equiv k + \frac{\ell-1}{2} \pmod{\ell - 1} \text{ and } k_0 + \frac{1}{2} \le \frac{1}{\ell}\left(k + \frac{\ell^2}{2}\right),$$

and there is $g_0 \in S_{k_0+\frac{1}{2}}(\Gamma_0(4))$ such that

$$g_0(z) \equiv (f - \Theta^{(\ell-1)/2}(f)) | U_\ell(z) \equiv \sum_{n=1}^{\infty} a_f(\ell n) q^n \pmod{v}.$$

Since $k < \frac{\ell-1}{2}$, we have $k_0 = 0$ and then $g_0 = 0$. Thus, $f \equiv \Theta^{(\ell-1)/2}(f) \pmod{v}$ when $k < \frac{\ell-1}{2}$. This implies that $f \equiv 0 \pmod{v}$ by Lemma 2.6. Hence, we conclude that Conjecture 1.2 is true when $k < \frac{\ell-1}{2}$.

We fix a prime ℓ with $5 \le \ell \le 2001$. Assume that there is $f \in S_{k+\frac{1}{2}}(\Gamma_0(4); O_K)$ having the form

$$\Theta(f)(z) \equiv \frac{1}{2} \sum_{\substack{n \in \mathbb{Z} \\ \ell \nmid n}} n^2 a_f(n^2) q^{n^2} \pmod{v}$$

such that

$$\Theta(f) \not\equiv \frac{a_f(1)}{2} \Theta^{r_\ell(k+2)/2}(T) \pmod{v}.$$

Then, $f \cdot E_{\ell-1} \in S_{k+\ell-\frac{1}{2}}(\Gamma_0(4); O_K)$ satisfies

$$\Theta(f \cdot E_{\ell-1}) \not\equiv \frac{a_f(1)}{2} \Theta^{r_\ell(k+\ell+1)/2}(T) \pmod{v}.$$

Thus, for a positive integer m_0, confirming Conjecture 1.2 for positive integers k such that $k \le m_0$ reduces to confirming Conjecture 1.2 for positive integers k such that $m_0 + 2 - \ell \le k \le m_0$.

When $\max(0, 1{,}002 - \ell) \le k \le 1{,}000$ and k is even, we obtain by numerical method

$$\dim \mathrm{Null}(B_{k,1{,}000}) = \mathbb{1}_{+}(\alpha(\ell, k)).$$

In the proof of Theorem 1.5, we have $\dim \mathrm{Null}(B_{k,m}) \ge \mathbb{1}_{+}(\alpha(\ell, k))$ for any positive integer m. Since $\dim \mathrm{Null}(B_{k,m}) \le \dim \mathrm{Null}(B_{k,1{,}000})$ for $m \ge 1{,}000$, we have

$$\lim_{m\to\infty} \dim \mathrm{Null}(B_{k,m}) = \mathbb{1}_{+}(\alpha(\ell, k))$$

when $\max(0, 1{,}002 - \ell) \le k \le 1{,}000$. By Theorem 1.5, we conclude that Conjecture 1.2 is true when $k \le 1{,}000$.

The proofs for the cases when $\ell \in \{5, 7, 11, 13, 17, 19\}$ and $k \le 10{,}000$ are similar to the proof of the previous case. So, we skip it. □

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