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Research Article

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B-Fredholm elements in primitive C*-algebras

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Abstract: Let $\mathcal{A}$ be a unital primitive C*-algebra. This article studies the properties of the B-Fredholm elements, the B-Weyl elements and the B-Browder elements in $\mathcal{A}$. Particularly, this article describes the B-Fredholm element as the sum of a Fredholm element and a nilpotent element. In addition, the socle of $\mathcal{A}$ is characterized by the B-Fredholm elements.

Keywords: primitive C*-algebra, B-Fredholm elements, B-Browder elements, socle

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1 Introduction

In 1900, Fredholm published his famous article "On a new method for the solution of Dirichlet's problem," which changed the study of the solution of integral equations. In 1918, inspired by Fredholm, Riesz established Fredholm's abstract methods in the form of compact operators, which initiated what is now known as Fredholm theory for operators [1]. Fredholm theory, as an important branch of operator algebra theory, not only has significant applications in solving integral equations but also is the foundation of the K-theory of C*-algebra. Numerous investigations and studies have been developed on Fredholm theory. For example, Aiena studied Fredholm theory and Fredholm spectral theory and gave applications to multipliers and Weyl-type theorems [2,3]. Furthermore, Berkani introduced B-Fredholm operator theory similar to Fredholm theory formulated by Aiena. Meanwhile, the studies of semi B-Fredholm and B-Weyl operators are developed by Berkani and Koliha [4–8]. In addition, some classes of operators evolved by Fredholm operators and B-Fredholm operators theories have been considered by several authors, among which we can mention Berkani, Aiena, Triolo [9], Schmoeger, and so on [10–17].

To extend Fredholm theory in a more abstract setting, Barnes [18] called Fredholm elements of a ring as the elements that are invertible modulo the socle. Subsequently, Männle and Schmoeger argued the Fredholm elements and the generalized Fredholm elements in an unital semisimple Banach algebra. Furthermore, Berkani introduced and studied B-Fredholm elements in a primitive Banach algebra. Studies in this direction have been expanded by Berkani [4–6,19,20] and Grobler et al. [21] and Smyth [22,23–25], involving Fredholm theory with respect to a Banach algebra homomorphism.

It is important to mention that Berkani established a connection between B-Fredholm element and B-Fredholm operator by means of left regular representation in an unital primitive Banach algebra. But there exists a drawback of the left regular representation for a general primitive Banach algebra. Based on Berkani's work, this article follows the same line of research as the articles referenced earlier, but now we

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consider a particular primitive Banach algebra, that is, primitive C*-algebra, which has more useful left regular representation, because in this case, the left regular representation is a faithful irreducible *-representation.

As a continuation and a development of [20], this article describes the B-Fredholm element as the sum of a Fredholm element and a nilpotent element, and B-Weyl element as the sum of a Weyl element and a nilpotent element. Finally, we present an application of B-Fredholm theory in primitive C*-algebra, which provides a new method to characterize the socle of the primitive C*-algebra.

2 Preliminaries

Throughout this article, all algebras will be infinite-dimensional over the field of complex numbers. Suppose that P is an ideal of an algebra $\mathcal{A}$, P is called a primitive ideal if it is the kernel of an irreducible representation of $\mathcal{A}$. Recall that an algebra $\mathcal{A}$ is called primitive if $\{0\}$ is a primitive ideal of $\mathcal{A}$, that is to say, $\mathcal{A}$ possesses a faithful irreducible representation [25, page 29]. Let $B(X)$ (resp. $B(H)$) denote the algebra of bounded linear operators on an infinite dimensional Banach space X (resp. Hilbert space H). It is evident that $B(H)$ is a primitive C*-algebra [26, page 711].

Suppose that $\mathcal{A}$ is a semi-prime algebra, the socle of $\mathcal{A}$ is the algebraic sum of all the minimal left ideals of $\mathcal{A}$ (which equals to the sum of all the minimal right ideals), or $\{0\}$ if $\mathcal{A}$ has no minimal left ideals. Also, the socle of $\mathcal{A}$ (if it exists) denoted by $\mathrm{Soc}(\mathcal{A})$ is an ideal in $\mathcal{A}$ [2, page 245]. Recall also that it is well known that a primitive Banach algebra is a semi-prime algebra [20, page 6].

Definition 1. [2, page 244] Let $\mathcal{A}$ be any complex algebra. e_0 is called a minimal idempotent element if $e_0 \neq 0$ and $e_0^2 = e_0$ such that $e_0\mathcal{A}e_0$ is a division algebra. Denote the set of all minimal idempotents of $\mathcal{A}$ by $\mathrm{Min}(\mathcal{A})$.

Next, we review the concepts of the Fredholm elements and the generalized Fredholm elements in a semisimple Banach algebra.

Definition 2. [24, page 5] Suppose that $\mathcal{A}$ is a unital semisimple Banach algebra. The element $a \in \mathcal{A}$ is called a Fredholm element if a is invertible modulo $\mathrm{Soc}(\mathcal{A})$. In other words, $a + \mathrm{Soc}(\mathcal{A})$ is an invertible element in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. The set of the Fredholm elements is denoted by $\Phi(\mathcal{A})$.

Definition 3. [24, page 8] Assume that $\mathcal{A}$ is a semisimple Banach algebra with a unit e. An element $a \in \mathcal{A}$ is called a generalized Fredholm element if there exists $b \in \mathcal{A}$ such that $aba = a$ and $e - ab - ba$ is a Fredholm element. The set of the generalized Fredholm elements is labeled as $\Phi_g(\mathcal{A})$.

Recall that an element $a \in \mathcal{A}$ is said to be Drazin invertible if there exists a unique $b \in \mathcal{A}$ and some $k \in \mathbf{N}$ such that $bab = b$, $ab = ba$, $a^k ba = a^k$ [27, page 3730].

The quotient algebra $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$ is written as $\hat{\mathcal{A}}$. Evidently, $\hat{\mathcal{A}}$ is not a Banach algebra since $\mathrm{Soc}(\mathcal{A})$ is not closed. For $a \in \mathcal{A}$, we denote the canonical homomorphism:

$$\begin{aligned} \pi : \mathcal{A} &\to \hat{\mathcal{A}} \\ a &\mapsto a + \mathrm{Soc}(\mathcal{A}) \end{aligned}$$

and write $\hat{a} = a + \mathrm{Soc}(\mathcal{A})$ for the coset of a in $\hat{\mathcal{A}}$. For any $a, b \in \mathcal{A}$, denote $ab - ba$ by $[a, b]$.

Definition 4. [20, Definition 1.2] Let $\mathcal{A}$ be a unital semi-prime Banach algebra. An element $a \in \mathcal{A}$ is called a B-Fredholm element of $\mathcal{A}$ if a is Drazin invertible modulo $\mathrm{Soc}(\mathcal{A})$. In other words, $\hat{a}$ is Drazin invertible in $\hat{\mathcal{A}}$.

Definition 5. [6, Definition 2.1] Let I be an ideal in a Banach algebra $\mathcal{A}$. A function $\tau : I \to \mathbb{C}$ is called a trace on I if
(1) $\tau(p) = 1$ if $p \in I$ is an idempotent of rank one.
(2) $\tau(a + b) = \tau(a) + \tau(b)$ for all $a, b \in I$.
(3) $\tau(\alpha a) = \alpha\tau(a)$ for all $\alpha \in \mathbb{C}$ and $a \in I$.
(4) $\tau(ab) = \tau(ba)$ for all $a \in I$ and $b \in \mathcal{A}$.

Definition 6. [6, Definition 2.2] Let τ be a trace on $\mathrm{Soc}(\mathcal{A})$ of a unital primitive Banach algebra $\mathcal{A}$. The index of a B-Fredholm element $a \in \mathcal{A}$ is defined by

$$i(a) = \tau(aa_0 - a_0a) = \tau[a, a_0],$$

where a_0 is a Drazin inverse of a modulo the socle of $\mathcal{A}$.

From [6, Theorem 2.3], the index of a B-Fredholm element $a \in \mathcal{A}$ is well defined and is independent of the Drazin inverse a_0 of a modulo the ideal $\mathrm{Soc}(\mathcal{A})$. Since invertible elements are always Drazin invertible, it follows immediately that Fredholm elements are B-Fredholm elements. In the following, the B-Weyl elements will be presented.

Definition 7. [20, Definition 3.3] Suppose that $\mathcal{A}$ is a unital primitive Banach algebra and $a \in \mathcal{A}$. If the element a is a B-Fredholm element of index 0, then it is called a B-Weyl element.

From now on, we always assume that $\mathcal{A}$ is a unital infinite dimensional primitive C*-algebra if there are no special instructions. For an element a in a primitive C*-algebra $\mathcal{A}$, the rest of this section aims to recall the nullity, defect, ascent, and descent of a. Evidently, a primitive Banach algebra must be semisimple [25, page 29]. For $a \in \mathcal{A}$, set

$$R(a) = \{x \in \mathcal{A} : ax = 0\}, \quad L(a) = \{x \in \mathcal{A} : xa = 0\}.$$

Suppose that $J \subseteq \mathcal{A}$ is a right(resp. left) ideal of $\mathcal{A}$. J is called having finite order if it can be written as the sum of a finite number of minimal right(resp. left) ideals of $\mathcal{A}$. The order $\Theta(J)$ of J is defined to be the smallest number of minimal right(resp. left) ideals satisfying the condition that the sum of them equals to J. Set $\Theta(\{0\}) = 0$, and $\Theta(J) = +\infty$ if J does not have finite order [18, page 84].

Definition 8. [24, page 5] Suppose that $\mathcal{A}$ is a semisimple Banach algebra. For $a \in \mathcal{A}$, the nullity of a is defined by

$$\mathrm{nul}(a) = \Theta(R(a)),$$

and the defect of a is defined by

$$\mathrm{def}(a) = \Theta(L(a)).$$

An element $a \in \mathcal{A}$ is called relatively regular if $aba = a$ for some $b \in \mathcal{A}$. In this case, b is called a pseudo-inverse of a. From [24, Proposition 3.5], it follows that $a \in \mathcal{A}$ is a Fredholm element if and only if it is relatively regular and $\mathrm{nul}(a) < \infty$, $\mathrm{def}(a) < \infty$. An element $a \in \mathcal{A}$ is called upper semi-Fredholm element if it is a left invertible element modulo $\mathrm{Soc}(\mathcal{A})$. The set of these elements is denoted by Φ_+. An element $a \in \mathcal{A}$ is called lower semi-Fredholm element if it is a right invertible element modulo $\mathrm{Soc}(\mathcal{A})$. The set of these elements is denoted by Φ_-. Similarly, $a \in \mathcal{A}$ is an upper (resp. a lower) semi-Fredholm element if and only if it is relatively regular and $\mathrm{nul}(a) < \infty$ (resp. $\mathrm{def}(a) < \infty$) [24, Proposition 3.5]. An element is called a semi-Fredholm element if it is an upper semi-Fredholm element or lower semi-Fredholm element. It is necessary to recall the definitions of the ascent and descent of an element. Let $T \in B(X)$, where X is a Banach space, and we already acquainted with the ascent of T denoted by $\alpha(T)$, the descent of T denoted by $\beta(T)$, and the index of T denoted by $\mathrm{ind}(T)$.

Suppose that $\mathcal{A}$ is a unital primitive C*-algebra. Let $a \in \mathcal{A}$, and let the linear operator $L_a : \mathcal{A}p \to \mathcal{A}p$ be defined by $L_a(x) = ax$ for any $x \in \mathcal{A}p$, where p is the minimal idempotent in $\mathcal{A}$. Put

$$p_l(a) = \alpha(L_a) \quad \text{and} \quad q_l(a) = \beta(L_a).$$

We call $p_l(a)$ the ascent of a and $q_l(a)$ the descent of a.

Associated with Definition 2 and Definition 4, $\Phi(\mathcal{A})$ equals to the set of invertible elements in $\mathcal{A}$ when $\mathrm{Soc}(\mathcal{A}) = \{0\}$. The set of all B-Fredholm elements is equal to the set of Drazin invertible elements in $\mathcal{A}$ when $\mathrm{Soc}(\mathcal{A}) = \{0\}$. To avoid appearing that particular and trivial circumstance, we will assume that the socle of $\mathcal{A}$ is not reduced to $\{0\}$, and so in this case, $\mathcal{A}$ possesses minimal idempotents [25]. Let p be a minimal idempotent in $\mathcal{A}$. The set of bounded linear operators on $\mathcal{A}p$ is denoted by $B(\mathcal{A}p)$. The left regular representation of $\mathcal{A}$ on the Banach space $\mathcal{A}p$ is defined by $\Gamma : \mathcal{A} \to B(\mathcal{A}p)$, such that $\Gamma(a) = L_a$ for any $a \in \mathcal{A}$. Recall that

$$\Gamma(\mathrm{Soc}(\mathcal{A})) = F(\mathcal{A}p) \subseteq B(\mathcal{A}p)$$

by [25, Theorem F.4.3], where $F(\mathcal{A}p)$ denotes the set of all finite rank operators on $\mathcal{A}p$. From [25, F.2.1], it follows that the nullity and defect of the operator $L_a \in B(\mathcal{A}p)$ are independent of the choice of $p \in \mathrm{Min}(\mathcal{A})$.

In [20], Berkani established a connection between B-Fredholm element and B-Fredholm operator. In detail, for a primitive Banach algebra $\mathcal{A}$, let p be a minimal idempotent in $\mathcal{A}$, if $a \in \mathcal{A}$ is a B-Fredholm element, then the left multiplication operator L_a is a B-Fredholm operator, where L_a is defined by $L_a : x \in \mathcal{A}p \to ax \in \mathcal{A}p$. But the converse is not true in general from [25]. This exhibits a drawback of the left regular representation $\Gamma : a \to L_a$ for a general primitive Banach algebra. However, if $\mathcal{A}$ is a primitive C*-algebra, the left regular representation is more useful, which is an isometric faithful irreducible *-representation [28]. One can see that $a \in \mathcal{A}$ is a Fredholm (resp. B-Fredholm) element if and only if L_a is a Fredholm (resp. B-Fredholm) operator on $\mathcal{A}p$ when $\mathcal{A}$ is a primitive C*-algebra, which is the reason of considering the primitive C*-algebra in this article.

A multitude of indispensable notations and concepts have been listed. In the following, the properties of the B-Fredholm elements, the B-Weyl elements, and the B-Browder elements are presented in a primitive C*-algebra.

3 Properties of B-Fredholm elements, B-Weyl elements, and B-Browder elements

In this section, we demonstrate the properties of B-Fredholm elements, B-Weyl elements, and B-Browder elements in a primitive C*-algebra $\mathcal{A}$.

3.1 Properties of B-Fredholm elements and generalized Fredholm elements

The section discusses the algebraic properties of the set of all B-Fredholm elements in a primitive C*-algebra $\mathcal{A}$. Meanwhile, similar to the decompositions of B-Fredholm operators and the generalized Fredholm operators given by [4], this section describes the B-Fredholm element as the sum of a Fredholm element in $\mathcal{A}$ and a nilpotent element in $\mathcal{A}$, and the generalized Fredholm element as the sum of a Fredholm element in $\mathcal{A}$ and a nilpotent element in $\mathrm{Soc}(\mathcal{A})$. In addition, it also establishes the relation between the Fredholm elements and the generalized Fredholm elements.

In 1999, Berkani provided the decomposition of the B-Fredholm operator as the direct sum of a Fredholm operator and a nilpotent operator [4, Theorem 2.7]. Motivated by Berkani, in the following, we describe the B-Fredholm element as the sum of a Fredholm element and a nilpotent element.

Suppose that $\mathcal{A}$ is a unital primitive C$*$-algebra, recall that an element $a \in \mathcal{A}$ is called a nilpotent element if there exists $k \in \mathbf{N}^*$ such that $a^k = 0$, denote the set of all nilpotent elements in $\mathcal{A}$ by $\mathcal{A}^N$. $N(\mathcal{A}p)$ means the set of all nilpotent operators in $B(\mathcal{A}p)$, where p is the minimal idempotent in $\mathcal{A}$. According to [28, page 903], Γ is an isometric irreducible $*$-representation of primitive C$*$-algebra $\mathcal{A}$. Some necessary lemmas will be stated as follows.

Lemma 1. [20, Example 3.11] *If $\mathcal{A}$ is a unital primitive C$*$-algebra, then $a \in \mathcal{A}$ is a Fredholm (B-Fredholm) element if and only if L_a is a Fredholm (B-Fredholm) operator.*

Theorem 1. *Suppose that $\mathcal{A}$ is a unital primitive C$*$-algebra with $\Gamma(\mathcal{A}^N) \supseteq N(\mathcal{A}p)$. If $a \in \mathcal{A}$ is a B-Fredholm element, then there exist $b \in \Phi(\mathcal{A})$ and $c \in \mathcal{A}^N$ such that $a = b + c$.*

Proof. If $a \in \mathcal{A}$ is a B-Fredholm element, then L_a is a B-Fredholm operator on $\mathcal{A}p$ from Lemma 1. Associated with Theorem 2.7 in [4], there exist a Fredholm operator S in $B(\mathcal{A}p)$ and a nilpotent operator F in $B(\mathcal{A}p)$ such that $L_a = S + F$. It follows that there exists $c \in \mathcal{A}^N$ such that $L_c = F$, and hence, $S = L_{a-c}$. Since S is a Fredholm operator, it follows that $a - c$ is a Fredholm element. Set $b = a - c$, then $b \in \Phi(\mathcal{A})$ and $c \in \mathcal{A}^N$, and it is clear that $a = b + c$. This completes the proof. □

Inspired by the operator situation [14, Proposition 1.2, Corollary 1.3], some properties of the B-Fredholm elements are considered.

Proposition 2. *Suppose that $\mathcal{A}$ is a primitive Banach algebra. If $a \in \mathcal{A}$ is a B-Fredholm element and if $n \in \mathbf{N}^*$, then a^n is a B-Fredholm element and $i(a^n) = n \cdot i(a)$.*

Proof. If $a \in \mathcal{A}$ is a B-Fredholm element, then $\pi(a)$ is a Drazin invertible element in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. From [7, Proposition 2.6], it follows that

$$\pi(a)^n = \pi(a^n) \text{ is Drazin invertible},$$

which implies a^n is a B-Fredholm element. From Lemma 1, one can obtain that L_{a^n} is a B-Fredholm operator. Applying [6, Lemma 3.2] and [14, Proposition 1.2], one can prove that

$$i(a^n) = \mathrm{ind}\big(L_{a^n}\big) = \mathrm{ind}(L_a)^n = n \cdot \mathrm{ind}(L_a),$$

because L_a is a B-Fredholm operator. Therefore, $i(a^n) = n \cdot \mathrm{ind}(L_a) = n \cdot i(a)$. □

Corollary 1. *Let $\mathcal{A}$ be a unital primitive Banach algebra, $a \in \mathcal{A}$, and let*

$$f(x) = (x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \cdots (x - \lambda_m)^{n_m},$$

where $\lambda_i \in \mathbf{C}$ for $i = 1, 2, \ldots, m$. If $a - \lambda_i$ is a B-Fredholm element for any $i = 1, 2, \ldots, m$, then $f(a) = (a - \lambda_1)^{n_1}(a - \lambda_2)^{n_2} \cdots (a - \lambda_m)^{n_m}$ is a B-Fredholm element and

$$i(f(a)) = \sum_{i=1}^{m} n_i \cdot i(a - \lambda_i).$$

Proof. Let $g(x) = (x - \lambda_j)^{n_j}$, where $1 \le j \le m$, $h(x) = (x - \lambda_1)^{n_1}(x - \lambda_2)^{n_2} \cdots (x - \lambda_{j-1})^{n_{j-1}}(x - \lambda_{j+1})^{n_{j+1}}(x - \lambda_{j+2})^{n_{j+2}} \cdots (x - \lambda_m)^{n_m}$. It is evident that $g(x)$ and $h(x)$ are prime each other. Hence, there exist two polynomials $u(x)$ and $v(x)$ such that

$$u(x)g(x) + v(x)h(x) = 1.$$

It implies that $u(a)g(a) + v(a)h(a) = e$, where e is the identity. And one can see that $u(a)$, $g(a)$, $v(a)$, and $h(a)$ are two by two commuting elements in $\mathcal{A}$. Associated with [6, Proposition 3.3], it can be proved that $g(a)h(a) = f(a)$ is a B-Fredholm element and

$$i(f(a)) = i(g(a)) + i(h(a)) = n_j \cdot i(a - \lambda_j) + i(h(a)) = \sum_{i=1}^{m} n_i \cdot i(a - \lambda_i).$$ □

Corollary 2. *Suppose that $\mathcal{A}$ is a unital primitive C^*-algebra and $a \in \mathcal{A}$. The following statements are equivalent*:
(1) *a is a B-Fredholm element.*
(2) *a^m is a B-Fredholm element for each integer $m \in \mathbf{N}^*$.*
(3) *a^m is a B-Fredholm element for some integer $m \in \mathbf{N}^*$.*

Proof. (1) ⇒ (2): According to Proposition 2, it is clear to prove that.
(2) ⇒ (3): Clearly.
(3) ⇒ (1): If a^m is a B-Fredholm element for some $m \in \mathbf{N}^*$, then it follows that $(L_a)^m$ is a B-Fredholm operator from [20, Theorem 3.6]. Since $L_a \in B(\mathcal{A}p)$, one can obtain that L_a is a B-Fredholm operator from [4, Proposition 2.8]. Applying Lemma 1, it is clear that a is a B-Fredholm element. □

Remark 1. Suppose that $\mathcal{A}$ is a unital primitive C∗-algebra and $a \in \mathcal{A}$. If there exists $n \in \mathbf{N}^*$ such that a^n is a generalized Fredholm element, then a must be a B-Fredholm element. Indeed, if there exists n such that a^n is a generalized Fredholm element, then $(L_a)^n$ is a generalized Fredholm operator. Associated with [7, Proposition 3.3], one can obtain that L_a is a B-Fredholm operator, which implies that a is a B-Fredholm element from Lemma 1.

Recall that an element a in $\mathcal{A}$ is called a generalized invertible element if a is relatively regular, and for some pseudo-inverse b of a, the element $e - ab - ba$ is invertible in $\mathcal{A}$ [24, page 10]. The following aims to study the relation between the B-Fredholm elements and the generalized Fredholm elements, so let us give a proposition for that.

Proposition 3. *If $\mathcal{A}$ is a primitive C∗-algebra and $a \in \mathcal{A}$, then a is a generalized Fredholm element if and only if there exist $b \in \Phi(\mathcal{A})$ and $c \in \mathrm{Soc}(\mathcal{A}) \bigcap \mathcal{A}^N$ such that $a = b + c$.*

Proof. If there exist $b \in \Phi(\mathcal{A})$ and $c \in \mathrm{Soc}(\mathcal{A}) \bigcap \mathcal{A}^N$ such that $a = b + c$, then $b + c$ is a Fredholm element. Since the Fredholm elements must be the generalized Fredholm elements, one can see that $a = b + c$ is a generalized Fredholm element.

Now we prove the other direction. If a is a generalized Fredholm element, then there exists $b \in \mathcal{A}$ such that $aba = a$ and $e - ab - ba$ is a Fredholm element, where e is the unit. Due to $aba = a$, then $L_{aba} = L_a$, which means that $L_a L_b L_a = L_a$; hence, L_a is relatively regular. Because $L_{e-ab-ba}$ is a Fredholm operator, it is clear that $L_e - L_a L_b - L_b L_a$ is a Fredholm operator in $B(\mathcal{A}p)$. In other words, L_a is a generalized Fredholm operator on $\mathcal{A}p$. It follows from [29, Theorem 1.1] that there exist a Fredholm operator T in $B(\mathcal{A}p)$, and a finite rank operator $S \in N(\mathcal{A}p)$ such that the relation $L_a = S + T$ holds. Since $\mathcal{A}$ is a primitive C∗-algebra [28, page 903], there exists $s \in \mathrm{Soc}(\mathcal{A})$ such that $S = L_s$. Associated with the fact that S is a nilpotent operator on $\mathcal{A}p$, there exists n such that $S^n = 0$, that is to say, $(L_s)^n = 0$, and hence, $L_{s^n} = 0$. Note that $\mathcal{A}$ is a primitive C∗-algebra, which implies that the left regular representation of $\mathcal{A}$ is isometric [28, page 903]. It follows that $s^n = 0$, which means $s \in \mathrm{Soc}(\mathcal{A}) \bigcap \mathcal{A}^N$. In this case, notice that $L_a = L_s + T$, so one can see that $T = L_{a-s}$ is a Freholm operator. From Lemma 1, $a - s$ is a Fredholm element in $\mathcal{A}$. Consequently, it is not difficult to see that

$$a = s + (a - s), \quad \text{where } s \in \mathrm{Soc}(\mathcal{A}) \cap \mathcal{A}^N, \quad a - s \in \Phi(\mathcal{A}).$$

Considering $a - s = b$ and $s = c$, the proof is completed. □

Proposition 4. *If $\mathcal{A}$ is a primitive C∗-algebra with a unit e, then the following statements are equivalent*:
(1) *$a \in \mathcal{A}$ is a B-Fredholm element.*
(2) *There exists an integer $k \in \mathbf{N}^*$ such that a^k is a generalized Fredholm element.*

Proof. (1) $\Rightarrow$ (2). If a is a B-Fredholm element, then $\hat{a}$ is Drazin invertible in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. In other words, there exist $b \in \mathcal{A}$, $k \in \mathbf{N}^*$ such that

$$\hat{b}\hat{a}\hat{b} = \hat{b}, \quad \hat{a}\hat{b} = \hat{b}\hat{a}, \quad (\hat{a})^{k+1}\hat{b} = (\hat{a})^k.$$

It follows that $(\hat{a})^k(\hat{b})^k = (\hat{b})^k(\hat{a})^k$, $(\hat{b})^k(\hat{a})^k(\hat{b})^k = (\hat{b})^k$. And one can check that $(\hat{a})^k(\hat{b})^k(\hat{a})^k = (\hat{a})^k$. If $c = b^k$, then $\hat{e} - (\hat{a})^k\hat{c} - \hat{c}(\hat{a})^k = \hat{e}$ is invertible in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. It leads to the conclusion that $(\hat{a})^k$ is a generalized invertible element in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. In other words, $\widehat{a^k}$ is a generalized invertible element in $\mathcal{A}/\mathrm{Soc}(\mathcal{A})$. Hence, a^k is a generalized Fredholm element [24, page 11].

(2) $\Rightarrow$ (1). Similar to the proof in [20, Theorem 2.6], it is easy to prove this direction. □

The aforementioned proof has demonstrated the properties of the B-Fredholm elements. Subsequently, we will discuss the properties of the B-Weyl elements.

3.2 Properties of B-Weyl elements

This section aims to provide a characterization of the B-Weyl elements in a primitive C∗-algebra. Meanwhile, similar to the decomposition of B-Weyl operators given by [5, Lemma 4.1], this section also describes the B-Weyl element as the sum of a Weyl element and a nilpotent element.

Proposition 5. *Suppose that $\mathcal{A}$ is a unital primitive C∗-algebra satisfying $\Gamma(\mathcal{A}^N) \supseteq N(\mathcal{A}p)$. If $a \in \mathcal{A}$ is a B-Weyl element, then there exist a Weyl element $b \in \mathcal{A}$ and $c \in \mathcal{A}^N$ such that $a = b + c$.*

Proof. If $a \in \mathcal{A}$ is a B-Weyl element, then it is a B-Fredholm element. From Lemma 1, it follows that L_a is a B-Fredholm operator and $\mathrm{ind}(L_a) = i(a) = 0$. Therefore, L_a is a B-Weyl operator, namely, a B-Fredholm operator with index 0. It follows from [5, Lemma 4.1] that there exist a Weyl operator S and a nilpotent operator F such that $L_a = S + F$. It is evident that $F \in N(\mathcal{A}p)$, which implies that $F \in \Gamma(\mathcal{A}^N)$ by using $\Gamma(\mathcal{A}^N) \supseteq N(\mathcal{A}p)$. Hence, there exists $t \in \mathcal{A}^N$ such that $L_t = F$. Therefore, $L_a = S + L_t$. This leads to the conclusion that $S = L_{a-t}$ is a Weyl operator, which implies that $S = L_{a-t}$ is a Fredholm operator. Applying Lemma 1, $a - t$ is a Fredholm element. Associated with [6, Lemma 3.2], it is evident that

$$i(a - t) = \mathrm{ind}(L_{a-t}) = 0.$$

Hence, $a - t$ is a Weyl element. Since t is a nilpotent element, $a - t$ is a Weyl element and $a = (a - t) + t$. If $b = a - t$, $c = t$, then it is clear that $a = b + c$. This completes the proof. □

Remark 2. If $\mathcal{A}$ is a unital primitive C∗-algebra and $a \in \mathcal{A}$, then a is a B-Weyl element if and only if there exist a Drazin invertible element $b \in \mathcal{A}$ and $c \in \mathrm{Soc}(\mathcal{A})$ such that $a = b + c$. Notice that $\Gamma(\mathcal{A})$ is Drazin inverse closed in $B(\mathcal{A}p)$ and

$$\Gamma(\mathrm{Soc}(\mathcal{A})) = F(\mathcal{A}p),$$

[28, page 903]. Then one can prove from [20, Theorem 3.4] that the element $a \in \mathcal{A}$ is a B-Weyl element if and only if there exist a Drazin invertible element $b \in \mathcal{A}$ and $c \in \mathrm{Soc}(\mathcal{A})$ such that $a = b + c$.

Suppose that $K \subseteq \mathbf{C}$, denote iso$K$ by the set of all isolated points in K. $\sigma(a)$ means the spectrum of an element $a \in \mathcal{A}$. Assume that $\mathcal{A}$ is a Banach algebra with a unit e, $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ containing e, and $a \in \mathcal{A}$. We call the element a invertible in $\mathcal{A}$ (resp. in $\mathcal{B}$) if there exists an element $b \in \mathcal{A}$ (resp. $b \in \mathcal{B}$) such that $ab = ba = e$. Evidently, if a is invertible in $\mathcal{B}$, then it must be invertible in $\mathcal{A}$. But, if a is invertible in $\mathcal{A}$, we cannot deduce that a is invertible in $\mathcal{B}$. Notice that $\Gamma(\mathcal{A}) \subseteq B(\mathcal{A}p)$. For $L_a \in \Gamma(\mathcal{A})$, where $a \in \mathcal{A}$, we call that $\Gamma(\mathcal{A})$ is inverse closed (resp. Drazin inverse closed) in $B(\mathcal{A}p)$ if L_a is invertible (resp. Drazin invertible) in $\Gamma(\mathcal{A})$ when L_a is invertible (resp. Drazin invertible) in $B(\mathcal{A}p)$. In the following, an equivalent description of the B-Weyl elements is illustrated.

Proposition 6. *Suppose that $\mathcal{A}$ is a unital primitive $C*$-algebra and $a \in \mathcal{A}$. If $0 \in \operatorname{iso}\sigma(a)$, then a is a B-Weyl element if and only if a is Drazin invertible.*

Proof. Notice that Γ is a isometric homomorphism and $\Gamma(\mathcal{A})$ is inverse closed in $B(\mathcal{A}p)$ since $\mathcal{A}$ is a primitive C*-algebra [25, Theorem F.4.3], one can show that if $0 \in \operatorname{iso}\sigma(a)$, then $0 \in \operatorname{iso}\sigma(L_a)$. If a is a B-Weyl element, from Lemma 1 and [6, Lemma 3.2], it can be indicated that L_a is a B-Weyl operator. Applying [5, Theorem 4.2], it follows that L_a is Drazin invertible. Hence, a is Drazin invertible because Γ is injective and $\Gamma(\mathcal{A})$ is Drazin inverse closed in $B(\mathcal{A}p)$.

Conversely, if a is Drazin invertible, then L_a is also Drazin invertible. This leads to the conclusion that L_a is a B-Weyl operator [5, Theorem 4.2]. Therefore, a is a B-Weyl element because $\mathcal{A}$ is a primitive C*-algebra [20, page 9]. □

The following example aims to illustrate that the condition "$0 \in \operatorname{iso}\sigma(a)$" in Proposition 6 is essential.

Example 1. Let $T_1, T_2 \in B(l^2)$ be given by

$$T_1(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, \ldots),$$
$$T_2(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

Let $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$. One can calculate that $\operatorname{ind}(T) = 0$, $0 \notin \operatorname{iso}\sigma(T)$ and T is a Weyl operator. But the ascent and descent of T are not finite, which implies that T is not Drazin invertible. However, T is a B-Weyl operator because it is a Weyl operator.

3.3 Properties of B-Browder elements

Suppose that $\mathcal{A}$ is a primitive C*-algebra and p is a minimal idempotent in $\mathcal{A}$. For $a \in \mathcal{A}$, recall that L_a is defined by $L_a : x \in \mathcal{A}p \to ax \in \mathcal{A}p$. This section aims to give the characterizations of the B-Browder elements and the B-Browder spectrum. Meanwhile, the relation between the B-Browder element $a \in \mathcal{A}$ and the B-Browder operator L_a on $\mathcal{A}p$ is established.

Definition 9. Assume that $\mathcal{A}$ is a unital semisimple Banach algebra. We say that $a \in \mathcal{A}$ is a B-Browder element if there exist a Drazin invertible element $b \in \mathcal{A}$, and $c \in \operatorname{Soc}(\mathcal{A})$ such that $bc = cb$ and $a = b + c$.

Definition 10. Suppose that $a \in \mathcal{A}$. The Drazin spectrum of a, the B-Fredholm spectum of a, and the B-Browder spectrum of a are defined by:

$$\sigma_D(a) = \{\lambda \in \mathbf{C} : a - \lambda e \text{ is not a Drazin invertible element}\};$$
$$\sigma_{BF}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \text{ is not a B-Fredholm element}\};$$
$$\sigma_{BB}(a) = \{\lambda \in \mathbf{C} : a - \lambda e \text{ is not a B-Browder element}\},$$

respectively, where e is the identity. Note that $a - \lambda e$ is always abbreviated to $a - \lambda$.

Correspondingly, set $\rho_D(a) = \mathbf{C}\backslash\sigma_D(a)$, $\rho_{BF}(a) = \mathbf{C}\backslash\sigma_{BF}(a)$, and $\rho_{BB}(a) = \mathbf{C}\backslash\sigma_{BB}(a)$.

Next, an equivalent description of the B-Browder elements will be illustrated. It is necessary to give the following proposition.

Proposition 7. *If $\mathcal{A}$ is a unital primitive $C*$-algebra and $a \in \mathcal{A}$, then $\sigma_{BB}(a) = \bigcap_{b\in\operatorname{Soc}(\mathcal{A}), ab=ba}\sigma_D(a + b)$.*

Proof. First, we prove that $\sigma_{BB}(a) \subseteq \bigcap_{b\in\operatorname{Soc}(\mathcal{A}), ab=ba}\sigma_D(a + b)$. It suffices to prove that

$$\text{if } 0 \notin \bigcap_{b\in\operatorname{Soc}(\mathcal{A}), ab=ba}\sigma_D(a + b), \text{ then } 0 \notin \sigma_{BB}(a).$$

If $0 \notin \bigcap_{b\in \mathrm{Soc}(\mathcal{A}),ab=ba}\sigma_D(a+b)$, then there exists $b_0 \in \mathrm{Soc}(\mathcal{A})$ such that $b_0a = ab_0$ and $a + b_0$ is a Drazin invertible element. Evidently, it follows that L_{b_0} is a finite rank operator in $B(\mathcal{A}p)$ because $\mathcal{A}$ is a primitive C∗-algebra [28, page 903]. And it is clear that $L_{b_0}L_a = L_aL_{b_0}$. One can check that L_{a+b_0} is a Drazin invertible operator. Applying [19, Theorem 2.7], it follows that L_a is a Drazin invertible operator. Note that the left regular representation is faithful because $\mathcal{A}$ is a primitive C∗-algebra [25, page 30]. One can check that a is Drazin invertible by the fact that $\Gamma(\mathcal{A})$ is Drazin inverse closed in $B(\mathcal{A}p)$. Thus, $a \in \mathcal{A}$ is a B-Browder element. That is to say, $0 \notin \sigma_{BB}(a)$.

Subsequently, we prove that $\bigcap_{b\in \mathrm{Soc}(\mathcal{A}),ab=ba}\sigma_D(a+b) \subseteq \sigma_{BB}(a)$. It suffices to show that

$$\text{if } 0 \in \bigcap_{b\in \mathrm{Soc}(\mathcal{A}),ab=ba} \sigma_D(a+b), \text{ then } 0 \in \sigma_{BB}(a).$$

Suppose $0 \in \bigcap_{b\in \mathrm{Soc}(\mathcal{A}),ab=ba}\sigma_D(a+b)$, then for any $b \in \mathrm{Soc}(\mathcal{A})$ with $ab = ba$, $a + b$ is not a Drazin invertible element. One can assert that for any finite rank operator $F = L_t$ in $B(\mathcal{A}p)$, where $t \in \mathrm{Soc}(\mathcal{A})$ with $L_tL_a = L_aL_t$, $L_a + L_t$ is not a Drazin invertible operator. Otherwise, there exists

$$L_{t_1} \in F(\mathcal{A}p), \text{ where } t_1 \in \mathrm{Soc}(\mathcal{A}) \quad \text{and} \quad L_{t_1}L_a = L_aL_{t_1},$$

such that $L_a + L_{t_1}$ is a Drazin invertible operator. It can be proved that $a + t_1$ is a Drazin invertible element using the fact that the left regular representation is faithful since $\mathcal{A}$ is a primitive C∗-algebra [28, page 903]. Notice that $at_1 = t_1a$ and $t_1 \in \mathrm{Soc}(\mathcal{A})$, and these are contradict to the assumption that $a + b$ is not a Drazin invertible element for any $b \in \mathrm{Soc}(\mathcal{A})$ with $ab = ba$. Therefore, L_a is not a Drazin invertible operator from [19, Theorem 2.7], and hence, $0 \in \sigma_{BB}(L_a)$ from [12, Theorem 2.2]. In other words, L_a is not a B-Browder operator. Consequentlly, a is not a B-Browder element. Indeed, if a is a B-Browder element, according to the definition of the B-Browder element, it leads to a conclusion that L_a is Drazin invertible by [19, Theorem 2.7]. It is a contradiction with the fact that L_a is not a B-Browder operator [12, Theorem 2.2]. So from the aforementioned arguement, one can obtain that $0 \in \sigma_{BB}(a)$. □

For a primitive C∗-algebra $\mathcal{A}$, if $a \in \mathcal{A}$ is a B-Browder element, it must be a B-Weyl element from Remark 2 and Definition 9. Also, Definition 7 indicates that if $a \in \mathcal{A}$ is a B-Weyl element, then it must be a B-Fredholm element. Next a characterization of the B-Browder elements will be given by the B-Fredholm elements using the following lemma.

Lemma 2. *If $\mathcal{A}$ is a unital primitive C∗-algebra, then $a \in \mathcal{A}$ is a B-Browder element if and only if L_a is a B-Browder operator on $\mathcal{A}p$.*

Proof. Suppose that a is a B-Browder element. Then

$$0 \notin \sigma_{BB}(a) = \bigcap_{s\in \mathrm{Soc}(\mathcal{A}),as=sa} \sigma_D(a+s)$$

from Proposition 7. Hence, there exists $s_0 \in \mathrm{Soc}(\mathcal{A})$ such that $as_0 = s_0a$ and $a + s_0$ is a Drazin invertible element. It follows that L_{s_0} is a finite rank operator on $\mathcal{A}p$ and $L_aL_{s_0} = L_{s_0}L_a$ because $\mathcal{A}$ is a primitive C∗-algebra [28, page 903]. One can check that $L_a + L_{s_0}$ is a Drazin invertible operator on $\mathcal{A}p$. It is easy to find that L_a is a Drazin invertible operator [19, Theorem 2.7], which implies that $0 \notin \sigma_{BB}(L_a)$ by [12, Theorem 2.2]. Therefore, and L_a is a B-Browder operator.

Conversely, if L_a is a B-Browder operator, then $0 \notin \sigma_{BB}(L_a)$. From [12, Theorem 2.2], one can obtain that L_a is a Drazin invertible operator. Since $\mathcal{A}$ is a primitive C∗-algebra, then $\Gamma(\mathcal{A})$ is Drazin inverse closed in $B(\mathcal{A}p)$ according to [25, Theorem F.4.3] and [30, Corollary 6]. Since $\mathcal{A}$ is a primitive C∗-algebra, it follows from [28, page 903] that the left regular representation is faithful, and hence, a is Drazin invertible. From the definition of the B-Browder elements, it follows that $0 \notin \sigma_{BB}(a)$. Consequently, a is a B-Browder element. This completes the proof. □

Theorem 8. *If $\mathcal{A}$ is a unital primitive C∗-algebra, then $a \in \mathcal{A}$ is a B-Browder element if and only if a is a B-Fredholm element and $0 \in \mathrm{iso}\sigma(a)\bigcup\rho(a)$.*

Proof. It suffices to prove the situation that a is a B-Browder element but not invertible. If $a \in \mathcal{A}$ is a B-Browder element, then L_a is a B-Browder operator from Lemma 2. Applying [13, Theorem 2.7] and [12, Theorem 2.2], it follows that L_a is a B-Fredholm operator and $0 \in \text{iso}\sigma(L_a)$. Next we prove that a is a B-Fredholm element and $0 \in \text{iso}\sigma(a)$. Since a is a B-Browder element, it is clear that a is a B-Fredholm element. It suffices to prove that $0 \in \text{iso}\sigma(a)$. Since $0 \in \text{iso}\sigma(L_a)$, there exists $\delta > 0$ such that $L_a - \lambda$ is an invertible operator on $\mathcal{A}p$ when $0 < |\lambda| < \delta$. Hence, $L_a - L_\lambda$ is invertible in $\Gamma(\mathcal{A})$ because $\mathcal{A}$ is a primitive C*-algebra [20, Example 3.5]. One can show that $a - \lambda$ is invertible when $0 < |\lambda| < \delta$ from the faithfullity of the left regular representation because $\mathcal{A}$ is a primitive C*-algebra [28, page 903]. Therefore, $0 \in \text{iso}\sigma(a)$.

Conversely, suppose that $0 \in \rho(a)$, in other words, a is an invertible element, which implies that a is a B-Browder element. Therefore, there is no harm in assuming that $0 \in \text{iso}\sigma(a)$.

Since a is a B-Fredholm element, from [20, Theorem 3.6] one can obtain that L_a is a B-Fredholm operator. Since $0 \in \text{iso}\sigma(a)$, one can check that L_a is not an invertible operator because $\mathcal{A}$ is a primitive C*-algebra [20, Example 3.5]. In other words, $0 \in \sigma(L_a)$. Evidently, there exists $\delta > 0$ such that $a - \lambda$ is an invertible element when $0 < |\lambda| < \delta$. Hence,

$$L_a - L_\lambda = L_a - \lambda L_e \text{ is an invertible operator in } \Gamma(\mathcal{A}),$$

where e is the identity of $\mathcal{A}$. Then it is invertible in $B(\mathcal{A}p)$. This leads to the conclusion that $L_a - \lambda L_e$ is invertible when $0 < |\lambda| < \delta$. So one can see that $0 \in \text{iso}\sigma(L_a)$. Associated with [13, Theorem 2.7] and [12, Theorem 2.2], it follows that L_a is a B-Browder operator. By Lemma 2, a is a B-Browder element. This completes the proof. □

The condition "primitive C*-algebra" in Theorem 8 is essential. Otherwise, (1) and (2) in Theorem 8 are not equivalent as can be seen in the following example.

Example 2. Let $l^2(\mathbb{Z}^+)$ and $l^2(\mathbb{Z})$ be the square summable sequences. Let H be the Hilbert space direct sum $H = l^2(\mathbb{Z}^+) \oplus l^2(\mathbb{Z})$. Let $\mathcal{A}$ be the subalgebra of $B(H)$ consisting of the operators, which leave each of the two direct summands invariant, constructed as in [31, Example 1.8]. For two operators $U \in B(l^2(\mathbb{Z}^+))$ and $W \in B(l^2(\mathbb{Z}))$, we define

$$U \oplus W(\{x_i\}_{i=1}^{\infty}, \{y_j\}_{j=-\infty}^{\infty}) = (U\{x_i\}_{i=1}^{\infty}, W\{y_j\}_{j=-\infty}^{\infty}).$$

Then, $U \oplus W$ in $\mathcal{A}$. Define T and F as follows:

$$T(x_1, x_2, \ldots) = \left(0, \frac{x_1}{2}, \frac{x_2}{3}, \ldots\right),$$
$$F(\cdots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = (\ldots, x_{-3}, x_{-2}, x_{-1}, x_0, x_1, \ldots).$$

One can show that $T \oplus F \in \Phi(\mathcal{A})$ and $\sigma(T \oplus F) = \{0\} \bigcup \{\lambda : |\lambda| = 1\}$. Therefore, $0 \in \text{iso}\sigma(T \oplus F)$. Suppose that $T \oplus F$ is a B-Browder element. Then there exist a $V \in \text{Soc}(\mathcal{A})$ such that $(T \oplus F) - V$ is Drazin invertible in $\mathcal{A}$. Let V_1 and V_2 be the restrictions of V to the first and second coordinate spaces, respectively. Then $V = V_1 \oplus V_2$ and $V_1 \in B(l^2(\mathbb{Z}^+))$, $V_2 \in B(l^2(\mathbb{Z}))$. Since $(T \oplus F) - V = (T - V_1) \oplus (F - V_2)$ is Drazin invertible, it follows that each of the operators $T - V_1$ and $F - V_2$ is Drazin invertible. Hence, T is Drazin invertible in $B(l^2(\mathbb{Z}^+))$ and F is Drazin invertible in $B(l^2(\mathbb{Z}))$. It is a contradiction. Consequently, $T \oplus F$ is not a B-Browder element.

For the semisimple Banach algebra $\mathcal{A}$, recall that the socle of $\mathcal{A}$ denoted by $\text{Soc}(\mathcal{A})$ is the algebraic sum of all the minimal left ideals of $\mathcal{A}$ (which equals to the sum of all the minimal right ideals) [32]. As we know, more and more scholars devoted to the characterization of the socle. A characterization of $\text{Soc}(\mathcal{A})$ can be found in [34], where the notion of degenerate elements in a semisimple Banach algebra is defined. Indeed, the set of all degenerate elements in $\mathcal{A}$ is equal to $\text{Soc}(\mathcal{A})$. In the same article, it is shown that if S is the set of all similarity classes of minimal idempotents in $\text{Soc}(\mathcal{A})$, then the ideal $\mathcal{A}_s$ generated by an element $p_s \in s \in S$ is the same for any choice of $p_s \in s$. It is then shown that $\text{Soc}(\mathcal{A}) = \sum_{s \in S} \mathcal{A}_s$. Some other characterizations of the socle can refer to [21,33]. We have discussed the properties of the B-Fredholm elements, the B-Weyl elements and the B-Browder elements in a primitive C*-algebra. The final section of the article will give another characterization of the socle of a primitive C*-algebra by the B-Fredholm elements.

4 The characterization of Soc($\mathcal{A}$)

Let Φ_g (resp. Φ) be the set of the generalized Fredholm operators (resp. Fredholm operators) on H. Notice that

$$F(H) = \{T \in B(H) : T + S \in \Phi_g \quad \text{for all } S \in \Phi_g\},$$

where $F(H)$ means the set of all finite rank operators in $B(H)$ [35, Theorem 3.24]. Furthermore, in [36], the authors characterize the socle of a primitive C∗-algebra by using the concept of generalized Fredholm elements. In this section, we give a characterization of the socle of a primitive C∗-algebra by using the concept of B-Fredholm elements.

Suppose that $\mathcal{A}$ is a unital primitive C∗-algebra, denote the set of all generalized Fredholm elements in $\mathcal{A}$ by $\Phi_g(\mathcal{A})$. Define the generalized Fredholm spectrum $\sigma_{\Phi_g}(a)$ of an element $a \in \mathcal{A}$ as follows:

$$\sigma_{\Phi_g}(a) = \{\lambda \in \mathbf{C} : a - \lambda \text{ is not a generalized Fredholm element}\}.$$

Denote $N_s(\mathcal{A}) = \mathcal{A}^N \bigcap \mathrm{Soc}(\mathcal{A})$. $BF(\mathcal{A})$ means the set of all B-Fredholm elements in $\mathcal{A}$. Now we come to the final result of this article, which gives the characterization of the socle of a primitive C∗-algebra $\mathcal{A}$.

Theorem 9. *If $\mathcal{A}$ is a unital primitive C^*-algebra with $\Gamma(N_s(\mathcal{A})) \supseteq N(\mathcal{A}p)$, then*

$$\begin{aligned}\mathrm{Soc}(\mathcal{A}) &= \{x \in \mathcal{A} : x + y \in BF(\mathcal{A}) \textit{ for all } y \in BF(\mathcal{A})\}\\ &= \{x \in \mathcal{A} : \sigma_{BF}(x + y) = \sigma_{BF}(y) \textit{ for all } y \in \mathcal{A}\}.\end{aligned}$$

Before proving Theorem 9, we need a lemma first, which is of interest by it own means.

Lemma 3. *Suppose $\mathcal{A}$ is a unital primitive C^*-algebra with $\Gamma(N_s(\mathcal{A})) \supseteq N(\mathcal{A}p)$, then the B-Fredholm elements in $\mathcal{A}$ are equivalent to the generalized Fredholm elements in $\mathcal{A}$.*

Proof. If $a \in \mathcal{A}$ is a B-Fredholm element, then L_a is a B-Fredholm operator from Lemma 1. Using [4, Theorem 2.7], there exist a Fredholm operator S in $B(\mathcal{A}p)$ and a nilpotent operator F in $B(\mathcal{A}p)$ such that $L_a = S + F$. Hence, there exists

$$c \in N_s(\mathcal{A}) = \mathcal{A}^N \bigcap \mathrm{Soc}(\mathcal{A}) \text{ such that } L_c = F.$$

One can find that $S = L_{a-c}$. Since S is a Fredholm operator, it follows that $a - c$ is a Fredholm element using Lemma 1. Set $b = a - c$, then $b \in \Phi(\mathcal{A})$ and $c \in \mathcal{A}^N \bigcap \mathrm{Soc}(\mathcal{A})$, and it is evident that $a = b + c$. Applying Lemma 3, one can obtain that a is a generalized Fredholm element.

Suppose that $a \in \mathcal{A}$ is a generalized Fredholm element. Associated with Lemma 3, one can see that there exist $b \in \mathcal{A}^N \bigcap \mathrm{Soc}(\mathcal{A})$ and $c \in \Phi(\mathcal{A})$ such that $a = b + c$. According to Theorem 3.6 in [24], it follows that $c + b = a$ is a Fredholm element. Therefore, a is a B-Fredholm element. This completes the proof. ☐

At the end of this article, we present the proof of Theorem 9.

The proof of Theorem 9. According to [6, Proposition 3.3], it follows that $\mathrm{Soc}(\mathcal{A}) \subseteq \{x \in \mathcal{A} : x + y \in BF(\mathcal{A}) \text{ for all } y \in BF(\mathcal{A})\}$. Therefore, it suffices to prove that $\{x \in \mathcal{A} : x + y \in BF(\mathcal{A}) \text{ for all } y \in BF(\mathcal{A})\} \subseteq \mathrm{Soc}(\mathcal{A})$. Suppose that $x \in \mathcal{A}$ and $x + y \in BF(\mathcal{A})$ for all $y \in BF(\mathcal{A})$. Thus, x is a B-Fredholm element, which implies x is a generalized Fredholm element by Lemma 3. Let $\lambda \neq 0$ and $z \in \overline{\mathrm{Soc}(\mathcal{A})}$. Set $y = z - \lambda e$, then $y \in \Phi(\mathcal{A}) \subseteq \Phi_g(\mathcal{A})$. Therefore, y is a B-Fredholm element. This leads to the conclusion that $x + y = (x - \lambda e) + z \in BF(\mathcal{A})$, and hence, $x + y = (x - \lambda e) + z \in \Phi_g(\mathcal{A})$ according to Lemma 3. Similar to the method of [35, Theorem 3.22], one can check that $x - \lambda e \in \Phi(\mathcal{A})$ for $\lambda \neq 0$. Hence, $\sigma(\hat{x}) = \{0\}$. From [24, Theorem 6.7], it implies that $x \in \mathrm{Soc}(\mathcal{A})$.

Next, we prove that $\mathrm{Soc}(\mathcal{A}) \subseteq \{x \in \mathcal{A} : \sigma_{BF}(x + y) = \sigma_{BF}(y) \text{ for all } y \in \mathcal{A}\}$. If $k \in \mathrm{Soc}(\mathcal{A})$ and $y \in \mathcal{A}$, then from [6, Proposition 3.3] that

$$\lambda \notin \sigma_{BF}(k + y) \Leftrightarrow k + y - \lambda \in BF(\mathcal{A}) \Leftrightarrow y - \lambda \in BF(\mathcal{A}) \Leftrightarrow \lambda \notin \sigma_{BF}(y).$$

Conversely, if $x_0 \in \{x \in \mathcal{A} : \sigma_{BF}(x + y) = \sigma_{BF}(y) \text{ for all } y \in \mathcal{A}\}$, then for all $y \in BF(\mathcal{A})$, one can check that $0 \notin \sigma_{BF}(x_0 + y)$. In other words, $x_0 + y$ is a B-Fredholm element for all $y \in BF(\mathcal{A})$. Consequently,

$$x_0 \in \{x \in \mathcal{A} : x + y \in BF(\mathcal{A}) \text{ for all } y \in BF(\mathcal{A})\} \subseteq \mathrm{Soc}(\mathcal{A}).$$

This completes the proof.

Remark 3. From the aforementioned theorem, one can see that $N_s(\mathcal{A}) \neq \varnothing$ because $0 \in N_s(\mathcal{A})$. When $\mathcal{A} = B(H)$, we have $N_s(\mathcal{A})$ is the set of all finite rank nilpotent operators on H. Hence, $N_s(\mathcal{A})$ is meaningful.

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