Research Article

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Existence of positive solutions of discrete third-order three-point BVP with sign-changing Green's function

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Abstract: In this article, we consider a discrete nonlinear third-order boundary value problem

$$\begin{cases}\Delta^3 u(k-1) = \lambda a(k) f(k, u(k)), \quad k \in [1, N-2]_{\mathbb{Z}},\\ \Delta^2 u(\eta) = \alpha \Delta u(N-1), \Delta u(0) = -\beta u(0), \quad u(N) = 0,\end{cases}$$

where $N > 4$ is an integer, $\lambda > 0$ is a parameter. $f : [1, N-2]_{\mathbb{Z}} \times [0, +\infty) \to [0, +\infty)$ is continuous, $a : [1, N-2]_{\mathbb{Z}} \to (0, +\infty)$, $\alpha \in \left[0, \frac{1}{N-1}\right)$, $\beta \in \left[0, \frac{2(1-\alpha(N-1))}{N(2-\alpha(N-1))}\right)$, and $\eta \in \left[\left[\frac{N-2}{2}\right] + 1, N-2\right]_{\mathbb{Z}}$. With the sign-changing Green's function, we obtain not only the existence of positive solutions but also the multiplicity of positive solutions to this problem.

Keywords: third-order difference equation, three-point boundary conditions, sign-changing Green's function, Guo-Krasnoselskii's fixed-point theorem

MSC 2020: 39A10, 39A12

1 Introduction

For any integers c and d with $c \le d$, let $[c, d]_{\mathbb{Z}} = \{c, c+1, \ldots, d\}$. In this article, we study existence and multiplicity of positive solutions for the following discrete nonlinear third-order boundary value problem (BVP)

$$\begin{cases}\Delta^3 u(k-1) = \lambda a(k) f(k, u(k)), \quad k \in [1, N-2]_{\mathbb{Z}},\\ \Delta^2 u(\eta) = \alpha \Delta u(N-1), \quad \Delta u(0) = -\beta u(0), \ u(N) = 0,\end{cases} \tag{1.1}$$

where $N > 4$ is an integer and $\lambda > 0$ is a parameter. $f : [1, N-2]_{\mathbb{Z}} \times [0, +\infty) \to [0, +\infty)$ is continuous, $a : [1, N-2]_{\mathbb{Z}} \to (0, +\infty)$, $\alpha \in \left[0, \frac{1}{N-1}\right)$, $\beta \in \left[0, \frac{2(1-\alpha(N-1))}{N(2-\alpha(N-1))}\right)$, and η satisfies the condition:

$$(\mathrm{H}_0) : \eta \in \left[\left[\frac{N-2}{2}\right] + 1, N-2\right]_{\mathbb{Z}}.$$

BVPs for difference equations have been widely studied for different disciplines, such as the computer sciences, applied mathematics, economics, mechanical engineering and control systems, and so on, see

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[1–3]. Great effort has been made to study the existence, multiplicity, and uniqueness of solutions of BVPs by using fixed-point theorem [4–12], monotone iterative [13], degree theory [14], the method of upper and lower solutions [15], critical point theorem [16,17], and so forth. In recent years, the existence of positive solutions for third-order three-point BVPs has also been discussed by several authors, and see [8–11,18,19]. However, it is necessary to point out that Green's functions in the aforementioned literature are positive. As we all know, the positivity of Green's function guarantees the positivity of the corresponding difference operator. Now, the natural question is: When the Green's function changes its sign, how could we obtain the existence of positive solutions? Fortunately, there have been some papers on positive solution for third-order BVPs when the corresponding Green's functions are sign-changing, and see [5–7,12,13,20]. For example, Gao and Geng [6] considered positive solutions of the following discrete nonlinear third-order three-point eigenvalue problem:

$$\begin{cases}\Delta^3 u(k-1) = \lambda a(k) f(k, u(k)), \quad k \in [1, T-2]_{\mathbb{Z}},\\ \Delta u(0) = u(T) = \Delta^2 u(\eta) = 0,\end{cases}$$

where $\eta \in \left\{\frac{T-1}{2}, \ldots, T-2\right\}$ for odd T and $\eta \in \left\{\frac{T-2}{2}, \ldots, T-2\right\}$ for even T, and the Green's function is sign changing. Furthermore, by using Guo-Krasnoselskii's fixed-point theorem, Xu et al. [12] considered the existence of positive solutions for third-order three-point BVP:

$$\begin{cases}\Delta^3 u(k-1) = a(k) f(k, u(k)), \quad k \in [1, T-2]_{\mathbb{Z}},\\ \Delta^2 u(\eta) - \alpha \Delta u(T-1) = 0, \Delta u(0) = u(T) = 0,\end{cases} \tag{1.2}$$

where $\eta \in \left[\left[\frac{T-2}{2}\right] + 1, T-2\right]_{\mathbb{Z}}$ and the Green's function is sign changing. Clearly, BVP (1.2) is a special case of BVP (1.1), so BVP (1.1) is parallel to BVP (1.2) but more general. Recently, Cao et al. [5] discussed the existence of positive solutions for the following BVP:

$$\begin{cases}\Delta^3 u(k-1) = -\lambda a(k) f(k, u(k)), \quad k \in [1, T-1]_{\mathbb{Z}},\\ u(0) = 0, \Delta^2 u(\eta) = \alpha \Delta u(T), \Delta u(T) = \beta u(T+1),\end{cases}$$

where both the weight function $a(t)$ and the Green's function $G(t, s)$ change their sign. Inspired by the aforementioned works, in this article, we try to study problem (1.1).

The rest of this article is organized as follows. In Section 2, we study the linear problem. In particular, we will point out that the Green's function changes its sign. Meanwhile, we also give a typical example to explain why we choose $\eta \in \left[\left[\frac{N-2}{2}\right] + 1, N-2\right]_{\mathbb{Z}}$. In Section 3, we impose some conditions to obtain existence and multiplicity of positive solutions to problem (1.1). In Section 4, we add several specific examples to verify our main results. The main tool we will use is the following Guo-Krasnoselskii's fixed-point theorem.

Theorem 1.1. [21] *Let E be a Banach space and $K \subset E$ be a cone. Assume that Ω_1 and Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$. If $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is a completely continuous operator such that*
(i) $\|Tu\| \le \|u\|$, $u \in K \cap \partial\overline{\Omega}_1$ and $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\overline{\Omega}_2$,
or
(ii) $\|Tu\| \ge \|u\|$, $u \in K \cap \partial\overline{\Omega}_1$, and $\|Tu\| \le \|u\|$, $u \in K \cap \partial\overline{\Omega}_2$,
then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

2 Linear problem

Let us study the linear problem

$$\begin{aligned}\Delta^3 u(k-1) &= h(k), \quad k \in [1, N-2]_{\mathbb{Z}},\\ \Delta^2 u(\eta) &= \alpha \Delta u(N-1), \quad \Delta u(0) = -\beta u(0), \ u(N) = 0,\end{aligned} \tag{2.1}$$

with $\frac{1}{N-1} > \alpha \geq 0$, $\frac{2(1-\alpha(N-1))}{N(2-\alpha(N-1))} > \beta \geq 0$. We denote

$$\rho_k = 2(1-\beta k)(1-\alpha(N-1)) - \alpha\beta k(k-1)$$

and

$$\rho = 2(1-\beta N)(1-\alpha(N-1)) - \alpha\beta N(N-1).$$

Lemma 2.1. *Problem* (2.1) *has a unique solution*

$$u(k) = \sum_{s=1}^{N-2} G(k,s)h(s),$$

where $G(k,s) = u_1(k,s) + u_2^*(k,s) + u_3^*(k,s)$ *with*

$$u_1(k,s) = \frac{\alpha(N-s-1)[k(k-1) - \frac{\rho_k}{\rho}N(N-1)]}{2-2\alpha(N-1)} - \frac{\rho_k(N-s)(N-s-1)}{2\rho},$$

$$u_2^*(k,s) = \begin{cases} u_2(k,s) := \dfrac{\frac{\rho_k}{\rho}N(N-1) - k(k-1)}{2-2\alpha(N-1)}, & s \leq \eta, \\ 0, & s > \eta, \end{cases}$$

and

$$u_3^*(k,s) = \begin{cases} u_3(k,s) := \dfrac{(k-s)(k-s-1)}{2}, & 0 < s \leq k-2 \leq N-2, \\ 0, & 0 \leq k-2 < s \leq N-2. \end{cases}$$

Proof. Summing both sides of $\Delta^3 u(s-1) = h(s)$ from $s = 1$ to $s = k-1$, we obtain

$$\Delta^2 u(k-1) = \Delta^2 u(0) + \sum_{s=1}^{k-1} h(s), \tag{2.2}$$

and

$$\Delta u(k-1) = \Delta u(0) + (k-1)\Delta^2 u(0) + \sum_{s=1}^{k-2}(k-s-1)h(s). \tag{2.3}$$

Summing both sides of the aforementioned equation from $\tau = 1$ to $\tau = k$, we deduce that

$$u(k) = u(0) + k\Delta u(0) + \frac{k(k-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s). \tag{2.4}$$

Next, refer equations (2.2) and (2.3) when $\Delta^2 u(\eta) = \alpha\Delta u(N-1)$ is rewritten as follows:

$$\Delta^2 u(0) + \sum_{s=1}^{\eta} h(s) = \alpha\left(\Delta u(0) + (N-1)\Delta^2 u(0) + \sum_{s=1}^{N-2}(N-s-1)h(s)\right),$$

and thus,

$$\begin{aligned}\Delta^2 u(0) &= \frac{\alpha\left(\Delta u(0) + \sum_{s=1}^{N-2}(N-s-1)h(s)\right) - \sum_{s=1}^{\eta} h(s)}{1-\alpha(N-1)} \\ &= \frac{\alpha\Delta u(0)}{1-\alpha(N-1)} + \sum_{s=1}^{N-2}\frac{\alpha(N-s-1)}{1-\alpha(N-1)}h(s) - \sum_{s=1}^{\eta}\frac{1}{1-\alpha(N-1)}h(s).\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$\begin{aligned}u(k) &= u(0) + k\Delta u(0) + \frac{k(k-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s)\\ &= u(0) + k\Delta u(0) + \frac{k(k-1)}{2}\frac{\alpha\Delta u(0)}{1-\alpha(N-1)} + \sum_{s=1}^{N-2}\frac{k(k-1)}{2}\frac{\alpha(N-s-1)}{1-\alpha(N-1)}h(s)\\ &\quad - \sum_{s=1}^{\eta}\frac{k(k-1)}{2}\frac{1}{1-\alpha(N-1)}h(s) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s).\end{aligned}$$

By using boundary condition $\Delta u(0) = -\beta u(0)$, we deduce that

$$u(0) + k\Delta u(0) + \frac{k(k-1)}{2}\frac{\alpha\Delta u(0)}{1-\alpha(N-1)} = u(0)\left(\frac{\rho_k}{2(1-\alpha(N-1))}\right).$$

Then, by using $u(N) = 0$, we obtain that

$$u(0) + N\Delta u(0) + \frac{N(N-1)}{2}\Delta^2 u(0) + \sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)}{2}h(s) = 0,$$

which is the same as follows:

$$\begin{aligned}u(0)(1-\beta N) &= -\frac{N(N-1)}{2}\Delta^2 u(0) - \sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)}{2}h(s)\\ &= -\frac{N(N-1)}{2}\left(\frac{-\alpha\beta u(0)}{1-\alpha(N-1)} + \sum_{s=1}^{N-2}\frac{\alpha(N-s-1)}{1-\alpha(N-1)}h(s) - \sum_{s=1}^{\eta}\frac{1}{1-\alpha(N-1)}h(s)\right)\\ &\quad - \sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)}{2}h(s).\end{aligned}$$

Hence,

$$u(0)\left(\frac{\rho}{2(1-\alpha(N-1))}\right) = \sum_{s=1}^{\eta}\frac{N(N-1)}{2(1-\alpha(N-1))}h(s) - \sum_{s=1}^{N-2}\left(\frac{\alpha N(N-1)(N-s-1)}{2(1-\alpha(N-1))} + \frac{(N-s)(N-s-1)}{2}\right)h(s)$$

and

$$u(0) = \sum_{s=1}^{\eta}\frac{N(N-1)}{\rho}h(s) - \sum_{s=1}^{N-2}\frac{(N-s-1)[\alpha N(N-1) + (N-s)(1-\alpha(N-1))]}{\rho}h(s).$$

Finally, we obtain that

$$\begin{aligned}u(k) &= \frac{\rho_k u(0)}{2(1-\alpha(N-1))} + \sum_{s=1}^{N-2}\frac{k(k-1)}{2}\frac{\alpha(N-s-1)}{1-\alpha(N-1)}h(s)\\ &\quad - \sum_{s=1}^{\eta}\frac{k(k-1)}{2}\frac{1}{1-\alpha(N-1)}h(s) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s)\\ &= \frac{\rho_k}{\rho}\sum_{s=1}^{\eta}\frac{N(N-1)}{2-2\alpha(N-1)}h(s) - \frac{\rho_k}{\rho}\sum_{s=1}^{N-2}\frac{\alpha N(N-1)(N-s-1)}{2-2\alpha(N-1)}h(s)\\ &\quad - \frac{\rho_k}{\rho}\sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)}{2}h(s) + \sum_{s=1}^{N-2}\frac{\alpha k(k-1)(N-s-1)}{2-2\alpha(N-1)}h(s)\\ &\quad - \sum_{s=1}^{\eta}\frac{k(k-1)}{2-2\alpha(N-1)}h(s) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s)\end{aligned}$$

$$= \sum_{s=1}^{N-2}\left(\frac{\alpha(N-s-1)[k(k-1)-\frac{\rho_k}{\rho}N(N-1)]}{2-2\alpha(N-1)} - \frac{\rho_k(N-s)(N-s-1)}{2\rho}\right)h(s)$$
$$+ \sum_{s=1}^{\eta}\frac{\frac{\rho_k}{\rho}N(N-1)-k(k-1)}{2-2\alpha(N-1)}h(s) + \sum_{s=1}^{k-2}\frac{(k-s)(k-s-1)}{2}h(s)$$
$$:= \sum_{s=1}^{N-2}u_1(k,s)h(s) + \sum_{s=1}^{\eta}u_2(k,s)h(s) + \sum_{s=1}^{k-2}u_3(k,s)h(s). \qquad \square$$

Remark 2.2. We point out that under conditions for α and β, we have $2(1-\alpha(N-1))(1-\beta N) - \alpha\beta N(N-1) > 0$. Moreover, if $\beta = 0$ and $\lambda = 1$, we obtain the expression given in [12].

Lemma 2.3. *The Green's function $G(k,s)$ satisfies the following properties*:
(*i*) *If $s \in [1,\eta]_{\mathbb{Z}}$, then $G(k,s)$ is nonincreasing with respect to $k \in [0,N]_{\mathbb{Z}}$. If $s \in [\eta+1, N-2]_{\mathbb{Z}}$, then $G(k,s)$ is nondecreasing with respect to $k \in [0,N]_{\mathbb{Z}}$.*
(*ii*) *$G(k,s)$ changes its sign on $[0,N]_{\mathbb{Z}} \times [1,N-2]_{\mathbb{Z}}$. In details, $G(k,s) \ge 0$ for $(k,s) \in [0,N]_{\mathbb{Z}} \times [1,\eta]_{\mathbb{Z}}$, $G(k,s) \le 0$ for $(k,s) \in [0,N]_{\mathbb{Z}} \times [\eta+1,N-2]_{\mathbb{Z}}$.*
(*iii*) *If $s > \eta$, then $\max_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(N,s) = 0$ and*

$$\begin{aligned}\min_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(0,s) &= -\frac{(N-s-1)(\alpha N(N-1)+(1-\alpha(N-1))(N-s))}{\rho}\\ &\ge -\frac{(N-\eta-1)(\alpha N(N-1)+(1-\alpha(N-1))(N-\eta))}{\rho}\\ &= -\frac{(N-\eta-1)(\alpha(N-1)\eta+(N-\eta))}{\rho}.\end{aligned}$$

If $s \le \eta$, then $\min_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(N,s) = 0$ and

$$\begin{aligned}\max_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(0,s) &= \frac{N(N-1)-(N-s-1)(\alpha N(N-1)+(1-\alpha(N-1))(N-s))}{\rho}\\ &\le \frac{N(N-1)-(N-\eta-1)(\alpha N(N-1)+(1-\alpha(N-1))(N-\eta))}{\rho}\\ &= \frac{\eta(2N-\eta-1)-\alpha\eta(N-\eta-1)(N-1)}{\rho}.\end{aligned}$$

Proof. Now, we will study the sign properties of function G.

Firstly, suppose that $s > \eta$ and $s > k-2$. In this case, we obtain $G(k,s) = u_1(k,s)$ and

$$\Delta_k G(k,s) = \Delta_k u_1(k,s) = \frac{\alpha(N-s-1)\left(2k - \frac{N(N-1)}{\rho}\Delta\rho_k\right)}{2-2\alpha(N-1)} - \frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho}.$$

Notice that $\Delta\rho_k = -2\beta(1-\alpha(N-1)) - 2\alpha\beta k < 0$, we obtain that $\Delta_k G(k,s) \ge 0$ for $s > \eta$ and $s > k-2$.

Secondly, suppose that $s > \eta$ and $s \le k-2$. In this case, it is fulfilled that $G(k,s) = u_1(k,s) + u_3(k,s)$ and

$$\Delta_k G(k,s) = \Delta_k u_1(k,s) + \Delta_k u_3(k,s) = \Delta_k u_1(k,s) + k - s > 0.$$

As a result, we deduce that $\Delta_k G(k,s) \ge 0$ for $s > \eta$, which give us that

$$\max_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(N,s) = 0.$$

Now, if $s \le \eta$ and $s \le k-2$, we have $G(k,s) = u_1(k,s) + u_2(k,s) + u_3(k,s)$ and

$$\begin{aligned}\Delta_k G(k,s) &= \Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s)\\ &= \frac{-\alpha sk + s(\alpha(N-1)-1)}{1-\alpha(N-1)} - \frac{(\alpha(N-s-1)-1)N(N-1)\Delta\rho_k}{(2-2\alpha(N-1))\rho} - \frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho}\\ &= \frac{1}{2\rho(1-\alpha(N-1))}\{[-\alpha sk + s(\alpha(N-1)-1)]2\rho - (\alpha(N-s-1)-1)N(N-1)\Delta\rho_k\\ &\quad - (N-s)(N-s-1)(1-\alpha(N-1))\Delta\rho_k\}\\ &= \frac{1}{2\rho(1-\alpha(N-1))}\{[1-\alpha(N-1)]\Delta\rho_k[N(N-1)-(N-s)(N-s-1)] + \alpha sN(N-1)\Delta\rho_k\\ &\quad + [-\alpha sk + s(\alpha(N-1)-1)]2\rho\}\\ &\le 0.\end{aligned}$$

Finally, suppose that $s \le \eta$ and $s > k-2$. We obtain that

$$\begin{aligned}\Delta_k G(k,s) &= \Delta_k u_1(k,s) + \Delta_k u_2(k,s)\\ &= \frac{(\alpha(N-s-1)-1)(2k - \frac{N(N-1)\Delta\rho_k}{\rho})}{2-2\alpha(N-1)} - \frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho}\\ &= \frac{1}{2\rho(1-\alpha(N-1))}\{[1-\alpha(N-1)][N(N-1)-(N-s)(N-s-1)]\Delta\rho_k + 2k\rho[\alpha(N-1)-1]\\ &\quad - \alpha s[2k\rho - N(N-1)\Delta\rho_k]\}\\ &\le 0.\end{aligned}$$

As a result, we have $\Delta_k G(k,s) \le 0$ for $s \le \eta$. Moreover,

$$\min_{k\in[0,N]_{\mathbb{Z}}} G(k,s) = G(N,s) = 0.$$

Summarizing the aforementioned results, if $s \in [1,\eta]_{\mathbb{Z}}$, then $G(k,s) \ge 0$ and $G(k,s)$ is nonincreasing. Moreover, if $s \in [\eta+1, N-2]_{\mathbb{Z}}$, then $G(k,s) \le 0$ and $G(k,s)$ is nondecreasing. □

Remark 2.4. Let us give some reasons for why we choose

$$\eta \in \left[\left[\frac{N-2}{2}\right] + 1, N-2\right]_{\mathbb{Z}}.$$

To obtain it, let us consider the following BVP:

$$\begin{aligned}&\Delta^3 u(k-1) = 1, \quad k \in [1, N-2]_{\mathbb{Z}},\\ &\Delta^2 u(\eta) = \alpha\Delta u(N-1), \quad \Delta u(0) = -\beta u(0), u(N) = 0.\end{aligned}$$

From Lemma 2.1, we obtain

$$u(k) = \sum_{s=1}^{N-2} u_1(k,s) + \sum_{s=1}^{\eta} u_2(k,s) + \sum_{s=1}^{k-2} u_3(k,s) = \frac{\varphi(k)}{12\rho(1-\alpha(N-1))},$$

where

$$\begin{aligned}\varphi(k) = {}& 3\alpha(N-1)(N-2)[k(k-1)\rho - \rho_k N(N-1)] - 2N(N-1)(N-2)(1-\alpha(N-1))\rho_k\\ &+ 6\eta[\rho_k N(N-1) - k(k-1)\rho] + 2k(k-1)(k-2)\rho(1-\alpha(N-1)).\end{aligned}$$

Clearly, $u(k) \ge 0$ is equivalent to $\varphi(k) \ge 0$, and

$$\begin{aligned}\Delta\varphi(k) &= 3\alpha(N-1)(N-2)[2k\rho - \Delta\rho_k N(N-1)] - 2N(N-1)(N-2)(1-\alpha(N-1))\Delta\rho_k\\ &\quad + 6\eta[\Delta\rho_k N(N-1) - 2k\rho] + 6k(k-1)\rho(1-\alpha(N-1))\\ &= 6\rho k[(k-1)(1-\alpha(N-1)) + \alpha(N-1)(N-2) - 2\eta] + \Delta\rho_k N(N-1)[6\eta - (N-2)(\alpha(N-1)+2)].\end{aligned}$$

Obviously, if $\Delta\varphi(0) \le 0$ and $\Delta\varphi(N-1) \le 0$, then $\Delta\varphi(k) \le 0, k \in [0, N-1]_{\mathbb{Z}}$. By direct computation, if $\Delta\varphi(0) \le 0$, then

$$\eta \ge \frac{[\alpha(N-1)+2](N-2)}{6} := C_1.$$

And if $\Delta\varphi(N-1) \le 0$, then

$$\eta \ge \frac{[3\rho + \beta N(\alpha(N-1)+2)](N-2)}{6(\rho + \beta N)} := C_2.$$

Combining with the fact $C_i \le \frac{N-2}{2}$, $i = 1, 2$, we obtain $\eta \in \left[\left[\frac{N-2}{2}\right] + 1, N-2\right]_{\mathbb{Z}}$, $\Delta\varphi(k) \le 0, k \in [0, N-1]_{\mathbb{Z}}$ and $\varphi(k) \ge 0, k \in [0, N]_{\mathbb{Z}}$. Hence, we have $u(k) \ge 0, k \in [0, N]_{\mathbb{Z}}$ and $\Delta u(k) \le 0, k \in [0, N-1]_{\mathbb{Z}}$.

Let

$$E = \{u : [0, N]_{\mathbb{Z}} \to \mathbb{R} | \Delta^2 u(\eta) = \alpha \Delta u(N-1), \Delta u(0) = -\beta u(0), u(N) = 0\}.$$

Then E is a Banach space with the norm $\|u\| = \max_{k \in [0,N]_{\mathbb{Z}}} |u(k)|$. Let

$$K_0 = \{h \in E : h(k) \ge 0, k \in [0, N]_{\mathbb{Z}}; \Delta h(k) \le 0, \quad k \in [0, N-1]_{\mathbb{Z}}\}.$$

Then K_0 is a cone in E.

Lemma 2.5. *Assume* (H_0) *hold. If* $h \in K_0$, *then the solution* $u(k)$ *of* (2.1) *belongs to* K_0, *i.e.,* $u \in K_0$. *Furthermore,* u *is concave on* $[0, \eta+1]_{\mathbb{Z}}$.

Proof. Suppose that $0 \le k-2 \le \eta$. Then

$$\begin{aligned} u(k) &= \sum_{s=1}^{N-2} u_1(k,s)h(s) + \sum_{s=1}^{\eta} u_2(k,s)h(s) + \sum_{s=1}^{k-2} u_3(k,s)h(s) \\ &= \sum_{s=1}^{k-2} (u_1(k,s) + u_2(k,s) + u_3(k,s))h(s) + \sum_{s=k-1}^{\eta} (u_1(k,s) + u_2(k,s))h(s) + \sum_{s=\eta+1}^{N-2} u_1(k,s)h(s). \end{aligned}$$

By using the fact that $u_3(k+1, k-1) = 1$, we obtain that

$$\begin{aligned} \Delta u(k) &= \sum_{s=1}^{k-2} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s))h(s) + (u_1(k+1,k-1) + u_2(k+1,k-1) \\ &\quad + u_3(k+1,k-1))h(k-1) + \sum_{s=k-1}^{\eta} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s))h(s) - (u_1(k+1,k-1) \\ &\quad + u_2(k+1,k-1))h(k-1) + \sum_{s=\eta+1}^{N-2} \Delta_k u_1(k,s)h(s) \\ &= \sum_{s=1}^{k-2} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s))h(s) + h(k-1) \\ &\quad + \sum_{s=k-1}^{\eta} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s))h(s) + \sum_{s=\eta+1}^{N-2} \Delta_k u_1(k,s)h(s). \end{aligned}$$

Now, we will show that $\Delta u(k) < 0$ for $0 \le k-2 \le \eta$. If $k-1 \le \eta$, then

$$\begin{aligned} \Delta u(k) &= \sum_{s=1}^{k-1} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s))h(s) + (u_1(k,k-1) + u_2(k,k-1) + u_3(k,k-1))h(k-1) \\ &\quad + \sum_{s=k}^{\eta} (\Delta_k u_1(k,s) + \Delta_k u_2(k,s))h(s) - (u_1(k,k-1) + u_2(k,k-1))h(k-1) + \sum_{s=\eta+1}^{N-2} \Delta_k u_1(k,s)h(s) \end{aligned}$$

$$= \sum_{s=1}^{k-1}(\Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s))h(s) + \sum_{s=k}^{\eta}(\Delta_k u_1(k,s) + \Delta_k u_2(k,s))h(s) + \sum_{s=\eta+1}^{N-2}\Delta_k u_1(k,s)h(s)$$

$$\le h(\eta)\left[\sum_{s=1}^{k-1}(\Delta_k u_1(k,s) + \Delta_k u_2(k,s) + \Delta_k u_3(k,s)) + \sum_{s=k}^{\eta}(\Delta_k u_1(k,s) + \Delta_k u_2(k,s)) + \sum_{s=\eta+1}^{N-2}\Delta_k u_1(k,s)\right]$$

$$= h(\eta)\left[\sum_{s=1}^{N-2}\Delta_k u_1(k,s) + \sum_{s=1}^{\eta}\Delta_k u_2(k,s) + \sum_{s=1}^{k-1}\Delta_k u_3(k,s)\right]$$

$$= h(\eta)\left[\sum_{s=1}^{N-2}\frac{\alpha(N-s-1)\left(2k - \frac{N(N-1)}{\rho}\Delta\rho_k\right)}{2-2\alpha(N-1)} - \sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho} + \sum_{s=1}^{\eta}\frac{\frac{N(N-1)}{\rho}\Delta\rho_k - 2k}{2-2\alpha(N-1)} + \sum_{s=1}^{k-1}(k-s)\right].$$

If $k-1 > \eta$, then $h(k-1) < h(\eta)$, so no matter either $k-1 < \eta$ or $k-1 > \eta$, we always have

$$\Delta u(k) \le h(\eta)\left[\sum_{s=1}^{N-2}\frac{\alpha(N-s-1)(2k - \frac{N(N-1)}{\rho}\Delta\rho_k)}{2-2\alpha(N-1)} - \sum_{s=1}^{N-2}\frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho} + \sum_{s=1}^{\eta}\frac{\frac{N(N-1)}{\rho}\Delta\rho_k - 2k}{2-2\alpha(N-1)} + \sum_{s=1}^{k-1}(k-s)\right]$$
$$= \frac{h(\eta)}{12\rho(1-\alpha(N-1))}[6\rho k((k-1)(1-\alpha(N-1)) + \alpha(N-1)(N-2) - 2\eta) + \Delta\rho_k N(N-1)(6\eta - (N-2)(\alpha(N-1)+2))].$$

By using Remark 2.4, we can deduce that $\Delta u(k) \le 0$.

Moreover, if $k-1 \le \eta$, then

$$\Delta^2 u(k-1) = \sum_{s=1}^{k-1}(\Delta_k^2 u_1(k-1,s) + \Delta_k^2 u_2(k-1,s) + \Delta_k^2 u_3(k-1,s))h(s) + \sum_{s=k}^{\eta}(\Delta_k^2 u_1(k-1,s) + \Delta_k^2 u_2(k-1,s))h(s) + \sum_{s=\eta+1}^{N-2}\Delta_k^2 u_1(k-1,s)h(s)$$
$$= \sum_{s=1}^{k-1}\left(\frac{-\alpha s}{1-\alpha(N-1)} - \frac{(\alpha(N-s-1)-1)N(N-1)\Delta^2\rho_{k-1}}{(2-2\alpha(N-1))\rho} - \frac{(N-s)(N-s-1)\Delta^2\rho_{k-1}}{2\rho}\right)h(s)$$
$$- \sum_{s=k}^{\eta}\frac{(1-\alpha(N-s-1))(2 - \frac{N(N-1)\Delta^2\rho_{k-1}}{\rho})}{2-2\alpha(N-1)}h(s) - \sum_{s=k}^{\eta}\frac{(N-s)(N-s-1)\Delta^2\rho_{k-1}}{2\rho}h(s)$$
$$+ \sum_{s=\eta+1}^{N-2}\left(\frac{\alpha(N-s-1)[2 - \frac{N(N-1)}{\rho}\Delta^2\rho_{k-1}]}{2-2\alpha(N-1)} - \frac{(N-s)(N-s-1)\Delta^2\rho_{k-1}}{2\rho}\right)h(s)$$
$$\le h(\eta)\left[\sum_{s=1}^{N-2}\left(\frac{\alpha(N-s-1)[2 - \frac{N(N-1)}{\rho}\Delta^2\rho_{k-1}]}{2-2\alpha(N-1)} - \frac{(N-s)(N-s-1)\Delta^2\rho_{k-1}}{2\rho}\right) + \sum_{s=1}^{\eta}\frac{\frac{N(N-1)}{\rho}\Delta^2\rho_{k-1} - 2}{2-2\alpha(N-1)} + \sum_{s=1}^{k-1}1\right]$$
$$= \frac{h(\eta)}{2\rho(1-\alpha(N-1))}\left[\rho\{-2\eta + \alpha(N-1)(N-2) + 2(k-1)(1-\alpha(N-1))\} + \Delta^2\rho_{k-1}N(N-1)\{-\frac{(N-2)(1-\alpha(N-1))}{3} - \frac{\alpha(N-1)(N-2)}{2} + \eta\}\right].$$

Since $\eta > \frac{N-2}{2}$, it is enough to show that

$$\begin{aligned}\Delta^2 u(k-1) \le & \frac{h(\eta)}{2\rho(1-\alpha(N-1))}\Big[\rho\{-2\eta + \alpha(N-1)(N-2) + 2\eta(1-\alpha(N-1))\} \\ & + \Delta^2\rho_{k-1}N(N-1)\{-\frac{(N-2)(1-\alpha(N-1))}{3} - \frac{\alpha(N-1)(N-2)}{2} + \eta\}\Big] \\ \le & \, 0.\end{aligned}$$

Suppose that $\eta \le k-2 \le N-2$. We have

$$u(k) = \sum_{s=1}^{\eta}(u_1(k,s) + u_2(k,s) + u_3(k,s))h(s) + \sum_{s=\eta+1}^{N-2} u_1(k,s)h(s) + \sum_{s=\eta+1}^{k-2} u_3(k,s)h(s).$$

We will show that $\Delta u(k) < 0$ for $\eta \le k-2 \le N-2$. Indeed, using the same arguments as mentioned earlier, we obtain that

$$\begin{aligned}\Delta u(k) = & \sum_{s=1}^{\eta}\left[\frac{-\alpha sk + s(\alpha(N-1)-1)}{1-\alpha(N-1)} - \frac{(\alpha(N-s-1)-1)N(N-1)\Delta\rho_k}{(2-2\alpha(N-1))\rho} - \frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho}\right]h(s) \\ & + \sum_{s=\eta+1}^{k-1}(k-s)h(s) + \sum_{s=\eta+1}^{N-2}\left(\frac{\alpha(N-s-1)[2k - \frac{N(N-1)}{\rho}\Delta\rho_k]}{2-2\alpha(N-1)} - \frac{(N-s)(N-s-1)\Delta\rho_k}{2\rho}\right)h(s) \\ \le & \frac{h(\eta)}{12\rho(1-\alpha(N-1))}\{6\rho k((k-1)(1-\alpha(N-1)) + \alpha(N-1)(N-2) - 2\eta) \\ & + \Delta\rho_k N(N-1)[6\eta - (N-2)(\alpha(N-1)+2)]\} \\ \le & \, 0.\end{aligned}$$

So, $\Delta u(k) \le 0$ for $k \in [0, N-1]_{\mathbb{Z}}$, which implies that $u(k)$ is nonincreasing. Since $u(N) = 0$, we have $u(k) \ge 0$, $k \in [0, N]_{\mathbb{Z}}$. For $k \in [1, \eta]_{\mathbb{Z}}$, $\Delta^2 u(k-1) \le 0$, we found that u is concave on $[0, \eta+1]_{\mathbb{Z}}$. ☐

Lemma 2.6. *Let* (H_0) *hold. Assume that* $h \in K_0$ *and* u *is the solution of* (2.1). *Then*

$$\min_{k\in[N-\theta,\theta]_{\mathbb{Z}}} u(k) \ge \theta^* \|u\|,$$

where $\theta^* = \frac{\eta+1-\theta}{\eta+1}$, $\theta \in \left[\left[\frac{N}{2}\right] + 1, \eta + 1\right]_{\mathbb{Z}}$.

Proof. It follows from Lemma 2.5 that $u(k)$ is concave on $[0, \eta+1]_{\mathbb{Z}}$. Therefore,

$$\frac{u(k) - u(0)}{k} \ge \frac{u(\eta+1) - u(0)}{\eta+1}, \quad k \in [0, \eta+1]_{\mathbb{Z}}.$$

Meanwhile, $u(k)$ is nonincreasing on $[0, N]_{\mathbb{Z}}$, which implies that $u(0) = \|u\|$. Therefore,

$$u(k) \ge \frac{\eta+1-k}{\eta+1}u(0) = \frac{\eta+1-k}{\eta+1}\|u\|.$$

Thus,

$$\min_{k\in[N-\theta,\theta]_{\mathbb{Z}}} u(k) = u(\theta) \ge \frac{\eta+1-\theta}{\eta+1}\|u\|.$$ ☐

3 Main results

In this section, we consider the existence of at least one positive solution of the problem (1.1). Assume that

(H_1) $f:[1,N-2]_{\mathbb{Z}}\times[0,\infty)\to[0,\infty)$ is continuous, the mapping $k\to f(k,u)$ is decreasing for each $u\in[0,\infty)$ and the mapping $u\to f(k,u)$ is increasing for each $k\in[1,N-2]_{\mathbb{Z}}$;

(H_2) $a:[1,N-2]_{\mathbb{Z}}\to[0,+\infty)$ is decreasing.

Define the cone K by

$$K=\{u\in K_0|\min_{k\in[N-\theta,\theta]_{\mathbb{Z}}}u(k)\ge\theta^*\|u\|\}$$

and the operator $T_\lambda:K\to E$ by

$$T_\lambda u(k)=\lambda\sum_{s=1}^{N-2}G(k,s)a(s)f(s,u(s)).$$

Lemma 3.1. *$T_\lambda:K\to K$ is a completely continuous operator.*

Proof. It is obvious that $T_\lambda:K\to E$ is a completely continuous operator. Now, let us prove that $T_\lambda:K\to K$. We will show that for any $u\in K$, we have $T_\lambda u\in K$.

Let $u\in K$. Then $u\in K_0$, which gives us that $\Delta u(k)\le 0$. In other words u is decreasing. Then, by (H1), we have that $f(k,u)$ is also decreasing on k. Denote $y(k):=a(k)f(k,u(k))$. By using (H1) and (H2), we obtain that $y(k)\ge 0$ and $y(k)$ is decreasing. Thus, $y\in K_0$.

Moreover, by using the definition of T_λ, one can check that

$$\Delta^3(T_\lambda u)(k-1)=\lambda y(k),\quad k\in[1,N-2]_{\mathbb{Z}}$$

and

$$\Delta^2(T_\lambda u)(\eta)=\alpha\Delta(T_\lambda u)(N-1),\quad \Delta(T_\lambda u)(0)=-\beta(T_\lambda u)(0),\quad (T_\lambda u)(N)=0.$$

Therefore, T_λ satisfies problem (2.1). Now, using similar arguments to the ones given in the proof of Lemma 2.5 and the fact that $y\in K_0$, we know that $T_\lambda u\in K_0$ and $T_\lambda u$ is concave on $[0,\eta+1]_{\mathbb{Z}}$.

Furthermore, by Lemma 2.6 and $T_\lambda u\in K_0$, we deduce that

$$\min_{k\in[N-\theta,\theta]_{\mathbb{Z}}}(T_\lambda u)(k)\ge\theta^*\|T_\lambda u\|.$$

Therefore, $T_\lambda u\in K$ and $T_\lambda:K\to K$ is a completely continuous operator. □

Set

$$\tau=\max\left\{\frac{\eta(2N-\eta-1)-\alpha\eta(N-\eta-1)(N-1)}{\rho},\frac{(N-\eta-1)(\alpha(N-1)\eta+(N-\eta))}{\rho}\right\},$$
$$A=\sum_{s=1}^{N-2}\tau a(s),\quad B=\sum_{s=N-\theta}^{\theta}G(N-\theta,s)a(s).$$

Theorem 3.2. *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If there exist two positive constants r and R with $r\ne R$ such that*

(A_1) $f(k,u)\le\frac{r}{\lambda A}$, $(k,u)\in[1,N-2]_{\mathbb{Z}}\times[0,r]$,

(A_2) $f(k,u)\ge\frac{R}{\lambda B}$, $(k,u)\in[1,N-2]_{\mathbb{Z}}\times[\theta^*R,R]$,

then problem (1.1) *has at least one positive solution $u\in K$ with* $\min\{r,R\}\le\|u\|\le\max\{r,R\}$.

Proof. We only deal with the case $r<R$, since the case that $r>R$ could be treated similarly.

Let $\Omega_1=\{u\in E:\|u\|<r\}$. From Lemma 2.3 (ii), we know that $G(k,s)\le 0$ for $s\in[\eta+1,N-2]_{\mathbb{Z}}$ and $G(k,s)\ge 0$ for $s\in[1,\eta]_{\mathbb{Z}}$. Then by (A_1), for $u\in K\cap\partial\Omega_1$, it is obvious that

$$\begin{aligned}\|T_\lambda u\| &= \lambda \max_{k\in[0,N]_{\mathbb{Z}}} \left|\sum_{s=1}^{N-2} G(k,s)a(s)f(s,u(s))\right| \\ &\le \lambda \max_{k\in[0,N]_{\mathbb{Z}}} \left|\sum_{s=1}^{\eta} G(k,s)a(s)f(s,u(s))\right| + \lambda \max_{k\in[0,N]_{\mathbb{Z}}} \left|\sum_{s=\eta+1}^{N-2} G(k,s)a(s)f(s,u(s))\right| \\ &\le \lambda \sum_{s=1}^{\eta} \frac{\eta(2N-\eta-1)-\alpha\eta(N-\eta-1)(N-1)}{\rho} a(s)f(s,u(s)) \\ &\quad + \lambda \sum_{s=\eta+1}^{N-2} \frac{(N-\eta-1)(\alpha(N-1)\eta+(N-\eta))}{\rho} a(s)f(s,u(s)) \\ &\le \lambda \sum_{s=1}^{N-2} \tau a(s)f(s,u(s)) \\ &\le r.\end{aligned}$$

Therefore,

$$\|T_\lambda u\| \le \|u\|, \quad u \in K \cap \partial\Omega_1. \tag{3.1}$$

Let $\Omega_2 = \{u \in E : \|u\| < R\}$. For $u \in K \cap \partial\Omega_2$ and $k \in [N-\theta, \theta]_{\mathbb{Z}}$, we have

$$T_\lambda u(N-\theta) = \lambda \sum_{s=1}^{N-2} G(N-\theta,s)a(s)f(s,u(s)) \ge \lambda \sum_{s=N-\theta}^{\theta} G(N-\theta,s)a(s)f(s,u(s)). \tag{3.2}$$

In fact,

$$\begin{aligned}&\sum_{s=1}^{N-\theta-1} G(N-\theta,s)a(s)f(s,u(s)) + \sum_{s=\theta+1}^{N-2} G(N-\theta,s)a(s)f(s,u(s)) \\ &\quad \ge \sum_{s=1}^{N-\theta-1} G(N-\theta,s)a(s)f(s,u(s)) + \sum_{s=\eta+1}^{N-2} G(N-\theta,s)a(s)f(s,u(s)) \\ &\quad \ge a(\eta)f(\eta,u(\eta))\left[\sum_{s=1}^{N-\theta-1} G(N-\theta,s) + \sum_{s=\eta+1}^{N-2} G(N-\theta,s)\right].\end{aligned}$$

Furthermore,

$$\begin{aligned}&\sum_{s=1}^{N-\theta-1} G(N-\theta,s) + \sum_{s=\eta+1}^{N-2} G(N-\theta,s) \\ &\ge \sum_{s=1}^{N-\theta-1} \frac{\alpha(N-s-1)[(N-\theta)(N-\theta-1) - \frac{\rho_{N-\theta}}{\rho}N(N-1)]}{2-2\alpha(N-1)} - \sum_{s=1}^{N-\theta-1} \frac{\rho_{N-\theta}(N-s)(N-s-1)}{2\rho} \\ &\quad + \sum_{s=1}^{N-\theta-1} \frac{\frac{\rho_{N-\theta}}{\rho}N(N-1) - (N-\theta)(N-\theta-1)}{2-2\alpha(N-1)} + \sum_{s=1}^{N-\theta-1} \frac{(N-\theta-s)(N-\theta-s-1)}{2} \\ &\quad + \sum_{s=\theta}^{N-2} \frac{\alpha(N-s-1)[(N-\theta)(N-\theta-1) - \frac{\rho_{N-\theta}}{\rho}N(N-1)]}{2-2\alpha(N-1)} - \sum_{s=\theta}^{N-2} \frac{\rho_{N-\theta}(N-s)(N-s-1)}{2\rho} \\ &= \frac{\rho_{N-\theta}N(N-1) - (N-\theta)(N-\theta-1)\rho}{(2-2\alpha(N-1))\rho} \frac{[2-\alpha(N-2+\theta)](N-\theta-1)}{2} \\ &\quad - \frac{\rho_{N-\theta}(N-\theta-1)}{2\rho}\left[N(N-1) - \frac{(N-\theta)(2N+\theta-1)}{3}\right] \\ &\quad + \frac{(N-\theta)(N-\theta-1)(N-\theta-2)}{6} - \frac{\rho_{N-\theta}(N-\theta)(N-\theta+1)(N-\theta-1)}{6\rho} \\ &\quad + \frac{(N-\theta)(N-\theta-1)\rho - N(N-1)\rho_{N-\theta}}{(2-2\alpha(N-1))\rho} \frac{\alpha(N-\theta)(N-\theta-1)}{2}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho_{N-\theta}N(N-1)-(N-\theta)(N-\theta-1)\rho}{2\rho}(N-\theta-1)-\frac{\rho_{N-\theta}(N-\theta-1)}{2\rho}\left[N(N-1)-\frac{(N-\theta)(2N+\theta-1)}{3}\right]\\
&\quad+\frac{(N-\theta)(N-\theta-1)(N-\theta-2)}{6}-\frac{\rho_{N-\theta}(N-\theta)(N-\theta+1)(N-\theta-1)}{6\rho}\\
&= \frac{\rho_{N-\theta}}{6\rho}(N-\theta-1)(N-\theta)(N+2\theta-2)+\frac{(N-\theta)(N-\theta-1)(-2N+2\theta+1)}{6}\\
&\geq \frac{(N-\theta-1)(N-\theta)}{6}(-N+4\theta-1)\\
&\geq 0.
\end{aligned}
$$

Therefore, (3.2) holds, which implies that

$$T_\lambda u(N-\theta) \geq \lambda \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s)\frac{R}{\lambda B} = R.$$

So, for $u \in K \cap \partial\Omega_2$, we have

$$\|T_\lambda u\| \geq \|u\|. \tag{3.3}$$

From Theorem 1.1(i), (3.1), and (3.3), we found that T_λ has a fixed point $u \in K \cap (\overline{\Omega}_2 \backslash \Omega_1)$, and then u is a positive solution of (1.1) with $r \leq \|u\| \leq R$. □

Theorem 3.3. *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If one of the following conditions holds*:

(A_3) $f^0 := \lim_{u\to 0^+} \max_{k\in[1,N-2]_{\mathbb{Z}}} \frac{f(k,u)}{u} = 0$, $f_\infty := \lim_{u\to\infty} \min_{k\in[1,N-2]_{\mathbb{Z}}} \frac{f(k,u)}{u} = \infty$, (*superlinear case*), *or*

(A_4) $f_0 := \lim_{u\to 0^+} \min_{k\in[1,N-2]_{\mathbb{Z}}} \frac{f(k,u)}{u} = \infty$, $f^\infty := \lim_{u\to\infty} \max_{k\in[1,N-2]_{\mathbb{Z}}} \frac{f(k,u)}{u} = 0$, (*sublinear case*),

then for any $\lambda \in (0, \infty)$, *problem* (1.1) *has at least one positive solution.*

Proof. (*Superlinear case*). From (A_3), $f^0 = 0$, then there exists a constant $R_1 > 0$ such that

$$f(k,u) \leq \frac{R_1}{\lambda A}, \quad (k,u) \in [1, N-2]_{\mathbb{Z}} \times [0, R_1].$$

Furthermore, since $f_\infty = \infty$, there exists $R_2 > R_1$ such that

$$f(k,u) \geq \frac{u}{\theta^*\lambda B} \geq \frac{\theta^* R_2}{\theta^*\lambda B} = \frac{R_2}{\lambda B}, \quad (k,u) \in [1, N-2]_{\mathbb{Z}} \times [\theta^* R_2, R_2].$$

Now, by Theorem 3.2, problem (1.1) has a positive solution $u \in K$.

(*Sublinear case*). On the one hand, since $f_0 = \infty$, there exists $r_1 > 0$ such that

$$f(k,u) \geq \frac{u}{\theta^*\lambda B}, \quad (k,u) \in [1, N-2]_{\mathbb{Z}} \times [0, r_1].$$

Set $\Omega_1 = \{u \in E : \|u\| < r_1\}$. If $u \in K \cap \partial\Omega_1$, then

$$\min_{s\in[N-\theta,\theta]_{\mathbb{Z}}} u(s) \geq \theta^*\|u\| = \theta^* r_1.$$

We obtain

$$T_\lambda u(N-\theta) = \lambda \sum_{s=1}^{N-2} G(N-\theta, s)a(s)f(s, u(s)) \geq \lambda \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s)\frac{u(s)}{\theta^*\lambda B} \geq r_1.$$

This implies that

$$\|T_\lambda u\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1.$$

On the other hand, since $f^\infty = 0$, there exists $r_2 > 0$ such that

$$f(k,u) \le \frac{u}{\lambda A}, \quad (k,u) \in [1, N-2]_{\mathbb{Z}} \times [r_2, \infty).$$

We consider two cases: f is bounded and f is unbounded. If f is bounded, i.e., there exists a constant $M > 0$ such that $f \le M$, then we take $r_3 = \max\{2r_2, \lambda MA\}$. If f is unbounded, then we take $r_3 > \max\{2r_1, r_2\}$ such that $f(k,u) \le f(k, r_2)$, $(k,u) \in [1, N-2]_{\mathbb{Z}} \times [0, r_2]$. Set $\Omega_2 = \{u \in E : \|u\| < r_3\}$. Now, similar to the proof of (3.1), we obtain

$$\|T_\lambda u\| \le \|u\|, \quad u \in K \cap \partial\Omega_2.$$

Therefore, by Theorem 1.1, we obtain a positive solution $u \in K$ of problem (1.1). □

Theorem 3.4. *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If* $0 < Af^0 < \theta^* Bf_\infty < \infty$, *then for each* $\lambda \in \left(\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0}\right)$, *problem* (1.1) *has at least one positive solution.*

Proof. For any $\lambda \in \left(\frac{1}{\theta^* Bf_\infty}, \frac{1}{Af^0}\right)$, there exists $\varepsilon > 0$ such that

$$\frac{1}{\theta^* B(f_\infty - \varepsilon)} \le \lambda \le \frac{1}{A(f^0 + \varepsilon)}. \tag{3.4}$$

By the definition of f^0, there exists $r_4 > 0$ such that $f(k,u) \le (f^0 + \varepsilon)u$, for $(k,u) \in [1, N-2]_{\mathbb{Z}} \times [0, r_4]$. Let $\Omega_3 = \{u \in E : \|u\| < r_4\}$, using similar arguments to these given in the proof of (3.1), we can deduce that

$$\|T_\lambda u\| \le \lambda \sum_{s=1}^{N-2} \tau a(s)(f^0 + \varepsilon)u.$$

Furthermore, by (3.4), we obtain

$$\|T_\lambda u\| \le \|u\|, \quad u \in K \cap \partial\Omega_3.$$

By the definition of f_∞, there exists R_3 such that $f(k,u) \ge (f_\infty - \varepsilon)u$, $(k,u) \in [1, N-2]_{\mathbb{Z}} \times [R_3, \infty)$. Let $R_4 = \max\{2r_4, \frac{R_3}{\theta^*}\}$ and $\Omega_4 = \{u \in E : \|u\| < R_4\}$. If $u \in K$ with $\|u\| = R_4$, then $\min_{s \in [N-\theta, \theta]_{\mathbb{Z}}} u(s) \ge \theta^* \|u\|$. Therefore, similar to the discussion from (3.2) and (3.3), we obtain

$$T_\lambda u(N-\theta) \ge \lambda \sum_{s=N-\theta}^{\theta} G(N-\theta, s)(f_\infty - \varepsilon)\theta^* \|u\| a(s) = \lambda \theta^* B(f_\infty - \varepsilon)\|u\|.$$

By (3.9), we obtain

$$\|T_\lambda u\| \ge \|u\|, \quad u \in K \cap \partial\Omega_4. \tag{3.5}$$

By Theorem 1.1 (i), problem (1.1) has at least one positive solution $u \in K$. □

Similar to the discussion of Theorems 3.2–3.4, we could obtain the following two theorems. So, we just state them here without any proof.

Theorem 3.5. *Suppose that* (H_0), (H_1), *and* (H_2) *hold. If* $0 < Af^\infty < \theta^* Bf_0 < \infty$, *then for each* $\lambda \in \left(\frac{1}{\theta^* Bf_0}, \frac{1}{Af^\infty}\right)$, *problem* (1.1) *has at least one positive solution.*

Theorem 3.6. *Suppose* (H_0), (H_1), *and* (H_2) *hold. Then the following results hold.*

(1) *If* $f_\infty = \infty$, $0 < f^0 < \infty$, *then for each* $\lambda \in \left(0, \frac{1}{Af^0}\right)$, *problem* (1.1) *has at least one positive solution.*

(2) *If* $f_0 = \infty$, $0 < f_\infty < \infty$, *then for each* $\lambda \in \left(0, \frac{1}{Af^\infty}\right)$, *problem* (1.1) *has at least one positive solution.*

(3) *If* $f^0 = 0$, $0 < f_\infty < \infty$, *then for each* $\lambda \in \left(\frac{1}{\theta^* Bf_\infty}, \infty\right)$, *problem* (1.1) *has at least one positive solution.*

(4) *If* $f^{\infty} = 0$, $0 < f_0 < \infty$, *then for each* $\lambda \in \left(\frac{1}{\theta^* B f_0}, \infty\right)$, *problem* (1.1) *has at least one positive solution.*

Theorem 3.7. *Assume that* (H_0), (H_1), *and* (H_2) *hold. If*

(F_1) $f_0 := \lim_{u\to 0^+}\min_{k\in[1,N-2]_{\mathbb{Z}}}\frac{f(k,u)}{u} = +\infty$, $f_\infty := \lim_{u\to\infty}\min_{k\in[1,N-2]_{\mathbb{Z}}}\frac{f(k,u)}{u} = +\infty$, *and*

(F_2) *There exists a constant* $p > 0$ *such that* $f(s,u) \le \delta p$ *for* $0 \le u \le p$ *and* $s \in [1, N-2]_{\mathbb{Z}}$, *where*

$$\delta = \left(\lambda \sum_{s=1}^{N-2} \tau a(s)\right)^{-1},$$

then problem (1.1) *has at least two positive solutions* u_1 *and* u_2 *with* $0 \le \|u_1\| \le p \le \|u_2\|$.

Proof. Since $f_0 = +\infty$, there exists a constant r with $0 < r < p$ such that $f(t,u) \ge Mu$ for $0 \le u \le r$, where $M > 0$ satisfies

$$\lambda M\theta^* \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s) \ge 1.$$

Let $K_r = \{u \in K : \|u\| < r\}$. Then, for $\forall u \in \partial K_r$, we have

$$\begin{aligned} T_\lambda u(N-\theta) &= \lambda \sum_{s=1}^{N-2} G(N-\theta, s)a(s)f(s,u(s)) \\ &\ge \lambda \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s)f(s,u(s)) \\ &\ge \lambda M\theta^* \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s)\|u\| \\ &\ge \|u\|. \end{aligned}$$

Since $f_\infty = +\infty$, there exists a constant $R_1 \ge 0$ such that $f(t,u) \ge Mu$ for $u \ge R_1$. Let $K_R = \{u \in K : \|u\| < R\}$. Choose $R \ge \max\left\{p, \frac{R_1}{\theta^*}\right\}$, then for $u \in \partial K_R$, $\min_{t\in[N-\theta,\theta]_{\mathbb{Z}}} u(t) \ge \theta^*\|u\| > R_1$. Similar to the aforementioned proof, we have $\|Tu\| \ge \|u\|$ for $u \in \partial K_R$. Let $K_p = \{u \in K : \|u\| < p\}$. From (F_2), for $u \in \partial K_p$,

$$\|T_\lambda u\| = \lambda \max_{k\in[0,N]_{\mathbb{Z}}} \left|\sum_{s=1}^{N-2} G(k,s)a(s)f(s,u(s))\right| \le \lambda \sum_{s=1}^{N-2} \tau a(s)f(s,u(s)) \le \|u\|.$$

Therefore, for $u \in \partial K_p$, $\|Tu\| \le \|u\|$. So, T has a fixed point u_1 in $K_p \backslash \mathring{K}_r$ and anther fixed point u_2 in $K_R \backslash \mathring{K}_p$. Therefore, problem (1.1) has at least two positive solutions u_1 and u_2 with $0 \le \|u_1\| \le p \le \|u_2\|$. □

Similar to the discussion of Theorem 3.7, we could obtain the following theorem. So, we just state it here without any proof.

Theorem 3.8. *Assume that* (H_0), (H_1), *and* (H_2) *hold. If*

(L_1) $f^0 := \lim_{u\to 0^+}\max_{k\in[1,N-2]_{\mathbb{Z}}}\frac{f(k,u)}{u} = 0$, $f^\infty := \lim_{u\to\infty}\max_{k\in[1,N-2]_{\mathbb{Z}}}\frac{f(k,u)}{u} = 0$, *and*

(L_2) *there exists a constant* $q > 0$ *such that* $f(s,u) > \beta q$ *for* $\theta^* q \le u \le q$ *and* $s \in [1, N-2]_{\mathbb{Z}}$, *where*

$$\beta = \left(\lambda\theta^* \sum_{s=N-\theta}^{\theta} G(N-\theta, s)a(s)\right)^{-1},$$

then problem (1.1) *has at least two solutions* u_1 *and* u_2 *with* $0 \le \|u_1\| \le q \le \|u_2\|$.

4 Example

Example 4.1. Consider the following discrete third-order three-point BVP:

$$\begin{cases}\Delta^3 u(k-1)=\lambda a(k)f(k,u(k)), & k\in[1,9]_{\mathbb{Z}},\\ \Delta^2 u(\eta)=\dfrac{1}{12}\Delta u(10),\ \Delta u(0)=-\dfrac{1}{44}u(0), & u(11)=0,\end{cases}\tag{4.1}$$

where $\lambda=\frac{4}{5}$, $a(k)=\frac{10-k}{10}$, $f(k,u)=\sqrt[3]{u}+20-2k$. Now, according to Remark 2.4, let $\eta=6$. Furthermore, we obtain the expression of $G(t,s)$ as follows.

If $s>6$, then

$$G(k,s)=\begin{cases}\dfrac{(10-s)(77-s)}{44}k^2+\dfrac{(10-s)(187-3s)}{44}k-(10-s)(264-4s), & s>k-2,\\ \dfrac{s^2-87s+792}{44}k^2+\dfrac{3s^2-261s+1848}{44}k-\dfrac{7}{2}s^2+\dfrac{609}{2}s-2640, & s\le k-2.\end{cases}$$

If $s\le 6$, then

$$G(k,s)=\begin{cases}\dfrac{s^2-87s-22}{44}k^2+\dfrac{3s^2-217s+22}{44}k-4s^2+304s, & s>k-2,\\ \dfrac{s^2-87s}{44}k^2+\dfrac{3s^2-261s}{44}k-\dfrac{7}{2}s^2+\dfrac{609}{2}s, & s\le k-2.\end{cases}$$

Obviously, if $(k,s)\in[0,11]_{\mathbb{Z}}\times[1,6]_{\mathbb{Z}}$, then $\Delta_k G(k,s)\le 0$. This implies that $G(k,s)$ is nonincreasing with respect to k in this case. Hence, $\max_{k\in[0,N]_{\mathbb{Z}}}G(k,s)=G(0,s)=-4s+304s\le 1{,}680$, $\min_{k\in[0,N]_{\mathbb{Z}}}G(k,s)=G(N,s)=0$ and $G(k,s)\ge 0$. If $(k,s)\in[0,11]_{\mathbb{Z}}\times[7,9]_{\mathbb{Z}}$, then $\Delta_k G(k,s)\ge 0$. This implies that $G(k,s)$ is nondecreasing with respect to k in this case. Therefore, $\min_{k\in[0,N]_{\mathbb{Z}}}G(k,s)=G(0,s)=-(10-s)(264-4s)\ge -960$, $\max_{k\in[0,N]_{\mathbb{Z}}}G(k,s)=G(N,s)=0$, and $G(k,s)\le 0$.

Now, by direct calculation, we choose $\theta=6$, then $\theta^*=\frac{\eta+1-\theta}{\eta+1}=\frac{1}{7}$. So

$$\begin{aligned}\tau&=\max\{1{,}680,960\}=1{,}680,\\ A&=\sum_{s=1}^{N-2}\tau a(s)=1{,}680\sum_{s=1}^{9}\frac{10-s}{10}=7{,}560,\\ B&=\sum_{s=N-\theta}^{\theta}G(N-\theta,s)a(s)=\sum_{s=5}^{6}\left(\frac{25(s^2-87s-22)}{44}+\frac{5(3s^2-217s+22)}{44}-4s^2+304s\right)\frac{10-s}{10}=\frac{22{,}757}{22}.\end{aligned}$$

We choose $R=448$, $r=1{,}000{,}000$, and from Theorem 3.2, problem (4.1) has at least one positive solution.

Example 4.2. Consider the following discrete third-order three-point BVP:

$$\begin{cases}\Delta^3 u(k-1)=\lambda a(k)f(k,u(k)), & k\in[1,5]_{\mathbb{Z}},\\ \Delta^2 u(\eta)=\dfrac{1}{7}\Delta u(6),\ \Delta u(0)=-\dfrac{1}{56}u(0), & u(7)=0,\end{cases}\tag{4.2}$$

where $\eta=4$, $a(k)=\frac{9-k}{7}$, $f(k,u)=\frac{(8-k)u^2\log_3(u+1)}{11}$. By direct computation, we have $f^0=0, f_\infty=\infty$. Furthermore, by Theorem 3.3, for $\lambda\in(0,\infty)$, (4.2) has at least one positive solution.

Example 4.3. In this example, we continue to discuss problem (4.1) with

$$f(k,u)=\begin{cases}\dfrac{u^2+(10-k)}{6{,}323}, & (k,u)\in[1,9]_{\mathbb{Z}}\times[0,1],\\ \dfrac{\sqrt{u^3}+(10-k)}{6{,}323}, & (k,u)\in[1,9]_{\mathbb{Z}}\times[1,\infty).\end{cases}$$

We take $a(k) = \frac{10-k}{10}$, $\lambda = \frac{1}{120}$, and $\eta = 6$. Then

$$\delta = \left(\lambda \sum_{s=1}^{N-2} \tau a(s)\right)^{-1} = \frac{1}{63}.$$

Furthermore, if we choose $p = 1$, then $f(k, u) \le \delta p$ for $0 \le u \le p$. From Theorem 3.7, problem (4.1) has at least two positive solutions u_1 and u_2 with $0 \le \|u_1\| \le p \le \|u_2\|$.

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