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Review Article

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Multiple periodic solutions for discrete boundary value problem involving the mean curvature operator

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Abstract: In this article, by using critical point theory, we prove the existence of multiple T-periodic solutions for difference equations with the mean curvature operator:

$$-\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda f(t, u(t)), \quad t \in \mathbb{Z},$$

where $\mathbb{Z}$ is the set of integers. As a T-periodic problem, it does not require the nonlinear term is unbounded or bounded, and thus, our results are supplements to some well-known periodic problems. Finally, we give one example to illustrate our main results.

Keywords: difference equations, mean curvature operator, periodic solutions, critical point theory

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1 Introduction

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, $\mathbb{Z}(a, b)$ denotes the discrete interval $\{a, a+1 \ldots, b\}$ if $a \le b$.

In this article, we consider the following nonlinear difference equations with mean curvature operator

$$-\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda f(t, u(t)), \quad t \in \mathbb{Z}, \tag{1}$$

where λ is a positive real parameter, Δ is the forward difference operator defined by $\Delta u(t) = u(t+1) - u(t)$, T is an integer, $q(t) : \mathbb{Z} \to \mathbb{R}^+$ and is T-periodic function, $f(t, \cdot) \in C^1(\mathbb{R}, \mathbb{R})$ satisfies $f(t, 0) = 0$ for each $k \in \mathbb{Z}$, $f(t, u) = f(t+T, u)$, and ϕ_c is the mean curvature operator defined by $\phi_c(s) = \frac{s}{\sqrt{1+\kappa s^2}} : \mathbb{R} \to \left(-\frac{1}{\sqrt{\kappa}}, \frac{1}{\sqrt{\kappa}}\right)$, where $\kappa > 0$. For general background on the mean curvature operator, we refer to [1–4].

Difference equations have been widely used in various research fields such as computer science, economics, biology, and other fields [5–7]. Recently, many excellent results for difference equations have been achieved, for example, positive solutions [8–12], homoclinic solutions [13,14], and ground-state solutions [15]. To study the existence of solutions for the discrete boundary value problems of difference equations, many authors give some important tools, such as the fixed point theory, critical point theory, and upper and lower solution techniques [3,8,16,17].

Problem (1) may be regarded as the discrete analog of the following one-dimensional prescribed mean curvature equation:

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$$-(\phi_c(u'))' + q(t)u = \lambda f(t, u), \tag{2}$$

which has also been investigated by many authors. For example, see [1,3,4] and the references therein.

If $\kappa = 0$, $q(t) = 0$ for $t \in \mathbb{Z}$ and $\lambda = 1$, then problem (1) is degenerated to

$$-\Delta^2 u(t-1) = f(t, u(t)), \quad t \in \mathbb{Z}. \tag{3}$$

The existence and multiplicity results of periodic and subharmonic solutions for problem (3) have been obtained by means of the variational methods in [18].

In addition, if $\kappa = 0$ and $\lambda = 1$, many authors considered the existence results for the following difference equation by means of the variational methods,

$$\Delta[p(t)\Delta u(t-1)] + q(t)u(t) = f(t, u(t)), \quad t \in \mathbb{Z}. \tag{4}$$

For example, Yu et al. in [23] studied the existence of periodic solution of problem (4), where they considered that the nonlinear term was unbounded or bounded, respectively.

Assume that N is a positive integer. If $\kappa = 1$, $q(t) = 0$ for $t \in \mathbb{Z}(1, N)$, Zhou and Ling [11] first used the critical point theory to study the following discrete boundary value problem:

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) = \lambda f(t, u(t)), & t \in \mathbb{Z}(1, N), \\ u(0) = u(N+1) = 0. \end{cases} \tag{5}$$

They proved the existence of infinitely many positive solutions of problem (5) under the suitable oscillating behavior of the nonlinear term f at infinity.

In this article, our main aim is to use the critical point theory to establish the existence of multiple T-periodic solutions of problem (1). We consider that problem (1) is a T-periodic problem, and hence, problem (1) reduces to the following periodic boundary value problem:

$$\begin{cases} -\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) = \lambda f(t, u(t)), & t \in \mathbb{Z}(1, T). \\ u(0) = u(T), u(1) = u(T+1). \end{cases} \tag{6}$$

Obviously, problem (6) is a more general difference equation with mean curvature operator. Although many excellent results have been worked out on the existence of periodic solutions for difference equations [18–21,23], the multiple periodic solutions of the discrete boundary value problem involving the mean curvature operator show that no similar results were obtained in the literature.

This article is organized as follows. In Section 2, we present some definitions and results of the critical point theory. We establish the variational framework of problem (6) and transfer the existence of periodic solutions of problem (6) into the existence of critical points of the corresponding functional. In Section 3, we obtain the existence and multiple periodic solutions for boundary value problem (6), and we give an example to illustrate our main results.

2 Preliminaries

In this section, our aim is to establish the existence of multiple periodic solutions of problem (6) by means of critical point theory. First, we recall some basic definitions and known results from the critical point theory.

Consider the T-dimensional Banach space:

$$S = \{u : [0, T+1] \to \mathbb{R} \quad \text{such that } u(0) = u(T), u(1) = u(T+1)\}$$

endowed with the norm

$$\|u\| = \left(\sum_{t=1}^{T} |u(t)|^2\right)^{\frac{1}{2}}.$$

Let E be a real Banach space and $J \in C^1(E, \mathbb{R})$. A sequence $\{u_n\} \subset E$ is called a Palais-Smale sequence (P.S. sequence) for J if $\{J(u_n)\}$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$. We say J satisfies the Palais-Smale condition (P.S. condition) if any P.S. sequence for J possesses a convergent subsequence in E.

Let E be a finite dimensional real Banach space and $J_\lambda : E \to \mathbb{R}$ be a function satisfying the following structure hypothesis:

(H) Assume that λ is a real positive parameter. $J_\lambda = \Phi(u) + \lambda\Psi(u)$ for all $u \in E$, where $\Phi, \Psi \in C^1(E, \mathbb{R})$, Φ is coercive, that is, $\lim_{\|u\|\to\infty}\Phi(u) = +\infty$.

Put

$$\varphi_1(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\Psi(u) - \inf_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r - \Phi(u)},$$

and

$$\varphi_2(r) = \inf_{u\in\Phi^{-1}(-\infty,r)} \sup_{v\in\Phi^{-1}(r,+\infty)} \frac{\Psi(u) - \Psi(v)}{\Phi(v) - \Phi(u)}.$$

The following lemma will be used to prove our main results.

Lemma 2.1. [22] *Assume that* (H) *and the following conditions hold,*

(a_1) *For each* $\lambda > 0$*, the functional* $J_\lambda = \Phi(u) + \lambda\Psi(u)$ *satisfies the P.S. condition and it is bounded from below*;

(a_2) *There exists* $r > \inf_E\Phi$ *such that* $\varphi_1(r) < \varphi_2(r)$.

Then, for $\lambda \in \left(\frac{1}{\varphi_2(r)}, \frac{1}{\varphi_1(r)}\right)$, J_λ *has at least three critical points.*

For every $u \in S$, put

$$\Phi(u) = \sum_{t=1}^{T}\left(\left(\frac{\sqrt{1 + \kappa(\Delta u(t))^2} - 1}{\kappa}\right) + \frac{q(t)u(t)^2}{2}\right), \Psi(u) = -\sum_{t=1}^{T}F(t, u(t)), \tag{7}$$

and

$$J_\lambda(u) = \Phi(u) + \lambda\Psi(u), \tag{8}$$

where $F(t, \xi) = \int_0^\xi f(t, s)ds$, $(t, \xi) \in \mathbb{Z}(1, T) \times \mathbb{R}$, then $J_\lambda \in C^1(S, \mathbb{R})$. By using $u(0) = u(T)$, $u(1) = u(T + 1)$, we can compute the Frećhet derivative as follows:

$$\langle J_\lambda'(u), v\rangle = \sum_{t=1}^{T}(-\Delta(\phi_c(\Delta u(t-1))) + q(t)u(t) - \lambda f(t, u(t)))v(t), \quad t \in \mathbb{Z}(1, T)$$

for all $u, v \in S$. It is clear that the critical points of J_λ are the solutions of problem (6).

Lemma 2.2. *Assume that*

(i) *There exists a positive constant* $q \in [1, 2]$ *such that*

$$\limsup_{|\xi|\to\infty}\frac{F(t, \xi)}{|\xi|^q} = 0, \quad \forall t \in \mathbb{Z}(1, T).$$

Then J_λ *satisfies the P.S. condition, and it is coercive on* S.

Proof. For any sequence $\{u_n\} \subset S$, with $\{J_\lambda(u_n)\}$ is bounded and $J_\lambda'(u_n) \to 0$ as $n \to +\infty$, there exists a positive constant $C \in \mathbb{R}$ such that $|J_\lambda(u_n)| \le C$. We shall prove the sequence $\{u_n\}$ is bounded.

If not, we assume $\|u_n\| \to +\infty$ as $n \to \infty$. From the condition (i), we take $\varepsilon \in \left(0, \frac{q_*}{2\lambda}\right)$, there exists $M > 0$ such that

$$|F(t, \xi)| \le \varepsilon|\xi|^q + M \quad \text{for each } t \in \mathbb{Z}(1, T), \xi \in \mathbb{R}, \tag{9}$$

where $q_* = \min_{t\in\mathbb{Z}(1,T)} q(t)$. Thus, we have

$$\begin{aligned} C &\ge J_\lambda(u_n) \\ &= \sum_{t=1}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u_n(t))^2}-1}{\kappa}\right) + \frac{q(t)(u_n(t))^2}{2}\right) - \lambda\sum_{t=1}^{T} F(t, u_n(t)) \\ &\ge \sum_{t=1}^{T}\frac{(\Delta u_n(t))^2}{2\sqrt{1+\kappa(\Delta u_n(t))^2}} + \frac{q_*}{2}\sum_{t=1}^{T}|u_n(t)|^2 - \varepsilon\lambda\sum_{t=1}^{T}|u_n(t)|^q - \lambda MT \\ &\ge \frac{q_*}{2}\|u_n\|^2 - \varepsilon\lambda(T)^{\frac{2-q}{2}}\|u_n\|^q - \lambda MT - \frac{T}{\sqrt{2\kappa}} \to +\infty \text{ as } n \to \infty. \end{aligned}$$

This contradicts the fact $|J_\lambda(u_n)| \le C$. Thus, the sequence $\{u_n\}$ is bounded in S and the Bolzano-Weierstrass theorem implies that $\{u_n\}$ has a convergent subsequence.

In fact,

$$J_\lambda(u) \ge \frac{q_*}{2}\|u\|^2 - \varepsilon\lambda(T)^{\frac{2-q}{2}}\|u\|^q - \lambda MT - \frac{T}{\sqrt{2\kappa}} \to +\infty \text{ as } \|u\| \to \infty.$$

Then J_λ is coercive on S. The proof is complete. □

Remark 2.1. Since S be a finite dimensional real Banach space, if J_λ is coercive on S, then the conclusion (a_1) of Lemma 2.1 holds. From the condition (i), the nonlinear term can be unbounded or bounded, and it does not require any asymptotic condition or a superlinear growth at infinity. It is different from the conditions of the literature [23].

3 Main results

Put

$$q^* = \max_{t\in\mathbb{Z}(1,T)} q(t), \qquad Q = \sum_{t=1}^{T} q(t).$$

Theorem 3.1. *Assume that the condition* (i) *is satisfied and there exist two positive constants* c *and* d *with* $0 < \sqrt{\frac{4+q^*}{Q}}\,c < d$ *such that*

$$\frac{\sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{c^2} < \frac{4+q^*}{Q}\,\frac{\sum_{t=1}^{T}F(t,d) - \sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{d^2}, \tag{10}$$

Then, for each $\lambda \in \left(\frac{Q}{2}\frac{d^2}{\sum_{t=1}^{T}F(t,d) - \sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}, \frac{4+q^*}{2}\frac{c^2}{\sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}\right)$, *problem* (6) *admits at least three periodic solutions.*

Proof. Our aim is to apply Lemma 2.1 to prove our conclusion.

We take Φ and Ψ defined as in (7) on the space S. It is easy to verify that Φ and Ψ satisfy the assumptions required in (H). In addition, it follows from Lemma 2.2 that (a_1) of Lemma 2.1 hold. It remains to verify (a_2).

Put

$$r = \frac{(4+q^*)c^2}{2}.$$

Let $u \in S$, we have

$$\sum_{t=1}^{T}(\Delta u(t))^2 \le \sum_{t=1}^{T}(|u(t+1)| + |u(t)|)^2 \le 2\left(\sum_{t=1}^{T}|u(t+1)|^2 + \sum_{t=1}^{T}|u(t)|^2\right) \le 4\|u\|^2,$$

since

$$\begin{aligned}
\Phi(u) &= \sum_{t=1}^{T}\left(\left(\frac{\sqrt{1+\kappa(\Delta u(t))^2} - 1}{\kappa}\right) + \frac{q(t)u(t)^2}{2}\right)\\
&\le \sum_{t=1}^{T}\frac{(\Delta u(t))^2}{\sqrt{1+\kappa(\Delta u(t))^2} + 1} + \frac{q^*}{2}\|u\|^2\\
&\le \frac{1}{2}\sum_{t=1}^{T}(\Delta u(t))^2 + \frac{q^*}{2}\|u\|^2\\
&\le \frac{4+q^*}{2}\|u\|^2.
\end{aligned}$$

If $\frac{4+q^*}{2}\|u\|^2 < r$, then we have

$$|u(t)| \le \max_{j\in\mathbb{Z}(1,T)}\{|u(j)|\} \le \|u\| < \left(\frac{2r}{4+q^*}\right)^{\frac{1}{2}} = c \tag{11}$$

for each $t \in \mathbb{Z}(1, T)$.

By (11), we obtain

$$\begin{aligned}
\varphi_1(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)}\frac{\Psi(u) - \inf_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r - \Phi(u)}\\
&\le \frac{-\inf_{u\in\Phi^{-1}(-\infty,r)}\Psi(u)}{r}\\
&\le \frac{\sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{\frac{(4+q^*)c^2}{2}}\\
&\le \frac{2}{4+q^*}\frac{\sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{c^2}.
\end{aligned}$$

Let $w(t) = d$ for every $t \in \mathbb{Z}(0, T+1)$, clearly, $w \subset S$. Since $d > \sqrt{\frac{4+q^*}{Q}}c$, we have $\Phi(w) = \frac{Q}{2}d^2 > \frac{4+q^*}{2}c^2 = r$.

Hence, we obtain

$$\begin{aligned}
\varphi_2(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)}\sup_{v\in\Phi^{-1}(r,+\infty)}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)}\\
&\ge \inf_{u\in\Phi^{-1}(-\infty,r)}\frac{\sum_{t=1}^{T}F(t,d) - \sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{\frac{Q}{2}d^2 - \Phi(u)}\\
&> \frac{2}{Q}\frac{\sum_{t=1}^{T}F(t,d) - \sum_{t=1}^{T}\max_{|\xi|\le c}F(t,\xi)}{d^2}.
\end{aligned} \tag{12}$$

By (10), we have $\varphi_1(r) < \varphi_2(r)$. The proof is complete. □

Remark 3.1. If $f(t,\cdot):\mathbb{R}\to\mathbb{R}$ is a continuous odd function and the conditions of Theorem 3.1 hold. Then problem (6) admits at least five periodic solutions. In fact, let u be one solution of problem (6). We see that $-u$ is also solution of problem (6) when $f(t,\cdot)$ is odd. If the conditions of Theorem 3.1 are satisfied and $u=0$ is the trivial solution of problem (6), then problem (6) admits at least four different nontrivial periodic solutions and one zero solution. Moreover, if $u=0$ is not the solution of problem (6), then problem (6) admits at least six different nontrivial periodic solutions.

Corollary 3.1. *If the conditions* (i) *holds and there exist two positive constants c and d with* $0<c<d$ *such that*

$$\frac{F(t,\xi)}{c^2}<\frac{q^*}{2Q}\frac{F(t,d)}{d^2}\quad \text{for } \xi\in[-c,c]\quad\text{and}\quad t\in\mathbb{Z}(1,T). \tag{13}$$

Then, for each $\lambda\in\left(\frac{Q}{2}\frac{d^2}{\sum_{t=1}^T F(t,d)-\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)},\frac{4+q^*}{2}\frac{c^2}{\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)}\right)$, *problem* (6) *admits at least three periodic solutions.*

Proof. In fact, in view of $0<c<d$ and (13) holds, we obtain

$$\begin{aligned}\frac{\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)}{c^2}&\le\frac{q^*}{2Q}\frac{\sum_{t=1}^TF(t,d)}{d^2}\\&<\frac{q^*}{Q}\frac{\sum_{t=1}^TF(t,d)}{d^2}-\frac{\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)}{d^2}\\&<\frac{q^*}{Q}\frac{\sum_{t=1}^TF(t,d)-\frac{Q}{q^*}\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)}{d^2}\\&<\frac{4+q^*}{Q}\frac{\sum_{t=1}^TF(t,d)-\sum_{t=1}^T\max_{|\xi|\le c}F(t,\xi)}{d^2},\end{aligned}$$

and hence, hypotheses of Theorem 3.1 are satisfied and our corollary holds. □

Theorem 3.2. *Let* $u\in S$, *assume that there exist two positive constants c and d with* $0<\sqrt{\frac{4+q^*}{Q}}c<d$ *and the condition* (i) *holds; moreover, we suppose that*

(ii) $\max_{|\xi|\le c}F(t,\xi)\le 0$ *for each* $t\in\mathbb{Z}(1,T)$;

(iii) $\sum_{t=1}^TF(t,d)>0$.

Then, for each $\lambda\in\left(\frac{Q}{2}\frac{d^2}{\sum_{t=1}^TF(t,d)},+\infty\right)$, *problem* (6) *admits at least three periodic solutions.*

Proof. We note that $f(t,0)=0$ for every $t\in\mathbb{Z}(1,T)$ by (ii), let $r=\frac{(4+q^*)c^2}{2}$ be the above, then it follows from $\Phi(u)\le\frac{4+q^*}{2}\|u\|^2<r$ that $\max_{t\in\mathbb{Z}(1,T)}\{|u(t)|\}<c$. Since (ii), we have $\inf_{\Phi^{-1}(-\infty,r)}\Psi=0$, which implies $\varphi_1(r)=0$. By choosing $w\subset S$ as Theorem 3.1, we obtain

$$\begin{aligned}\varphi_2(r)&=\inf_{u\in\Phi^{-1}(-\infty,r)}\sup_{v\in\Phi^{-1}(r,+\infty)}\frac{\Psi(u)-\Psi(v)}{\Phi(v)-\Phi(u)}\\&\ge\inf_{u\in\Phi^{-1}(-\infty,r)}\frac{\sum_{t=1}^TF(t,d)-\sum_{t=1}^T\max_{|\xi|\le c}\int_0^\xi f(t,s)ds}{\frac{Q}{2}d^2-\Phi(u)}\\&>\frac{2}{Q}\frac{\sum_{t=1}^TF(t,d)}{d^2}>0.\end{aligned}\tag{14}$$

Thus, we have $\varphi_1(r)=0<\varphi_2(r)$. By Lemma 2.1, we know that problem (6) admits at least two nontrivial periodic solutions for each $\lambda\in\left(\frac{Q}{2}\frac{d^2}{\sum_{t=1}^TF(t,d)},+\infty\right)$. The proof is complete. □

Finally, we give an example to illustrate our results.

Example 3.1. We consider the periodic problem (6) with

$$f(t,u)=\begin{cases}0 & \text{if } u<-1,\\ \frac{\pi}{2}\cos^2\left(\frac{\pi t}{T}\right)\cos(\frac{\pi}{2}u), & \text{if } |u|\le 1,\\ e^{-u}u^3(4-u)-3e^{-1}, & \text{if } u>1,\end{cases}$$

for all $t\in\mathbb{Z}(1,T)$, T is a given positive integer, obviously, $f(t,u)=f(t+T,u)$. Then

$$F(t,u)=\begin{cases}-\cos^2\left(\frac{\pi t}{T}\right) & \text{if } u<-1,\\ \cos^2\left(\frac{\pi t}{T}\right)\sin(\frac{\pi}{2}u), & \text{if } |u|\le 1,\\ e^{-u}u^4-3e^{-1}u+\cos^2\left(\frac{\pi t}{T}\right)+2e^{-1}, & \text{if } u>1.\end{cases}$$

For each $t\in\mathbb{Z}(1,T)$, there exists $q=2$ such that

$$\limsup_{u\to-\infty}\frac{F(t,u)}{|u|^2}=\lim_{u\to-\infty}\frac{-\cos^2\left(\frac{\pi t}{T}\right)}{|u|^2}\to 0,$$

and

$$\limsup_{u\to+\infty}\frac{F(t,u)}{|u|^2}=\lim_{u\to+\infty}\frac{e^{-u}u^4-3e^{-1}u+\cos^2\left(\frac{\pi t}{T}\right)+2e^{-1}}{|u|^2}\to 0.$$

In addition, let $T=2$, $Q=1$, $q^*=\frac{1}{2}$, $\kappa=1$, $c=1$, and $d=3$, we have $0<\sqrt{\frac{4+q^*}{Q}}c=\frac{3\sqrt{2}}{2}<d=3$. Thus,

$$\frac{\sum_{t=1}^{2}\max_{|\xi|\le 1}F(t,\xi)}{1^2}=1<\frac{9}{2}\,\frac{\sum_{t=1}^{2}F(t,3)-\sum_{t=1}^{2}\max_{|\xi|\le 1}F(t,\xi)}{3^2}=3^4\cdot e^{-3}-7\cdot e^{-1}\approx 1.4.$$

All conditions of Theorem 3.1 hold, we note that $f(2,0)=\frac{\pi}{2}\neq 0$, and problem (6) admits at least three nontrivial periodic solutions for each $\lambda\in\left(\frac{45}{28},\frac{9}{4}\right)$.

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