Research Article

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Exponential stability of traveling waves for a nonlocal dispersal SIR model with delay

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Abstract: This article is concerned with the nonlinear stability of traveling waves of a delayed susceptible-infective-removed (SIR) epidemic model with nonlocal dispersal, which can be seen as a continuity work of Li et al. [*Traveling waves for a nonlocal dispersal SIR model with delay and external supplies*, Appl. Math. Comput. **247** (2014), 723–740]. We prove that the traveling wave solution is exponentially stable when the initial perturbation around the traveling wave is relatively small in a weighted norm. The time decay rate is also obtained by weighted-energy estimates.

Keywords: stability, delayed reaction-diffusion system, traveling waves, weighted-energy method

MSC 2020: 35C07, 92D25, 35B35

1 Introduction

In this article, we investigate the exponential stability of traveling waves in the following nonlocal dispersal delayed susceptible-infective-removed (SIR) epidemic model:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_S\left[\displaystyle\int_{\mathbb{R}} J(x-y)S(y,t)\mathrm{d}y - S(x,t)\right] + \Lambda - \sigma S(x,t) - F(S(x,t), I(x,t-\tau)), \\ \dfrac{\partial I(x,t)}{\partial t} = d_I\left[\displaystyle\int_{\mathbb{R}} J(x-y)I(y,t)\mathrm{d}y - I(x,t)\right] + F(S(x,t), I(x,t-\tau)) - (\mu+\gamma)I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_R\left[\displaystyle\int_{\mathbb{R}} J(x-y)R(y,t)\mathrm{d}y - R(x,t)\right] + \gamma I(x,t) - \varrho R(x,t), \end{cases} \tag{1.1}$$

where $d_S, d_I, d_R, \Lambda, \sigma, \mu, \gamma$ and ϱ are the positive constants. Here, $S(x,t)$, $I(x,t)$ and $R(x,t)$ stand for the densities of susceptible, infective, and removed individuals at position x and time t, respectively. The parameters d_S, d_I and d_R describe the spatial motility of each class; the constant $\Lambda > 0$ represents the entering flux of the susceptible; $\gamma > 0$ is the recovery rate of the infective population; σ, μ, and ϱ are all positive parameters representing the death rates for all the susceptible, infective, and removed population, respectively; $\tau > 0$ denotes the latent period of the disease. Moreover, $J(y)$ denotes the probability distribution of rates of dispersal over distance y and $\int_{\mathbb{R}} J(x-y)v(y,t)\mathrm{d}y - v(x,t)$ can be interpreted as the net rate of increase due to dispersal of class v, where $\int_{\mathbb{R}} J(x-y)v(y,t)\mathrm{d}y$ is the standard convolution with

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space invariable x and v can be either S, I or R. In practical use, there are various types of the incidence term $F(S, I)$. The common types include bilinear incidence βSI, standard incidence $\frac{\beta SI}{S+I+R}$ and saturated incidence $\frac{\beta SI}{1+\alpha I}$.

In this article, we focus on the case of saturated incidence, and therefore, we assume that $F(S, I) = \frac{\beta SI}{1+\alpha I}$. Observing that the first two equations of (1.1) form a closed system and the function R can be derived as long as both S and I are solved, from now on, we only consider the first two equations of (1.1). Mathematically, for convenience, letting

$$\tilde{u}_1 = \frac{\sigma}{\Lambda}S, \quad \tilde{u}_2 = \frac{\sigma}{\Lambda}I, \quad \tilde{x} = x, \quad \tilde{t} = d_I t, \quad \tilde{\tau} = d_I\tau, \quad \tilde{d} = \frac{d_S}{d_I},$$

$$\tilde{\alpha} = \frac{\Lambda\alpha}{\sigma}, \quad \tilde{\beta} = \frac{\Lambda\beta}{\sigma d_I}, \quad \tilde{\sigma} = \frac{\sigma}{d_I}, \quad \tilde{\mu} = \frac{\mu}{d_I}, \quad \tilde{\gamma} = \frac{\gamma}{d_I},$$

scaling the spatial time variables, and absorbing the appropriate constant into u_1 and u_2 in (1.1), we rewrite (1.1) in the following form (dropping the tildes on d, β and α for notational convenience).

$$\begin{cases} \dfrac{\partial u_1(x,t)}{\partial t} = d\left[\displaystyle\int_{\mathbb{R}} J(x-y)u_1(y,t)\mathrm{d}y - u_1(x,t)\right] + \sigma - \sigma u_1(x,t) - \dfrac{\beta u_1(x,t)u_2(x,t-\tau)}{1+\alpha u_2(x,t-\tau)}, \\ \dfrac{\partial u_2(x,t)}{\partial t} = \left[\displaystyle\int_{\mathbb{R}} J(x-y)u_2(y,t)\mathrm{d}y - u_2(x,t)\right] + \dfrac{\beta u_1(x,t)u_2(x,t-\tau)}{1+\alpha u_2(x,t-\tau)} - (\mu+\gamma)u_2(x,t). \end{cases} \tag{1.2}$$

Note that the infection-free equilibrium state (1, 0) always exists in system (1.2). Besides, when the basic reproduction number $R_0 = \frac{\beta}{\gamma+\mu} > 1$, there also exists a positive endemic equilibrium state (u_1^*, u_2^*), where

$$u_1^* = \frac{\alpha\sigma + (\mu+\gamma)}{\alpha\sigma+\beta} \quad \text{and} \quad u_2^* = \frac{\sigma[\beta-(\mu+\gamma)]}{(\mu+\gamma)(\alpha\sigma+\beta)}.$$

It is easy to see that (1, 0) is unstable, and (u_1^*, u_2^*) is stable for the corresponding homogenous system for (1.2).

Throughout this article, we assume that (1.2) satisfies the initial conditions

$$\begin{cases} u_1(x,0) = u_{10}(x), x \in \mathbb{R}, \\ u_2(x,s) = u_{20}(x,s), (x,s) \in \mathbb{R}\times[-\tau,0]. \end{cases} \tag{1.3}$$

We make the following conditions which are needed in the sequel.

(A1) $J \in C^1(\mathbb{R})$, $J(x) = J(-x) \geq 0$, $\int_{\mathbb{R}} J(x)\mathrm{d}x = 1$, and J is compactly supported.

The theory of traveling wave solutions of reaction-diffusion systems has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics. For system (1.1), the spatial dynamics of some special cases have been extensively studied. System (1.1) is a nonlocal version of the following SIR epidemic model:

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_S\dfrac{\partial^2 S(x,t)}{\partial x^2} + \Lambda - \sigma S(x,t) - \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_I\dfrac{\partial^2 I(x,t)}{\partial x^2} + \dfrac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma)I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_R\dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t) - \varrho R(x,t). \end{cases} \tag{1.4}$$

Yang et al. [1] derived the existence of a traveling wave connecting the disease-free steady state and the endemic steady state of (1.4) by the cross-iteration method and Schauder's fixed point theorem. Later, by the upper-lower solution method and Schauder's fixed point theorem, Li et al. [2] further obtained the existence and nonexistence of traveling waves of the subsystem

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_S \dfrac{\partial^2 S(x,t)}{\partial x^2} + \Lambda - \sigma S(x,t) - \dfrac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_I \dfrac{\partial^2 I(x,t)}{\partial x^2} + \dfrac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma) I(x,t), \end{cases} \tag{1.5}$$

also, the minimal wave speed is established. When the natural death rate for all the susceptible, infective, and removed population are the same constant, system (1.4) is rewritten as

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_S \dfrac{\partial^2 S(x,t)}{\partial x^2} + \Lambda - \mu S(x,t) - \dfrac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_I \dfrac{\partial^2 I(x,t)}{\partial x^2} + \dfrac{\beta S(x,t) I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma) I(x,t), \\ \dfrac{\partial R(x,t)}{\partial t} = d_R \dfrac{\partial^2 R(x,t)}{\partial x^2} + \gamma I(x,t) - \mu R(x,t). \end{cases} \tag{1.6}$$

Applying the Schauder fixed point theorem to construct a family of solutions for the truncated problems and via the limiting argument, Fu [3] obtained the noncritical waves and critical waves of system (1.6). For the nonlocal system (1.1), Li et al. [4] proved the existence, nonexistence, and minimal wave speed of traveling waves; moreover, they discussed how the latency of infection and the spatial movement of the infective individuals affect the minimal wave speed.

Among the basic problems in the theory of traveling wave solutions, the stability of traveling wave solutions is an extremely important subject. Let us draw the background on the progress of the study in this subject. By the spectral analysis, Sattinger [5] considered a reaction-diffusion system without delay and proved that the traveling wavefronts were stable to perturbations in some exponentially weighted L^∞ spaces. Using the semigroup estimates, Kapitula [6] also studied a reaction-diffusion system without delay and obtained that the traveling wavefronts are stable in polynomially weighted L^∞ spaces. By the elementary super- and subsolution comparison and squeezing methods developed by Chen [7] (see also Wang et al. [8,9] for this technique), Smith and Zhao [10] studied the global asymptotic stability, Liapunov stability, and uniqueness of traveling wave solutions for a bistable quasimonotone delayed reaction-diffusion bistable equation on $\mathbb{R}$. For the monostable case, since the unstable equilibrium, the study of the stability of traveling waves is more difficult than the bistable case. The first study of this case was obtained by Mei et al. [11] by using the weighted-energy method. They investigated a time-delayed diffusive Nicholson's blowflies equation and proved that, under a weighted L^2 norm, if the solution is sufficiently close to a traveling wavefront initially, it converges exponentially to the wavefront as $t \to \infty$. By means of the weighted-energy method and the comparison principle, Lin and Mei [12] considered Nicholson's blowflies equation with diffusion and found that the wavefront is time-asymptotically stable when the delay time is sufficiently small and the initial perturbation around the wavefront decays to zero exponentially in space as $x \to \infty$, but it can be large in other locations. In [13], Huang et al. used the anti-weighted-energy method developed by Chern et al. [14], considered a nonlocal dispersion equation with time delay, and proved that all noncritical traveling waves (the wave speed is greater than the minimum speed), including those oscillatory waves, are time-exponentially stable when the initial perturbations around the waves are small. For the nonlocal version, by means of the weighted-energy method combined with the comparison principle, Lv and Wang [15] studied a nonlocal delayed reaction-diffusion equation and proved that traveling wavefronts are exponentially stable to perturbation in some exponentially weighted L^∞ spaces. Later, Yu et al. [16] extended this method to investigate the stability of invasion traveling waves for a competition system with nonlocal dispersals and proved that the invasion traveling waves are exponentially stable. For other related results on the stability of traveling wave solutions, one can refer to [17–27].

Recently, Li et al. [28] investigated the delayed SIR epidemic model (1.4) and proved the exponential stability of traveling waves when the delay τ is less than some constant τ_0 by using the weighted-energy method and nonlinear Halanay's inequality.

Encouraged by papers [11], [28], and [29], in this article, we will further consider the nonlocal dispersal delayed SIR epidemic model (1.1) by using the weighted energy method. But due to the effect of the nonlocal dispersal and the lack of quasi-monotonicity, we are not clear whether the weighted-energy method can also be used to solve the stability of traveling waves of this model. As a result, with the help of some technology, we successfully apply this method to prove that all noncritical traveling wave solutions with sufficiently large $c \gg 1$ and arbitrarily large delays are exponentially stable. However, the shortcoming of this article is that we do not prove any nonlinear stability result for the slower waves with $c > c_{\min}$ (c can be arbitrarily close to $c_{\min}$), where $c_{\min}$ denotes critical wave speed, and particularly, the case for the critical traveling waves with $c_{\min}$. We leave this problem for further research.

The rest of this article is organized as follows. In Section 2, we give some preliminary lemmas and state our main stability result. In Section 3, we reformulate system (1.2) into the corresponding perturbed system around a given traveling wave solution and state the stability theorem of the new perturbed system. In Section 4, we devote to establish the *a priori* estimates, which are the core of this article.

2 Preliminaries and main result

First, we introduce some notations throughout this article. Let $C > 0$ denote a generic constant and $C_i > 0$ $(i = 1, 2, \ldots)$ be a specific constant. I is an interval, typically $I = \mathbb{R}$. Denote by $L^2(I)$ the space of square integrable functions defined on I and $H^k(I)(k \geq 0)$ the Sobolev space of the L^2-function $f(x)$ defined on the interval I whose derivatives $\frac{\mathrm{d}^i}{\mathrm{d}x^i}f$ $(i = 1, 2, \ldots, k)$ also belong to $L^2(I)$. $L_w^2(I)$ denotes the weighted L^2-space with a weight function $w(x) > 0$, and its norm is defined by

$$\|f\|_{L_w^2} = \left(\int_I w(x)|f(x)|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

Let $H_w^k(I)$ be the weighted Sobolev space with the norm

$$\|f\|_{H_w^k} = \left(\sum_{i=0}^{k} \int_I w(x) \left| \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x) \right|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

If $T > 0$ is a number and $\mathcal{B}$ is a Banach space, we denote by $C^0([0, T], \mathcal{B})$ the space of the $\mathcal{B}$-valued continuous function on $[0, T]$ and by $L^2([0, T], \mathcal{B})$ the space of the $\mathcal{B}$-valued L^2-functions on $[0, T]$. The corresponding spaces of the $\mathcal{B}$-valued L^2-functions on $[0, \infty)$ are defined similarly.

Throughout this article, we always assume that $R_0 > 1$.

A traveling wave solution of (1.2) is a solution with the form

$$u_1(x, t) = \phi_1(\xi), \quad u_2(x, t) = \phi_2(\xi), \quad \xi = x + ct,$$

where $c > 0$ is called the wave speed, and $(\phi_1, \phi_2) \in C^0(\mathbb{R}, \mathbb{R})$ is called the profile function. Furthermore, (ϕ_1, ϕ_2) with $c > 0$ satisfies

$$\begin{cases} c\phi_1'(\xi) = d\int\limits_{\mathbb{R}} J(\xi - y)[\phi_1(y) - \phi_1(\xi)]\mathrm{d}y + \sigma - \sigma\phi_1(\xi) - \dfrac{\beta\phi_1(\xi)\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)}, \\ c\phi_2'(\xi) = \int\limits_{\mathbb{R}} J(\xi - y)[\phi_2(y) - \phi_2(\xi)]\mathrm{d}y + \dfrac{\beta\phi_1(\xi)\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - (\mu + \gamma)\phi_2(\xi), \end{cases} \tag{2.1}$$

and the following asymptotic boundary conditions

$$\lim_{\xi\to-\infty}(\phi_1(\xi), \phi_2(\xi)) = (1, 0), \ \lim_{\xi\to+\infty}(\phi_1(\xi), \phi_2(\xi)) = (u_1^*, u_2^*). \tag{2.2}$$

By using the upper-lower solutions and Schauder's fixed point theorem, in [4], Li et al. proved the existence of traveling wave solutions of system (1.2).

Theorem 2.1. (Existence of traveling waves). *Assume that* (*A*1) *holds. Then, there exists a constant* $c_{\min} > 0$ *such that for any* $c > c_{\min}$ *or* $c = c_{\min}$ *and* $R_0 > \frac{\beta}{\alpha\sigma} + 1$, *system* (1.2) *has a nontrivial traveling wave solution* $\Phi(\xi) = (\phi_1(\xi), \phi_2(\xi))$, $\xi = x + ct$, *which satisfies the asymptotic boundary conditions* (2.2). *However, for* $R_0 < 1$ *and* $c > 0$ *or* $R_0 > 1$ *and* $0 < c < c_{\min}$, *there exists no traveling wave solution of system* (1.2) *with* (2.2).

Remark 2.1. From the proof of Theorem 2.1, we can obtain that $\phi_1(\xi)$ and $\phi_2(\xi)$ are uniformly bounded for any $\xi \in \mathbb{R}$. Also, we can verify that $\frac{\alpha\sigma}{\alpha\sigma+\beta}$ is a lower solution of $\phi_1(\xi)$ and $\frac{1}{\alpha}\left\{\frac{\beta}{\mu+\gamma} - 1\right\}$ is an upper solution of $\phi_2(\xi)$, i.e.,

$$\frac{\alpha\sigma}{\alpha\sigma + \beta} \le \phi_1(\xi) \le 1, \quad \max\left\{\mathrm{e}^{\lambda_1\xi}(1 - M\mathrm{e}^{\chi\xi}), 0\right\} \le \phi_2(\xi) \le \frac{1}{\alpha}\left\{\frac{\beta}{\mu + \gamma} - 1\right\},$$

where M and χ are suitable parameters.

In order to state our stability result, we need a technical assumption.

(A2)

$$\begin{cases} \sigma\beta^2 - \beta\sigma(2\alpha + 1)(\mu + \gamma) + (3\alpha\sigma + \beta)(\mu + \gamma)^2 < 0, \\ 2\alpha\sigma^2\beta + \sigma\beta^2 + \beta\sigma(\mu + \gamma) - (\alpha\sigma + \beta)(\mu + \gamma)^2 > 0. \end{cases} \tag{2.3}$$

Remark 2.2. In fact, this condition is easy to meet. For example, we choose $\mu + \gamma = 1$ and $\sigma = 1$, then $\sigma\beta^2 - \beta\sigma(2\alpha + 1)(\mu + \gamma) + (3\alpha\sigma + \beta)(\mu + \gamma)^2 = -(2\beta - 3)\alpha + \beta^2 < 0$ and $2\alpha\sigma^2\beta + \sigma\beta^2 + \beta\sigma(\mu + \gamma) - (\alpha\sigma + \beta)(\mu + \gamma)^2 = (2\beta - 1)\alpha + \beta^2 > 0$ hold for arbitrary given $\beta > \frac{3}{2}$ and α large enough. Later, we will see condition (A2) ensures that Lemma 4.1 holds.

Define two functions on η as follows:

$$f_1(\eta) = 2\sigma + \frac{d}{2} + \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} - d\int\limits_{-\infty}^{0} e^{-\eta y}J(y)\mathrm{d}y$$

and

$$f_2(\eta) = 2\mu + 2\gamma + \frac{1}{2} - \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{3(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} - \int\limits_{-\infty}^{0} e^{-\eta y}J(y)\mathrm{d}y.$$

According to (A2), it is easily checked

$$f_1(0) = 2\sigma + \frac{d}{2} + \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} - d\int_{-\infty}^{0} J(y)\mathrm{d}y$$
$$= 2\sigma + \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} > 0$$

and

$$f_2(0) = 2\mu + 2\gamma + \frac{1}{2} - \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{3(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} - \int_{-\infty}^{0} J(y)\mathrm{d}y$$
$$= 2\mu + 2\gamma - \frac{\sigma[\beta - (\mu + \gamma)]}{\mu + \gamma + \alpha\sigma} - \frac{3(\mu + \gamma)^2(\alpha\sigma + \beta)}{(\mu + \gamma + \alpha\sigma)\beta} > 0.$$

Therefore, by continuity, there exists $\eta^* > 0$ such that $f_1(\eta^*) > 0$ and $f_2(\eta^*) > 0$. In addition, we also define other four functions on ξ as follows:

$$g_1(\xi) = 2\sigma + \frac{d}{2} + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{0} e^{-\eta^* y}J(y)\mathrm{d}y,$$

$$g_2(\xi) = 2\mu + 2\gamma + \frac{1}{2} - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} - \int_{-\infty}^{0} e^{-\eta^* y}J(y)\mathrm{d}y,$$

$$g_3(\xi) = g_1(\xi) - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3},$$

and

$$g_4(\xi) = g_2(\xi) - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3},$$

where (ϕ_1, ϕ_2) is a traveling wave solution given in Theorem 2.1. Then, we have the following lemma.

Lemma 2.1. *Assume that* (A1) *and* (A2) *hold. Then, there exists* $\xi^* > 0$ *such that for each* $\xi \geq \xi^*$, *we have*

$$g_i(\xi) \geq f_1(\eta^*) - \varepsilon, \quad i = 1, 3 \quad \text{and} \quad g_j(\xi) \geq f_2(\eta^*) - \varepsilon, \quad j = 2, 4, \tag{2.4}$$

where $\varepsilon < \min\left\{\frac{f_1(\eta^*)}{2}, \frac{f_2(\eta^*)}{2}\right\}$.

Proof. Applying the L. Hospital's rule, we can easily prove

$$\lim_{\xi\to+\infty} g_i(\xi) = f_1(\eta^*) > 0, \quad i = 1, 3, \quad \lim_{\xi\to+\infty} g_j(\xi) = f_2(\eta^*) > 0, \quad j = 2, 4,$$

which imply that there exists $\xi^* \in \mathbb{R}_+$ such that for all $\xi \geq \xi^*$, (2.4) is obtained. □

We define a weighted function as

$$w(\xi) = \begin{cases} \mathrm{e}^{-\eta^*(\xi - \xi^*)}, & \xi \leq \xi^*, \\ 1, & \xi > \xi^*. \end{cases} \tag{2.5}$$

Now, we present the corresponding stability theorem for the Cauchy problems (1.2) and (1.3) as follows.

Theorem 2.2. (Stability). *Assume that* (A1) *and* (A2) *hold. For any given traveling wave of* (1.2) *with the wave speed* $c > \max\{c_{\min}, \bar{c}, \tilde{c}\}$, *where*

$$\bar{c} = \frac{K_2 + d\int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y + (K_3 + K_4)M_2 + K_3M_1 - 2\sigma}{\eta^*}$$

and

$$\tilde{c} = \frac{K_1 + 3K_2 + \int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y + (K_3 + K_4)M_2 + K_3M_1 - 2(\mu + \gamma)}{\eta^*}$$

and

$$K_1 := \sup_{\xi\in\mathbb{R}}\left\{\frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)}\right\}, \qquad K_2 := \sup_{\xi\in\mathbb{R}}\left\{\frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2}\right\}, \qquad M_1 := \sup_{\xi\in\mathbb{R}}|\phi_1'(\xi)|,$$
$$K_3 := \sup_{\xi\in\mathbb{R}}\left\{\frac{\beta}{(1 + \alpha\phi_2(\xi - c\tau))^2}\right\}, \qquad K_4 := \sup_{\xi\in\mathbb{R}}\left\{\frac{2\alpha\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^3}\right\}, \qquad M_2 := \sup_{\xi\in\mathbb{R}}|\phi_2'(\xi)|,$$

if the initial perturbations satisfy

$$\begin{cases} u_{10}(x) - \phi_1(x) \in H_w^1(\mathbb{R}), \quad x \in \mathbb{R}, \\ u_{20}(x, s) - \phi_2(x + cs) \in C^0([-\tau, 0], H_w^1(\mathbb{R})) \bigcap L^2([-\tau, 0], H_w^1(\mathbb{R})), \quad (x, s) \in \mathbb{R} \times [-\tau, 0], \end{cases}$$

where $w(x)$ is the weighted function given in (2.5), *there exist positive constants δ_0 and κ such that when*

$$\sup_{s\in[-\tau,0]} ||(u_2 - \phi_2)(s)||_{H_w^1(\mathbb{R})} + ||(u_1 - \phi_1)(0)||_{H_w^1(\mathbb{R})} \le \delta_0,$$

then the solution $u(x, t) = (u_1(x, t), u_2(x, t))$ of the Cauchy problems (1.2) *and* (1.3) *uniquely and globally exists in time and satisfies*

$$u_i(x, t) - \phi_i(x + ct) \in C^0([0, +\infty), H_w^1(\mathbb{R})) \bigcap L^2([0, +\infty), H_w^1(\mathbb{R})), \quad i = 1, 2.$$

Moreover, the solution $u(x, t) = (u_1(x, t), u_2(x, t))$ converges to the traveling wave $(\phi_1(x + ct), \phi_2(x + ct))$ exponentially in time t, i.e.,

$$\sup_{x\in\mathbb{R}}|u_i(x, t) - \phi_i(x + ct)| = O(1)e^{-\kappa t}, \quad i = 1, 2$$

for all $t \ge 0$.

3 Reformulation of the problem

Let $(u_1(x, t), u_2(x, t))$ be the solution of Cauchy problems (1.2) and (1.3), and $(\phi_1(x + ct), \phi_2(x + ct))$ be a given traveling wave solution of (1.2). Let $\xi = x + ct$ and

$$\begin{cases} U_i(\xi, t) = u_i(x, t) - \phi_i(\xi), \quad i = 1, 2, \\ U_1(\xi, 0) = u_{10}(x) - \phi_1(x), \quad x \in \mathbb{R}, \\ U_2(\xi, s) = u_{20}(x, s) - \phi_2(\xi), \quad (x, s) \in \mathbb{R} \times [-\tau, 0]. \end{cases} \tag{3.1}$$

Then, problems (1.2) and (1.3) can be reformulated as

$$\begin{aligned} \frac{\partial U_1(\xi, t)}{\partial t} + c\frac{\partial U_1(\xi, t)}{\partial \xi} = {} & d\left[\int_{\mathbb{R}} J(\xi - y)U_1(y, t)\mathrm{d}y - U_1(\xi, t)\right] - \left(\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)}\right)U_1(\xi, t) \\ & - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2}U_2(\xi - c\tau, t - \tau) - R(\xi - c\tau, t - \tau) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}\frac{\partial U_2(\xi,t)}{\partial t}+c\frac{\partial U_2(\xi,t)}{\partial \xi}=&\left[\int_{\mathbb{R}}J(\xi-y)U_2(y,t)\mathrm{d}y-U_2(\xi,t)\right]+\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}U_1(\xi,t)\\&-(\mu+\gamma)U_2(\xi,t)+\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}U_2(\xi-c\tau,t-\tau)+R(\xi-c\tau,t-\tau)\end{aligned}\tag{3.3}$$

with the initial condition

$$U_1(\xi,0)=U_{10}(\xi),\quad U_2(\xi,s)=U_{20}(\xi,s),\quad (\xi,s)\in\mathbb{R}\times[-\tau,0],\tag{3.4}$$

where

$$\begin{aligned}R(\xi-c\tau,t-\tau)=&\frac{\beta(U_1(\xi,t)+\phi_1(\xi))(U_2(\xi-c\tau,t-\tau)+\phi_2(\xi-c\tau))}{1+\alpha(U_2(\xi-c\tau,t-\tau)+\phi_2(\xi-c\tau))}-\frac{\beta\phi_1(\xi)\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\\&-\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}U_1(\xi,t)-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}U_2(\xi-c\tau,t-\tau).\end{aligned}$$

Let

$$\begin{aligned}\Omega(r-\tau,r+T)=\{&(U_1,U_2)|U_1\in C^0([r,r+T],H_w^1(\mathbb{R}))\bigcap L^2([r,r+T],H_w^1(\mathbb{R})),U_2\in C^0([r-\tau,r+T],H_w^1(\mathbb{R}))\\&\bigcap L^2([r-\tau,r+T],H_w^1(\mathbb{R}))\}\end{aligned}$$

endowed with the norm

$$N_r(T)=\max\left\{\sup_{t\in[r,r+T]}(\|U_1(t)\|^2_{H_w^1(\mathbb{R})}+\|U_2(t)\|^2_{H_w^1(\mathbb{R})})^{\frac12},\ \sup_{t\in[r-\tau,r]}\|U_2(t)\|_{H_w^1(\mathbb{R})}\right\},$$

where $\tau\geq 0$ and $T>0$. When $r=0$, we denote $N(T)=N_0(T)$.

Now, we state the stability result for the perturbed Cauchy problems (3.2)–(3.4), which automatically implies Theorem 2.2.

Theorem 3.1. (Stability). *Assume that* (*A*1) *and* (*A*2) *hold. For any given traveling wave of* (1.2) *with the wave speed* $c>\max\{c_{\min},\bar{c},\tilde{c}\}$, *if the initial perturbations satisfies*

$$\begin{cases}U_{10}(\xi)\in H_w^1(\mathbb{R}), & x\in\mathbb{R},\\ U_{20}(\xi,s)\in C^0([-\tau,0],H_w^1(\mathbb{R}))\bigcap L^2([-\tau,0],H_w^1(\mathbb{R})), & (x,s)\in\mathbb{R}\times[-\tau,0],\end{cases}$$

where $w(x)$ *is the weighted function given in* (2.5) *and* $\bar{c}$ *and* $\tilde{c}$ *are as defined in* Theorem 2.2, *there exist positive constants* δ_0 *and* κ *such that when* $N(0)\leq\delta_0$, *then the solution* $(U_1(\xi,t),U_2(\xi,t))$ *of the Cauchy problems* (3.2)–(3.4) *uniquely and globally exists in* $\Omega(-\tau,+\infty)$ *and satisfies*

$$\sup_{\xi\in\mathbb{R}}|U_i(\xi,t)|\leq Ce^{-\kappa t},\ t\geq 0$$

for some positive constant C.

By using the continuity extension method [11,25], the global existence of $(U_1(\xi,t),U_2(\xi,t))$ and its exponential decay estimate given in Theorem 3.1 directly follows from the local existence result and the *a priori* estimate given below.

Lemma 3.1. (Local existence). *Assume that* (*A*1) *and* (*A*2) *hold. Consider the Cauchy problem with the initial time* $r\geq 0$

$$\begin{cases} \dfrac{\partial U_1(\xi,t)}{\partial t} + c\dfrac{\partial U_1(\xi,t)}{\partial \xi} = d\left[\displaystyle\int_{\mathbb{R}} J(\xi-y)U_1(y,t)\mathrm{d}y - U_1(\xi,t)\right] - \left(\sigma + \dfrac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\right)U_1(\xi,t) \\ \quad - \dfrac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}U_2(\xi-c\tau,t-\tau) - R(\xi-c\tau,t-\tau), \\ \dfrac{\partial U_2(\xi,t)}{\partial t} + c\dfrac{\partial U_2(\xi,t)}{\partial \xi} = \left[\displaystyle\int_{\mathbb{R}} J(\xi-y)U_2(y,t)\mathrm{d}y - U_2(\xi,t)\right] + \dfrac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}U_1(\xi,t) \\ \quad - (\mu+\gamma)U_2(\xi,t) + \dfrac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}U_2(\xi-c\tau,t-\tau) + R(\xi-c\tau,t-\tau), \\ U_{10}(\xi) = u_{10}(x,s) - \phi_1(\xi) = U_{1r}(\xi,0), \\ U_{20}(\xi,s) = u_{20}(x,s) - \phi_2(\xi) = U_{2r}(\xi,s),\ (\xi,s)\in\mathbb{R}\times[r-\tau,r]. \end{cases} \tag{3.5}$$

If $U_{1r}(\xi,0)$, $U_{2r}(\xi,s)\in\Omega(r-\tau,r)$, and $N_r(0)\le\delta_1$ for a given positive constant δ_1, then there exists a constant $t_0=t_0(\delta_1)>0$ such that $(U_1(\xi,t),U_2(\xi,t))\in\Omega(r-\tau,r+t_0)$ and $N_r(t_0)\le\sqrt{2(1+\tau)}N_r(0)$.

The proof of Lemma 3.1 can be given by the elementary energy method, see [11,25] for details and we omit it here. Now, we state the *a priori* estimate as follows:

Lemma 3.2. (*A priori* estimate). *Assume that* (A1) *and* (A2) *hold. Let* $(U_1(\xi,t),U_2(\xi,t))\in\Omega(-\tau,T_0)$ *be a local solution of* (3.2)–(3.4). *Then, there exist positive constants $\delta_2>0$, $\kappa>0$, and $C_0>1$ independent of T_0 such that, when $N(T_0)\le\delta_2$, it holds that*

$$\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})} \le C_0e^{-2\kappa t}\left(\sum_{i=1}^{2}\|U_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \int_{-\tau}^{0}\|U_{20}(s)\|^2_{H^1_w(\mathbb{R})}\mathrm{d}s\right). \tag{3.6}$$

The proof for the *a priori* estimate of the solution in the space $\Omega(-\tau,T_0)$ plays a crucial role in this article and is discussed in the next section.

Proof of Theorem 3.1. Let δ_0, κ and C_0 be constants given in Lemma 3.2 independent of T_0. Set

$$\delta_0 = \min\left\{\frac{\delta_2}{\sqrt{2(1+\tau)}}, \frac{\delta_2}{\sqrt{2C_0}(1+\tau)}\right\},\quad \delta_1 = \max\{\sqrt{C_0(1+\tau)}N(0),\delta_2\}, \tag{3.7}$$

and

$$N(0)\le\delta_0<\delta_2\le\delta_1. \tag{3.8}$$

By Lemma 3.1, there exists a constant $t_0=t_0(\delta_1)>0$ such that $(U_1(\xi,t),U_2(\xi,t))\in\Omega(-\tau,t_0)$ and

$$N(t_0)\le\sqrt{2(1+\tau)}N(0)\le\sqrt{2(1+\tau)}\delta_0\le\delta_2. \tag{3.9}$$

Applying Lemma 3.2 on the interval $[-\tau,t_0]$, then for all $t\in[0,t_0]$, we have

$$\begin{aligned} \sup_{0\le t\le t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}} &\le \sup_{0\le t\le t_0}\left[C_0e^{-2\kappa t}\left(\sum_{i=1}^{2}\|U_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \int_{-\tau}^{0}\|U_{20}(s)\|^2_{H^1_w(\mathbb{R})}\mathrm{d}s\right)\right]^{\frac{1}{2}} \\ &\le \sqrt{C_0(1+\tau)}N(0) \le \sqrt{C_0(1+\tau)}\delta_0 \\ &\le \frac{\delta_2}{\sqrt{2(1+\tau)}} \le \delta_1. \end{aligned} \tag{3.10}$$

Consider the Cauchy problem (3.5) with the initial time $r = t_0$. Combining (3.7) with (3.10), we obtain

$$\begin{aligned} N_{t_0}(0) &= \max\left\{\sup_{t_0-\tau\le t\le t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \left(\sum_{i=1}^{2}\|U_i(t_0)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \max\left\{\sup_{-\tau\le t\le 0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \max\{\delta_0, \delta_1\} = \delta_1. \end{aligned} \tag{3.11}$$

Combining Lemma 3.1 with (3.10), we have the same t_0 such that $(U_1(\xi, t), U_2(\xi, t)) \in \Omega(-\tau, t_0)$ and

$$\begin{aligned} N_{t_0}(t_0) &\le \sqrt{2(1+\tau)}\,N_{t_0}(0) \\ &\le \sqrt{2(1+\tau)}\max\left\{\sup_{t_0-\tau\le t\le t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \left(\sum_{i=1}^{2}\|U_i(t_0)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \sqrt{2(1+\tau)}\max\left\{\sup_{-\tau\le t\le 0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \sqrt{2(1+\tau)}\max\left\{\delta_0, \frac{\delta_2}{\sqrt{2(1+\tau)}}\right\} = \delta_2. \end{aligned} \tag{3.12}$$

Consequently,

$$\begin{aligned} N(2t_0) &\le \max\left\{\sup_{-\tau\le t\le 0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le 2t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \max\left\{\sup_{-\tau\le t\le 0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}, \sup_{t_0\le t\le 2t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \max\left\{N(0), \frac{\delta_2}{\sqrt{2(1+\tau)}}, \delta_2\right\} = \delta_2. \end{aligned} \tag{3.13}$$

Applying Lemma 3.2 on the interval $[-\tau, 2t_0]$ again, then for all $t \in [0, 2t_0]$, we obtain (3.6) and

$$\begin{aligned} \sup_{0\le t\le 2t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}} &\le \sup_{0\le t\le 2t_0}\left[C_0\left(\sum_{i=1}^{2}\|U_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \int_{-\tau}^{0}\|U_{20}(s)\|^2_{H^1_w(\mathbb{R})}\mathrm{d}s\right)\right]^{\frac{1}{2}} e^{-\kappa t} \\ &\le \sqrt{C_0(1+\tau)}\,N(0) \le \sqrt{C_0(1+\tau)}\,\delta_0 \\ &\le \frac{\delta_2}{\sqrt{2(1+\tau)}} \le \delta_1. \end{aligned} \tag{3.14}$$

Consider the Cauchy problem (3.5) with the initial time $r = 2t_0$, we obtain

$$\begin{aligned} N_{2t_0}(0) &= \max\left\{\sup_{2t_0-\tau\le t\le 2t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \left(\sum_{i=1}^{2}\|U_i(2t_0)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\} \\ &\le \max\left\{\sup_{-\tau\le t\le 0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le 2t_0}\|U_2(t)\|_{H^1_w(\mathbb{R})}, \sup_{0\le t\le 2t_0}\left(\sum_{i=1}^{2}\|U_i(t)\|^2_{H^1_w(\mathbb{R})}\right)^{\frac{1}{2}}\right\}. \end{aligned}$$

Repeating this procedure, we can prove that the solution $(U_1(\xi,t), U_2(\xi,t))$ of the Cauchy problems (3.2)–(3.4) uniquely and globally exists in $\Omega(-\tau,+\infty)$ with the relation (3.6) for $t \in [0,+\infty)$. □

4 *A priori* estimate

In this section, we are going to prove Lemma 3.2. We need the following important lemma.

Define

$$
\begin{aligned}
A_w^1(\xi) &= -c\left(\frac{w'}{w}\right) + 2\sigma + d + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{\mathbb{R}} J(y)\frac{w(\xi + y)}{w(\xi)}\mathrm{d}y,\\
A_w^2(\xi) &= -c\left(\frac{w'}{w}\right) + 2\mu + 2\gamma + 1 - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2}\\
&\quad - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2}\frac{w(\xi + c\tau)}{w(\xi)} - \int_{\mathbb{R}} J(y)\frac{w(\xi + y)}{w(\xi)}\mathrm{d}y,\\
A_w^3(\xi) &= A_w^1(\xi) - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3},\\
A_w^4(\xi) &= A_w^2(\xi) - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3},\\
B_{\kappa,w}^1(\xi) &= A_w^1(\xi) - 2\kappa, \quad B_{\kappa,w}^2(\xi) = A_w^2(\xi) - 2\beta(e^{2\kappa\tau} - 1)\frac{w(\xi + c\tau)}{w(\xi)}\frac{\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} - 2\kappa,\\
B_{\kappa,w}^3(\xi) &= A_w^3(\xi) - 2\kappa, \quad B_{\kappa,w}^4(\xi) = A_w^4(\xi) - 2\beta(e^{2\kappa\tau} - 1)\frac{w(\xi + c\tau)}{w(\xi)}\frac{\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} - 2\kappa.
\end{aligned}
$$

Lemma 4.1. (Key inequality). *Assume that* $(A1)$ *and* $(A2)$ *hold. Let* $w(\xi)$ *be the weighted function given by* (2.5), *then for any* $c > \max\{c_{\min}, \bar{c}, \tilde{c}\}$, *there exist positive constants* C_i $(i = 1, 2, 3, 4)$ *such that*

$$A_w^i(\xi) \geq C_i \ (i = 1, 2, 3, 4). \tag{4.1}$$

Moreover, we have

$$B_{\kappa,w}^i(\xi) \geq \widetilde{C}_0(\kappa) = \min_{1\leq i\leq 4}\{\widetilde{C}_i(\kappa)\} \tag{4.2}$$

for all $\xi \in \mathbb{R}$ *and* $0 < \kappa < \kappa_0 = \min\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\}$, *where* $\kappa_i > 0$ *is the unique solution to the equation* $\widetilde{C}_i(\kappa) = 0$ $(i = 1, 2, 3, 4)$ *and*

$$\widetilde{C}_i(\kappa) = C_i - 2\kappa (i = 1, 3), \quad \widetilde{C}_i(\kappa) = C_i - 2K_2(e^{2\kappa\tau} - 1) - 2\kappa (i = 2, 4).$$

Proof. Since $c > \max\{c_{\min}, \bar{c}, \tilde{c}\}$, we have

$$c\eta^* > K_2 + d\int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y + (K_3 + K_4)M_2 + K_3M_1 - 2\sigma$$

and

$$c\eta^* > K_1 + 3K_2 + \int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y + (K_3 + K_4)M_2 + K_3M_1 - 2(\mu + \gamma).$$

First, we prove that $A_w^1(\xi) \geq C_1$ holds.

Case 1. $\xi \le \xi^*$. From (2.5), we have $w(\xi) = e^{-\eta^*(\xi-\xi^*)}$. Using the fact that $w(\xi)$ is nonincreasing, we obtain

$$\begin{aligned}
A_w^1(\xi) &= -c\left(\frac{w'}{w}\right) + 2\sigma + d + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{\mathbb{R}} J(y)\frac{w(\xi + y)}{w(\xi)}\mathrm{d}y \\
&\ge c\eta^* + 2\sigma + d - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{\xi^*-\xi} J(y)e^{-\eta^* y}\mathrm{d}y - d\int_{\xi^*-\xi}^{+\infty} J(y)e^{\eta^*(\xi-\xi^*)}\mathrm{d}y \\
&\ge c\eta^* + 2\sigma + d - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{\mathbb{R}} J(y)(e^{-\eta^* y} + 1)\mathrm{d}y \\
&\ge c\eta^* + 2\sigma - K_2 - d\int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y \\
&\ge (K_3 + K_4)M_2 + K_3M_1 > 0.
\end{aligned}$$

Case 2. $\xi > \xi^*$. In this case, $w(\xi) = w(\xi + c\tau) = 1$ and $w'(\xi) = 0$. Thus, by Lemma 2.2, we have

$$\begin{aligned}
A_w^1(\xi) &= 2\sigma + d + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{\mathbb{R}} J(y)\frac{w(\xi + y)}{w(\xi)}\mathrm{d}y \\
&\ge 2\sigma + d + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{\xi^*-\xi} J(y)e^{-\eta^*(\xi+y-\xi^*)}\mathrm{d}y - d\int_{\xi^*-\xi}^{+\infty} J(y)\mathrm{d}y \\
&= 2\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{\xi^*-\xi} J(y)(e^{-\eta^*(\xi+y-\xi^*)} - 1)\mathrm{d}y \\
&\ge 2\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{\xi^*-\xi} J(y)(e^{-\eta^* y} - 1)\mathrm{d}y \\
&\ge 2\sigma + \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - d\int_{-\infty}^{0} J(y)(e^{-\eta^* y} - 1)\mathrm{d}y \\
&= g_1(\xi) \ge \frac{f_1(\eta^*)}{2} > 0.
\end{aligned}$$

Let $C_1 = \min\left\{(K_3 + K_4)M_2 + K_3M_1, \frac{f_1(\eta^*)}{2}\right\}$, then $A_w^1(\xi) \ge C_1$ holds.

Next, we prove that $A_w^2(\xi) \ge C_2$ holds.

Case 1. $\xi \le \xi^*$. It follows from $w(\xi) = e^{-\eta^*(\xi-\xi^*)}$ and the monotonicity of $w(\xi)$ that

$$\begin{aligned}
A_w^2(\xi) &= c\eta^* + 2\mu + 2\gamma + 1 - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2}\frac{w(\xi + c\tau)}{w(\xi)} \\
&\quad - \int_{-\infty}^{\xi^*-\xi} J(y)e^{-\eta^* y}\mathrm{d}y - \int_{\xi^*-\xi}^{+\infty} J(y)e^{\eta^*(\xi-\xi^*)}\mathrm{d}y \\
&\ge c\eta^* + 2\mu + 2\gamma + 1 - (K_1 + 3K_2) - \int_{\mathbb{R}} J(y)(e^{-\eta^* y} + 1)\mathrm{d}y \\
&\ge c\eta^* + 2\mu + 2\gamma - (K_1 + 3K_2) - \int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y \\
&\ge (K_3 + K_4)M_2 + K_3M_1 > 0.
\end{aligned}$$

Case 2. $\xi > \xi^*$. From (2.5), we have $w(\xi) = w(\xi + c\tau_2) = 1$ and $w'(\xi) = 0$. By Lemma 2.2, we obtain

$$\begin{aligned}
A_w^2(\xi) &= 2\mu + 2\gamma + 1 - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2}\frac{w(\xi + c\tau)}{w(\xi)} \\
&\quad - \int_{-\infty}^{\xi^*-\xi} J(y)e^{-\eta^*(\xi+y-\xi^*)}\mathrm{d}y - \int_{\xi^*-\xi}^{+\infty} J(y)\mathrm{d}y \\
&= 2\mu + 2\gamma - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2}\frac{w(\xi + c\tau)}{w(\xi)} \\
&\quad - \int_{-\infty}^{\xi^*-\xi} J(y)(e^{-\eta^*(\xi+y-\xi^*)} - 1)\mathrm{d}y \\
&\geq 2\mu + 2\gamma - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} - \int_{-\infty}^{0} J(y)(e^{-\eta^* y} - 1)\mathrm{d}y \\
&\geq 2\mu + 2\gamma + \frac{1}{2} - \frac{\beta\phi_2(\xi - c\tau)}{1 + \alpha\phi_2(\xi - c\tau)} - \frac{\beta\phi_1(\xi)}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\beta\phi_1(\xi + c\tau)}{(1 + \alpha\phi_2(\xi))^2} - \int_{-\infty}^{0} e^{-\eta^* y}J(y)\mathrm{d}y \\
&= g_2(\xi) \geq \frac{f_2(\eta^*)}{2} > 0.
\end{aligned}$$

Let $C_2 = \min\{(K_3 + K_4)M_2 + K_3M_1, \frac{f_2(\eta^*)}{2}\}$, then $A_w^2(\xi) \geq C_2$ holds.

The proofs of $A_w^3(\xi)$ and $A_w^4(\xi)$ are similar, and we only sketch the outline. When $\xi \leq \xi^*$, we obtain

$$\begin{aligned}
A_w^3(\xi) &\geq c\eta^* + 2\sigma - K_2 - d\int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} \\
&\quad - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3} \\
&\geq c\eta^* + 2\sigma - K_2 - d\int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y - (K_3 + K_4)M_2 - K_3M_1 \\
&= C_5 > 0
\end{aligned}$$

and

$$\begin{aligned}
A_w^4(\xi) &\geq c\eta^* + 2\mu + 2\gamma - (K_1 + 3K_2) - \int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y - \frac{\beta|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} \\
&\quad - \frac{\beta|\phi_1'(\xi)|}{(1 + \alpha\phi_2(\xi - c\tau))^2} - \frac{2\alpha\beta\phi_1(\xi)|\phi_2'(\xi - c\tau)|}{(1 + \alpha\phi_2(\xi - c\tau))^3} \\
&\geq c\eta^* + 2\mu + 2\gamma - (K_1 + 3K_2) - \int_{\mathbb{R}} J(y)e^{-\eta^* y}\mathrm{d}y - (K_3 + K_4)M_2 - K_3M_1 \\
&= C_6 > 0.
\end{aligned}$$

When $\xi > \xi^*$, we obtain

$$A_w^3(\xi) \geq g_3(\xi) \geq \frac{f_1(\eta^*)}{2} > 0 \quad \text{and} \quad A_w^4(\xi) \geq g_4(\xi) \geq \frac{f_2(\eta^*)}{2} > 0.$$

Let $C_3 = \min\left\{C_5, \frac{f_1(\eta^*)}{2}\right\}$ and $C_4 = \min\left\{C_6, \frac{f_2(\eta^*)}{2}\right\}$, then $A_w^3(\xi) \geq C_3$ and $A_w^4(\xi) \geq C_4$ hold.

We estimate $B^2_{\kappa,w}(\xi)$ as

$$\begin{aligned}B^2_{\kappa,w}(\xi) &= A^2_w(\xi) - 2\beta(e^{2\kappa\tau}-1)\frac{w(\xi+c\tau)}{w(\xi)}\frac{\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2} - 2\kappa\\ &\geq C_2 - 2K_2(e^{2\kappa\tau}-1) - 2\kappa\\ &\geq \widetilde{C}_0(\kappa) > 0\end{aligned}$$

by selecting a constant κ sufficiently small such that $0 < \kappa < \kappa_2$, where $\kappa_2 \in (0,\infty)$ is the unique root of the equation $\widetilde{C}_2(\kappa) = 0$. The others are similar. □

Next, we are going to derive the *a priori* estimate for $U_1(\xi,t)$ and $U_2(\xi,t)$ in the weighted Sobolev space $H^1_w(\mathbb{R})$.

Lemma 4.2. *Assume that (A1) and (A2) hold. Let* $(U_1(\xi,t), U_2(\xi,t)) \in \Omega(-\tau, T_0)$ *be a local solution of* (3.2)–(3.4). *Then, for any* $c > \max\{c_{\min}, \bar{c}, \tilde{c}\}$, *there exists a positive constant C, such that*

$$\sum_{i=1}^{2}\|U_i(t)\|^2_{L^2_w(\mathbb{R})} + \sum_{i=1}^{2}\int_0^t e^{-2\kappa(t-s)}\|U_i(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s \leq Ce^{-2\kappa t}\left(\sum_{i=1}^{2}\|U_{i0}(0)\|^2_{L^2_w(\mathbb{R})} + \int_{-\tau}^{0}\|U_{20}(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s\right), \quad t \geq 0, \tag{4.3}$$

provided $N(T_0) \ll 1$, *where the weighted function* $w(\xi)$ *is defined by* (2.5) *and* κ *is given as in* Lemma 4.1.

Proof. Multiplying (3.2) and (3.3) by $e^{2\kappa t}w(\xi)U_1(\xi,t)$ and $e^{2\kappa t}w(\xi)U_2(\xi,t)$, respectively, we obtain

$$\begin{aligned}&\left\{\frac{1}{2}e^{2\kappa t}wU_1^2\right\}_t + \left\{\frac{1}{2}ce^{2\kappa t}wU_1^2\right\}_\xi + \left\{-\kappa - \frac{1}{2}c\left(\frac{w'}{w}\right) + \sigma + d + \frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\right\}e^{2\kappa t}wU_1^2\\ &= de^{2\kappa t}wU_1\int_{\mathbb{R}}J(y)U_1(\xi-y,t)\mathrm{d}y - \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2\kappa t}w(\xi)U_1(\xi,t)U_2(\xi-c\tau,t-\tau)\\ &\quad - e^{2\kappa t}w(\xi)U_1(\xi,t)R(\xi-c\tau,t-\tau)\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}&\left\{\frac{1}{2}e^{2\kappa t}wU_2^2\right\}_t + \left\{\frac{1}{2}ce^{2\kappa t}wU_2^2\right\}_\xi + \left\{-\kappa - \frac{1}{2}c\left(\frac{w'}{w}\right) + \mu + \gamma + 1\right\}e^{2\kappa t}wU_2^2\\ &= e^{2\kappa t}wU_2\int_{\mathbb{R}}J(y)U_2(\xi-y,t)\mathrm{d}y + \frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2\kappa t}w(\xi)U_1(\xi,t)U_2(\xi,t)\\ &\quad + \frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}e^{2\kappa t}w(\xi)U_2(\xi,t)U_2(\xi-c\tau,t-\tau) + e^{2\kappa t}w(\xi)U_2(\xi,t)R(\xi-c\tau,t-\tau).\end{aligned}\tag{4.5}$$

Integrating (4.4) and (4.5) over $\mathbb{R}\times[0,t]$ with respect to ξ and t, respectively, we have

$$\begin{aligned}&e^{2\kappa t}\|U_1(t)\|^2_{L^2_w(\mathbb{R})} + \int_0^t\int_{\mathbb{R}}\left\{-2\kappa - c\left(\frac{w'}{w}\right) + 2\sigma + 2d + \frac{2\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\right\}e^{2\kappa s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &= \|U_1(0)\|^2_{L^2_w(\mathbb{R})} + H_{11}(t) + H_{21}(t) + H_{31}(t)\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}&e^{2\kappa t}\|U_2(t)\|^2_{L^2_w(\mathbb{R})} + \int_0^t\int_{\mathbb{R}}\left\{-2\kappa - c\left(\frac{w'}{w}\right) + 2\mu + 2\gamma + 2\right\}e^{2\kappa s}w(\xi)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &= \|U_2(0)\|^2_{L^2_w(\mathbb{R})} + H_{12}(t) + H_{22}(t) + H_{32}(t) + H_4(t),\end{aligned}\tag{4.7}$$

where

$$H_{1i}(t) = 2d_i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) U_i(\xi, s) \int_{\mathbb{R}} J(y) U_i(\xi - y, s) \mathrm{d}y \mathrm{d}\xi \mathrm{d}s,$$

$$H_{2i}(t) = 2(-1)^i \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) U_i(\xi, s) U_2(\xi - c\tau, s - \tau) \mathrm{d}\xi \mathrm{d}s,$$

$$H_{3i}(t) = 2(-1)^i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) U_i(\xi, s) R(\xi - c\tau, s - \tau) \mathrm{d}\xi \mathrm{d}s,$$

$$H_4(t) = 2 \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_2(\xi - c\tau)}{1 + \alpha \phi_2(\xi - c\tau)} e^{2\kappa s} w(\xi) U_1(\xi, s) U_2(\xi, s) \mathrm{d}\xi \mathrm{d}s$$

and $d_i = d$, $i = 1$, and $d_i = 1$, $i = 2$. By the Cauchy inequality $xy \le \varepsilon x^2 + \frac{y^2}{4\varepsilon}$ and $\varepsilon = \frac{1}{2}$, we obtain

$$\begin{aligned} |H_{1i}(t)| &\le 2d_i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) |U_i(\xi, s)| \int_{\mathbb{R}} J(y) |U_i(\xi - y, s)| \mathrm{d}y \mathrm{d}\xi \mathrm{d}s \\ &\le d_i \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\kappa s} w(\xi) J(y) [U_i^2(\xi, s) + U_i^2(\xi - y, s)] \mathrm{d}y \mathrm{d}\xi \mathrm{d}s \\ &= d_i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + d_i \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\kappa s} w(\xi) J(y) U_i^2(\xi - y, s) \mathrm{d}y \mathrm{d}\xi \mathrm{d}s \\ &= d_i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + d_i \int_0^t \int_{\mathbb{R}} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \int_{\mathbb{R}} J(y) \frac{w(\xi + y)}{w(\xi)} \mathrm{d}y \mathrm{d}\xi \mathrm{d}s, \end{aligned} \tag{4.8}$$

$$\begin{aligned} |H_{2i}(t)| &\le 2 \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) |U_i(\xi, s)| |U_2(\xi - c\tau, s - \tau)| \mathrm{d}\xi \mathrm{d}s \\ &\le \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) [U_i^2(\xi, s) + U_2^2(\xi - c\tau, s - \tau)] \mathrm{d}\xi \mathrm{d}s \\ &\le \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) U_2^2(\xi \\ &\quad - c\tau, s - \tau) \mathrm{d}\xi \mathrm{d}s \\ &= \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + \int_{-\tau}^{t-\tau} e^{2\kappa(s+\tau)} \int_{\mathbb{R}} \frac{\beta \phi_1(\xi + c\tau)}{(1 + \alpha \phi_2(\xi))^2} w(\xi \\ &\quad + c\tau) U_2^2(\xi, s) \mathrm{d}\xi \mathrm{d}s \\ &\le \int_0^t \int_{\mathbb{R}} \frac{\beta \phi_1(\xi)}{(1 + \alpha \phi_2(\xi - c\tau))^2} e^{2\kappa s} w(\xi) U_i^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + \int_{-\tau}^{0} e^{2\kappa(s+\tau)} \int_{\mathbb{R}} \frac{\beta \phi_1(\xi + c\tau)}{(1 + \alpha \phi_2(\xi))^2} w(\xi \\ &\quad + c\tau) U_{20}^2(\xi, s) \mathrm{d}\xi \mathrm{d}s + \int_0^t e^{2\kappa(s+\tau)} \int_{\mathbb{R}} \frac{\beta \phi_1(\xi + c\tau)}{(1 + \alpha \phi_2(\xi))^2} w(\xi + c\tau) U_2^2(\xi, s) \mathrm{d}\xi \mathrm{d}s, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned}|H_4(t)| &\le 2\int_0^t\int_{\mathbb{R}}\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2\kappa s}w(\xi)|U_1(\xi,s)||U_2(\xi,s)|\mathrm{d}\xi\mathrm{d}s\\ &\le \int_0^t\int_{\mathbb{R}}\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2\kappa s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+\int_0^t\int_{\mathbb{R}}\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2\kappa s}w(\xi)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s.\end{aligned}\tag{4.10}$$

Substituting (4.8)–(4.10) into (4.6) and (4.7), respectively, we can obtain

$$\begin{aligned}&e^{2\kappa t}\|U_1(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}\Bigg\{-2\kappa-c\left(\frac{w'}{w}\right)+2\sigma+d+\frac{2\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}-d\int_{\mathbb{R}}J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\\ &\quad-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}\Bigg\}e^{2\kappa s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s-\int_0^t e^{2\kappa(s+\tau)}\int_{\mathbb{R}}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}w(\xi+c\tau)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\le\|U_1(0)\|^2_{L^2_w(\mathbb{R})}+\int_{-\tau}^0 e^{2\kappa(s+\tau)}\int_{\mathbb{R}}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}w(\xi+c\tau)U_{20}^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+H_{31}(t)\end{aligned}\tag{4.11}$$

and

$$\begin{aligned}&e^{2\kappa t}\|U_2(t)\|^2_{L^2_w(\mathbb{R})}-\int_0^t\int_{\mathbb{R}}\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}e^{2\kappa s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}}\Bigg\{-2\kappa-c\left(\frac{w'}{w}\right)+2\mu+2\gamma+1-\int_{\mathbb{R}}J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y\\ &\quad-\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}-e^{2\kappa\tau}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}\frac{w(\xi+c\tau)}{w(\xi)}\Bigg\}e^{2\kappa s}w(\xi)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\le\|U_2(0)\|^2_{L^2_w(\mathbb{R})}+\int_{-\tau}^0 e^{2\kappa(s+\tau)}\int_{\mathbb{R}}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}w(\xi+c\tau)U_{20}^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+H_{32}(t).\end{aligned}\tag{4.12}$$

From (4.11) and (4.12), we obtain

$$\begin{aligned}&e^{2\kappa t}\|U_1(t)\|^2_{L^2_w(\mathbb{R})}+e^{2\kappa t}\|U_2(t)\|^2_{L^2_w(\mathbb{R})}+\int_0^t\int_{\mathbb{R}}\Bigg\{-2\kappa-c\left(\frac{w'}{w}\right)+2\sigma+d+\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\\ &\quad-d\int_{\mathbb{R}}J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}\Bigg\}e^{2\kappa s}w(\xi)U_1^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\quad+\int_0^t\int_{\mathbb{R}}\Bigg\{-2\kappa-c\left(\frac{w'}{w}\right)+2\mu+2\gamma+1-\int_{\mathbb{R}}J(y)\frac{w(\xi+y)}{w(\xi)}\mathrm{d}y-\frac{\beta\phi_2(\xi-c\tau)}{1+\alpha\phi_2(\xi-c\tau)}\\ &\quad-\frac{\beta\phi_1(\xi)}{(1+\alpha\phi_2(\xi-c\tau))^2}-2e^{2\kappa\tau}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}\frac{w(\xi+c\tau)}{w(\xi)}\Bigg\}e^{2\kappa s}w(\xi)U_2^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\le\sum_{i=1}^2\|U_i(0)\|^2_{L^2_w(\mathbb{R})}+2\int_{-\tau}^0 e^{2\kappa(s+\tau)}\int_{\mathbb{R}}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}w(\xi+c\tau)U_{20}^2(\xi,s)\mathrm{d}\xi\mathrm{d}s+H_{31}(t)+H_{32}(t),\end{aligned}\tag{4.13}$$

i.e.,

$$\begin{aligned}&\sum_{i=1}^{2} e^{2\kappa t}\|U_i(t)\|^2_{L^2_w(\mathbb{R})} + \sum_{i=1}^{2}\int_0^t\int_{\mathbb{R}} B^i_{\mu,w}(\xi)e^{2\kappa s}w(\xi)U_i^2(\xi,s)\mathrm{d}\xi\mathrm{d}s\\ &\quad\le \sum_{i=1}^{2}\|U_i(0)\|^2_{L^2_w(\mathbb{R})} + 2\int_{-\tau}^{0} e^{2\kappa(s+\tau)}\int_{\mathbb{R}}\frac{\beta\phi_1(\xi+c\tau)}{(1+\alpha\phi_2(\xi))^2}w(\xi+c\tau)U_{20}^2(\xi,s)\mathrm{d}\xi\mathrm{d}s - H_{31}(t) + H_{32}(t),\end{aligned} \tag{4.14}$$

where $B^i_{\mu,w}(\xi)$ are given in the beginning of this section. For the nonlinearity $R(\xi-c\tau, s-\tau)$, using Taylor's formula, we have

$$\begin{aligned}|R(\xi-c\tau,s-\tau)| &= |F(U_1(\xi,s)+\phi_1(\xi), U_2(\xi-c\tau,s-\tau)+\phi_2(\xi-c\tau)) - F(\phi_1(\xi),\phi_2(\xi-c\tau))\\ &\qquad - \partial_1F(\phi_1(\xi),\phi_2(\xi-c\tau))U_1(\xi,s) - \partial_2F(\phi_1(\xi),\phi_2(\xi-c\tau))U_2(\xi-c\tau,s-\tau)|\\ &\le\Bigg|\frac{\beta}{(1+\alpha(\phi_2(\xi-c\tau)+\theta_1U_2(\xi-c\tau,s-\tau)))^2}U_1(\xi,s)U_2(\xi-c\tau,s-\tau)\\ &\qquad - \frac{\alpha\beta(\phi_1(\xi)+\theta_1U_1(\xi,s))}{(1+\alpha(\phi_2(\xi-c\tau)+\theta_1U_2(\xi-c\tau,s-\tau)))^3}U_2^2(\xi-c\tau,s-\tau)\Bigg|\\ &\le L_1|U_1(\xi,s||U_2(\xi-c\tau,s-\tau)| + L_2|U_2^2(\xi-c\tau,s-\tau)|,\end{aligned}$$

where $\theta_1\in(0,1)$ and

$$\begin{aligned}L_1 &= \sup_{\xi\in\mathbb{R}}\frac{\beta}{(1+\alpha(\phi_2(\xi-c\tau)+\theta_1U_2(\xi-c\tau,s-\tau)))^2},\\ L_2 &= \sup_{\xi\in\mathbb{R}}\frac{\alpha\beta(\phi_1(\xi)+\theta_1U_1(\xi,s))}{(1+\alpha(\phi_2(\xi-c\tau)+\theta_1U_2(\xi-c\tau,s-\tau)))^3}.\end{aligned}$$

Using the Sobolev's embedding theorem $H^1(\mathbb{R})\hookrightarrow C(\mathbb{R})$ and the embedding inequality $H^1_w(\mathbb{R})\hookrightarrow H^1(\mathbb{R})$ since $w(\xi)\ge 1$, we can immediately obtain

$$\sup_{\xi\in\mathbb{R}}|U_i(\xi,s)| \le C_7\|U_i(s)\|_{H^1(\mathbb{R})} \le C_8\|U_i(s)\|_{H^1_w(\mathbb{R})} \le C_8N(T_0). \tag{4.15}$$

Then, we have

$$\begin{aligned}|H_{31}(t)| &\le 2\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)|U_1(\xi,s)||R(\xi-c\tau,s-\tau)|\mathrm{d}\xi\mathrm{d}s\\ &\le 2\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)[L_1U_1^2(\xi,s)|U_2(\xi-c\tau,s-\tau)| + L_2|U_1(\xi,s)|U_2^2(\xi-c\tau,s-\tau)]\mathrm{d}\xi\mathrm{d}s\\ &\le CN(T_0)\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)[U_1^2(\xi,s)+U_2^2(\xi-c\tau,s-\tau)]\mathrm{d}\xi\mathrm{d}s\\ &\le CN(T_0)\left[\sum_{i=1}^{2}\int_0^t e^{2\kappa s}\|U_i(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s + \int_{-\tau}^{0} e^{2\kappa s}\|U_{20}(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s\right]\end{aligned} \tag{4.16}$$

and

$$\begin{aligned}|H_{32}(t)| &\le 2\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)|U_2(\xi,s)||R(\xi-c\tau,s-\tau)|\mathrm{d}\xi\mathrm{d}s\\ &\le 2\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)[L_1|U_1(\xi,s)||U_2(\xi,s)||U_2(\xi-c\tau,s-\tau)| + L_2|U_2(\xi,s)|U_2^2(\xi-c\tau,s-\tau)]\mathrm{d}\xi\mathrm{d}s\end{aligned}$$

$$\begin{aligned}
&\le CN(T_0)\int_0^t\int_{\mathbb{R}} e^{2\kappa s}w(\xi)[U_2^2(\xi, s) + U_2^2(\xi - c\tau, s - \tau)]\mathrm{d}\xi\mathrm{d}s \\
&\le CN(T_0)\left[\int_0^t e^{2\kappa s}\|U_2(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s + \int_{-\tau}^0 e^{2\kappa s}\|U_{20}(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s\right].
\end{aligned} \tag{4.17}$$

According to Lemma 4.1 and substituting (4.16) and (4.17) into (4.14), we obtain

$$\begin{aligned}
&\sum_{i=1}^2 e^{2\kappa t}\|U_i(t)\|^2_{L^2_w(\mathbb{R})} + \int_0^t [\widetilde{C}_0(\kappa) - C_9N(T_0)]e^{2\kappa s}\|U_1(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s + \int_0^t [\widetilde{C}_0(\kappa) - C_{10}N(T_0)]e^{2\kappa s}\|U_2(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s \\
&\le C\left(\sum_{i=1}^2 \|U_i(0)\|^2_{L^2_w(\mathbb{R})} + \int_{-\tau}^0 e^{2\kappa s}\|U_{20}(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s\right).
\end{aligned} \tag{4.18}$$

Letting $0 < N(T_0) < \min\left\{\frac{\widetilde{C}_0(\kappa)}{C_9}, \frac{\widetilde{C}_0(\kappa)}{C_{10}}\right\}$, we immediately obtain (4.3). □

Similar to that in Lemma 4.2, we derive the estimates for $U_{1\xi}(\xi, t)$ and $U_{2\xi}(\xi, t)$.

Lemma 4.3. *Assume that* (A1) *and* (A2) *hold. Let* $(U_1(\xi, t), U_2(\xi, t)) \in \Omega(-\tau, T_0)$ *be a local solution of* (3.2)–(3.4). *Then, for any* $c > \max\{c_{\min}, \bar{c}, \tilde{c}\}$, *there exists a positive constant C, such that*

$$\sum_{i=1}^2 \|U_{i\xi}(t)\|^2_{L^2_w(\mathbb{R})} + \sum_{i=1}^2\int_0^t e^{-2\kappa(t-s)}\|U_{i\xi}(s)\|^2_{L^2_w(\mathbb{R})}\mathrm{d}s \le Ce^{-2\kappa t}\left(\sum_{i=1}^2 \|U_{i0}(0)\|^2_{H^1_w(\mathbb{R})} + \int_{-\tau}^0 \|U_{20}(s)\|^2_{H^1_w(\mathbb{R})}\mathrm{d}s\right), \quad t \ge 0, \tag{4.19}$$

provided $N(T_0) \ll 1$, *where the weighted function* $w(\xi)$ *is defined by* (2.5) *and* κ *is given as in* Lemma 4.1.

Thus, the *a priori* estimate Lemma 3.2 immediately obtained from Lemmas 4.2 and 4.3.

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