DE GRUYTER Open Mathematics 2022; 20: 1215–1228

Research Article

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Dependence of eigenvalues of Sturm-Liouville problems on time scales with eigenparameter-dependent boundary conditions

https://doi.org/10.1515/math-2022-0507
received October 30, 2021; accepted September 20, 2022

Abstract: In this article, we study the eigenvalue dependence of Sturm-Liouville problems on time scales with spectral parameter in the boundary conditions. We obtain that the eigenvalues not only continuously but also smoothly depend on the parameters of the problem. Moreover, the differential expressions of the eigenvalues with respect to the data are given.

Keywords: Sturm-Liouville problems, time scales, derivative formulas, eigenparameter-dependent boundary

MSC 2020: 34B08, 34N99, 34L05

1 Introduction

It is well known, Stefan Hilger, a German mathematician, first proposed the concept of time scales in his doctoral dissertation. Since then there are a considerable number of studies on the problems on time scales, and here we refer to [1–11]. Time scale organically unifies continuous systems and discrete systems. Therefore, the research results on time scales are more general and have a wide application prospect.

In 1999, Agarwal et al. studied the Sturm-Liouville (S-L) problem $y^{\Delta\Delta} + qy^{\sigma} + \lambda y^{\sigma} = 0$ under separated boundary conditions and proved the existence of the eigenvalues, and the number of generalized zeros of the eigenfunctions [1]. In 2008, Kong considered the S-L problem in the general form and discussed the dependence of eigenvalues of the S-L problem on boundary conditions [3]. In 2011, Zhang and Yang studied the eigenvalues of S-L problem with coupled boundary conditions on time scales [4]. The inverse spectral problems for S-L operators on time scales have been studied by Kuznetsova et al. in [12–14] and their references. Besides the aforementioned contents, the self-adjoint even-order differential equations on time scales have also been given, e.g., in [5,6] and their references.

In classical S-L problems, the dependence of the eigenvalues on the problem is widely studied by many authors [15–19]. These studies play an important role in the fundamental theory of differential operators and the numerical computation of spectra. For general theory and methods on these problems, the readers may refer to [18,19].

In most recent years, the research on the dependence of the eigenvalues of a differential operator or boundary value problem on the problem has been extended in various aspects. In 2015, Zhang and Wang

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showed the eigenvalues of an S-L problem with interface conditions and obtained that the eigenvalues depend not only continuously but also smoothly on the coefficient functions, boundary conditions, and interface conditions [20]. In 2016, Zhu and Shi generalized the problem to the discrete case and considered the dependence of eigenvalues of discrete S-L problems on the problem [21]. They also considered the eigenvalue dependence problems for singular S-L problems in [22]. For higher order boundary value problems, there are several literature on the problems, for example, in [23–29].

In recent years, there has been a lot of interest in the literature on boundary value problems with eigenparameter-dependent boundary conditions, for example, in some physical problems such as heat conduction problems and vibrating string problems. In particular, the spectral problems having boundary conditions depending on the eigenparameter arising in mechanical engineering can be found in the well-known textbook [30] of Collatz. For such problems arising in applications, including an extensive bibliography and historical notes, also see [31–33]. The recent important achievements on such problems can be found in many literature, and here, we only refer to some of them, for example, in [34–37].

In 2020, Zhang and Li studied the regular S-L problems with eigenparameter-dependent boundary conditions [38]. They obtained that the eigenvalues not only continuously but also smoothly depend on the parameters of the problem, and further, the differential expressions of the eigenvalues with respect to the data are given.

There are many conclusions about the problems of differential equations on time scales and the dependence of eigenvalues of differential equations; however, to our best knowledge, few people have studied the dependence of eigenvalues of differential equations on time scales. Therefore, it is very meaningful to consider the eigenvalue dependence of S-L problems on time scales with spectral parameter boundary conditions.

This article is organized as follows. In Section 2, we introduce the problems studied here and show the continuous dependence of the eigenvalues on the problem. In Section 3, the differential properties of eigenvalues with respect to the data of the problem are given, and in particular, the derivative formulas are listed.

2 Continuous dependence of eigenvalues and eigenfunctions

Before presenting the main results, we recall the following concepts related to time scales for the convenience of the reader. For further knowledge on time scales, the reader may refer to [1–10] and the references therein.

Definition 1. A time scale $\mathbb{T}$ is a closed subset of $\mathbb{R}$ with the inherited Euclidean topology. For $t \in \mathbb{T}$, we define the forward-jump operator σ and the backward-jump operator ρ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

where $\inf\varnothing := \sup\mathbb{T}$ and $\sup\varnothing := \inf\mathbb{T}$. If $\sigma(t) > t$, t is said to be right-scattered; otherwise, it is right-dense. If $\rho(t) < t$, t is said to be left-scattered; otherwise, it is left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is then defined by $\mu(t) := \sigma(t) - t$.

In this article, we use the notation $f^{\sigma}(t) = f(\sigma(t))$ for any function f defined on a time scale $\mathbb{T}$.

Definition 2. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$ (if $t = \sup\mathbb{T}$, assume t is not left-scattered), define the Δ-derivative $f^{\Delta}(t)$ of $f(t)$ to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood $\mathcal{M}$ of t such that

$$|[f^{\sigma}(t) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|,$$

for all $s \in \mathcal{M}$.

The following formula involving the graininess function is valid for all points at which $f^\Delta(t)$ exists

$$f^\sigma(t) = f(t) + f^\Delta(t)\mu(t).$$

Definition 3. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. We say that $f \in C_{rd}$ if it is continuous at each right-dense point in $\mathbb{T}$ and $\lim_{s\to t^-} f(s)$ exists as a finite number for all left-dense points in $\mathbb{T}$.

Definition 4. If $F^\Delta(t) = f(t)$, then we define the integral of f on $[a, b] \cap \mathbb{T}$ by

$$\int_a^b f(\tau)\Delta\tau = F(b) - F(a).$$

It has been shown that if f is rd-continuous on $[a, b] \cap \mathbb{T}$, i.e. $f \in C_{rd}([a, b] \cap \mathbb{T})$, then $\int_a^b f(\tau)\Delta\tau$ exists.

Consider the differential equation

$$-(py^\Delta)^\Delta + qy^\sigma = \lambda w y^\sigma \quad \text{on } [a, b] \cap \mathbb{T}, \tag{2.1}$$

with boundary conditions

$$\cos\alpha y(\rho(a)) - \sin\alpha (py^\Delta)(\rho(a)) = 0, \tag{2.2}$$

$$(\beta_1\lambda + \gamma_1)y(b) - (\beta_2\lambda + \gamma_2)(py^\Delta)(b) = 0. \tag{2.3}$$

Where $-\infty < a < b < \infty$, q is real valued function and $\lambda \in \mathbb{C}$ is the spectral parameter with coefficients satisfying:

$$1/p,\, q,\, w \in C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}),\, p, w > 0,\, \beta_i, \gamma_i \in \mathbb{R},\, i = 1, 2, \quad \alpha \in [\hat{0}, \hat{\pi}), \tag{2.4}$$

$$\eta := \gamma_2\beta_1 - \gamma_1\beta_2 > 0, \quad \hat{0} = \begin{cases} 0, & \rho(a) = a \\ -\arctan\dfrac{\mu(\rho(a))}{p(\rho(a))}, & \rho(a) < a \end{cases} \quad \text{and} \quad \hat{\pi} = \pi + \hat{0}. \tag{2.5}$$

Remark 1. When $\alpha = \hat{0}$ or $\hat{\pi}$, (2.2) is equivalent to $y(a) = 0$.

In fact, this is obviously true when $\rho(a) = a$. Now we assume $\rho(a) < a$. Then when $\alpha = \hat{0}$, (2.2) means that

$$\begin{aligned} 0 &= y(\rho(a)) - \tan\hat{0}\, p(\rho(a))[y(a) - y(\rho(a))]/\mu(\rho(a)) \\ &= y(\rho(a)) + [y(a) - y(\rho(a))] = y(a). \end{aligned}$$

Similarly for $\alpha = \hat{\pi}$.

Let the weighted space be defined as follows:

$$\mathcal{H}_1 = L^2_w([\rho(a), b] \cap \mathbb{T}) = \left\{ y : \int_{\rho(a)}^{b} (y^\sigma)^2(t) w(t)\Delta t < \infty \right\},$$

with the inner product $\langle x, y\rangle_{\mathcal{H}_1} = \int_{\rho(a)}^b x^\sigma(t) y^\sigma(t) w(t)\Delta t$ for any $x, y \in \mathcal{H}_1$. Define

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathbb{R},$$

with the inner product

$$\langle (x, x_1)^T, (y, y_1)^T \rangle = \langle x, y\rangle_{\mathcal{H}_1} + \frac{1}{\eta} x_1 y_1,$$

for any $(x, x_1)^T, (y, y_1)^T \in \mathcal{H}$. Define an operator $\mathbf{T}$ as

$$\mathbf{T}\begin{pmatrix} y \\ \beta_1 y(b) - \beta_2(py^\Delta)(b) \end{pmatrix} = \begin{pmatrix} w^{-1}(-(py^\Delta)^\Delta + qy^\sigma) \\ -\gamma_1 y(b) + \gamma_2(py^\Delta)(b) \end{pmatrix},$$

with the domain

$$\mathbf{D} = \{(y, y_1) \in \mathcal{H} : y, py^\Delta \in C_{rd}([\rho(a), b] \cap \mathbb{T}), y_1 = \beta_1 y(b) - \beta_2(py^\Delta)(b) \in \mathbb{R},$$
$$\cos\alpha y(\rho(a)) - \sin\alpha(py^\Delta)(\rho(a)) = 0\}.$$

Lemma 1. [10] *Equation* (2.1) *is equivalent to the following form*:

$$\begin{pmatrix} y \\ py^\Delta \end{pmatrix}^\Delta = \begin{pmatrix} 0 & \frac{1}{p(t)} \\ q(t) - \lambda w(t) & 0 \end{pmatrix}\begin{pmatrix} y^\sigma \\ py^\Delta \end{pmatrix},$$

or

$$Y^\Delta = A(t)Y,$$

where $Y = \begin{pmatrix} y \\ py^\Delta \end{pmatrix}$ *and* $A(t) = \begin{pmatrix} 0 & \frac{1}{p(t)} \\ q(t) - \lambda w(t) & [q(t) - \lambda w(t)]\frac{\mu(t)}{p(t)} \end{pmatrix}$.

Lemma 2. [10] $\forall t_0 \in [a, b] \cap \mathbb{T}$, $A(t) \in C_{rd}$ is $n \times n$ *functional matrix, and* $\forall t \in [a, t_0] \cap \mathbb{T}$, $I + \mu(t)A(t)$ *is invertible matrix, then the initial value problem*

$$Y^\Delta = A(t)Y, \quad Y(t_0) = Y_0, \quad Y_0 \in \mathbb{R}^n$$

has unique solution $Y \in C_{rd}$.

Let $\phi_\lambda(t)$ and $\chi_\lambda(t)$ be the fundamental solutions of the S-L equation (2.1) satisfying the initial conditions

$$\phi_\lambda(\rho(a)) = 1, \quad \chi_\lambda(\rho(a)) = 0;$$
$$(p\phi_\lambda^\Delta)(\rho(a)) = 0, \quad (p\chi_\lambda^\Delta)(\rho(a)) = 1.$$

Define

$$\Phi_\lambda(t) := \begin{pmatrix} \phi_\lambda(t) & \chi_\lambda(t) \\ (p\phi_\lambda^\Delta)(t) & (p\chi_\lambda^\Delta)(t) \end{pmatrix},$$

then Φ_λ is the fundamental solution matrix of equation (2.1) satisfying initial conditions $\Phi_\lambda(\rho(a)) = I$ on $[\rho(a), b] \cap \mathbb{T}$.

Lemma 3. *The complex number* λ *is an eigenvalue of the S-L problem* (2.1)–(2.5) *if and only if the equality*

$$\Delta(\lambda) = \det[A + B_\lambda \Phi_\lambda(b)] = 0$$

holds, where

$$A = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix}, B_\lambda = \begin{pmatrix} 0 & 0 \\ \beta_1\lambda + \gamma_1 & -(\beta_2\lambda + \gamma_2) \end{pmatrix}.$$

Proof. Let

$$y(t, \lambda) = c_1\phi_\lambda(t) + c_2\chi_\lambda(t), \tag{2.6}$$

where $c_1, c_2 \in \mathbb{R}$. If λ is an eigenvalue of the problem (2.1)–(2.5), then there exists $(c_1, c_2)^T \neq \mathbf{0}$ such that y satisfies (2.1)–(2.5). It follows from the boundary conditions (2.2) and (2.3) that

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix}\left(c_1\begin{pmatrix} \phi_\lambda(\rho(a)) \\ (p\phi_\lambda^\Delta)(\rho(a)) \end{pmatrix} + c_2\begin{pmatrix} \chi_\lambda(\rho(a)) \\ (p\chi_\lambda^\Delta)(\rho(a)) \end{pmatrix}\right) + \begin{pmatrix} 0 & 0 \\ \beta_1\lambda + \gamma_1 & -(\beta_2\lambda + \gamma_2) \end{pmatrix}\left(c_1\begin{pmatrix} \phi_\lambda(b) \\ (p\phi_\lambda^\Delta)(b) \end{pmatrix} + c_2\begin{pmatrix} \chi_\lambda(b) \\ (p\chi_\lambda^\Delta)(b) \end{pmatrix}\right) = \mathbf{0}.$$

According to the aforementioned initial conditions, we have

$$\left[\begin{pmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta_1\lambda + \gamma_1 & -(\beta_2\lambda + \gamma_2) \end{pmatrix}\Phi_\lambda(b)\right]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}. \tag{2.7}$$

Because $(c_1, c_2)^T \neq \mathbf{0}$, so $\Delta(\lambda) = 0$.

Conversely, if $\Delta(\lambda) = 0$, then there exists $(c_1, c_2)^T \neq \mathbf{0}$ such that (2.7) holds. If the solution $y(t, \lambda)$ described by (2.6) is chosen, then $y(t, \lambda)$ satisfies the S-L problem (2.1)–(2.5); hence, $y(t, \lambda)$ is an eigenfunction and λ is the eigenvalue of the S-L problem (2.1)–(2.5). This completes the proof. □

Lemma 4. *The spectrum of* $\mathbf{T}$ *consists of isolated eigenvalues, which coincide with those of the S-L problem* (2.1)–(2.5). *Moreover, all eigenvalues are simple, real, bounded below and can be ordered as follows*:

$$-\infty < \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots, \quad n \in \mathbb{N}_0^k, \tag{2.8}$$

where $\mathbb{N}_0^k := \begin{cases} \{0, 1, 2, \ldots\}, & k = \infty \\ \{0, 1, 2, \ldots, k-1\}, & k < \infty. \end{cases}$

Proof. Let $y(t, \lambda)$ be a nontrivial solution of the problem (2.1) and (2.2). The eigenvalues of the problem (2.1)–(2.5) are the roots of the equation:

$$(\beta_1\lambda + \gamma_1)y(b, \lambda) - (\beta_2\lambda + \gamma_2)(py^\Delta)(b, \lambda) = 0. \tag{2.9}$$

Let λ_* be a nonreal eigenvalue of the S-L problem (2.1)–(2.5). Then $\overline{\lambda_*}$ is also an eigenvalue of this problem, since $p(t), q(t), w(t), \alpha, \beta_1, \beta_2, \gamma_1$, and γ_2 are real; moreover, $y(t, \overline{\lambda_*}) = \overline{y(t, \lambda_*)}$. Let ν be another eigenvalue of the S-L problem (2.1)–(2.5), by virtue of (2.1), we have

$$(py^\Delta(t, \lambda))^\Delta y^\sigma(t, \nu) - (py^\Delta(t, \nu))^\Delta y^\sigma(t, \lambda) = (\nu - \lambda)w(t)y^\sigma(t, \nu)y^\sigma(t, \lambda).$$

By integrating this relation from $\rho(a)$ to b, and using the formula for the integration by parts, and taking into account the condition (2.2), we obtain

$$py^\Delta(b, \lambda)y(b, \nu) - py^\Delta(b, \nu)y(b, \lambda) = (\nu - \lambda)\int_{\rho(a)}^{b} [w(t)y^\sigma(t, \nu)y^\sigma(t, \lambda)]\Delta t. \tag{2.10}$$

Setting $\nu = \overline{\lambda_*}$ and $\lambda = \lambda_*$ in (2.10), we obtain

$$py^\Delta(b, \lambda_*)\overline{y(b, \lambda_*)} - \overline{py^\Delta(b, \lambda_*)}y(b, \lambda_*) = (\overline{\lambda_*} - \lambda_*)\int_{\rho(a)}^{b} w(t)|y^\sigma(t, \lambda_*)|^2\Delta t. \tag{2.11}$$

Since λ_* is a root of equation (2.9), we have the relation

$$py^\Delta(b, \lambda_*) = \frac{\beta_1\lambda_* + \gamma_1}{\beta_2\lambda_* + \gamma_2}y(b, \lambda_*).$$

In view of this relation, from (2.11), we obtain

$$\frac{(\overline{\lambda_*}-\lambda_*)(\beta_2\gamma_1-\beta_1\gamma_2)}{|\beta_2\lambda_*+\gamma_2|^2}|y(b,\lambda_*)|^2=(\overline{\lambda_*}-\lambda_*)\int\limits_{\rho(a)}^{b} w(t)|y^\sigma(t,\lambda_*)|^2\Delta t.$$

Since $\overline{\lambda_*}\neq\lambda_*$, we have the relation

$$\frac{-\eta|y(b,\lambda_*)|^2}{|\beta_2\lambda_*+\gamma_2|^2}=\int\limits_{\rho(a)}^{b} w(t)|y^\sigma(t,\lambda_*)|^2\Delta t,$$

which contradicts the condition $\eta>0$. Therefore, $\lambda_*\in\mathbb{R}$.

The entire function occurring on the left-hand side in equation (2.9) does not vanish for nonreal λ. Consequently, it does not vanish identically. Therefore, its zeros form an at most countable set without finite limit points [18,39].

Let us show that equation (2.9) has only simple roots. Indeed, if λ_* is a multiple root of equation (2.9), then

$$(\beta_1\lambda_*+\gamma_1)y(b,\lambda_*)-(\beta_2\lambda_*+\gamma_2)(py^\Delta)(b,\lambda_*)=0, \tag{2.12}$$

$$\beta_1 y(b,\lambda_*)+(\beta_1\lambda_*+\gamma_1)\frac{\partial}{\partial\lambda}y(b,\lambda_*)-\beta_2(py^\Delta)(b,\lambda_*)-(\beta_2\lambda_*+\gamma_2)\frac{\partial}{\partial\lambda}(py^\Delta)(b,\lambda_*)=0. \tag{2.13}$$

Dividing both sides of (2.10) with $(\nu-\lambda)(\nu\neq\lambda)$ and by passing to the limit as $\nu\to\lambda$, we obtain

$$py^\Delta(b,\lambda)\frac{\partial}{\partial\lambda}y(b,\lambda)-\frac{\partial}{\partial\lambda}py^\Delta(b,\lambda)y(b,\lambda)=\int\limits_{\rho(a)}^{b}[w(t)y^\sigma(t,\lambda)y^\sigma(t,\lambda)]\Delta t. \tag{2.14}$$

Since $\eta>0$, we have $(\beta_1\lambda_*+\gamma_1)^2+(\beta_2\lambda_*+\gamma_2)^2\neq0$. Suppose that $\beta_1\lambda_*+\gamma_1\neq0$. Then, by expressing $y(b,\lambda_*)$ and $\frac{\partial}{\partial\lambda}y(b,\lambda_*)$ from (2.12) and (2.13), respectively, and by substituting them into relation (2.14) for $\lambda=\lambda_*$, we obtain

$$\frac{-\eta}{(\beta_1\lambda_*+\gamma_1)^2}(py^\Delta)^2(b,\lambda_*)=\int\limits_{\rho(a)}^{b}[w(t)y^\sigma(t,\lambda_*)y^\sigma(t,\lambda_*)]\Delta t,$$

which is impossible in view of condition $\eta>0$.

The case in which $\beta_2\lambda_*+\gamma_2\neq0$ can be proved in a similar way.

The proof of formula (2.8) is the same as in [1], only to note together with the proof in [38]. The proof of Lemma 4 is completed. □

Lemma 5. *Let* (2.4) *and* (2.5) *hold, and let* $s\in[a,b]\cap\mathbb{T}$ *and* $h,l\in\mathbb{R}$. *Consider the initial value problem consisting of equation* (2.1) *and the initial conditions*:

$$y(s)=h,\quad (py^\Delta)(s)=l.$$

Then the unique solution $y=y(\cdot,s,h,l,\frac{1}{p},q,w)$ *is a continuous function of all its variables. More precisely, given* $\varepsilon>0$ *and any compact subinterval of* $[a,b]\cap\mathbb{T}$, *there exists a* $\delta>0$ *such that if*

$$|s-s_0|+|h-h_0|+|l-l_0|+\int\limits_{\rho(a)}^{b}\left(|\frac{1}{p}-\frac{1}{p_0}|+|q-q_0|+|w-w_0|\right)\Delta t<\delta,$$

then

$$|y\left(t,s,h,l,\frac{1}{p},q,w\right)-y\left(t,s_0,h_0,l_0,\frac{1}{p_0},q_0,w_0\right)|<\varepsilon,$$

and

$$\left|(py^{\Delta})\left(t, s, h, l, \frac{1}{p}, q, w\right) - (py^{\Delta})\left(t, s_0, h_0, l_0, \frac{1}{p_0}, q_0, w_0\right)\right| < \varepsilon,$$

for all $t \in [a, b] \cap \mathbb{T}$.

Proof. This follows from Theorem 2.6.1 in [9]. □

Let

$$M = \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix}$$

be the boundary condition parameter matrix and consider the following Banach spaces

$$\mathfrak{B}_1 := C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus \mathbb{R}^5;$$
$$\mathfrak{B}_2 := C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus C_{rd}([\rho(a), \sigma(b)] \cap \mathbb{T}) \oplus M_{2\times 2}(\mathbb{R}) \oplus \mathbb{R},$$

equipped with the norms

$$\|\dot{\omega}\| := \int_{\rho(a)}^{b} \left|\frac{1}{p}\right| \Delta t + \int_{\rho(a)}^{b} |q|\Delta t + \int_{\rho(a)}^{b} |w|\Delta t + |\beta_1| + |\beta_2| + |\gamma_1| + |\gamma_2| + |\alpha|;$$
$$\|\ddot{\omega}\| := \int_{\rho(a)}^{b} \left|\frac{1}{p}\right| \Delta t + \int_{\rho(a)}^{b} |q|\Delta t + \int_{\rho(a)}^{b} |w|\Delta t + \|M\| + |\alpha|,$$

for any $\dot{\omega} = \left(\frac{1}{p}, q, w, \beta_1, \beta_2, \gamma_1, \gamma_2, \alpha\right) \in \mathfrak{B}_1$ and $\ddot{\omega} = \left(\frac{1}{p}, q, w, M, \alpha\right) \in \mathfrak{B}_2$, respectively. Let

$$\Omega_1 = \{\dot{\omega} \in \mathfrak{B}_1 : (2.4), (2.5) \text{ hold}\};$$
$$\Omega_2 = \{\ddot{\omega} \in \mathfrak{B}_2 : (2.4), (2.5) \text{ hold}\}.$$

Then we obtain the continuous dependence of the eigenvalues on the parameters in the S-L problem (2.1)–(2.5).

Theorem 1. *Let* $\tilde{\omega} = \left(\frac{1}{\tilde{p}}, \tilde{q}, \tilde{w}, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\gamma}_1, \tilde{\gamma}_2, \tilde{\alpha}\right) \in \Omega_1$. *Let* $\lambda = \lambda(\dot{\omega})$ *be an eigenvalue of the S-L problem* (2.1)–(2.5). *Then* λ *is continuous at* $\tilde{\omega}$. *That is, for any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that if* $\dot{\omega} = \left(\frac{1}{p}, q, w, \beta_1, \beta_2, \gamma_1, \gamma_2, \alpha\right) \in \Omega_1$ *satisfies*

$$\|\dot{\omega} - \tilde{\omega}\| = \int_{\rho(a)}^{b} \left(\left|\frac{1}{p} - \frac{1}{\tilde{p}}\right| + |q - \tilde{q}| + |w - \tilde{w}|\right)\Delta t + |\beta_1 - \tilde{\beta}_1| + |\beta_2 - \tilde{\beta}_2| + |\gamma_1 - \tilde{\gamma}_1| + |\gamma_2 - \tilde{\gamma}_2| + |\alpha - \tilde{\alpha}| < \delta,$$

then

$$|\lambda(\dot{\omega}) - \lambda(\tilde{\omega})| < \varepsilon.$$

Proof. According to Lemma 3, the complex number $\lambda(\tilde{\omega})$ is an eigenvalue of the S-L problem (2.1)–(2.5) if and only if $\Delta(\tilde{\omega}, \lambda(\tilde{\omega})) = 0$ holds. For any $\dot{\omega} \in \Omega_1$, B_λ and $\Phi_\lambda(b)$ are all entire functions of λ and are continuous in $\dot{\omega}$, then $\Delta(\dot{\omega}, \lambda(\dot{\omega}))$ is an entire function of λ and it is continuous in $\dot{\omega}$ [18,39]. Because **T** is a self-adjoint operator, we know that $\lambda(\tilde{\omega})$ is an isolated eigenvalue, so $\Delta(\tilde{\omega}, \lambda)$ is not a constant value function about λ. Hence, there exists $\rho_0 > 0$ such that $\Delta(\tilde{\omega}, \lambda) = 0$ for $\lambda(\dot{\omega}) \in S_{\rho_0} = \{\lambda(\dot{\omega}) : |\lambda(\dot{\omega}) - \lambda(\tilde{\omega})| = \rho_0\}$. By the well-known theorem on continuity of the roots of an equation as a function of parameters [40], the result follows. □

Lemma 6. *Let the hypotheses and notation of* Theorem 1 *hold. Assume that $\lambda(\dot{\omega})$ be a simple eigenvalue of the S-L problem* (2.1)–(2.5) *of $\dot{\omega}$. Then there exists a neighborhood U of $\dot{\omega}$ in Ω_1 such that $\lambda(\tilde{\omega})$ is simple for every $\tilde{\omega} \in U$.*

Proof. For $\tilde{\omega} \in U$, let

$$D(\tilde{\omega}) = A + B_\lambda \Phi_\lambda(b).$$

Note that $D(\tilde{\omega})$ depends continuously on $\tilde{\omega}$. Solutions of the S-L problem (2.1)–(2.5) are the first components of the vector functions $\Phi(\cdot,\tilde{\omega})\mathbf{d}$ with $\mathbf{d} \in \mathbb{R}^2$, $D(\tilde{\omega})\mathbf{d} = 0$, and these functions are not identically zero if $\mathbf{d} \neq 0$. Hence, the geometric multiplicity of $\lambda(\tilde{\omega})$ equals to the dimension of the kernel $N(D(\tilde{\omega}))$ of $D(\tilde{\omega})$. In our case, this means that $\operatorname{rank}(D(\dot{\omega})) = 1$. Since $D(\tilde{\omega})$ depends continuously on $\tilde{\omega}$, $D(\tilde{\omega})$ must have rank of 1 in a neighborhood of $\dot{\omega}$. This completes the proof. □

Definition 5. By a normalized eigenvector $(y, y_1)^T \in \mathcal{H}$, we mean y satisfies the S-L problem (2.1)–(2.5), $y_1 = \beta_1 y(b) - \beta_2 (py^\Delta)(b)$, and

$$\|(y, y_1)^T\|^2 = \langle (y, y_1)^T, (y, y_1)^T \rangle = \int_{\rho(a)}^{b} (y^\sigma)^2 w \Delta t + \frac{1}{\eta} y_1^2 = 1.$$

Then we have the continuity of the corresponding eigenvector.

Theorem 2. *Assume $\lambda(\dot{\omega})$ is an eigenvalue of the S-L problem* (2.1)–(2.5) *for some $\dot{\omega} \in \Omega_1$ and let $(u, u_1)^T \in \mathcal{H}$ denote a normalized eigenvector of $\lambda(\dot{\omega})$, then there exists a normalized eigenvector $(v, v_1)^T \in \mathcal{H}$ of $\lambda(\tilde{\omega})$ for $\tilde{\omega} \in \Omega_1$ such that when $\tilde{\omega} \to \dot{\omega}$ in Ω_1, we have*

$$v(t) \to u(t), \quad pv^\Delta(t) \to pu^\Delta(t), \tag{2.15}$$

both uniformly on $[a, b] \cap \mathbb{T}$, and $v_1 \to u_1$.

Proof. For each $\dot{\omega} \in \Omega_1$, set

$$D(\dot{\omega}) = \begin{pmatrix} D_{11}(\dot{\omega}) & D_{12}(\dot{\omega}) \\ D_{21}(\dot{\omega}) & D_{22}(\dot{\omega}) \end{pmatrix} := A(\dot{\omega}) + B_{\lambda(\dot{\omega})}(\dot{\omega})\Phi_{\lambda(\dot{\omega})}(b),$$

where $D_{ij}(\dot{\omega}) \in \mathbb{R}$, $1 \le i, j \le 2$, and $A(\dot{\omega})$ and $B_{\lambda(\dot{\omega})}(\dot{\omega})$ are 2×2 matrices. Assume that $(y(t, \dot{\omega}), y_1(t, \dot{\omega}))^T$ is an eigenvector for $\lambda(\dot{\omega})$ with

$$\|y(t, \dot{\omega})\| = \int_{\rho(a)}^{b} (y^\sigma(t, \dot{\omega}))^2 w(t)\Delta t = 1.$$

Rewrite

$$y(t, \dot{\omega}) = c_1 \phi_{\lambda(\dot{\omega})}(t, \dot{\omega}) + c_2 \chi_{\lambda(\dot{\omega})}(t, \dot{\omega}),$$

where $c_1, c_2 \in \mathbb{R}$. Inserting $y(t, \dot{\omega})$ into the boundary conditions (2.2) and (2.3), one obtains that

$$D(\dot{\omega}) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0.$$

Since $\lambda(\dot{\omega})$ has multiplicity 1, $\operatorname{rank}(D(\dot{\omega})) = 1$. For the given $\dot{\omega}$, without loss of generality, we assume that $D_{11}(\dot{\omega}) \neq 0$. It is clear that there exists $\nu \neq 0$ such that

$$y(t, \dot{\omega}) = \nu D_{12}(\dot{\omega})\phi_{\lambda(\dot{\omega})}(t, \dot{\omega}) - \nu D_{11}(\dot{\omega})\chi_{\lambda(\dot{\omega})}(t, \dot{\omega}).$$

Since λ is continuous, by Lemma 6, there exists a neighborhood $U_0 \subset \Omega_1$ of $\dot{\omega}$ such that $D_{11}(\tilde{\omega}) \neq 0$ for every $\tilde{\omega} \in U_0$, and $D(\tilde{\omega}) \to D(\dot{\omega})$ as $U_0 \ni \tilde{\omega} \to \dot{\omega}$. So

$$y(t, \tilde{\omega}) \to y(t, \dot{\omega}), \quad (py^{\Delta})(t, \tilde{\omega}) \to (py^{\Delta})(t, \dot{\omega}), \tag{2.16}$$

as $\tilde{\omega} \to \dot{\omega}$ both uniformly on $[a, b] \cap \mathbb{T}$. Therefore, we have

$$y_1(\tilde{\omega}) \to y_1(\dot{\omega}), \quad \text{as } \tilde{\omega} \to \dot{\omega}. \tag{2.17}$$

Let

$$(u, u_1)^T = \frac{(y(t,\dot{\omega}), y_1(\dot{\omega}))^T}{\|(y(t,\dot{\omega}), y_1(\dot{\omega}))^T\|}, \quad (v, v_1)^T = \frac{(y(t,\tilde{\omega}), y_1(\tilde{\omega}))^T}{\|(y(t,\tilde{\omega}), y_1(\tilde{\omega}))^T\|};$$
$$pu^{\Delta} = \frac{(py^{\Delta})(t,\dot{\omega})}{\|(y(t,\dot{\omega}), y_1(\dot{\omega}))^T\|}, \quad pv^{\Delta} = \frac{(py^{\Delta})(t,\tilde{\omega})}{\|(y(t,\tilde{\omega}), y_1(\tilde{\omega}))^T\|}.$$

Then (2.15) holds by (2.16) and (2.17). □

Remark 2. The statements and proofs of Theorem 1, Lemma 6, and Theorem 2 still hold for $\tilde{\omega} \in \Omega_2$. Since the proofs are similar to those $\tilde{\omega} \in \Omega_1$, we omit the details here.

3 Differential expressions of eigenvalues

In this section, we shall show the eigenvalues determined in Theorem 1 are differentiable, and in particular, we give the derivative formulas of the eigenvalues for all parameters.

The Lagrange sesquilinear form is introduced by

$$[y, z] = y(pz^{\Delta}) - (py^{\Delta})z,$$

where $y, z, py^{\Delta}, pz^{\Delta} \in C_{rd}([\rho(a), b] \cap \mathbb{T})$. Recall the definition of the Frechet derivative:

Definition 6. A map $\mathcal{T}$ from a Banach space X into another Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear operator $\mathrm{d}\mathcal{T}_x : X \to Y$ such that for $h \in X$

$$|\mathcal{T}(x + h) - \mathcal{T}(x) - \mathrm{d}\mathcal{T}_x(h)| = o(h), \quad \text{as } h \to 0.$$

Theorem 3. *Let $\lambda(\dot{\omega})$ be an eigenvalue for the S-L problem* (2.1)–(2.5) *with $\dot{\omega} \in \Omega_1$, and $(u, u_1)^T$ be a normalized eigenvector for $\lambda(\dot{\omega})$, then λ is differentiable with respect to all the parameters in $\dot{\omega}$, and more precisely, the derivative formulas of $\beta_1, \beta_2, \gamma_1$, and γ_2 are given as follows:*

(1) *Fix the parameters of $\dot{\omega}$ except β_1 and let $\lambda = \lambda(\beta_1)$ be the eigenvalue of the S-L problem* (2.1)–(2.5), *and $(u(\cdot,\beta_1), u_1)^T$ be the normalized eigenvector. Then*

$$\lambda'(\beta_1) = -\frac{\lambda}{\beta_2\lambda + \gamma_2} u^2(b),$$

where $\beta_2\lambda + \gamma_2 \neq 0$.

(2) *Fix the parameters of $\dot{\omega}$ except γ_1 and let $\lambda = \lambda(\gamma_1)$ be the eigenvalue of the S-L problem* (2.1)–(2.5), *and $(u(\cdot,\gamma_1), u_1)^T$ be the normalized eigenvector. Then*

$$\lambda'(\gamma_1) = -\frac{1}{\beta_2\lambda + \gamma_2} u^2(b),$$

where $\beta_2\lambda + \gamma_2 \neq 0$.

(3) *Fix the parameters of $\dot{\omega}$ except β_2 and let $\lambda = \lambda(\beta_2)$ be the eigenvalue of the S-L problem* (2.1)–(2.5), *and $(u(\cdot,\beta_2), u_1)^T$ be the normalized eigenvector. Then*

$$\lambda'(\beta_2) = \frac{\lambda}{\beta_1\lambda + \gamma_1}(pu^\Delta)^2(b),$$

where $\beta_1\lambda + \gamma_1 \neq 0$.

(4) *Fix the parameters of $\dot{\omega}$ except γ_2 and let $\lambda = \lambda(\gamma_2)$ be the eigenvalue of the S-L problem* (2.1)–(2.5), *and $(u(\cdot,\gamma_2), u_1)^T$ be the normalized eigenvector. Then*

$$\lambda'(\gamma_2) = \frac{1}{\beta_1\lambda + \gamma_1}(pu^\Delta)^2(b),$$

where $\beta_1\lambda + \gamma_1 \neq 0$.

Proof. We fix the data except β_1 on $\dot{\omega}$, and for arbitrary small ε, let $(u, u_1)^T$, $(v, v_1)^T$ be the corresponding eigenvectors for $\lambda(\beta_1)$, $\lambda(\beta_1 + \varepsilon)$, respectively, and here, $u = u(\cdot,\beta_1)$, $v = u(\cdot,\beta_1 + \varepsilon)$. Then

$$\begin{aligned}[\lambda(\beta_1+\varepsilon) - \lambda(\beta_1)]\langle u, v\rangle &= [\lambda(\beta_1+\varepsilon) - \lambda(\beta_1)]\left[\int_{\rho(a)}^{b} u^\sigma v^\sigma w \Delta t + \frac{1}{\eta}u_1 v_1\right] \\ &= -[u, v](b) + \frac{1}{\eta}[\beta_1 u(b) - \beta_2(pu^\Delta)(b)][-\gamma_1 v(b) + \gamma_2(pv^\Delta)(b)] \\ &\quad - \frac{1}{\eta}[-\gamma_1 u(b) + \gamma_2(pu^\Delta)(b)][(\beta_1+\varepsilon)v(b) - \beta_2(pv^\Delta)(b)] \\ &= -[u, v](b) + [u, v](b) + \frac{1}{\eta}[\gamma_1\varepsilon u(b)v(b) - \gamma_2\varepsilon v(b)(pu^\Delta)(b)] \\ &= \frac{1}{\eta}\left[\gamma_1\varepsilon u(b)v(b) - \gamma_2\varepsilon\frac{\beta_1\lambda+\gamma_1}{\beta_2\lambda+\gamma_2}v(b)u(b)\right] \\ &= -\varepsilon\frac{\lambda}{\beta_2\lambda+\gamma_2}v(b)u(b).\end{aligned} \tag{3.1}$$

Dividing both sides of (3.1) by ε and taking the limit as $\varepsilon \to 0$, by Theorem 2, we obtain

$$\lambda'(\beta_1) = -\frac{\lambda}{\beta_2\lambda + \gamma_2}u^2(b),$$

where $\beta_2\lambda + \gamma_2 \neq 0$.

Parts 2–4 can be proved in the same way, and here, we omit the details. □

Theorem 4. *Let $\lambda(\ddot{\omega})$ be an eigenvalue for the S-L problem* (2.1)–(2.5) *with $\ddot{\omega} \in \Omega_2$, and $(u, u_1)^T$ be a normalized eigenvector for $\lambda(\ddot{\omega})$, and then λ is differentiable with respect to the parameters in $\ddot{\omega}$, and more precisely, the derivative formulas of λ are given as follows*:

(1) *Fix the parameters of $\ddot{\omega}$ except α and let $\lambda = \lambda(\alpha)$ be the eigenvalue of the S-L problem* (2.1)–(2.5), *and $(u(\cdot,\alpha), u_1)^T$ be the normalized eigenvector. Then,*

$$\lambda'(\alpha) = -\sec^2\alpha(pu^\Delta)^2(\rho(a)), \quad \alpha \in [\hat{0}, \hat{\pi}).$$

(2) *Fix the parameters of $\ddot{\omega}$ except the boundary condition parameter matrix*

$$M = \begin{pmatrix} \beta_1 & \gamma_1 \\ \beta_2 & \gamma_2 \end{pmatrix},$$

and let $\lambda = \lambda(M)$ be the eigenvalue of the S-L problem (2.1)–(2.5), *and $(u(\cdot,M), u_1)^T$ be the normalized eigenvector. Then*

$$\mathrm{d}\lambda_M(H) = -(u(b), -(pu^\Delta)(b))[I - M(M+H)^{-1}]\begin{pmatrix}(pu^\Delta)(b)\\ u(b)\end{pmatrix}, \quad H \to 0,$$

for all H *satisfying* $\det(M+H) = \det M = \eta$.

(3) *Fix the parameters of* $\ddot{\omega}$ *except* p *and let* $\lambda = \lambda\left(\frac{1}{p}\right)$ *be the eigenvalue of the S-L problem* (2.1)–(2.5), *and* $\left(u\left(\cdot,\frac{1}{p}\right), u_1\right)^T$ *be the normalized eigenvector. Then*

$$\mathrm{d}\lambda_{\frac{1}{p}}(h) = -\int_{\rho(a)}^{b} (pu^\Delta)^2 h\Delta t, \quad h \to 0, \quad h \in C_{rd}([\rho(a), b] \cap \mathbb{T}).$$

(4) *Fix the parameters of* $\ddot{\omega}$ *except* q *and let* $\lambda = \lambda(q)$ *be the eigenvalue of the S-L problem* (2.1)–(2.5), *and* $(u(\cdot,q), u_1)^T$ *be the normalized eigenvector. Then*

$$\mathrm{d}\lambda_q(h) = \int_{\rho(a)}^{b} (u^\sigma)^2 h\Delta t, \quad h \to 0, \quad h \in C_{rd}([\rho(a), b] \cap \mathbb{T}).$$

(5) *Fix the parameters of* $\ddot{\omega}$ *except* w *and let* $\lambda = \lambda(w)$ *be the eigenvalue of the S-L problem* (2.1)–(2.5), *and* $(u(\cdot,w), u_1)^T$ *be the normalized eigenvector. Then*

$$\mathrm{d}\lambda_w(h) = -\lambda\int_{\rho(a)}^{b} (u^\sigma)^2 h\Delta t, \quad h \to 0, \quad h \in C_{rd}([\rho(a), b] \cap \mathbb{T}).$$

Proof. Fix the parameters of $\ddot{\omega}$ except one and denote by $\lambda(\tilde{\omega})$ the eigenvalue satisfying Theorem 1 when $\|\tilde{\omega} - \ddot{\omega}\| < \varepsilon$ for sufficiently small $\varepsilon > 0$. For parts 1–5, we replace $\lambda(\tilde{\omega})$ by $\lambda(\alpha+\varepsilon)$, $\lambda(M+H)$, $\lambda\left(\frac{1}{p}+h\right)$, $\lambda(q+h)$, and $\lambda(w+h)$, respectively.

1. We fix the data except α on $\ddot{\omega}$, and let $(u(\cdot,\alpha), u_1)^T$, $(u(\cdot,\alpha+\varepsilon), u_1)^T$ be the normalized eigenvectors corresponding to the eigenvalues $\lambda(\alpha)$, $\lambda(\alpha+\varepsilon)$, respectively, and denote $(v, v_1)^T = (u(\cdot,\alpha+\varepsilon), u_1)^T$. Then

$$(\lambda(\alpha+\varepsilon) - \lambda(\alpha))\int_{\rho(a)}^{b} u^\sigma v^\sigma w\Delta t = [u,v](\rho(a)) - [u,v](b). \tag{3.2}$$

By the boundary condition (2.3), we obtain

$$\lambda(\alpha+\varepsilon)[\beta_1 v(b) - \beta_2(pv^\Delta)(b)] = -\gamma_1 v(b) + \gamma_2(pv^\Delta)(b); \quad \lambda(\alpha)[\beta_1 u(b) - \beta_2(pu^\Delta)(b)] = -\gamma_1 u(b) + \gamma_2(pu^\Delta)(b),$$

and thus,

$$[\lambda(\alpha+\varepsilon) - \lambda(\alpha)]u_1 v_1\frac{1}{\eta} = \frac{1}{\eta}(\beta_1\gamma_2 - \beta_2\gamma_1)[u(b)(pv^\Delta)(b) - v(b)(pu^\Delta)(b)] = [u,v](b). \tag{3.3}$$

Combining (3.2) and (3.3) and using the boundary condition (2.2), we obtain

$$\begin{aligned}[\lambda(\alpha+\varepsilon) - \lambda(\alpha)]\left[\int_{\rho(a)}^{b} u^\sigma v^\sigma w\Delta t + u_1 v_1\frac{1}{\eta}\right] &= [u,v](\rho(a))\\ &= u(\rho(a))(pv^\Delta)(\rho(a)) - v(\rho(a))(pu^\Delta)(\rho(a))\\ &= [\tan\alpha - \tan(\alpha+\varepsilon)](pu^\Delta)(\rho(a))(pv^\Delta)(\rho(a)).\end{aligned} \tag{3.4}$$

Dividing both sides of (3.4) by ε and taking the limit as $\varepsilon \to 0$, then by Theorem 2, we obtain

$$\lambda'(\alpha) = -\sec^2\alpha(pu^\Delta)^2(\rho(a)).$$

2. We fix the data except M on $\ddot{\omega}$, let $(u(\cdot,M),u_1)^T,(u(\cdot,M+H),v_1)^T$ be the normalized eigenvectors corresponding to eigenvalues $\lambda(M),\lambda(M+H)$, respectively and denote $(v,v_1)^T=(u(\cdot,M+H),v_1)^T$, then direct computation yields that

$$[\lambda(M+H)-\lambda(M)]\int_{\rho(a)}^{b}u^\sigma v^\sigma w\Delta t=-[u,v](b)=-[u(b)(pv^\Delta)(b)-v(b)(pu^\Delta)(b)]. \tag{3.5}$$

Let

$$M+H=\begin{pmatrix}\tilde{\beta}_1 & \tilde{\gamma}_1\\ \tilde{\beta}_2 & \tilde{\gamma}_2\end{pmatrix},$$

then

$$\begin{aligned}
&[\lambda(M+H)-\lambda(M)]\left[\int_{\rho(a)}^{b}u^\sigma v^\sigma w\Delta t+u_1v_1\frac{1}{\eta}\right]\\
&=[v(b)(pu^\Delta)(b)-u(b)(pv^\Delta)(b)]+[-\tilde{\gamma}_1v(b)+\tilde{\gamma}_2(pv^\Delta)(b)][\beta_1u(b)-\beta_2(pu^\Delta)(b)]\frac{1}{\eta}\\
&\quad-[-\gamma_1u(b)+\gamma_2(pu^\Delta)(b)][\tilde{\beta}_1v(b)-\tilde{\beta}_2(pv^\Delta)(b)]\frac{1}{\eta}\\
&=-(u(b),-(pu^\Delta)(b))\left[I-\frac{1}{\eta}\begin{pmatrix}\beta_1\tilde{\gamma}_2-\gamma_1\tilde{\beta}_2 & -\beta_1\tilde{\gamma}_1+\gamma_1\tilde{\beta}_1\\ \beta_2\tilde{\gamma}_2-\gamma_2\tilde{\beta}_2 & -\tilde{\gamma}_1\beta_2+\tilde{\beta}_1\gamma_2\end{pmatrix}\right]\begin{pmatrix}(pv^\Delta)(b)\\ v(b)\end{pmatrix}\\
&=-(u(b),-(pu^\Delta)(b))[I-M(M+H)^{-1}]\begin{pmatrix}(pv^\Delta)(b)\\ v(b)\end{pmatrix}.
\end{aligned} \tag{3.6}$$

Let $H\to 0$, then the desired result can be obtained by Theorem 2.

3. We fix the data except p on $\ddot{\omega}$, let $(u(\cdot,\frac{1}{p}),u_1)^T,(u(\cdot,\frac{1}{p}+h),u_1)^T$ be the normalized eigenvectors corresponding to eigenvalues $\lambda\left(\frac{1}{p}\right),\lambda\left(\frac{1}{p}+h\right)$, respectively, and denote $(v,v_1)^T=\left(u\left(\cdot,\frac{1}{p}+h\right),u_1\right)^T$, $\frac{1}{p}+h=\frac{1}{p_h}$. Then direct computation yields that

$$\left[\lambda\left(\frac{1}{p}+h\right)-\lambda\left(\frac{1}{p}\right)\right]\int_{\rho(a)}^{b}u^\sigma v^\sigma w\Delta t=[-u(p_hv^\Delta)+v(pu^\Delta)]|_{\rho(a)}^{b}+\int_{\rho(a)}^{b}[u^\Delta(p_hv^\Delta)-v^\Delta(pu^\Delta)]\Delta t. \tag{3.7}$$

By using the boundary condition (2.3), we obtain

$$\begin{aligned}
\left(\lambda\left(\frac{1}{p}+h\right)-\lambda\left(\frac{1}{p}\right)\right)u_1v_1\frac{1}{\eta}&=[\beta_1u(b)-\beta_2(pu^\Delta)(b)][-\gamma_1v(b)+\gamma_2(p_hv^\Delta)(b)]\frac{1}{\eta}\\
&\quad-[\beta_1v(b)-\beta_2(p_hv^\Delta)(b)][-\gamma_1u(b)+\gamma_2(pu^\Delta)(b)]\frac{1}{\eta}\\
&=u(b)(p_hv^\Delta)(b)-(pu^\Delta)(b)v(b),
\end{aligned} \tag{3.8}$$

and combining (3.7) and (3.8), we obtain

$$\left(\lambda\left(\frac{1}{p}+h\right)-\lambda\left(\frac{1}{p}\right)\right)\left[\int_{\rho(a)}^{b}u^\sigma v^\sigma w\Delta t+u_1v_1\frac{1}{\eta}\right]=-\int_{\rho(a)}^{b}(pu^\Delta)(p_hv^\Delta)h\Delta t. \tag{3.9}$$

Let $h\to 0$, then the desired result can be obtained by Theorem 2.

Parts 4 and 5 can be proved in the same way, and here, we omit the details. □

Acknowledgements: The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

Funding information: This work was supported by National Natural Science Foundation of China (Grant Nos. 12261066 and 11661059) and Natural Science Foundation of Inner Mongolia (Grant No. 2021MS01020).

Author contributions: The authors equally contributed this work. All authors read and approved the final manuscript.

Conflict of interest: The authors declare that they have no conflict of interest.

Data availability statement: No data were used to support this study.

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