Research Article

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Displacement structure of the DMP inverse

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Abstract: A matrix A is said to have the displacement structure if the rank of the Sylvester displacement $AU - VA$ or the Stein displacement $A - VAU$ is much smaller than the rank of A. In this article, we study the displacement structure of the DMP inverse $A^{d,\dagger}$. Estimations for the Sylvester displacement rank of DMP inverse are presented under some restrictions. The generalized displacement is also discussed. The general results are applied to the core inverse.

Keywords: DMP inverse, displacement rank, estimation

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1 Introduction

Let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices. For $A \in \mathbb{C}^{m\times n}$, $\mathcal{R}(A)$, $\mathrm{Ker}(A)$, A^*, and A^T denote the range, kernel, conjugate transpose, and transpose of A, respectively. I_n is the identity matrix of order n. The index of $A \in \mathbb{C}^{n\times n}$, denoted by $\mathrm{Ind}(A)$, is the smallest nonnegative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$. The present article is to study the displacement structure of the DMP inverse [1], which is motivated by [2–10]. To begin with, we shall recall definitions of some generalized inverses.

The Moore-Penrose inverse of a matrix $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n\times m}$ satisfying the following Penrose equations [11]:

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA.$$

The Drazin inverse of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $X = A^d$ such that

$$A^{k+1}X = A^k, \quad XAX = X, \quad AX = XA,$$

where k is the index of A. When $\mathrm{Ind}(A) = 1$, A^d is called the group inverse of A and is denoted by $A^\#$, and see [11].

In [12], Baksalary and Trenkler introduced the notion of core inverse for a square matrix of index 1. Let $A \in \mathbb{C}^{n\times n}$ be with $\mathrm{Ind}(A) = 1$. Then the core inverse of A is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying

$$AX = AA^\dagger, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A)$$

and is denoted by $X = A^{\textcircled{\#}}$. It is known that $A^{\textcircled{\#}} = A^\# AA^\dagger$.

Then three generalizations of the core inverse were recently introduced for $n \times n$ complex matrices, namely, core-EP inverse [13], BT inverse [14], and DMP inverse [15]. The DMP inverse [16] of $A \in \mathbb{C}^{n\times n}$, denoted as $A^{d,\dagger}$, is the unique solution to the system of equations:

$$XAX = X, \quad XA = A^dA, \quad A^kX = A^kA^\dagger,$$

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where $k = \text{Ind}(A)$. It was shown that $A^{d,\dagger} = A^d AA^\dagger$. Thus, if $\text{Ind}(A) = 1$, then $A^{d,\dagger} = A^\# AA^\dagger = A^{\oplus}$, the core inverse of A. For more properties of the DMP inverse, we refer to [17–20].

Furthermore, the definition of the DMP inverse of a square matrix was extended to rectangular matrices in [21], which is called the W-weighted DMP inverse.

Malik and Thome [15] gave the canonical form for the DMP inverse of a square matrix by using the Hartwig-Spindelböck decomposition (see [22]), which is useful in analyzing the properties of the DMP inverse. For any $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$, the Hartwig-Spindelböck decomposition is given by

$$A = S\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} S^*, \tag{1}$$

where $S \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \text{diag}(\sigma_1 I_{r_1}, \sigma_2 I_{r_2}, \ldots, \sigma_t I_{r_t})$ is a diagonal matrix, and the diagonal entries $\sigma_i (1 \le i \le t)$ are the singular values of A with $\sigma_1 > \sigma_2 > \cdots > \sigma_t > 0$, $r_1 + r_2 + \cdots + r_t = r$ and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Lemma 1.1. [15] *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1). *Then*

$$A^{d,\dagger} = S\begin{bmatrix} (\Sigma K)^d & 0 \\ 0 & 0 \end{bmatrix} S^*. \tag{2}$$

Lemma 1.2. [15] *Let $A \in \mathbb{C}^{n\times n}$ be a matrix of index m written as* (1). *Then* $\text{Ind}(\Sigma K) = m - 1$.

The concept of the displacement structure was presented in [23] for the inverse of an integral operator with a convolution kernel. A matrix A is said to have the displacement structure if we can find two-dimensionally compatible matrices U and V such that the rank of the Sylvester displacement $AU - VA$ or the Stein displacement $A - VAU$ is much smaller than the rank of A [24]. It is well known that fast inversion algorithms for a matrix A can be constructed if A is a matrix with a displacement structure. The displacement structure is commonly exploited in the computation of generalized inverses. In recent years, displacement structures of various generalized inverses, such as Moore-Penrose inverse, weighted Moore-Penrose inverse, group inverse, M-group inverse, Drazin inverse, W-weighted Drazin inverse, and core inverse, were studied, and see [2–6,8–10].

In this article, we will give an upper bound for the Sylvester displacement rank of the DMP inverse under some restrictions in Section 2. In Section 3, we will consider a more general displacement. An estimation for the generalized displacement rank of the DMP inverse under some restrictions is presented. The general results are applied to the core inverse.

2 Sylvester displacement rank

Let $A \in \mathbb{C}^{m\times n}$ and let $U \in \mathbb{C}^{n\times n}$, $V \in \mathbb{C}^{m\times m}$ be some fixed matrices. The operator

$$d(U, V)A = AU - VA$$

is called the Sylvester displacement of A. The rank of $d(U, V)A$ is called the Sylvester displacement rank of A. It is well known that the Sylvester displacement rank of a Toeplitz rank is at most 2. This low displacement rank property can be exploited to develop fast algorithms for triangular factorization, and inversion, among others [24]. For a nonsingular matrix A, the equality $AU - VA = A(UA^{-1} - A^{-1}V)A$ tells us that the Sylvester displacement rank of a nonsingular matrix A equals that of its inverse A^{-1}. In other words, if A is structured with respect to the Sylvester displacement rank associated with (U, V), then its inverse A^{-1} is also structured with respect to the Sylvester displacement rank associated with (V, U). It is natural to consider the displacement rank of generalized inverses. In this section, we will give an upper bound for the displacement rank of DMP inverse under some restrictions.

Lemma 2.1. [11] *Let $A, P \in \mathbb{C}^{n\times n}$ satisfy $P^2 = P$. Then*
(*i*) *$PA = A$ if and only if $\mathcal{R}(A) \subset \mathcal{R}(P)$.*
(*ii*) *$AP = A$ if and only if $\mathrm{Ker}(P) \subset \mathrm{Ker}(A)$.*

Define the matrices

$$M = A^{d,\dagger}A, \quad N = I - M, \quad M_* = AA^{d,\dagger}, \quad N_* = I - M_*.$$

Then it is easy to see that $A^{d,\dagger}(AU - VA)A^{d,\dagger} = (I - N)UA^{d,\dagger} - A^{d,\dagger}V(I - N_*)$. Thus, we obtain the following lemma.

Lemma 2.2. *Let $A, U, V \in \mathbb{C}^{n\times n}$. Then*

$$A^{d,\dagger}V - UA^{d,\dagger} = A^{d,\dagger}VN_* - NUA^{d,\dagger} - A^{d,\dagger}(AU - VA)A^{d,\dagger}. \tag{3}$$

By considering the ranks of both sides in equality (3), we can obtain an upper bound for the Sylvester displacement rank of $A^{d,\dagger}$.

Theorem 2.1. *The VU-displacement rank of $A^{d,\dagger}$ satisfies the following estimate*:

$$\mathrm{rank}(A^{d,\dagger}V - UA^{d,\dagger}) \le \mathrm{rank}(AU - VA) + \mathrm{rank}(M_*VN_*) + \mathrm{rank}(NUM). \tag{4}$$

Proof. It is clear that $\mathcal{R}(M) = \mathcal{R}(A^{d,\dagger}A) \subset \mathcal{R}(A^{d,\dagger})$. Since

$$\mathcal{R}(A^{d,\dagger}) = \mathcal{R}(A^dAA^\dagger) \subset \mathcal{R}(A^dA) = \mathcal{R}(A^{d,\dagger}A) = \mathcal{R}(M),$$

then $\mathcal{R}(M) = \mathcal{R}(A^{d,\dagger})$.

On the other hand, it is clear that $\mathrm{Ker}(A^{d,\dagger}) \subset \mathrm{Ker}(AA^{d,\dagger}) = \mathrm{Ker}(M_*)$. Conversely, for any $x \in \mathrm{Ker}(M_*)$, $AA^{d,\dagger}x = AA^dAA^\dagger x = 0$, which implies that $A^dAA^dAA^\dagger x = A^dAA^\dagger x = A^{d,\dagger}x = 0$, i.e., $\mathrm{Ker}(M_*) \subset \mathrm{Ker}(A^{d,\dagger})$. Therefore, $\mathrm{Ker}(A^{d,\dagger}) = \mathrm{Ker}(M_*)$.

It follows that

$$\begin{aligned}\mathrm{rank}(NUA^{d,\dagger}) &= \dim[\mathcal{R}(NUA^{d,\dagger})]\\ &= \dim[NU\mathcal{R}(A^{d,\dagger})]\\ &= \dim[NU\mathcal{R}(M)]\\ &= \dim[\mathcal{R}(NUM)]\\ &= \mathrm{rank}(NUM)\end{aligned}$$

and

$$\begin{aligned}\mathrm{rank}(A^{d,\dagger}VN_*) &= n - \dim[\mathrm{Ker}(A^{d,\dagger}VN_*)]\\ &= n - \dim[\mathrm{Ker}(M_*VN_*)]\\ &= \mathrm{rank}(M_*VN_*).\end{aligned}$$

Now, the inequality (4) follows immediately from Lemma 2.2. □

Next we give an upper bound for the sum of the second and third terms on the right-hand side of (4) under some restrictions.

Theorem 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and let*

$$U = S\begin{bmatrix} U_{11} & U_{12}\\ U_{21} & U_{22}\end{bmatrix}S^* \in \mathbb{C}^{n\times n},\ V = S\begin{bmatrix} V_{11} & V_{12}\\ V_{21} & V_{22}\end{bmatrix}S^* \in \mathbb{C}^{n\times n},$$

where $U_{11}, V_{11} \in \mathbb{C}^{r\times r}$. If $\mathcal{R}(U_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\mathrm{Ker}[(\Sigma K)^{p-1}] \subset \mathrm{Ker}(V_{11})$, then

$$\mathrm{rank}(M_*VN_*) + \mathrm{rank}(NUM) \le \mathrm{rank}(UG - GV), \tag{5}$$

where $G = A^pA^\dagger$, $p = \mathrm{Ind}(A)$.

Proof. We first determine the structure of G. If A has the form (1), then it is not difficult to see that

$$A^{\dagger} = S\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}S^*.$$

Thus,

$$\begin{aligned} G &= A^pA^{\dagger} \\ &= S\begin{bmatrix} (\Sigma K)^p & (\Sigma K)^{p-1}\Sigma L \\ 0 & 0 \end{bmatrix}S^*S\begin{bmatrix} K^*\Sigma^{-1} & 0 \\ L^*\Sigma^{-1} & 0 \end{bmatrix}S^* \\ &= S\begin{bmatrix} (\Sigma K)^pK^*\Sigma^{-1} + (\Sigma K)^{p-1}\Sigma LL^*\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}S^* \\ &= S\begin{bmatrix} (\Sigma K)^{p-1}\Sigma(KK^* + LL^*)\Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}S^* \\ &= S\begin{bmatrix} (\Sigma K)^{p-1} & 0 \\ 0 & 0 \end{bmatrix}S^*. \end{aligned}$$

Next, we determine the block representations of M_*VN_* and NUM. Denoted $P = (\Sigma K)(\Sigma K)^d$. Then

$$\begin{aligned} M_*VN_* &= AA^{d,\dagger}V(I - AA^{d,\dagger}) \\ &= S\begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}\begin{bmatrix} I - P & 0 \\ 0 & I \end{bmatrix}S^* \\ &= S\begin{bmatrix} PV_{11}(I - P) & PV_{12} \\ 0 & 0 \end{bmatrix}S^*. \end{aligned} \tag{6}$$

Since $\operatorname{Ind}(A) = p$, then by Lemma 1.2, $\operatorname{Ind}(\Sigma K) = p - 1$. If $\operatorname{Ker}[(\Sigma K)^{p-1}] \subset \operatorname{Ker}(V_{11})$, then $\operatorname{Ker}(P) = \operatorname{Ker}[(\Sigma K)^d] = \operatorname{Ker}[(\Sigma K)^{p-1}] \subset \operatorname{Ker}(V_{11})$. Thus, by Lemma 2.1, $V_{11}(I - P) = 0$. Now, it can be seen from (6) that

$$\begin{aligned} \operatorname{rank}(M_*VN_*) &= \operatorname{rank}(PV_{12}) \\ &= n - \dim \operatorname{Ker}(PV_{12}) \\ &= n - \dim \operatorname{Ker}[(\Sigma K)^{p-1}V_{12}] \\ &= \operatorname{rank}[(\Sigma K)^{p-1}V_{12}]. \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} NUM &= (I - A^{d,\dagger}A)U(A^{d,\dagger}A) \\ &= S\begin{bmatrix} I - P & -Q \\ 0 & I \end{bmatrix}\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix}S^* \\ &= S\begin{bmatrix} (I - P)U_{11}P - QU_{21}P & (I - P)U_{11}Q - QU_{21}Q \\ U_{21}P & U_{21}Q \end{bmatrix}S^*, \end{aligned} \tag{8}$$

where $Q = (\Sigma K)^d(\Sigma L)$.

We can see from (8) that

$$\begin{aligned} \operatorname{rank}(NUM) &= \operatorname{rank}\left(S\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}S^*NUM\right) \\ &= \operatorname{rank}\left(S\begin{bmatrix} (I - P)U_{11}P & (I - P)U_{11}Q \\ U_{21}P & U_{21}Q \end{bmatrix}S^*\right) \\ &= \operatorname{rank}\left(S\begin{bmatrix} (I - P)U_{11} & 0 \\ U_{21}P & 0 \end{bmatrix}\begin{bmatrix} P & Q \\ 0 & 0 \end{bmatrix}S^*\right) \\ &\le \operatorname{rank}\left(\begin{bmatrix} (I - P)U_{11} & 0 \\ U_{21}P & 0 \end{bmatrix}\right). \end{aligned} \tag{9}$$

Notice that $\mathcal{R}[(\Sigma K)^{p-1}] = \mathcal{R}[(\Sigma K)^{d}] = \mathcal{R}(P)$. If $\mathcal{R}(U_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$, then by Lemma 2.1, $PU_{11} = U_{11}$. It follows from (9) that

$$\operatorname{rank}(NUM) \le \operatorname{rank}(U_{21}P) = \operatorname{rank}[U_{21}(\Sigma K)^{d}] = \operatorname{rank}[U_{21}(\Sigma K)^{p-1}]. \tag{10}$$

Set $F = UG - GV$. Then

$$\begin{aligned} S^*FS &= S^*USS^*GS - S^*GSS^*VS \\ &= \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}\begin{bmatrix} (\Sigma K)^{p-1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} (\Sigma K)^{p-1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} U_{11}(\Sigma K)^{p-1} - (\Sigma K)^{p-1}V_{11} & -(\Sigma K)^{p-1}V_{12} \\ U_{21}(\Sigma K)^{p-1} & 0 \end{bmatrix}. \end{aligned}$$

It follows from [25], (7), and (10) that

$$\begin{aligned} &\operatorname{rank}\left(\begin{bmatrix} U_{11}(\Sigma K)^{p-1} - (\Sigma K)^{p-1}V_{11} & -(\Sigma K)^{p-1}V_{12} \\ U_{21}(\Sigma K)^{p-1} & 0 \end{bmatrix}\right) \\ &\ge \operatorname{rank}[(\Sigma K)^{p-1}V_{12}] + \operatorname{rank}[U_{21}(\Sigma K)^{p-1}] \\ &\ge \operatorname{rank}(M_*VN_*) + \operatorname{rank}(NUM), \end{aligned}$$

which completes the proof. □

We remark that if the two conditions $\mathcal{R}(U_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\operatorname{Ker}[(\Sigma K)^{p-1}] \subset \operatorname{Ker}(V_{11})$ were removed from Theorem 2.2, then the inequality (5) may not hold. Let's give an example to show this.

Example 2.1. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Then $\operatorname{rank}(A) = 4$ and $\operatorname{rank}(A^2) = \operatorname{rank}(A^3) = 3$. Thus, $\operatorname{Ind}(A) = 2$. The singular value decomposition of A is

$$\begin{aligned} A &= S\tilde{\Sigma}Q^T \\ &= \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ -y & 0 & 0 & x & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ -x & 0 & 0 & -y & 0 \end{bmatrix}\begin{bmatrix} \frac{\sqrt{5}+1}{2} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{\sqrt{5}-1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -x & 0 & 0 & y & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & -1 & 0 & 0 & 0 \\ -y & 0 & 0 & -x & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \\ &= S\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}Q^TSS^T \\ &= S\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K & L \\ M & N \end{bmatrix}S^T \\ &= S\begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix}S^T, \end{aligned}$$

where $x = \sqrt{\frac{5-\sqrt{5}}{10}}$, $y = \sqrt{\frac{5+\sqrt{5}}{10}}$, and $\Sigma K = \begin{bmatrix} 0 & \frac{(x+y)(\sqrt{5}+1)}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ -(x+y) & 0 & 0 & x-y \\ 0 & \frac{(x-y)(\sqrt{5}-1)}{2\sqrt{2}} & 0 & 0 \end{bmatrix}$.

Let

$$U = V = S\begin{bmatrix} I_4 & 0 \\ 0 & 0 \end{bmatrix}S^T = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then it can be seen from $\operatorname{rank}(\Sigma K) = 3$ and $\operatorname{rank}(U_{11}) = \operatorname{rank}(V_{11}) = 4$ that $\mathcal{R}(U_{11}) \not\subset \mathcal{R}[(\Sigma K)]$ and $\operatorname{Ker}[(\Sigma K)] \not\subset \operatorname{Ker}(V_{11})$.

A direct calculation shows that

$$M = A^{d,\dagger}A = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}, \quad M_* = AA^{d,\dagger} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

and

$$G = A^2A^\dagger = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.$$

Now, it can be seen that

$$\operatorname{rank}(M_*VN_*) = 3, \quad \operatorname{rank}(NUM) = 2, \quad \operatorname{rank}(UG - GV) = 2,$$

i.e., $\operatorname{rank}(M_*VN_*) + \operatorname{rank}(NUM) > \operatorname{rank}(UG - GV)$.

By combining Theorems 2.1 and 2.2 we have the following.

Corollary 2.1. *Let $A \in \mathbb{C}^{n\times n}$ be of the form* (1) *and let*

$$U = S\begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}S^* \in \mathbb{C}^{n\times n}, \quad V = S\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}S^* \in \mathbb{C}^{n\times n},$$

where $U_{11}, V_{11} \in \mathbb{C}^{r\times r}$. If $\mathcal{R}(U_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\operatorname{Ker}[(\Sigma K)^{p-1}] \subset \operatorname{Ker}(V_{11})$, then

$$\operatorname{rank}(A^{d,\dagger}V - UA^{d,\dagger}) \le \operatorname{rank}(AU - VA) + \operatorname{rank}(UG - GV),$$

where $G = A^pA^\dagger$, $p = \operatorname{Ind}(A)$.

Let $A \in \mathbb{C}^{n\times n}$ be of the form (1). If $\operatorname{Ind}(A) = 1$, it was shown in [22] that K is nonsingular. In this case, ΣK is also nonsingular. Then $(\Sigma K)^{p-1} = I$ and the two restrictions $\mathcal{R}(U_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\operatorname{Ker}[(\Sigma K)^{p-1}] \subset \operatorname{Ker}(V_{11})$ can be removed from Theorem 2.2. Hence, we can obtain an estimate for the VU-displacement rank of the core inverse.

Corollary 2.2. *For any $A \in \mathbb{C}^{n\times n}$ with $\operatorname{Ind}(A) = 1$, we have*

$$\operatorname{rank}(A^{\circledast}V - UA^{\circledast}) \le \operatorname{rank}(AU - VA) + \operatorname{rank}(UG - GV), \tag{11}$$

where $G = AA^\dagger$.

We remark that the author in [6] showed that (11) holds for $G = AA^*$, while we derive a different G such that the inequality (11) holds.

3 Generalized displacement rank

In this section, we will consider a more general displacement, and the results of Sylvester displacement in Section 2 will be extended.

Let $a = [a_{ij}]_0^1$ be a nonsingular 2×2 matrix. For any fixed $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$, the generalized $a(V, U)$ displacement of $A \in \mathbb{C}^{m\times n}$ is defined by

$$a(V, U)A = \sum_{i,j=0}^{1} a_{ij} V^i A U^j. \tag{12}$$

If we set

$$a = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

respectively, in (12), then we obtain the Sylvester displacement and the Stein displacement.

For $Z \in \mathbb{C}^{n\times n}$ and $a = [a_{ij}]_0^1 \in \mathbb{C}^{2\times 2}$, if $a_{00}I + a_{01}Z$ is nonsingular, then we denote by $f_a(Z)$ the matrix

$$f_a(Z) = (a_{00}I + a_{01}Z)^{-1}(a_{10}I + a_{11}Z).$$

We first give two lemmas, which are taken from [4].

Lemma 3.1. *Let a be a 2×2 nonsingular matrix and $U \in \mathbb{C}^{m\times m}$, $V \in \mathbb{C}^{n\times n}$. Then there exist 2×2 matrices $b = [b_{ij}]_0^1$ and $c = [c_{ij}]_0^1$ such that $b_{00}I + b_{01}V$ and $c_{00}I + c_{01}U$ are nonsingular and*

$$a = b^T dc \quad \text{for } d = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Lemma 3.2. *Let b and c be matrices satisfying the conditions in* Lemma 3.1. *Then for $A \in \mathbb{C}^{m\times n}$,*

$$a(V, U)A = (b_{00}I + b_{01}V)[Af_c(U) - f_b(V)A](c_{00}I + c_{01}U).$$

The following results play important roles in obtaining an extension of Corollary 2.1 for generalized $a(U, V)$ displacement.

Theorem 3.1.
(*a*) *If $[\psi_{ij}]_0^1$ is nonsingular and $\psi_{00}I + \psi_{01}V$ is nonsingular, then*

$$\operatorname{rank}(M_* V N_*) = \operatorname{rank}(M_* \tilde{V} N_*),$$

where $\tilde{V} = f_\psi(V)$.
(*b*) *If $[\phi_{ij}]_0^1$ is nonsingular and $\phi_{00}I + \phi_{01}U$ is nonsingular, then*

$$\operatorname{rank}(NUM) = \operatorname{rank}(N\tilde{U}M),$$

where $\tilde{U} = f_\phi(U)$.

Proof. Define

$$\Psi = \operatorname{Ker}(G) \cap \operatorname{Ker}(GV), \Psi_1 = \operatorname{Ker}(G) \ominus \Psi,$$

where $G = A^pA^\dagger$, $p = \operatorname{Ind}(A)$. We conclude that $\operatorname{Ker}(A^{d,\dagger}) = \operatorname{Ker}(G)$. Indeed, for any $x \in \operatorname{Ker}(A^{d,\dagger})$, $A^{d,\dagger}x = A^dAA^\dagger x = 0$; thus, $A^dA^dAA^\dagger x = A^dA^\dagger x = 0$. It follows that $A^\dagger x \in Ker(A^d) = \operatorname{Ker}(A^p)$. Hence, $A^pA^\dagger x = 0$, i.e.,

$\mathrm{Ker}(A^{d,\dagger}) \subset \mathrm{Ker}(G)$. Conversely, for any $x \in \mathrm{Ker}(A^pA^\dagger)$, we have $A^\dagger x \in \mathrm{Ker}(A^p) = \mathrm{Ker}(A^d)$. Then $A^dA^\dagger x = 0$, which gives that $AA^dA^\dagger x = A^dAA^\dagger x = 0$, i.e., $x \in \mathrm{Ker}(A^dAA^\dagger) = \mathrm{Ker}(A^{d,\dagger})$. Therefore, $\mathrm{Ker}(A^{d,\dagger}) = \mathrm{Ker}(G)$.

Next, we will show that M_*VN_* is one-to-one on Ψ_1. For any $x \in \mathrm{Ker}(M_*VN_*)$ and $x \in \Psi_1$, then $VN_*x \in \mathrm{Ker}(M_*)$. Since $\mathrm{Ker}(M_*) = \mathrm{Ker}(A^{d,\dagger}) = \mathrm{Ker}(G)$, then $GVN_*x = GVx = 0$. Now, it is easy to see that $x \in \Psi \cap \Psi_1 = \{0\}$, i.e., $x = 0$.

Furthermore, for any $x \in \Psi$, we have $A^{d,\dagger}x = 0$ and $A^{d,\dagger}Vx = 0$. Thus, $M_*VN_*x = M_*V(I - AA^{d,\dagger})x = M_*Vx = AA^{d,\dagger}Vx = 0$, i.e., $(M_*VN_*)|_\Psi = 0$. Now we conclude that $\mathrm{rank}(M_*VN_*) = \dim(\Psi_1)$.

Similarly, we define

$$\tilde{\Psi} = \mathrm{Ker}(G) \cap \mathrm{Ker}(G\tilde{V}),\ \tilde{\Psi}_1 = \mathrm{Ker}(G) \ominus \tilde{\Psi},$$

then it follows that $\mathrm{rank}(M_*\tilde{V}N_*) = \dim(\tilde{\Psi}_1)$.

Now we show that the nonsingular matrix $\psi_{00}I + \psi_{01}V$ bijectively maps Ψ onto $\tilde{\Psi}$. Suppose that $x \in \Psi$, then x, $Vx \in \mathrm{Ker}(G)$. Hence, both$y \equiv (\psi_{10}I + \psi_{11}V)x$ and $z \equiv (\psi_{00}I + \psi_{01}V)x$ are contained in $\mathrm{Ker}(G)$. Since

$$\begin{aligned} y &= (\psi_{10}I + \psi_{11}V)x \\ &= (\psi_{00}I + \psi_{01}V)^{-1}(\psi_{00}I + \psi_{01}V)(\psi_{10}I + \psi_{11}V)x \\ &= (\psi_{00}I + \psi_{01}V)^{-1}(\psi_{10}I + \psi_{11}V)(\psi_{00}I + \psi_{01}V)x \\ &= \tilde{V}z, \end{aligned}$$

then z, $\tilde{V}z \in \mathrm{Ker}(G)$, which implies that $z \in \tilde{\Psi}$. Conversely, in a similar way, we can obtain that $(\psi_{00}I + \psi_{01}V)^{-1}z \in \Psi$ for $z \in \tilde{\Psi}$.

Therefore,

$$\mathrm{rank}(M_*VN_*) = \dim(\Psi_1) = \dim[\mathrm{Ker}(G)] - \dim(\Psi) = \dim[\mathrm{Ker}(G)] - \dim(\tilde{\Psi}) = \dim(\tilde{\Psi}_1) = \mathrm{rank}(M_*\tilde{V}N_*),$$

which proves the assertion (a).

Assertion (b) can be proved analogously. ◻

Now we can generalize Corollary 2.1 for general $a(U, V)$ displacement.

Theorem 3.2. *Let a and b be 2×2 nonsingular matrices and let $A \in \mathbb{C}^{n\times n}$ be of the form* (1). *Suppose that*

$$f_w(U) = S\begin{bmatrix} \tilde{U}_{11} & \tilde{U}_{12} \\ \tilde{U}_{21} & \tilde{U}_{22} \end{bmatrix}S^*, \quad f_z(V) = S\begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}S^*.$$

If $\mathcal{R}(\tilde{U}_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\mathrm{Ker}[(\Sigma K)^{p-1}] \subset \mathrm{Ker}(\tilde{V}_{11})$, Then

$$\mathrm{rank}[a(U, V)A^{d,\dagger}] \le \mathrm{rank}[a^T(V, U)A] + \mathrm{rank}[b(U, V)G],$$

where $G = A^pA^\dagger$ and $p = \mathrm{Ind}(A)$.

Proof. By Lemma 3.1, there exist 2×2 matrices w, x, y, z such that $w_{00}I + w_{01}U$, $x_{00}I + x_{01}V$, $y_{00}I + y_{01}U$, and $z_{00}I + z_{01}V$ are nonsingular and $a = w^Tdz$, $b = x^Tdy$.

By Lemma 3.2,

$$\mathrm{rank}[a(U, V)A^{d,\dagger}] = \mathrm{rank}[A^{d,\dagger}f_z(V) - f_w(U)A^{d,\dagger}]$$

and

$$\mathrm{rank}[a^T(V, U)A] = \mathrm{rank}[Af_w(U) - f_z(V)A].$$

Moreover, it follows from Theorem 2.1 that

$$\mathrm{rank}[A^{d,\dagger}f_z(V) - f_w(U)A^{d,\dagger}] \le \mathrm{rank}[Af_w(U) - f_z(V)A] + \mathrm{rank}[M_*f_z(V)N_*] + \mathrm{rank}(Nf_w(U)M].$$

Hence, according to Corollary 2.1 and Theorems 2.2 and 3.1, we obtain

$$\begin{aligned}\operatorname{rank}[a(U,V)A^{d,\dagger}]-\operatorname{rank}[a^T(V,U)A] &= \operatorname{rank}[f_w(U)A^{d,\dagger}-A^{d,\dagger}f_z(V)]-\operatorname{rank}[Af_w(U)-f_z(V)A]\\ &\le \operatorname{rank}[Nf_w(U)M]+\operatorname{rank}[M_*f_z(V)N_*]\\ &= \operatorname{rank}[Nf_y(U)M]+\operatorname{rank}[M_*f_x(V)N_*]\\ &\le \operatorname{rank}[f_y(U)G-Gf_x(V)]\\ &= \operatorname{rank}[b(U,V)G].\end{aligned}$$

□

Corollary 3.1. *Let a and b be 2×2 nonsingular matrices. Then*

$$\operatorname{rank}[a(U,V)A^{\oplus}]\le \operatorname{rank}[a^T(V,U)A]+\operatorname{rank}[b(U,V)G], \tag{13}$$

where $G=AA^\dagger$.

Example 3.1. Denote

$$U=\begin{bmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\end{bmatrix},\quad V=\begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$$

as the shift-down matrix and the shift-up matrix, respectively. Let

$$A=\begin{bmatrix}0&1&0&0&0&1\\0&0&1&0&0&0\\1&0&0&1&0&0\\0&1&0&0&1&0\\1&0&1&0&0&1\\0&1&0&1&0&0\end{bmatrix}$$

be a singular Toeplitz matrix with $\operatorname{Ind}(A)=1$ and $\operatorname{rank}(A)=5$.

The core inverse of A is

$$A^{\oplus}=\begin{bmatrix}-\frac{1}{20}&-\frac{11}{20}&\frac{9}{20}&\frac{1}{4}&\frac{3}{10}&-\frac{9}{20}\\ \frac{7}{20}&-\frac{3}{20}&-\frac{3}{20}&\frac{1}{4}&-\frac{1}{10}&\frac{3}{20}\\ -\frac{1}{5}&\frac{4}{5}&-\frac{1}{5}&0&\frac{1}{5}&\frac{1}{5}\\ -\frac{3}{20}&\frac{7}{20}&\frac{7}{20}&-\frac{1}{4}&-\frac{1}{10}&\frac{13}{20}\\ -\frac{7}{20}&\frac{3}{20}&\frac{3}{20}&\frac{3}{4}&\frac{1}{10}&-\frac{3}{20}\\ \frac{9}{20}&-\frac{1}{20}&-\frac{1}{20}&-\frac{1}{4}&\frac{3}{10}&\frac{1}{20}\end{bmatrix}.$$

If we choose

$$a=b=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

in Corollary 3.1, then a direct computation shows that

$$\operatorname{rank}[a(U,V)A^{\oplus}]=4,\quad \operatorname{rank}[a^T(V,U)A]=2,\quad \operatorname{rank}[b(U,V)AA^\dagger]=5,$$

i.e., the estimate (13) in Corollary 3.1 holds.

4 Computation of the displacement

In this section, we study the explicit form of the displacement, and we give an expression for the displacement of the DMP inverse of A through DMP inverse solutions of some special linear systems of equations. For simplicity, we only consider the Sylvester displacement, and the starting point is (3).

(1) To compute $A^{d,\dagger}(AU - VA)A^{d,\dagger}$, first find a full-rank decomposition of $AU - VA$,

$$AU - VA = GF^* = \sum_{i=1}^{r} g_i f_i^*,$$

where, $G = [g_1, g_2, \ldots, g_r] \in \mathbb{C}^{n\times r}$ and $F = [f_1, f_2, \ldots, f_r] \in \mathbb{C}^{n\times r}$ are of rank r, and then compute the DMP inverse solutions $A^{d,\dagger}g_i$ and $f_i^* A^{d,\dagger}$.

(2) Next we show how to compute $A^{d,\dagger}V(I - AA^{d,\dagger})$. We start with a full-rank decomposition

$$A^{d,\dagger}V(I - AA^{d,\dagger}) = MN^*.$$

Now we determine the kernels of $A^p A^\dagger$ $(p = \operatorname{Ind}(A))$ and the matrix

$$C = \begin{bmatrix} A^p A^\dagger \\ N^* \end{bmatrix}$$

and a set of vectors $w_1, w_2, \ldots, w_p$ forming an orthogonal basis for the orthogonal complements of $\operatorname{Ker}(C)$ in $\operatorname{Ker}(A^p A^\dagger)$. We generate the matrix $W = [w_1, w_2, \ldots, w_p]$. Then we have

$$\operatorname{Ker}(A^p A^\dagger) = \operatorname{Ker}(C) \oplus \mathcal{R}(W).$$

The matrix $R = WW^*$ is an orthogonal projection onto $\mathcal{R}(W)$. We have

$$R(I - AA^{d,\dagger}) = WW^*(I - AA^{d,\dagger}) = WW^* = R$$

and

$$A^{d,\dagger}V(I - R)(I - AA^{d,\dagger}) = 0.$$

Hence,

$$A^{d,\dagger}V(I - AA^{d,\dagger}) = A^{d,\dagger}VR = A^{d,\dagger}VWW^*.$$

(3) We proceed analogously for $(I - A^{d,\dagger}A)UA^{d,\dagger}$. We obtain

$$(I - A^{d,\dagger}A)UA^{d,\dagger} = SUA^{d,\dagger} = ZZ^*UA^{d,\dagger},$$

where S is the orthogonal complement onto $\operatorname{Ker}(A) \ominus \operatorname{Ker}(C_*)$ with C_* defined by $C_*^* = [A^p A^\dagger, X^*]$ and Z being a matrix with columns forming an orthogonal basis of $\mathcal{R}(S)$.

To compute $A^{d,\dagger}V - UA^{d,\dagger}$, one has to find $2r$ DMP inverse solutions $A^{d,\dagger}g_i$ and $f_i^* A^{d,\dagger}$, where $r = \operatorname{rank}(AU - VA)$, and the $p + q$ DMP inverse solutions $A^{d,\dagger}Vw_i (1 \le i \le p)$ and $z_j^* UA^{d,\dagger} (1 \le j \le q)$, where $p + q \le \operatorname{rank}(A^p A^\dagger V - UA^p A^\dagger)$.

5 Concluding remark

In this article, we studied the displacement structure of the DMP inverse. An upper bound for the Sylvester displacement rank of the DMP inverse was given under some restrictions. An example was presented to show that these restrictions can not be removed. Estimations for the general displacement rank of the DMP inverse were also investigated. As corollaries, estimations for the Sylvester displacement rank and the general displacement rank of the core inverse without any restrictions were obtained.

Furthermore, the theorems obtained before can be applied to many classical structured matrices, such as Toeplitz matrices, Hankel matrices, and Cauchy matrices. For simplicity, we only consider close-to-Toeplitz matrices.

Close-to-Toeplitz matrices are a class of matrices whose UV-displacement ranks are small compared with the sizes of U and V being (forward or backward) (block) shifts, including Toeplitz matrices, Hankel matrices, and more general block matrices with Toeplitz or Hankel blocks, and sums, products, and inverses of these matrices.

Let U be the shift-down matrix

$$Z = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix},$$

V be the shift-up matrix Z^*.

If $\mathcal{R}(\tilde{U}_{11}) \subset \mathcal{R}[(\Sigma K)^{p-1}]$ and $\mathrm{Ker}[(\Sigma K)^{p-1}] \subset \mathrm{Ker}(\tilde{V}_{11})$, then choosing

$$a = b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

in Theorem 3.2 gives

$$\mathrm{rank}(A^{d,\dagger} - ZA^{d,\dagger}Z^*) \leq r_+ + r_{A^pA^\dagger}, \tag{14}$$

where $r_+ = \mathrm{rank}(A - Z^*AZ)$, $r_{A^pA^\dagger} = \mathrm{rank}(A^pA^\dagger - ZA^pA^\dagger Z^*)$, and $p = \mathrm{Ind}(A)$.

If the estimate of (14) is small for a close-to-Toeplitz matrix, then it leads to the famous representation Gohberg-Semencul type of $A^{d,\dagger}$:

$$A^{d,\dagger} = \sum_{k=1}^{r} L_k U_k,$$

where r is the displacement rank of $A^{d,\dagger}$, i.e., $r_+ + r_{A^pA^\dagger}$ in (14).

As a corollary of (14), for core inverse, we have

$$\mathrm{rank}(A^{\circledast} - ZA^{\circledast}Z^*) \leq r_+ + r_{AA^\dagger},$$

where $r_+ = \mathrm{rank}(A - Z^*AZ)$ and $r_{AA^\dagger} = \mathrm{rank}(AA^\dagger - ZAA^\dagger Z^*)$.

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