Research Article

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Dual uniformities in function spaces over uniform continuity

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Abstract: The notion of dual uniformity is introduced on $UC(Y, Z)$, the uniform space of uniformly continuous mappings between Y and Z, where $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ are two uniform spaces. It is shown that a function space uniformity on $UC(Y, Z)$ is admissible (resp. splitting) if and only if its dual uniformity on $\mathcal{U}_Z(Y) = \{f_2^{-1}(U) \mid f \in UC(Y, Z), U \in \mathcal{U}\}$ is admissible (resp. splitting). It is also shown that a uniformity on $\mathcal{U}_Z(Y)$ is admissible (resp. splitting) if and only if its dual uniformity on $UC(Y, Z)$ is admissible (resp. splitting). Using duality theorems, it is also proved that the greatest splitting uniformity and the greatest splitting family open uniformity exist on $\mathcal{U}_Z(Y)$ and $UC(Y, Z)$, respectively, and these two uniformities are mutually dual splitting uniformities.

Keywords: dual uniformity, uniform space, function space, splittingness, admissibility

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1 Introduction

In [1], Gupta et al. introduced the concept of uniform space over uniform continuity, wherein they introduced point-entourage uniformity and entourage-entourage uniformity on $UC(Y, Z)$, the family of uniformly continuous mappings between the uniform spaces $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ and gave a systematic study of properties of such uniform structures. Any metric on X generates a uniformity on X; similarly, a uniformity on X generates a topology on X. The reverse of either of them is not true. In that sense, uniform spaces are positioned between metric spaces and topological spaces. Hence, structures in one of them are expected to have their counterpart in the other and *vice versa*. This is further evident from the fact that in [1], various concepts of function space topologies including admissibility, splittings, etc. have been introduced and successfully investigated for uniformities over uniform spaces (see also [2–9]). Some more relevant literature can be found in [10–12]. Recently, it has been shown that the admissibility of the function space topology for a pair of topological vector spaces provides sufficient conditions for the existence of a solution to variational inequality problems [13,14]. This motivates us for further study of this concept for uniform spaces too.

In this article, we introduce and study the concept of dual uniformity for the uniformities on $UC(Y, Z)$. We have come up with a good number of results of interest. It is found that a uniformity on $UC(Y, Z)$ is admissible (resp. splitting) if and only if its dual uniformity on $\mathcal{U}_Z(Y)$ is admissible (resp. splitting).

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Similarly, a uniformity on $\mathcal{U}_Z(Y)$ is admissible (resp. splitting) if and only if its dual uniformity on $UC(Y, Z)$ is admissible (resp. splitting). We have also proved the existence of the greatest splitting uniformity on $\mathcal{U}_Z(Y)$ as well as the greatest splitting family open uniformity on $UC(Y, Z)$. In addition, it is shown that these two uniformities are mutually dual splitting uniformities. We also provide few examples to illustrate the concept of dual uniformity and how admissibility and splittingness of dual uniform spaces are connected with the uniform spaces.

2 Preliminaries

Definition 2.1. [15,16] A *uniform structure* or *uniformity* on a non-empty set X is a family $\mathcal{U}$ of subsets of $X \times X$ satisfying the following properties:

(2.1.1) if $U \in \mathcal{U}$, then $\Delta X \subseteq U$;
where $\Delta X = \{(x, x) \in X \times X \text{ for all } x \in X\}$;

(2.1.2) if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
where U^{-1} is called *inverse relation* of U and is defined as :

$$U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\};$$

(2.1.3) if $U \in \mathcal{U}$, then there exists some $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;
where the composition $U \circ V = \{(x, z) \in X \times X \mid \text{ for some } y \in X, (x, y) \in V \text{ and } (y, z) \in U\}$;

(2.1.4) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$;

(2.1.5) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a *uniform space* and the members of $\mathcal{U}$ are called *entourages*.

Definition 2.2. [17] A subfamily $\mathcal{B}$ of a uniformity $\mathcal{U}$ is called a *base* for $\mathcal{U}$ if each member of $\mathcal{U}$ contains a member of $\mathcal{B}$.

Definition 2.3. [17] A subfamily $\mathcal{S}$ of a uniformity $\mathcal{U}$ is called a *sub-base* for $\mathcal{U}$ if the family of finite intersections of members of $\mathcal{S}$ is a base for $\mathcal{U}$.

The finite intersection of the members of a sub-base generates a base. A uniformity is obtained by taking the collection of supersets of the members of its base.

Theorem 2.4. [17] *A non-empty family $\mathcal{U}$ of subsets of $X \times X$ is a base for some uniformity on X if and only if conditions* (2.1.1)–(2.1.4) *defined above hold.*

Theorem 2.5. [17] *A non-empty family $\mathcal{U}$ of subsets of $X \times X$ is a sub-base for some uniformity on X if and only if conditions* (2.1.1)–(2.1.3) *defined above hold.*

In particular, the union of any collection of uniformities on X forms a sub-base for a uniformity for X.

Definition 2.6. [17] Let $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ be two uniform spaces. A mapping $f : X \to Y$ is called *uniformly continuous* if for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $f_2[U] \subset V$ (where $f_2 : X \times X \to Y \times Y$ is a map corresponding to f defined as $f_2(x, x') = (f(x), f(x'))$ for $(x, x') \in X \times X$), that is, $f(U_1) \times f(U_2) \subseteq V$, where $U = U_1 \times U_2$.

The following concepts and definitions were introduced in [1]:

The collection of all uniformly continuous functions from X to Y is denoted by $UC(X, Y)$. Let $\mathcal{A}$ be a uniformity on $UC(X, Y)$. Then the pair $(UC(X, Y), \mathcal{A})$ is called a *uniform space over uniformly continuous mappings* or *uniform space over uniform continuity*.

Definition 2.7. Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces and let $(X, \mathcal{W})$ be another uniform space. Then for a map $g : X \times Y \to Z$, we define $g^* : X \to UC(Y, Z)$ by $g^*(x)(y) = g(x, y)$.

The mappings g and g^* related in this way are called *associated maps*.

Definition 2.8. [1] Let $(Y, \mathcal{U})$ and $(Z, \mathcal{V})$ be two uniform spaces. A uniformity $\mathcal{A}$ on $UC(Y, Z)$ is called
(1) *admissible* if for each uniform space $(X, \mathcal{W})$, uniform continuity of $g^* : X \to UC(Y, Z)$ implies uniform continuity of the associated map $g : X \times Y \to Z$;
(2) *splitting* if for each uniform space $(X, \mathcal{W})$, uniform continuity of $g : X \times Y \to Z$ implies uniform continuity of $g^* : X \to UC(Y, Z)$, where g^* is the associated map of g.

3 Main results

3.1 Dual uniformity concerning $UC(Y, Z)$

In this section, we introduce the concept of dual uniform spaces for the uniform spaces over $UC(Y, Z)$.

Definition 3.1. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then we define:

$$\mathcal{U}_Z(Y) = \{f_2^{-1}(U) \mid f \in UC(Y, Z), U \in \mathcal{U}\}.$$

Definition 3.2. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $UC(Y, Z)$ be the class of all uniformly continuous mappings from Y to Z. Then for subsets $\mathbb{H} \subset \mathcal{U}_Z(Y) \times \mathcal{U}_Z(Y)$, $\mathcal{H} \subset UC(Y, Z) \times UC(Y, Z)$, and $U \in \mathcal{U}$, we define:

$$\mathbb{H}_U = \{(f, g) \mid (f_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{H}, f, g \in UC(Y, Z)\},$$
$$\mathcal{H}_U = \{(f_2^{-1}(U), g_2^{-1}(U)) \mid (f, g) \in \mathcal{H}\}.$$

Definition 3.3. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $\mathfrak{U}$ be a sub-base for a uniformity on $\mathcal{U}_Z(Y)$. Then we define:

$$\mathcal{S}(\mathfrak{U}) = \{\mathbb{H}_U \mid \mathbb{H} \in \mathfrak{U}, U \in \mathcal{U}\}.$$

Theorem 3.4. *$\mathcal{S}(\mathfrak{U})$ is a sub-base for a uniformity on $UC(Y, Z)$.*

Proof. By Theorem 2.5, it is enough to show that $\mathcal{S}(\mathfrak{U})$ satisfies conditions (2.1.1)–(2.1.3).
(1) Let $\mathbb{H}_U \in \mathcal{S}(\mathfrak{U})$ and $f \in UC(Y, Z)$. Since $\mathbb{H} \in \mathfrak{U}$ and $\mathfrak{U}$ is a sub-base for a uniformity on $\mathcal{U}_Z(Y)$, we have $(f_2^{-1}(U), f_2^{-1}(U)) \in \mathbb{H}$ for all $f \in UC(Y, Z)$ and $U \in \mathcal{U}$. Thus, we have $(f, f) \in \mathbb{H}_U$ for all $f \in UC(Y, Z)$. Hence, $\Delta \subset \mathbb{H}_U$.
(2) Let $\mathbb{H}_U \in \mathcal{S}(\mathfrak{U})$ and $(f, g) \in \mathbb{H}_U$. Then we have $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{H}$. Since $\mathbb{H} \in \mathfrak{U}$, there exists $\mathbb{H}^{-1} \in \mathfrak{U}$. Thus, we have $(g_2^{-1}(U), f_2^{-1}(U)) \in \mathbb{H}^{-1}$. Hence, we have $(g, f) \in \mathbb{H}_U^{-1} \in \mathcal{S}(\mathfrak{U})$.
(3) Let $\mathbb{H}_U \in \mathcal{S}(\mathfrak{U})$. Since $\mathbb{H} \in \mathfrak{U}$ and $\mathfrak{U}$ is a sub-base for a uniformity on $\mathcal{U}_Z(Y)$, there exists an entourage $\mathbb{G} \in \mathfrak{U}$ such that $\mathbb{G} \circ \mathbb{G} \subset \mathbb{H}$. Now, we claim that $\mathbb{G}_U \circ \mathbb{G}_U \subset \mathbb{H}_U$, where $\mathbb{G}_U \in \mathcal{S}(\mathfrak{U})$. Let $(f, g) \in \mathbb{G}_U \circ \mathbb{G}_U$. Then there exists $h \in UC(Y, Z)$ such that $(f, h) \in \mathbb{G}_U$ and $(h, g) \in \mathbb{G}_U$. Then we have $(f_2^{-1}(U), h_2^{-1}(U)) \in \mathbb{G}$

and $(h_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{G}$ and so $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{G} \circ \mathbb{G}$. Since $\mathbb{G} \circ \mathbb{G} \subset \mathbb{H}$, we have $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{H}$, that is, $(f, g) \in \mathbb{H}_U$. Hence, we have $\mathbb{G}_U \circ \mathbb{G}_U \subset \mathbb{H}_U$.

Therefore, $\mathcal{S}(\mathfrak{U})$ forms a sub-base of a uniformity over $UC(Y, Z)$. □

Uniformity generated by the sub-base $\mathcal{S}(\mathfrak{U})$ is called the *dual uniformity* to $\mathfrak{U}$ and it is denoted by $\mathfrak{V}(\mathfrak{U})$. Similarly, we define:

Definition 3.5. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $\mathfrak{V}$ be a sub-base for a uniformity on $UC(Y, Z)$. Then we define:

$$\mathcal{S}(\mathfrak{V}) = \{\mathcal{H}_U \mid \mathcal{H} \in \mathfrak{V}, U \in \mathcal{U}\}.$$

Theorem 3.6. *Let $\mathfrak{V}$ be a sub-base for a uniformity on $UC(Y, Z)$. Then $\mathcal{S}(\mathfrak{V})$ is a sub-base for a uniformity on $\mathcal{U}_Z(Y)$.*

Proof. By Theorem 2.5, it is enough to show that $\mathcal{S}(\mathfrak{V})$ satisfies conditions (2.1.1)–(2.1.3).

(1) Let $\mathcal{H}_U \in \mathcal{S}(\mathfrak{V})$, and $f_2^{-1}(U) \in \mathcal{U}_Z(Y)$ so that $f \in UC(Y, Z)$. Since $\mathcal{H} \in \mathfrak{V}$ and $\mathfrak{V}$ is a sub-base for a uniformity on $UC(Y, Z)$, we have $(f, f) \in \mathcal{H}$. Thus, $(f_2^{-1}(U), f_2^{-1}(U)) \in \mathcal{H}_U$ for all $f_2^{-1}(U) \in \mathcal{U}_Z(Y)$ and (2.1.1) holds.
(2) Let $\mathcal{H}_U \in \mathcal{S}(\mathfrak{V})$ and $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathcal{H}_U$. Then we have $(f, g) \in \mathcal{H}$. Since $\mathcal{H} \in \mathfrak{V}$, there exists $\mathcal{H}^{-1} \in \mathfrak{V}$. Thus, we have $(g, f) \in \mathcal{H}^{-1}$. Hence, we have $(g_2^{-1}(U), f_2^{-1}(U)) \in \mathcal{H}_U^{-1} \in \mathcal{S}(\mathfrak{V})$.
(3) Let $\mathcal{H}_U \in \mathcal{S}(\mathfrak{V})$. Since $\mathcal{H} \in \mathfrak{V}$ and $\mathfrak{V}$ is a sub-base for a uniformity on $UC(Y, Z)$, there exists an entourage $\mathcal{A} \in \mathfrak{V}$ such that $\mathcal{A} \circ \mathcal{A} \subset \mathcal{H}$. Now, we claim that $\mathcal{A}_U \circ \mathcal{A}_U \subset \mathcal{H}_U$, where $\mathcal{A}_U \in \mathcal{S}(\mathfrak{V})$. Let $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathcal{A}_U \circ \mathcal{A}_U$. Then there exists $h \in UC(Y, Z)$ such that $(f_2^{-1}(U), h_2^{-1}(U)) \in \mathcal{A}_U$ and $(h_2^{-1}(U), g_2^{-1}(U)) \in \mathcal{A}_U$. Then we have $(f, h) \in \mathcal{A}$ and $(h, g) \in \mathcal{A}$ and so $(f, g) \in \mathcal{A} \circ \mathcal{A}$. Since $\mathcal{A} \circ \mathcal{A} \subset \mathcal{H}$, we have $(f, g) \in \mathcal{H}$, that is, $(f_2^{-1}(U), g_2^{-1}(U)) \in \mathcal{H}_U$. Hence, we have $\mathcal{A}_U \circ \mathcal{A}_U \subset \mathcal{H}_U$.

Therefore, $\mathcal{S}(\mathfrak{V})$ forms a sub-base of a uniformity over $\mathcal{U}_Z(Y)$. □

Uniformity generated by the sub-base $\mathcal{S}(\mathfrak{V})$ is called *dual uniformity* to $\mathfrak{V}$ and is denoted by $\mathfrak{U}(\mathfrak{V})$. We explain the above with the help of the following examples:

Example 3.7. Let $Y = \mathbb{R}$ be the set of all real numbers. Then, consider the family of subsets of $\mathbb{R} \times \mathbb{R}$

$$U_\varepsilon = \{(x, y) \mid |x - y| < \varepsilon\}$$

for $\varepsilon > 0$. The uniform structure generated by the subsets U_ε for $\varepsilon > 0$ is called the *Euclidean uniformity* of $\mathbb{R}$. We say, a subset $D \subseteq \mathbb{R} \times \mathbb{R}$ is an entourage if for some $\varepsilon > 0$, we have $U_\varepsilon \subseteq D$.

Let $Z = \mathbb{Z}$ be the set of integers. Then the *p-adic uniform* structure on $\mathbb{Z}$, for a given prime number p, is the uniformity $\mathcal{U}$ generated by the subsets $\mathbb{Z}_n$ of $\mathbb{Z} \times \mathbb{Z}$, for $n = 1, 2, 3, \ldots$, where $\mathbb{Z}_n$ is defined as follows:

$$\mathbb{Z}_n = \{(k, m) \mid k \equiv m \bmod p^n\}.$$

Let $Y = \mathbb{R}$ be the set of real numbers with Euclidean uniformity $\mathcal{V}$ and $Z = \mathbb{Z}$ be the set of integers with p-adic uniform structures $\mathcal{U}$. Let $UC(\mathbb{R}, \mathbb{Z})$ be the collection of all the uniformly continuous functions from the uniform space Y to Z. Consider the point-entourage uniformity for $UC(\mathbb{R}, \mathbb{Z})$ defined in [1], having a sub-base defined as:

$$\mathcal{S}_{p,\mathcal{U}} = \{(x, \mathbb{Z}_n) \mid x \in \mathbb{R}, \mathbb{Z}_n \in \mathcal{U}\},$$

where

$$(x, \mathbb{Z}_n) = \{(f, g) \in UC(Y, Z) \times UC(Y, Z) \mid (f(x), g(x)) \in \mathbb{Z}_n\}.$$

Here, structure of the point-entourage uniformity is the collection of all the pair (f, g) of uniformly continuous functions from $\mathbb{R}$ to $\mathbb{Z}$ such that $f(x) - g(x)$ is divisible by p^n for some $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now, we define the dual of the point-entourage uniformity as follows:

Consider

$$((x, \mathbb{Z}_n), \mathbb{Z}_m) = \{(f_2^{-1}(\mathbb{Z}_m), g_2^{-1}(\mathbb{Z}_m)) \mid (f, g) \in (x, \mathbb{Z}_n)\},$$

for some $n, m \in \mathbb{N}$.

Let $\mathfrak{V}$ denote a sub-base for the point-entourage uniformity defined as above, on $UC(Y, Z)$. Then, consider

$$\mathcal{S}(\mathfrak{V}) = \{((x, \mathbb{Z}_n), \mathbb{Z}_m) \mid \text{for some } n, m \in \mathbb{N} \quad \text{and} \quad x \in \mathbb{R}\}.$$

It can be easily verified that $\mathcal{S}(\mathfrak{V})$ satisfies the first three conditions of Definition 2.1. Thus, $\mathcal{S}(\mathfrak{V})$ forms a sub-base for the dual of the point-entourage uniformity.

Example 3.8. Let $Y = \mathbb{R}$ be the set of real numbers with Euclidean uniformity $\mathcal{V}$ and $Z = \mathbb{Z}$ be the set of integers with p-adic uniform structures $\mathcal{U}$. Let $UC(\mathbb{R}, \mathbb{Z})$ be the collection of all the uniform continuous functions from the uniform space Y to Z.

Now, consider the entourage-entourage uniformity for $UC(\mathbb{R}, \mathbb{Z})$ defined in [1] having a sub-base defined as:

$$\mathcal{S}_{\mathcal{V},\mathcal{U}} = \{(U_\varepsilon, \mathbb{Z}_n) \mid U_\varepsilon \in \mathcal{V}, \mathbb{Z}_n \in \mathcal{U}\},$$

where

$$(U_\varepsilon, \mathbb{Z}_n) = \{(f, g) \in UC(Y, Z) \times UC(Y, Z) \mid (f(U_1), g(U_2)) \subseteq \mathbb{Z}_n\} \bigcup \{(f, f) \mid f \in UC(Y, Z)\},$$

where $U_\varepsilon = U_1 \times U_2$.

Now, we define the dual of this entourage-entourage uniformity as follows.

Consider

$$((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m) = \{(f_2^{-1}(\mathbb{Z}_m), g_2^{-1}(\mathbb{Z}_m)) \mid (f, g) \in (U_\varepsilon, \mathbb{Z}_n)\},$$

for some $\varepsilon > 0$, and $n, m \in \mathbb{N}$.

Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers, respectively. Let $\mathfrak{V}_1$ be a sub-base for the entourage-entourage uniformity defined above on $UC(Y, Z)$.

Then, consider

$$\mathcal{S}(\mathfrak{V}_1) = \{((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m) \mid \text{for some } n, m \in \mathbb{N} \quad \text{and} \quad \varepsilon > 0\}.$$

It can be easily verified that $\mathcal{S}(\mathfrak{V}_1)$ satisfies the first three conditions of Definition 2.1. Thus, $\mathcal{S}(\mathfrak{V}_1)$ forms a sub-base for the dual of the entourage–entourage uniformity.

3.2 Duality theorems

Now we introduce the notion of admissibility and splittingness on $\mathcal{U}_Z(Y)$ and investigate them between a uniformity on $UC(Y, Z)$ and its dual.

Definition 3.9. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $(X, \mathcal{W})$ be another uniform space. Let $g : X \times Y \to Z$ and $g^* : X \to UC(Y, Z)$ be two associated maps. Then we define a map $\overline{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ as $\overline{g}(x, U) = [g^*(x)]_2^{-1}(U)$, for every $x \in X$ and $U \in \mathcal{U}$.

Definition 3.10. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $(X, \mathcal{W})$ be another uniform space. A map $M : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is called *uniformly continuous with respect to the first variable* if the map

$M_U : X \to \mathcal{U}_Z(Y)$ defined by $M_U(x) = M(x, U)$ is uniformly continuous for every $x \in X$ and for some fixed $U \in \mathcal{U}$.

Now we define the admissibility and splittingness of the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$.

Definition 3.11. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. The uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$ is
(1) *admissible* if for any uniform space $(X, \mathcal{W})$ and every map $g : X \times Y \to Z$, uniform continuity with respect to the first variable of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ implies the uniform continuity of the map $g : X \times Y \to Z$;
(2) *splitting* if for any uniform space $(X, \mathcal{W})$ and every map $g : X \times Y \to Z$, uniform continuity of the map $g : X \times Y \to Z$ implies uniform continuity with respect to the first variable of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$.

In this section, we investigate how duality links the admissibility and splittingness of a uniform space $UC(Y, Z)$ and that on $\mathcal{U}_Z(Y)$.

Theorem 3.12. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $\mathfrak{B}$ be a sub-base for a uniformity $\mathfrak{U}$ on $\mathcal{U}_Z(Y)$. Then the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$ is splitting if and only if the dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ generated by $\mathfrak{U}$ is splitting.*

Proof. Let the uniformity $\mathfrak{U}$ on $\mathcal{U}_Z(Y)$ be splitting, that is for every uniform space $(X, \mathcal{W})$, uniform continuity of the map $g : X \times Y \to Z$ implies uniform continuity with respect to the first variable of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$. We have to show that its dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is splitting. That is, uniform continuity of the map $g : X \times Y \to Z$ implies the uniform continuity of its associated map $g^* : X \to UC(Y, Z)$. Let $g : X \times Y \to Z$ be uniformly continuous. Since $(\mathcal{U}_Z(Y), \mathfrak{U})$ is splitting, by definition $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to the first variable. Let $\mathbb{H}_U$ be any entourage in the dual uniformity of $UC(Y, Z)$, where $\mathbb{H}$ is an entourage of the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$. Since the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to the first variable, that is, the map $\bar{g}_U : X \to \mathcal{U}_Z(Y)$ is uniformly continuous and $\mathbb{H} \in \mathfrak{U}$, we have the set $A = \{(x, x') \in X \times X \mid (\bar{g}_U(x), \bar{g}_U(x')) \in \mathbb{H}\} \in \mathcal{W}$. We have to show that the map $g^* : X \to UC(Y, Z)$ is uniformly continuous. It is sufficient to show that there exists a set $B = \{(x, x') \in X \times X \mid (g^*(x), g^*(x')) \in \mathbb{H}_U\}$ is an entourage in $(X, \mathcal{W})$. We claim that $A = B$. Let $(x, y) \in B$, that is, $(g^*(x), g^*(y)) \in \mathbb{H}_U$. Thus we have, $([g^*(x)]_2^{-1}(U), [g^*(y)]_2^{-1}(U)) \in \mathbb{H}$ and $(\bar{g}(x, U), \bar{g}(y, U)) \in \mathbb{H}$. Hence, we have $(\bar{g}_U(x), \bar{g}_U(y)) \in \mathbb{H}$ and $B \subset A$. Similarly, let $(x, x') \in A$. Therefore, we have $(\bar{g}_U(x), \bar{g}_U(x')) \in \mathbb{H}$. Thus, $(\bar{g}(x, U), \bar{g}(x', U)) \in \mathbb{H}$, which implies $([g^*(x)]_2^{-1}(U), [g^*(x')]_2^{-1}(U)) \in \mathbb{H}$. We have $(g^*(x), g^*(x')) \in \mathbb{H}_U$. Hence, we have $A \subset B$ and thus $A = B$. Therefore, the map g^* is uniformly continuous.

Conversely, suppose $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is splitting. We show that $(\mathcal{U}_Z(Y), \mathfrak{U})$ is splitting. Let $g : X \times Y \to Z$ be uniformly continuous. We will show that $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to the first variable. Since $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is splitting, by definition $g^* : X \to UC(Y, Z)$ is uniformly continuous. Let $U \in \mathcal{U}$ and $\mathbb{H}$ be any entourage in $(\mathcal{U}_Z(Y), \mathfrak{U})$. Therefore, $\mathbb{H}_U$ is an entourage in the dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$. As the map g^* is uniformly continuous, the set $\{(x, x') \in X \times X \mid (g^*(x), g^*(x')) \in \mathfrak{V}(\mathfrak{U})\}$ is an entourage in uniform space $(X, \mathcal{W})$. By applying similar logic as in the previous part, we obtain that the $\{(x, x') \in X \times X \mid (\bar{g}_U(x), \bar{g}_U(x')) \in \mathbb{H}\}$ is a member of $(X, \mathcal{W})$. Hence, the map $\bar{g}$ is uniformly continuous with respect to the first variable. Therefore, $(\mathcal{U}_Z(Y), \mathfrak{U})$ is splitting. □

Theorem 3.13. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces, $\mathfrak{V}$ be a sub-base for a uniformity $\mathfrak{A}$ on $UC(Y, Z)$. Then the uniform space $(UC(Y, Z), \mathfrak{A})$ is splitting if and only if the dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ generated by $\mathfrak{A}$ is splitting.*

Proof. Let $(UC(Y, Z), \mathfrak{A})$ be a splitting uniform space. We have to show that its dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ is also splitting. For this, it is sufficient to show that for any uniform space $(X, \mathcal{W})$, uniform continuity of the map $g : X \times Y \to Z$ implies uniform continuity with respect to the first variable of the map

$\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$. Suppose $g : X \times Y \to Z$ is uniformly continuous. Since $(UC(Y, Z), \mathfrak{A})$ is splitting, by definition $g^* : X \to UC(Y, Z)$ is uniformly continuous. We will show that $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to the first variable. Let $\mathcal{H}_U$ be any entourage in the dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$. Then we have $\mathcal{H} \in \mathfrak{U}$ is an entourage in uniform space $(UC(Y, Z), \mathfrak{A})$. Since the map $g^* : X \to UC(Y, Z)$ is uniformly continuous, there exists an entourage $A \subset X \times X$ such that $g_2^*(A) \subset \mathcal{H}$. Consider $(x, y) \in A$. Therefore, we have $(g^*(x), g^*(y)) \in \mathcal{H}$. Since $U \in \mathcal{U}$, we have $([g^*(x)]_2^{-1}(U), [g^*(y)]_2^{-1}(U)) \in \mathcal{H}_U$, which implies $(\bar{g}(x, U), \bar{g}(y, U)) \in \mathcal{H}_U$, which further implies $(\bar{g}_U(x), \bar{g}_U(y)) \in \mathcal{H}_U$. Hence, we have $[\bar{g}_U]_2(A) \subset \mathcal{H}_U$. Therefore, the map $\bar{g}$ is uniformly continuous with respect to the first variable. Hence, the result.

Conversely, suppose the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ is splitting. We show that the space $(UC(Y, Z), \mathfrak{A})$ is also splitting. Let $g : X \times Y \to Z$ be uniformly continuous. Since $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ is splitting, by definition the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to the first variable. We have to show that the map $g^* : X \to UC(Y, Z)$ is also uniformly continuous. Let $\mathcal{H} \in \mathfrak{A}$ be any entourage in $(UC(Y, Z), \mathfrak{A})$. For a fixed $U \in \mathcal{U}$, $\mathcal{H}_U$ is an entourage in the dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$. For this fixed $U \in \mathcal{U}$, the map $\bar{g}_U$ is uniformly continuous and since $\mathcal{H}_U$ is an arbitrary entourage, there exists an entourage $A \in \mathcal{W}$ of $(X, \mathcal{W})$ such that $[\bar{g}_U]_2(A) \subset \mathcal{H}_U$. Consider $(x, y) \in A$. We have $(\bar{g}_U(x), \bar{g}_U(y)) \in \mathcal{H}_U$, that is, $(\bar{g}(x, U), \bar{g}(y, U)) \in \mathcal{H}_U$. Hence, we have $([g^*(x)]_2^{-1}(U), [g^*(y)]_2^{-1}(U)) \in \mathcal{H}_U$, which implies $(g^*(x), g^*(y)) \in \mathcal{H}$ for all $(x, y) \in A$. Thus, we have $g_2^*(A) \subset \mathcal{H}$. Hence, the map g^* is uniformly continuous and the space $(UC(Y, Z), \mathfrak{A})$ is splitting. □

Now, we illustrate the above results with the help of the following result. Here, we prove that the dual of the point-entourage uniformity defined in Example 3.7 is splitting.

Proposition 3.14. *Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers, respectively. Let $\mathfrak{U}(\mathfrak{V})$ be the uniformity defined by the sub-base $\mathcal{S}(\mathfrak{V})$ on $\mathcal{U}_{\mathbb{Z}}(\mathbb{R})$. Then the space $(\mathcal{U}_{\mathbb{Z}}(\mathbb{R}), \mathfrak{U}(\mathfrak{V}))$ is splitting.*

Proof. Let $(X, \mathcal{W})$ be any uniform space. Since the point-entourage uniformity is splitting [1], it is sufficient to show that the uniform continuity of the map $g^* : X \to UC(\mathbb{R}, \mathbb{Z})$ implies uniform continuity of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_{\mathbb{Z}}(\mathbb{R})$.

Let for some $x \in \mathbb{R}$ and for some given $n, m \in \mathbb{N}$, $((x, \mathbb{Z}_n), \mathbb{Z}_m)$ be any entourage in $\mathcal{U}_{\mathbb{Z}}(\mathbb{R})$. Now, $(x, \mathbb{Z}_n)$ is an entourage in $UC(\mathbb{R}, \mathbb{Z})$ and the map g^* is uniformly continuous. Therefore, there exists an entourage $A \subseteq X \times X$ such that $g_2^*(A) \subseteq (x, \mathbb{Z}_n)$. Let $(a, b) \in A$, then we have, $(g^*(a), g^*(b)) \in (x, \mathbb{Z}_n)$. Since $\mathbb{Z}_m$ belongs to $\mathcal{U}$, we have $([g^*(a)]^{-1}(\mathbb{Z}_m), [g^*(b)]^{-1}(\mathbb{Z}_m)) \in ((x, \mathbb{Z}_n), \mathbb{Z}_m)$. Thus, $(\bar{g}(a, \mathbb{Z}_m), \bar{g}(b, \mathbb{Z}_m)) \in ((x, \mathbb{Z}_n), \mathbb{Z}_m)$, which implies $(\bar{g}_{\mathbb{Z}_m}(a), \bar{g}_{\mathbb{Z}_m}(b)) \in ((x, \mathbb{Z}_n), \mathbb{Z}_m)$. As $(a, b) \in A$ was chosen arbitrarily; therefore, we have $[\bar{g}_{\mathbb{Z}_m}]_2(A) \subseteq ((x, \mathbb{Z}_n), \mathbb{Z}_m)$. Hence, the map $\bar{g}$ is uniformly continuous with respect to the first variable. Therefore, the dual of point-entourage uniformity is also splitting. □

In the next set of theorem, we investigate that how admissibility links uniform space and its dual uniform space.

Theorem 3.15. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $\mathfrak{B}$ be a sub-base for a uniformity $\mathfrak{U}$ on $\mathcal{U}_Z(Y)$. Then the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$ is admissible if and only if its dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ generated by $\mathfrak{U}$ is admissible.*

Proof. Let the uniform space $(\mathcal{U}_Z(Y), \mathfrak{U})$ be admissible. We show that its dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is also admissible, that is, for any uniform space $(X, \mathcal{W})$ and the map $g : X \times Y \to Z$, uniform continuity of the map $g^* : X \to UC(Y, Z)$ implies the uniform continuity of its associated map $g : X \times Y \to Z$. Let $g^* : X \to UC(Y, Z)$ be uniformly continuous and $\mathbb{H} \in \mathfrak{B}$ be any entourage. Therefore, for a fixed $U \in \mathcal{U}$, $\mathbb{H}_U$ is an entourage in the dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$. Since the map $g^* : X \to UC(Y, Z)$ is uniformly continuous, there exists an entourage $A \in \mathcal{W}$ such that $g_2^*(A) \subset \mathbb{H}_U$. For $(x, x') \in A \subset X \times X$, we have $(g^*(x), g^*(x')) \in \mathbb{H}_U$. Therefore, $([g^*(x)]_2^{-1}(U), [g^*(x')]_2^{-1}(U)) \in \mathbb{H}$, which implies $(\bar{g}(x, U), \bar{g}(x', U)) \in \mathbb{H}$. Hence, we have $(\bar{g}_U(x), \bar{g}_U(x')) \in \mathbb{H}$. Thus, we have $\bar{g}_U(A) \subset \mathbb{H}$ and the map $\bar{g}$ is uniformly continuous with respect to the first variable. Then since $(\mathcal{U}_Z(Y), \mathfrak{U})$ is admissible, by definition $g : X \times Y \to Z$ is uniformly continuous and hence $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is admissible.

Conversely, suppose $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is admissible. We show that $(\mathcal{U}_Z(Y), \mathfrak{U})$ is admissible. Let the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ be uniformly continuous with respect to the first variable and $\mathbb{H}_U$ be any entourage in the dual uniform space $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$. For $U \in \mathcal{U}$, the map $\bar{g}_U : X \to \mathcal{U}_Z(Y)$ is uniformly continuous and $\mathbb{H} \in \mathfrak{B}$. Therefore, there exists an entourage $A \subset X \times X$ in $(X, \mathcal{W})$ such that $[\bar{g}_U]_2(A) \subset \mathbb{H}$. For $(x, x') \in A$, we have $(\bar{g}_U(x), \bar{g}_U(x')) \in \mathbb{H}$, that is, $(\bar{g}(x, U), \bar{g}(x', U)) \in \mathbb{H}$. Hence, we have $([g^*(x)]_2^{-1}(U),$ $[g^*(x')]_2^{-1}(U)) \in \mathbb{H}$, which implies $(g^*(x), g^*(x')) \in \mathbb{H}_U$ for all $(x, x') \in A$. Hence, $g_2^*(A) \subset \mathbb{H}_U$. Therefore, the map g^* is uniformly continuous and since $(UC(Y, Z), \mathfrak{V}(\mathfrak{U}))$ is admissible, by definition $g : X \times Y \to Z$ is uniformly continuous. Hence, the result. □

Theorem 3.16. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then a uniform space $(UC(Y, Z), \mathfrak{V})$ is admissible if and only if its dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$, generated by $\mathfrak{V}$, is admissible.*

Proof. Let the uniform space $(UC(Y, Z), \mathfrak{V})$ be admissible, that is, for each uniform space $(X, \mathcal{W})$, uniform continuity of $g^* : X \to UC(Y, Z)$ implies uniform continuity of the associated map $g : X \times Y \to Z$. We show that its dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ is admissible, that is, for every map $g^* : X \to UC(Y, Z)$, uniform continuity with respect to the first variable of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ implies the uniform continuity of the map $g : X \times Y \to Z$. Let $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ be uniformly continuous with respect to the first variable and $\mathcal{H} \in \mathfrak{V}$ be any entourage. Then, for a fixed $U \in \mathcal{U}$, $\mathcal{H}_U$ is an entourage in the dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$. Since the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_Z(Y)$ is uniformly continuous with respect to first variable, we have $U \in \mathcal{U}$ and the map $\bar{g}_U : X \to \mathcal{U}_Z(Y)$ is uniformly continuous. Thus, for entourage $\mathcal{H}_U$, there exists an entourage $A \in \mathcal{W}$ such that $[\bar{g}_u]_2(A) \subset \mathcal{H}_U$. For $(x, x') \in A \subset X \times X$, we have $(\bar{g}_U(x), \bar{g}_U(x')) \in \mathcal{H}_U$ for all $(x, x') \in A$. Therefore, $([g^*(x)]_2^{-1}(U), [g^*(x')]_2^{-1}(U)) \in \mathcal{H}_U$, for all $(x, x') \in A$, which implies $(g^*(x), g^*(x')) \in \mathcal{H}$. Hence, we have $g_2^*(A) \subseteq \mathcal{H}$. Thus, the map $g^* : X \to UC(Y, Z)$ is uniformly continuous. By definition $g : X \times Y \to Z$ is uniformly continuous and hence the result.

Conversely, suppose $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$ is admissible. We show that the dual space $(UC(Y, Z), \mathfrak{V})$ is admissible. Let $g^* : X \to UC(Y, Z)$ be uniformly continuous and $\mathcal{H}_U$ be any entourage in the dual uniform space $(\mathcal{U}_Z(Y), \mathfrak{U}(\mathfrak{V}))$. Then, for the entourage $\mathcal{H} \in \mathfrak{V}$ and uniform continuity of the map $g^* : X \to UC(Y, Z)$, there exists an entourage $A \subset X \times X$ in $(X, \mathcal{W})$ such that $g_2^*(A) \subset \mathcal{H}$. For $(x, x') \in A$, we have $(g^*(x), g^*(x')) \in \mathcal{H}$, that is, $([g^*(x)]_2^{-1}(U), [g^*(x')]_2^{-1}(U)) \in \mathcal{H}_U$. Hence, we have $[\bar{g}_U]_2(A) \subseteq \mathcal{H}_U$, which implies that the map $\bar{g}$ is uniformly continuous with respect to first variable. Then by definition $g : X \times Y \to Z$ is uniformly continuous and hence the result. □

In the following, we show that the dual of entourage-entourage uniformity defined in Example 3.8 is admissible.

Proposition 3.17. *Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers, respectively. Let $\mathfrak{U}(\mathfrak{V}_1)$ be the uniformity defined by the sub-base $\mathcal{S}(\mathfrak{V}_1)$ on $\mathcal{U}_{\mathbb{Z}}(\mathbb{R})$. Then the space $(\mathcal{U}_{\mathbb{Z}}(\mathbb{R}), \mathfrak{U}(\mathfrak{V}_1))$ is admissible.*

Proof. Let $Y = \mathbb{R}$ and $Z = \mathbb{Z}$ be the set of real numbers and integers, respectively. Let $\mathfrak{U}(\mathfrak{V}_1)$ be the uniformity defined by the sub-base $\mathcal{S}(\mathfrak{V}_1)$ on $\mathcal{U}_{\mathbb{Z}}(\mathbb{R})$. We have to show that the space $(\mathcal{U}_{\mathbb{Z}}(\mathbb{R}), \mathfrak{U}(\mathfrak{V}_1))$ is admissible.

Let $(X, \mathcal{W})$ be any uniform space. Then, we have to show that, for every map $g^* : X \to UC(\mathbb{R}, \mathbb{Z})$, uniform continuity with respect to the first variable of the map $\bar{g} : X \times \mathcal{U} \to \mathcal{U}_{\mathbb{Z}}(\mathbb{R})$ implies the uniform continuity of the map $g : X \times \mathbb{R} \to \mathbb{Z}$.

Since the space $UC(\mathbb{R}, \mathbb{Z})$ is admissible under entourage–entourage uniformity [1], it is sufficient to show that the map $g^* : X \to UC(\mathbb{R}, \mathbb{Z})$ is uniformly continuous. Let for some $\varepsilon > 0$, $n \in \mathbb{N}$, $(U_\varepsilon, \mathbb{Z}_n)$ be any entourage in $UC(\mathbb{R}, \mathbb{Z})$. Then for any given fixed $\mathbb{Z}_m$, $((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m)$ is an entourage in the dual space $\mathcal{U}_{\mathbb{Z}}(\mathbb{R})$. Since the map $\bar{g} : X \to \mathcal{U}_{\mathbb{Z}}(\mathbb{R})$ is uniformly continuous with respect to the first variable, thus map $\bar{g}_{\mathbb{Z}_m} : X \to \mathcal{U}_{\mathbb{Z}}(\mathbb{R})$ is uniformly continuous. Therefore, for the entourage $((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m)$, there exists an entourage $A \in \mathcal{W}$ such that $[\bar{g}_{\mathbb{Z}_m}]_2(A) \subseteq ((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m)$. Let $(a, b) \in A \subseteq X \times X$, we have $(\bar{g}_{\mathbb{Z}_m}(a), \bar{g}_{\mathbb{Z}_m}(b)) \in ((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m)$ for all $(a, b) \in A$. Therefore, $([g^*(a)]_2^{-1}(\mathbb{Z}_m), [g^*(b)]_2^{-1}(\mathbb{Z}_m)) \in ((U_\varepsilon, \mathbb{Z}_n), \mathbb{Z}_m)$. Thus, we have $(g^*(a), g^*(b)) \in (U_\varepsilon, \mathbb{Z}_n)$. Hence, $g_2^*(A) \subseteq (U_\varepsilon, \mathbb{Z}_n)$. Thus, the map $g^* : X \to UC(\mathbb{R}, \mathbb{Z})$ is uniformly continuous. Hence, the proof. □

3.3 Mutually dual uniformities

In this section, we discuss about the mutually dual uniformities over the uniform spaces.

Lemma 3.18. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then for $\mathbb{H} \subseteq \mathcal{U}_Z(Y) \times \mathcal{U}_Z(Y)$, we have $\mathbb{H} = \bigcup\{(\mathbb{H}_U)_U \mid U \in \mathcal{U}\}$.*

Proof. Let $U \in \mathcal{U}$. Then we have $(\mathbb{H}_U)_U = \{(f_2^{-1}(U), g_2^{-1}(U)) \mid (f, g) \in \mathbb{H}_U\} = \{(f_2^{-1}(U), g_2^{-1}(U)) \mid (f_2^{-1}(U), g_2^{-1}(U)) \in \mathbb{H}\} \subseteq \mathbb{H}$. Hence, $\bigcup\{(\mathbb{H}_U)_U \mid U \in \mathcal{U}\} \subset \mathbb{H}$.

Similarly, let $V \in \mathbb{H}$. Then for some $U \in \mathcal{U}$ and $f, g \in UC(Y, Z)$, we have $V = (f_2^{-1}(U), g_2^{-1}(U))$. Since $(f_2^{-1}(U), g_2^{-1}(U)) \in (\mathbb{H}_U)_U$, we have $\mathbb{H} \subset \bigcup\{(\mathbb{H}_U)_U \mid U \in \mathcal{U}\}$. Therefore, $\mathbb{H} = \bigcup\{(\mathbb{H}_U)_U\}$. □

Lemma 3.19. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be two uniformities on the set $\mathcal{U}_Z(Y)$ such that $\mathfrak{U}_1 \subset \mathfrak{U}_2$. Then $\mathfrak{V}(\mathfrak{U}_1) \subset \mathfrak{V}(\mathfrak{U}_2)$.*

Proof. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $\mathfrak{U}_1$, $\mathfrak{U}_2$ be two uniformities on the set $\mathcal{U}_Z(Y)$ such that $\mathfrak{U}_1 \subset \mathfrak{U}_2$. We have to show that $\mathfrak{V}(\mathfrak{U}_1) \subset \mathfrak{V}(\mathfrak{U}_2)$.

Let $\mathbb{H}_U$ be any entourage in the dual uniform space $\mathfrak{V}(\mathfrak{U}_1)$. Thus, $\mathbb{H}$ is an entourage in $\mathfrak{U}_1$. Therefore, $\mathbb{H} \in \mathfrak{U}_2$ as well. Hence, $\mathbb{H}_U \in \mathfrak{V}(\mathfrak{U}_2)$. Hence, the result. □

Lemma 3.20. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $\mathfrak{V}_1$, $\mathfrak{V}_2$ be two uniformities over $UC(Y, Z)$ such that $\mathfrak{V}_1 \subset \mathfrak{V}_2$. Then $\mathfrak{U}(\mathfrak{V}_1) \subset \mathfrak{U}(\mathfrak{V}_2)$.*

Proof. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $\mathfrak{V}_1$, $\mathfrak{V}_2$ be two uniformities over $UC(Y, Z)$ such that $\mathfrak{V}_1 \subset \mathfrak{V}_2$. Then we have to show that $\mathfrak{U}(\mathfrak{V}_1) \subset \mathfrak{U}(\mathfrak{V}_2)$.

Let $\mathcal{H}_U$ be any entourage in the corresponding dual uniform space $\mathfrak{U}(\mathfrak{V}_1)$. Thus, $\mathcal{H}$ is an entourage in $\mathfrak{V}_1$. Therefore, $\mathcal{H} \in \mathfrak{V}_2$ as well. Hence, $\mathcal{H}_U \in \mathfrak{U}(\mathfrak{V}_2)$. Hence, the result. □

Theorem 3.21. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and $\mathfrak{U}$ be a uniformity over $\mathcal{U}_Z(Y)$. Then we have the following*:
(1) $\mathfrak{U} \subset \mathfrak{U}(\mathfrak{V}(\mathfrak{U})) \subset \mathfrak{U}(\mathfrak{V}(\mathfrak{U}(\mathfrak{V}(\mathfrak{U}))))\subset \ldots\ldots$
(2) $\mathfrak{V}(\mathfrak{U}) \subset \mathfrak{V}(\mathfrak{U}(\mathfrak{V}(\mathfrak{U}))) \subset \mathfrak{V}(\mathfrak{U}(\mathfrak{V}(\mathfrak{U}(\mathfrak{V}(\mathfrak{U})))))\subset \ldots\ldots$

Proof.
(1) First, we show that $\mathfrak{U} \subset \mathfrak{U}(\mathfrak{V}(\mathfrak{U}))$. For this, let $\mathbb{H} \in \mathcal{U}_Z(Y) \times \mathcal{U}_Z(Y)$. Then from Lemma 3.18, we have $\mathbb{H} = \bigcup\{((\mathbb{H}_U), U) \mid U \in \mathcal{U}\}$. Therefore, $\mathfrak{U} \subset \mathfrak{U}(\mathfrak{V}(\mathfrak{U}))$. Thus, by using Lemmas 3.19 and 3.20, we obtained the required result.
(2) Can be proved in a similar way. □

Now, we provide our main theorem of this section.

Theorem 3.22. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. There exists the greatest splitting uniformity over $\mathcal{U}_Z(Y)$.*

Proof. Let $\{\mathfrak{U}_i \mid i \in I\}$ be the set of all splitting uniformities over $\mathcal{U}_Z(Y)$. Consider $\mathfrak{U} = \vee\{\mathfrak{U}_i \mid i \in I\}$. We claim that $\mathfrak{U}$ is the greatest splitting uniformity over $\mathcal{U}_Z(Y)$. From Theorem 3.12, it is sufficient to show that its dual uniform space $(UC(Y,Z), \mathfrak{V}(\mathfrak{U}))$ is splitting.

Since $(\mathcal{U}_Z(Y), \mathfrak{U}_i)$, $i \in I$ are splitting uniform spaces, by Theorem 3.12, their dual uniform spaces $(UC(Y,Z), \mathfrak{V}(\mathfrak{U}_i))$ are splitting. Let $\sigma = \bigcup\{\mathfrak{U}_i \mid i \in I\}$. Then σ forms a sub-base for $\mathfrak{U}$. In the light of Theorem 3.4, $\mathcal{S}(\sigma)$ is a a sub-base for a uniformity on $(UC(Y,Z), \mathfrak{V}(\mathfrak{U}))$. On the other hand, we have $\mathcal{S}(\sigma) = \bigcup\{(\mathcal{S}(\mathfrak{U}_i)) \mid i \in I\}$. Since $\mathcal{S}(\mathfrak{U}_i)$ is sub-base for $\mathfrak{V}(\mathfrak{U}_i)$, we have $\mathcal{S}(\sigma)$ is a sub-base for $\vee\{\mathfrak{V}(\mathfrak{U}_i) \mid i \in I\}$, thus we have $\mathfrak{V}(\mathfrak{U}) = \vee\{\mathfrak{V}(\mathfrak{U}_i) \mid i \in I\}$. Since $\mathfrak{V}(\mathfrak{U}_i)$ is splitting for every $i \in I$, $\vee\{\mathfrak{V}(\mathfrak{U}_i) \mid i \in I\}$ is also splitting. Therefore, $\mathfrak{V}(\mathfrak{U})$ is also splitting and hence $\mathfrak{U}$ is splitting. □

Definition 3.23. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then a uniformity on $UC(Y,Z)$ (respectively $\mathcal{U}_Z(Y)$) is said to be a *family-open uniformity* if it is a dual to a uniformity on $\mathcal{U}_Z(Y)$ (respectively $UC(Y,Z)$) ;

Definition 3.24. Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces and let $\mathfrak{U}$ and $\mathfrak{V}$ be uniformities over $\mathcal{U}_Z(Y)$ and $UC(Y,Z)$, respectively. Then the pair $(\mathfrak{U}, \mathfrak{V})$ is called a *pair of mutually dual uniformities* if $\mathfrak{U} = \mathfrak{U}(\mathfrak{V})$ and $\mathfrak{V} = \mathfrak{V}(\mathfrak{U})$, respectively.

Now we provide few more results regarding greatest splitting uniformities in the light of dual uniform spaces.

Theorem 3.25. *Let $(Y, \mathcal{V})$ and $(Z, \mathcal{U})$ be two uniform spaces. Then there exists the greatest splitting family-open uniformity on $UC(Y,Z)$.*

Proof. Let $\mathfrak{U}$ be the greatest splitting uniformity on $\mathcal{U}_Z(Y)$. Now, we claim that the family-open uniformity $\mathfrak{V}(\mathfrak{U})$ is the greatest splitting family open uniformity on $UC(Y,Z)$.

Let $\mathfrak{V}' = \mathfrak{V}(\mathfrak{U}')$ be a splitting family-open uniformity on $UC(Y,Z)$. Thus from Theorem 3.12, $\mathfrak{U}'$ is a splitting uniformity on $\mathcal{U}_Z(Y)$. Thus, $\mathfrak{U}' \subset \mathfrak{U}$. By Lemma 3.19, $\mathfrak{V}(\mathfrak{U}') \subset \mathfrak{V}(\mathfrak{U})$. Thus, $\mathfrak{V}(\mathfrak{U})$ is the greatest splitting family open uniformity. □

Theorem 3.26. *Let $(\mathcal{U}_Z(Y), \mathfrak{U})$ and $(UC(Y,Z), \mathfrak{V})$ be two uniform spaces. Let also $\mathfrak{U}$ be the greatest-splitting uniformity and $\mathfrak{V}$ be the greatest family-open uniformity on $UC(Y,Z)$. Then the pair $(\mathfrak{U}, \mathfrak{V})$ is a pair of mutually dual splitting uniformities.*

Proof. Let $\mathfrak{U}$ be the greatest-splitting uniformity on $\mathcal{U}_Z(Y)$ and $\mathfrak{V}$ be the greatest family-open uniformity on $UC(Y,Z)$. Then from the last theorem, we have $\mathfrak{V}(\mathfrak{U}) = \mathfrak{V}$. In the view of Theorem 3.21, we have $\mathfrak{U} \subset \mathfrak{U}(\mathfrak{V}(\mathfrak{U})) = \mathfrak{U}(\mathfrak{V})$. Thus by Theorem 3.13, the uniformity $\mathfrak{U}(\mathfrak{V})$ is also splitting. Thus, $\mathfrak{U}(\mathfrak{V}) \subset \mathfrak{U}$. Therefore, we have $\mathfrak{U} = \mathfrak{U}(\mathfrak{V})$. Hence, the pair $(\mathfrak{U}, \mathfrak{V})$ forms a pair of mutually dual splitting uniformities. □

4 Conclusion

This article is a sequel to the authors' earlier investigations on uniformities on the space of uniformly continuous mappings between uniform spaces [1]. Here, it is shown that properties of splittingness and admissibility of uniformity on $UC(Y,Z)$ implies the same for its dual uniformity on $\mathcal{U}_Z(Y)$ and *vice versa*. The existence of the greatest splitting uniformity on $UC(Y,Z)$ and the greatest family open splitting uniformity on $\mathcal{U}_Z(Y)$ are established and their duality relations are examined. Similar studies were carried out earlier for function space topologies [4], but not for uniformities. Our investigation has shown that the same can be achieved for function space uniformities as well. However, it still remains open to investigate the relationship between a uniformity on $UC(Y,Z)$ and that of the dual of its dual on $\mathcal{U}_Z(Y)$.

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